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Fundamentals for Symplectic $\mathcal{A}$-modules

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Abstract

In his [9], [10], the first author shows that the *sheaf theoretically* based *Abstract Differential Geometry incorporates and generalizes all the classical differential geometry*. Here, we undertake to partially explore the implications of Abstract Differential Geometry to classical *symplectic geometry*. The full investigation will be presented elsewhere.

Key Words: $\mathcal{A}$-module, vector sheaf, ordered $\mathbb{R}$-algebraized space, , symplectic $\mathcal{A}$-structure, symplectic group sheaf.

Introduction

Here is an attempt of taking the theory of *Abstract Differential Geometry* (à la Mallios), [9], [10], to new horizons, such as those related to the classical *symplectic geometry*. In this endeavor, we show the *extent* to which tools provided by Abstract Differential Geometry help (re)capture in a sheaf-theoretic manner fundamental notions and results which characterize the standard *symplectic algebra*. This endeavor will pave the way to rewrite and/or recover a great deal of classical symplectic geometry, with no use at all of any notion of "differentiability" (differentiability is here understood in the sense of the standard *differential geometry* of C^∞-manifolds). As a result of all *this*, we show that *sheaf-theory* *methods* turn out to be an appropriate way for the *algebraization* of classical symplectic geometry. As pointed out by the first author in [10], [11], this algebraic approach to differential (and symplectic) geometry is of a particular interest to theoretical physicists, for it has been for long the demand and/or wish of many to "*find a purely algebraic theory for the description of reality*" (i.e. physics according to our understanding) (A. Einstein, [[6], p.166]).

Our main reference, throughout the present account, is the first author's book [9], for which the reader is requested to have handy for we have skipped some necessary *basics* of Abstract Differential Geometry.

The paper is divided into four sections. §1 concerns with $\mathcal{A}$-*tensors*: these are the counterparts, in this framework, of *classical* tensors. Some results, pertaining to the standard *multilinear algebra machinery*, are hereby provided. $\mathcal{A}$-tensors constitute a precursor to the fundamental theory of *exterior* $\mathcal{A}$-k-*forms*, which are developed in §2. The *exterior algebra sheaf* is defined as the *direct sum of sheaves of germs of exterior* $\mathcal{A}$-k-*forms on an* $\mathcal{A}$-*module* $\mathcal{E}$; this sum is endowed with the exterior product $\wedge$. In §3, we show that given a *non-zero skew-symmetric non-degenerate* $\mathcal{A}$-*morphism* ω on the *standard free* $\mathcal{A}$-*module of rank* n, defined on a topological space X, there is a *basis* $\mathcal{B}$ of $\mathcal{A}^n(X)$, relative to which the matrix of ω is

$$\begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix (the diagonal entry 1 in I is the *global identity section*). It also turns out that this result holds for *vector sheaves* as well. We further introduce in this same section the notion of *symplectic group sheaf* of an arbitrary $\mathcal{A}$-module. In §4, we deal with *characteristic polynomial, eigenvector and eigenvalue sections* of an $\mathcal{A}$-module, and prove the corresponding version of the *Cayley-Hamilton* theorem.

1 $\mathcal{A}$-Tensors

Throughout this paper, the pair $(X, \mathcal{A})$, or just $\mathcal{A}$ will denote a fixed $\mathbb{C}$-*algebraized space*, where X is a topological space and $\mathcal{A}$ a *sheaf* (over X) of *unital, commutative algebras*. For more details about algebraized spaces, see [9].

Let

$$\mathcal{E} \equiv (\mathcal{E}, \pi, X)$$

be an $\mathcal{A}$-module on X. The (complete) *presheaves of sections* of sheaves $\mathcal{A}$ and $\mathcal{E}$ are denoted by

$$\Gamma(\mathcal{A}) \equiv (\Gamma(U, \mathcal{A}), \tau_V^U) \quad \text{and} \quad \Gamma(\mathcal{E}) \equiv (\Gamma(U, \mathcal{E}), \pi_V^U),$$

respectively. Let $\mathcal{E}^*$ be the dual $\mathcal{A}$-module of $\mathcal{E}$; so

$$\mathcal{E}^* = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$$

(see [9], p. 298). It is easy (cf. [9], p. 129) to see that the correspondence that associates with every open subset U of X the $\Gamma(U, \mathcal{A})$-module

$$\otimes^p \Gamma(U, \mathcal{E}) \otimes_{\Gamma(U,\mathcal{A})} \otimes^q \Gamma(U, \mathcal{E}^*),$$

where

$$\otimes^p \Gamma(U, \mathcal{E}) := \underbrace{\Gamma(U, \mathcal{E}) \otimes_{\Gamma(U,\mathcal{A})} \ldots \otimes_{\Gamma(U,\mathcal{A})} \Gamma(U, \mathcal{E})}_{p-\text{times}},$$

and

$$\otimes^q \Gamma(U, \mathcal{E}^*) := \underbrace{\Gamma(U, \mathcal{E}^*) \otimes_{\Gamma(U,\mathcal{A})} \ldots \otimes_{\Gamma(U,\mathcal{A})} \Gamma(U, \mathcal{E}^*)}_{q-\text{times}},$$

along with the obvious restriction morphisms, provides a *presheaf of* $\Gamma(\mathcal{A})$-*modules* on X. We denote this presheaf by

$$T^p_q \Gamma(\mathcal{E}) \equiv \otimes^p \Gamma(\mathcal{E}) \otimes_{\Gamma(\mathcal{A})} \otimes^q \Gamma(\mathcal{E}^*). \tag{1}$$

Definition 1.1 Given an $\mathcal{A}$-*module* $\mathcal{E}$ on a topological space X, we denote by

$$\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^*$$

the *sheaf* generated by the presheaf $T^p_q \Gamma(\mathcal{E})$, given in (1), i.e. one has

$$\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^* := \mathbf{S}(T^p_q \Gamma(\mathcal{E})) = \mathbf{S}(\otimes^p \Gamma(\mathcal{E}) \otimes_{\Gamma(\mathcal{A})} \otimes^q \Gamma(\mathcal{E}^*)).$$

We extend the definition of $\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^*$ to the cases $p = 0$ and $q = 0$ by setting

$$\otimes^0 \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^* = \otimes^q \mathcal{E}^*,$$

and

$$\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^0 \mathcal{E}^* = \otimes^p \mathcal{E}.$$

The elements of $\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^*$ are called **$\mathcal{A}$-tensors** over $\mathcal{E}$, and are said to be *contravariant of order* p and *covariant of order* q; or simply, of *type* (p, q).

The next lemma shows the analog of a classical result of (ordinary) tensors of type (p, q). Before we examine the result, let us recall that given an algebra sheaf $\mathcal{A}$ on a topological X, by a *vector sheaf* $\mathcal{E}$, *of a rank* n, on X, we mean a *locally free* $\mathcal{A}$-*module of rank* n on X; that is for every $x \in X$, there exists an open neighborhood U of $x \in X$ such that one has

$$\mathcal{E}|_U = \mathcal{A}^n|_U, \tag{2}$$

with the equality sign being actually an $\mathcal{A}|_U$-isomorphism of the $\mathcal{A}|_U$-modules $\mathcal{E}|_U$ and $\mathcal{A}^n|_U$. Any open set U in X for which (2) holds is called a local gauge of $\mathcal{E}$.

Lemma 1.1 If $\mathcal{E}$ is a *vector sheaf* on a topological space X, then

$$\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^* = \mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^* \otimes_{\mathcal{A}} \otimes^q \mathcal{E}, \mathcal{A}).$$

Proof. This is an easy verification. In fact, based on Mallios[[9], Comment (5.27), p. 132, Theorems 5.1, p. 299, 6.1, p. 302, 6.2, p. 304, and Corollary 6.2, p. 305], one has

$$\begin{aligned}
\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^* &= (\otimes^p \mathcal{E})^{**} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^* \\
&= \mathcal{H}om_{\mathcal{A}}((\otimes^p \mathcal{E})^*, \otimes^q \mathcal{E}^*) \\
&= \mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^*, (\otimes^q \mathcal{E})^*) \\
&= \mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^*, \mathcal{H}om_{\mathcal{A}}(\otimes^q \mathcal{E}, \mathcal{A})) \\
&= \mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^* \otimes_{\mathcal{A}} \otimes^q \mathcal{E}, \mathcal{A}).
\end{aligned}$$

□

A corollary that one can derive from Lemma 1.1 requires the following definitions.

Definition 1.2 Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$, $n \in \mathbb{N}$, and $\mathcal{F}$ be $\mathcal{A}$-*modules* on the same topological space X. The $\mathcal{A}$-morphism $\varphi : \mathcal{E}_1 \times \ldots \times \mathcal{E}_n \longrightarrow \mathcal{F}$ is called an $\mathcal{A}$-**multilinear morphism** if, for all *open subset* $U \subseteq X$,

$$\varphi_U : \Gamma(U, \mathcal{E}_1) \times_{\Gamma(U,\mathcal{A})} \ldots \times_{\Gamma(U,\mathcal{A})} \Gamma(U, \mathcal{E}_n) \longrightarrow \Gamma(U, \mathcal{F})$$

is a $\Gamma(U, \mathcal{A})$-*multilinear morphism* for the $\Gamma(U, \mathcal{A})$-*modules concerned*.

We are now set to *generalize the functor* $\mathcal{H}om_{\mathcal{A}}$ ($\mathcal{H}om_{\mathcal{A}}$ is a bifunctor $\mathcal{A}$-$\mathcal{M}od_X \longrightarrow \mathcal{A}$-$\mathcal{M}od_X$, where $\mathcal{A}$-$\mathcal{M}od_X$ is the category of $\mathcal{A}$-modules on a topological space X, (see [9], p. 133)), to *the functor* $\mathcal{L}^n_{\mathcal{A}}$, $n \in \mathbb{N}$, which we define below.

In effect, suppose that we are given $\mathcal{A}$-modules $\mathcal{E}_i$, $i = 1, \ldots, n$, and $\mathcal{F}$ on the same topological space X. For any open set U in X, let

$$\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{E}_1|_U \times \ldots \times \mathcal{E}_n|_U, \mathcal{F}|_U) \equiv L^n_{\mathcal{A}|_U}(\mathcal{E}_1|_U, \ldots, \mathcal{E}_n|_U; \mathcal{F}|_U) \tag{3}$$

be the set of $\mathcal{A}|_U$-*n-linear morphisms* of the $\mathcal{A}|_U$-module $\mathcal{E}_1 \times \ldots \times \mathcal{E}_n|_U$ into the $\mathcal{A}|_U$-module $\mathcal{F}|_U$.

Lemma 1.2 The set, in (3), is a *module* over $\mathcal{A}(U)$; hence, in particular, a $\mathbb{C}$-*vector space.*

Proof. The proof is similar to the proof of Statement 6.1, p. 133, [9]. □

On the other hand, it is readily verified that, given $\mathcal{A}$-modules $\mathcal{E}_1, \dots, \mathcal{E}_n$, and $\mathcal{F}$ on X as above, the correspondence

$$U \longmapsto L^n_{\mathcal{A}|_U}(\mathcal{E}_1|_U, \dots, \mathcal{E}_n|_U; \mathcal{F}|_U), \tag{4}$$

where U runs over the open subsets of X, along with the obvious *restriction maps* yields a *complete presheaf of* $\mathcal{A}$*-modules* on X.

Thus, we have

Definition 1.3 Let $\mathcal{E}_1, \dots, \mathcal{E}_n$ and $\mathcal{F}$ be $\mathcal{A}$-*modules* on a topological space X. By the **sheaf of germs of $\mathcal{A}$-n-linear morphisms** of $\mathcal{E}_1 \times \dots \times \mathcal{E}_n$ in $\mathcal{F}$, we mean the *sheaf*, on X, generated by the (complete) *presheaf*, defined by (4). We denote the induced sheaf by

$$\mathcal{L}^n_{\mathcal{A}}(\mathcal{E}_1, \dots, \mathcal{E}_n; \mathcal{F}).$$

We may now state

Corollary 1.1 Let $\mathcal{E}$ be an $\mathcal{A}$-*module* on X. Then,

$$\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^* = \mathcal{L}^{p+q}_{\mathcal{A}}(\underbrace{\mathcal{E}^*, \dots, \mathcal{E}^*}_{p\text{-times}}, \underbrace{\mathcal{E}, \dots, \mathcal{E}}_{q\text{-times}}; \mathcal{A}) \equiv \mathfrak{T}^p_q(\mathcal{E}),$$

where $\mathcal{L}^{p+q}_{\mathcal{A}}(\mathcal{E}^*, \dots, \mathcal{E}^*, \mathcal{E}, \dots, \mathcal{E}; \mathcal{A})$ is the $\mathcal{A}$-*module of* $\mathcal{A}$-$(p+q)$-*linear morphisms.*

Proof. Using Lemma 1.1 and Mallios[[9], Lemma 5.1, p. 132 and Definition 6.1, p. 134], one has, for every open set U in X,

$$\begin{aligned}\mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^* \otimes_{\mathcal{A}} \otimes^q \mathcal{E}, \mathcal{A})(U) \quad &= \mathrm{Hom}_{\mathcal{A}|_U}((\otimes^p \mathcal{E}^* \otimes_{\mathcal{A}} \otimes^q \mathcal{E})|_U, \mathcal{A}|_U) \\ &= \mathrm{Hom}_{\mathcal{A}|_U}(\otimes^p (\mathcal{E}^*|_U) \otimes_{\mathcal{A}|_U} \otimes^q (\mathcal{E}|_U), \mathcal{A}|_U) \\ &= L^{p+q}_{\mathcal{A}|_U}(\mathcal{E}^*|_U, \dots, \mathcal{E}^*|_U, \mathcal{E}|_U, \dots, \mathcal{E}|_U; \mathcal{A}|_U).\end{aligned}$$

Therefore,

$$\mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^* \otimes_{\mathcal{A}} \otimes^q \mathcal{E}, \mathcal{A}) = \mathcal{L}^{p+q}_{\mathcal{A}}(\mathcal{E}^*, \dots, \mathcal{E}^*, \mathcal{E}, \dots, \mathcal{E}; \mathcal{A}).$$

□

So, we come now to the following definition.

Definition 1.4 Let $\mathcal{E}$ be an $\mathcal{A}$-*module* on a topological space X, and let $\mathbf{t}_1 \in \mathfrak{T}^{i_1}_{j_1}(\mathcal{E})$ and $\mathbf{t}_2 \in \mathfrak{T}^{i_2}_{j_2}(\mathcal{E})$. The $\mathcal{A}$-**tensor product** of $\mathbf{t}_1$ and $\mathbf{t}_2$ is the $\mathcal{A}$-*tensor* $\mathbf{t}_1 \otimes \mathbf{t}_2 \in \mathfrak{T}^{i_1+i_2}_{j_1+j_2}(\mathcal{E})$, defined by

$$\begin{aligned}&\mathbf{t}_1 \otimes \mathbf{t}_2(s_1, \ldots, s_{i_1}, t_1, \ldots, t_{i_2}, u_1, \ldots, u_{j_1}, v_1, \ldots, v_{j_2})\\ &\quad = \mathbf{t}_1(s_1, \ldots, s_{i_1}, u_1, \ldots, u_{j_1})\mathbf{t}_2(t_1, \ldots, t_{i_2}, v_1, \ldots, v_{j_2})\end{aligned}$$

where $s_\alpha, t_\alpha \in \mathcal{E}^*(U)$ and $u_\beta, v_\beta \in \mathcal{E}(U)$, and where, for all $k = 1, 2$, the $\mathcal{A}-$tensor $\mathbf{t}_k$, viewed as a map on *sections* of the $\mathcal{A}$-modules of

$$\underbrace{\mathcal{E}^* \times_{\mathcal{A}} \ldots \times_{\mathcal{A}} \mathcal{E}^*}_{i_k} \times_{\mathcal{A}} \underbrace{\mathcal{E} \times_{\mathcal{A}} \ldots \times_{\mathcal{A}} \mathcal{E}}_{j_k} \quad \text{and } \mathcal{A},$$

is the $\mathcal{A}(U)$-$(i_k + j_k)$-linear morphism

$$\mathcal{E}^*(U) \times_{\mathcal{A}(U)} \ldots \times_{\mathcal{A}(U)} \mathcal{E}^*(U) \times_{\mathcal{A}(U)} \mathcal{E}(U) \times_{\mathcal{A}(U)} \ldots \times_{\mathcal{A}(U)} \mathcal{E}(U) \longrightarrow \mathcal{A}(U),$$

for all *open* subset $U \subseteq X$.

One can be assured that the standard *multilinear algebra machinery* can be appropriately reformulated within the present setting. For example, let us look at

Proposition 1.1 Let $\mathcal{E}$ be a *vector sheaf* of rank n on a topological space X. Then, for all $k, l \in \mathbb{N}$, the $\mathcal{A}$-module $\mathfrak{T}^k_l(\mathcal{E})$ is a vector sheaf of rank n^{k+l}.

Proof. The proof is based on relations (5.25), p. 132, (6.23), p. 137, and Statement (5.19), p. 301, all found in [9].

Let $x \in X$ and U an *open* neighborhood of x such that $\mathcal{E}|_U = \mathcal{A}^n|_U = \mathcal{E}^*|_U$. Then, for all open subset $V \subseteq U$, one has

$$\begin{aligned}(\mathfrak{T}^k_l(\mathcal{E})|_U)(V) &= (\mathcal{L}^{k+l}_{\mathcal{A}}(\underbrace{\mathcal{E}^*, \ldots, \mathcal{E}^*}_{k}, \underbrace{\mathcal{E}, \ldots, \mathcal{E}}_{l}; \mathcal{A})|_U)(V)\\
&= \mathcal{L}^{k+l}_{\mathcal{A}}(\mathcal{E}^*, \ldots, \mathcal{E}^*, \mathcal{E}, \ldots, \mathcal{E}; \mathcal{A})(V)\\
&= L^{k+l}_{\mathcal{A}|V}(\mathcal{E}^*|_V, \ldots, \mathcal{E}^*|_V, \mathcal{E}|_V, \ldots, \mathcal{E}|_V; \mathcal{A}|_V)\\
&= \mathrm{Hom}_{\mathcal{A}|_V}(\mathcal{E}^* \times \ldots \times \mathcal{E}^*|_V \times \mathcal{E}|_V \times \ldots \times \mathcal{E}|_V, \mathcal{A}|_V)\\
&= \mathrm{Hom}_{\mathcal{A}|_V}(\mathcal{A}^n|_V \times \ldots \times \mathcal{A}^n|_V \times \mathcal{A}^n|_V \times \ldots \times \mathcal{A}^n|_V, \mathcal{A}|_V)\\
&= \mathrm{Hom}_{\mathcal{A}|_V}((\mathcal{A}^n \otimes \ldots \otimes \mathcal{A}^n)|_V, \mathcal{A}|_V)\\
&= \mathcal{H}om_{\mathcal{A}}(\mathcal{A}^n \otimes \ldots \otimes \mathcal{A}^n, \mathcal{A})(V)\\
&= \mathcal{H}om_{\mathcal{A}}(\mathcal{A}^{n^{k+l}}, \mathcal{A})(V)\\
&= (\mathcal{A}^{n^{k+l}}|_U)(V),\end{aligned}$$

which shows that

$$\mathfrak{T}^k_l(\mathcal{E})|_U = \mathcal{A}^{n^{k+l}}|_U,$$

that is the $\mathcal{A}$-module $\mathfrak{T}^k_l(\mathcal{E})$ is a vector sheaf of rank n^{k+l}, as desired. □

Corollary 1.2 Let $\mathcal{E}$ be a *vector sheaf* of rank n on a topological space X, and $\{s_i\}_{1\leq i\leq n}$ a *basis* of the $\mathcal{A}(U)$-module $\Gamma(U,\mathcal{E})$, with U an open subset of X such that $\mathcal{E}|_U = \mathcal{A}^n|_U$. Then, for all $k, l \in \mathbb{N}$, a basis of the $\mathcal{A}(U)$-*module* $\Gamma(U, \mathfrak{T}^k_l(\mathcal{E}))$ is given by

$$\{s_{i_1} \otimes \ldots \otimes s_{i_k} \otimes s^{*j_1} \otimes \ldots \otimes s^{*j_l} |\ i_p, j_p = 1, \ldots, n\},$$

where $\{s^{*j}\}_{1\leq j\leq n}$ is the dual basis of $\{s_i\}_{1\leq i\leq n}$.

Proof. The proof is similar to the proof of Proposition 1.7.2, p. 53, [1]. □

We close this section with the following important definition, which will be of use in the sequel. (See also [[9]: p. 301, (5.22)- (5.24)].)

Definition 1.5 Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-*modules* on a topological space X. For $\varphi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})$, in keeping with the classical notation, see [3], p. 234, or [5], p. 68, we define the **transpose of** φ by

$$^t\varphi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{F}^*,\mathcal{E}^*)$$

such that

$$(^t\varphi)(u) := u \circ \varphi, \quad u \in \mathcal{F}^*(U) \tag{5}$$

with U *open* in X, i.e., in other words,

$$^t\varphi(u)(v) = u(\varphi(v)), \quad v \in \mathcal{E}(U).$$

(In (5), we have used the $^t\varphi$-*corresponding map on sections* of the $\mathcal{A}$-modules $\mathcal{E}^*$ and $\mathcal{F}^*$.)

2 Exterior $\mathcal{A}$-k-forms

As has been the case so far, we assume in this section as well that the triple $(\mathcal{A},\tau,X)$ stands for the *sheaf of commutative* $\mathbb{C}$-*algebras with an identity element* on a topological space X. Furthermore, we let

$$\Gamma(\mathcal{A}) \equiv (\Gamma(U,\mathcal{A}), \tau^U_V)$$

be the corresponding (complete) *presheaf of sections* of $\mathcal{A}$.

Now, let $\mathcal{E}$ be an $\mathcal{A}$-module on X. For any *open* set $U \subseteq X$, let

$$\Omega^k_{\mathcal{A}|_U}(\underbrace{\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U}_{k-\text{times}}, \mathcal{A}|_U)$$

be the *set of all skew-symmetric* $\mathcal{A}|_U$*-k-linear morphisms* of the $\mathcal{A}|_U$-modules $\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U$ and $\mathcal{A}|_U$. It is obvious that we have

$$\Omega^k_{\mathcal{A}|_U}(\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U) \subseteq \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U),$$

where for every open set $U \subseteq X$,

$$\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U)$$

is the set of all $\mathcal{A}|_U$*-k-linear morphisms* of $\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U$ into $\mathcal{A}|_U$.

As is naturally expected, the correspondence

$$U \longmapsto \Omega^k_{\mathcal{A}|_U}(\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U), \tag{6}$$

where U is open in X, along with the obvious *restriction maps*, yields a *complete presheaf of* $\mathcal{A}$*-modules* on X. The sheaf, on X, generated by the presheaf defined by (6) is called the **sheaf of germs of exterior $\mathcal{A}$-k-forms on $\mathcal{E}$**, and is denoted

$$\Omega^k(\mathcal{E}) \equiv \mathcal{L}^k_a(\mathcal{E} \oplus \ldots \oplus \mathcal{E}, \mathcal{A}) \equiv \mathcal{L}^k_a(\mathcal{E}, \mathcal{A}).$$

It is clear that, for every open set $U \subseteq X$, the set

$$\Omega^k_{\mathcal{A}|_U}(\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U)$$

is an $\mathcal{A}(U)$*-module*, i.e. a *module over the* $\mathbb{C}$*-algebra* $\mathcal{A}(U)$; hence a $\mathbb{C}$*-vector space*. Thus, based on [9], Proposition 1.1, p. 104 and Theorem 9.1, p. 41, we conclude that the *sheaf* $\Omega^k(\mathcal{E})$ is an $\mathcal{A}$*-module on* X. It follows, for every *open set* $U \subseteq X$, that

$$\Omega^k(\mathcal{E})(U) = \Omega^k_{\mathcal{A}|_U}(\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U),$$

within an $\mathcal{A}(U)$*-isomorphism* of the $\mathcal{A}(U)$-modules concerned. In particular, one has

$$\Omega^k(\mathcal{E})(X) = \Omega^k_{\mathcal{A}}(\mathcal{E} \oplus \ldots \oplus \mathcal{E}, \mathcal{A}),$$

where the equality actually means an $\mathcal{A}(X)$*-isomorphism* of the $\mathcal{A}(X)$-modules $\Omega^k(\mathcal{E})(X)$ and $\Omega^k_{\mathcal{A}}(\mathcal{E} \oplus \ldots \oplus \mathcal{E}, \mathcal{A})$.

Following the classical pattern, we set

$$\Omega^0(\mathcal{E}) = \mathcal{A}, \quad \text{and} \quad \Omega^1(\mathcal{E}) = \mathcal{E}^*,$$

as the *sheaf of germs of exterior* $\mathcal{A}$*-0-forms* and the *sheaf of germs of* $\mathcal{A}$*-1-forms*, on X, respectively.

The standard *exterior algebra of k-forms* can also be repeated here to some significant extent. Consider for instance the analogue, in this setting, of the usual *alternation* or *anti-symmetrizer map*, [1], p. 101, or [8], p. 85, or [12], p. 196, which we define below. To this end, suppose given an $\mathcal{A}$-*module* $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ on a topological space X. Let

$$\Gamma(\mathcal{E}) \equiv (\Gamma(U, \mathcal{E}), \pi_V^U)$$

be the corresponding (complete) *presheaf of sections* of $\mathcal{E}$. Instead of considering the *presheaf* $T_k^0\Gamma(\mathcal{E})$, $k \in \mathbb{N}$, of $\Gamma(\mathcal{A})$-*modules* on X, in order to define the *alternation morphism*

$$\mathbf{A} : \mathfrak{T}_k^0(\mathcal{E}) \longrightarrow \mathfrak{T}_k^0(\mathcal{E}),$$

where $\mathfrak{T}_k^0(\mathcal{E}) := \mathbf{S}(T_k^0\Gamma(\mathcal{E}))$, we deviate from this usual practice to defining $\mathbf{A}$ as the $\mathcal{A}$-morphism induced by maps

$$\mathbf{A}_x : (\mathfrak{T}_k^0(\mathcal{E}))_x = \mathcal{E}_x^* \otimes_{\mathcal{A}_x} \ldots \otimes_{\mathcal{A}_x} \mathcal{E}_x^* \longrightarrow \mathcal{E}_x^* \otimes_{\mathcal{A}_x} \ldots \otimes_{\mathcal{A}_x} \mathcal{E}_x^*, \quad x \in X, \tag{7}$$

such that

$$\mathbf{A}_x\mathbf{t}_x(s_{1,x}, \ldots, s_{k,x}) = \frac{1}{k!}\sum_{\sigma \in S_k}(sign\sigma)\ \mathbf{t}_x(s_{1,x}, \ldots, s_{k,x}),$$

where $s_{1,x}, \ldots, s_{k,x} \in \mathcal{E}_x$, $\mathbf{t}_x : \mathcal{E}_x \otimes_{\mathcal{A}_x} \ldots \otimes_{\mathcal{A}_x} \mathcal{E}_x \longrightarrow \mathcal{A}_x$ is $\mathcal{A}_x$-k-linear, and S_k is the permutation group on $\{1, \ldots, k\}$.

The equality $(\mathfrak{T}_k^0(\mathcal{E}))_x = \mathcal{E}_x^* \otimes_{\mathcal{A}_x} \ldots \otimes_{\mathcal{A}_x} \mathcal{E}_x^*$, $x \in X$, holds within an $\mathcal{A}_x$-*isomorphism*; for this purpose see [9], relation 5.9, p. 130.

The reason for this approach comes from the observation that the $\Gamma(\mathcal{A})$-*presheaf* defined by

$$U \longmapsto T_k^0\Gamma(\mathcal{E})(U) := \Gamma(U, \mathcal{E}^*) \otimes_{\Gamma(U,\mathcal{A})} \ldots \otimes_{\Gamma(U,\mathcal{A})} \Gamma(U, \mathcal{E}^*), \tag{8}$$

where $U \subseteq X$ is open, along with the restriction maps *is not always complete*, cf. [9], Statement 5.5, p. 129; therefore $\mathfrak{T}_k^0(\mathcal{E})(U)$ is not always $\mathcal{A}(U)$-isomorphic to the right hand side in the correspondence (8) above, i.e., to $T_k^0\Gamma(\mathcal{E})(U)$. Thus, in order to circumvent this obstacle, we resort, by virtue of [9], Lemma 8.1, p. 36, to anti-symmetrizers

$$\mathbf{A}_x : \mathfrak{T}_k^0(\mathcal{E})_x \longrightarrow \mathfrak{T}_k^0(\mathcal{E})_x, \quad x \in X,$$

from which the sought anti-symmetrizer $\mathcal{A}$-morphism $\mathbf{A}$ is obtained.

We may now define the **exterior product** as follows.

Definition 2.1 Let $\mathcal{E}$ be a *vector sheaf of rank n* on a topological space X, and let ξ and η be elements of $\Omega^k(\mathcal{E})$ and $\Omega^l(\mathcal{E})$, respectively. The *exterior product* of ξ_x and η_x, $x \in X$, is the *germ* $\xi_x \wedge \eta_x \in \Omega^{k+l}(\mathcal{E}_x)$, given by

$$\xi_x \wedge \eta_x = \frac{(k+l)!}{k!l!}\mathbf{A}_x(\mathcal{E}_x \otimes \eta_x),$$

that is, for all $s_{1,x}, \ldots, s_{k+l,x} \in \mathcal{E}_x$,

$$\xi_x \wedge \eta_x(s_{1,x}, \ldots, s_{k+l,x}) = \\ \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sign(\sigma)\ \xi_x(s_{\sigma(1),x}, \ldots, s_{\sigma k,x})\ \eta_x(s_{\sigma(k+1),x}, \ldots, s_{\sigma(k+l),x}).$$

In particular, for $\alpha_x \in \Omega^0(\mathcal{E}_x) \equiv \mathcal{A}_x$, $x \in X$, we put

$$\alpha_x \wedge \xi_x \equiv \xi_x \wedge \alpha_x \equiv \alpha_x \xi_x.$$

Finally, the $\mathcal{A}$-*morphism*

$$\xi \wedge \eta \in \Omega^{k+l}(\mathcal{E}),$$

obtained from germs $\xi_x \wedge \eta_x$, $x \in X$, above, by virtue of [9], Lemma 8.1, p. 36, is called the **exterior product of** ξ **and** η. Like earlier, for $\alpha \in \Omega^0(\mathcal{E}) \equiv \mathcal{A}$, we put

$$\alpha \wedge \xi \equiv \xi \wedge \alpha \equiv \alpha \xi.$$

Note that we do not index $\wedge$, when considering the exterior product $\xi_x \wedge \eta_x$, $x \in X$, for given $\xi \in \Omega^k(\mathcal{E})$ and $\eta \in \Omega^l(\mathcal{E})$, in order to avoid unnecessary meticulousness.

With this product, we define the **exterior algebra sheaf**, or the **Grassmann algebra sheaf** of the *vector sheaf* $\mathcal{E}$ of rank n, to be the $\mathcal{A}$-*module*

$$\Omega^*(\mathcal{E}) \equiv \Omega^0(\mathcal{E}) \oplus \Omega^1(\mathcal{E}) \oplus \ldots \oplus \Omega^n(\mathcal{E}), \tag{9}$$

such that

$$\Omega^*(\mathcal{E})_x \equiv \Omega^0(\mathcal{E})_x \oplus \Omega^1(\mathcal{E})_x \oplus \ldots \oplus \Omega^n(\mathcal{E})_x = \Omega^0(\mathcal{E}_x) \oplus \Omega^1(\mathcal{E}_x) \oplus \ldots \oplus \Omega^n(\mathcal{E}_x), \tag{10}$$

for all $x \in X$, where the last relation is valid, of course, within an $\mathcal{A}_x$-*isomorphism.*

The $\mathcal{A}_x$-isomorphism in (10) can be obtained in the following manner. In fact, one has

$$\Omega^1(\mathcal{E}) := \mathcal{L}^1(\mathcal{E}, \mathcal{A}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) = \mathcal{E}, \tag{11}$$

which implies that

$$\Omega^1(\mathcal{E})_x = \mathcal{E}_x, \quad x \in X.$$

(For the last relation in (11), see [9], relation (6.18), p. 136.) On the other hand, for all $x \in X$,

$$\Omega^1(\mathcal{E}_x) := \mathcal{L}^1(\mathcal{E}_x, \mathcal{A}_x) = \mathcal{H}om_{\mathcal{A}_x}(\mathcal{E}_x, \mathcal{A}_x) = \mathcal{E}_x,$$

because $\mathcal{E}$ is a vector sheaf of finite rank on X. Thus,

$$\Omega^1(\mathcal{E})_x = \Omega^1(\mathcal{E}_x),$$

for all $x \in X$. Likewise, for $k > 1$, we have

$$\Omega^k(\mathcal{E})_x =: \mathcal{L}^k_a(\mathcal{E} \oplus \ldots \oplus \mathcal{E}, \mathcal{A})_x = \mathcal{L}^k_a(\mathcal{E}_x \oplus \ldots \oplus \mathcal{E}_x, \mathcal{A}_x) := \Omega^k(\mathcal{E}_x).$$

Like in the classical theory, if α_i, $i = 1, \ldots, k$, are elements of $\Omega^1(\mathcal{E})$, where $\mathcal{E}$ is a vector sheaf of finite rank on a topological space X, then

$$(\alpha_{1,x} \wedge \ldots \wedge \alpha_{k,x})(s_{1,x}, \ldots, s_{k,x}) = \sum_\sigma sign(\sigma)\alpha_{1,x}(s_{\sigma(1),x}) \ldots \alpha_{k,x}(s_{\sigma k,x}),$$

where $s_{i,x} := s_i|_{\mathcal{E}_x}$, and $\alpha_{i,x} = \alpha_i|_{\Omega^1(\mathcal{E}_x)}$, for all $X \in X$ and $i = 1, \ldots, k$, so that

$$(\alpha_1 \wedge \ldots \wedge \alpha_k)(s_1, \ldots, s_k) = \sum_\sigma sign(\sigma)\alpha_1(s_{\sigma(1)}) \ldots \alpha_k(s_{\sigma k}).$$

3 Skew-Symmetric $\mathcal{A}$-bilinear forms

Definition 3.1 Let X be a topological space and $(X, \mathcal{A}, \mathcal{P})$ an *ordered* $\mathbb{R}$-*algebraized space* on X, (cf.[9], p. 316). A *section* $s \in \mathcal{A}(U)$, with U open in X, is called *strictly positive* if $s \in \mathcal{P}(U)$, and, given $x \in U$, $s(x) \neq 0_x$.

For the purpose of what lays ahead, we need the following notion.

Definition 3.2 Let $(X, \mathcal{A})$ be an algebraized space, and $\mathcal{E}$ an $\mathcal{A}$-module on X. An $\mathcal{A}$-*bilinear sheaf morphism*

$$\omega \equiv (\omega_U)_{X \supseteq U, open} : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$$

is called

- **skew symmetric** provided

  $$\omega_U(s, t) = -\omega_U(t, s),$$

  for all *sections* $s, t \in \mathcal{E}(U)$ over any *open* set $U \subseteq X$.
- **nondegenerate** if

  $$\omega_U(s, t) = 0 \quad \text{for all } t \in \mathcal{E}(U), \text{ with } U \text{ an arbitrary open set in } X,$$

  implies that $s = 0 \in \mathcal{E}(U)$.

In this Definition 3.2, we have identified the sheaf morphism $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ with the corresponding *presheaf morphism* $(\omega_U)_{X \supseteq U, open} : \Gamma(\mathcal{E} \oplus \mathcal{E}) \longrightarrow \Gamma(\mathcal{A})$ of (*complete*) *presheaves of sections* $\Gamma(\mathcal{E} \oplus \mathcal{E})$ and $\Gamma(\mathcal{A})$. This identification is based on the fact that, given a topological space X, we have

$$\Gamma : \mathcal{S}h_X \cong \mathcal{C}o\mathcal{P}\mathcal{S}h_X, \tag{12}$$

where $\mathcal{S}h_X$ is the category of sheaves over X, $\mathcal{C}o\mathcal{P}\mathcal{S}h_X$ is the category of complete presheaves over X, and Γ is the section functor. For suitable details, see [9], Theorem 13.1, p. 73.

For the purpose of the following theorem, we assume the following condition, referred to in the sequel as the **inverse-positive-section condition**:

> The ordered $\mathbb{R}$-algebraized $(X, \mathcal{A}, \mathcal{P})$ is such that all *strictly positive sections* of $\mathcal{A}$ are *invertible*. More explicitly, if $\mathcal{P}^*$ denotes the *subsheaf* of all strictly positive sections of $\mathcal{A}$, and $\mathcal{A}^\bullet$ the sheaf on X, generated by the presheaf
>
> $$U \longmapsto \mathcal{A}(U)^\bullet = \mathcal{A}^\bullet(U),$$
>
> where U is open in X, and $\mathcal{A}(U)^\bullet$ is the *group of units* of the unital **C**-algebra $\mathcal{A}(U)$, then
>
> $$\mathcal{P}^* \subset \mathcal{A}^\bullet. \tag{13}$$
>
> Section-wise, (13) would be understood in the following way: for any $s \in \mathcal{P}(U)$, where $U \subseteq X$ is open, such that $s(x) \neq 0_x \in \mathcal{A}_x$, $x \in U$, then there exists $s^{-1} \in \mathcal{P}(U)$ such that $s \cdot s^{-1} = s^{-1} \cdot s = 1_U \in \mathcal{A}(U)$.

Furthermore, we suppose that our *ordered algebraized space* $(X, \mathcal{A}, \mathcal{P})$, is also endowed with an *absolute value*, i.e., the following sheaf morphism,

$$|\cdot| : \mathcal{A} \longrightarrow \mathcal{A}^+ := \mathcal{P}, \tag{14}$$

having the analogous properties of the classical function; hence, for instance, *the property* that

$$|s| = \alpha \in \mathbb{R}^+ \subseteq \mathcal{A}^+(X) \Longleftrightarrow s = \pm\alpha \in \mathbb{R} \subseteq \mathcal{A}(X).$$

Now, the proof of the following theorem is based on the classical patterns, see e.g. [15], [1], [2], [14], within, of course, the present *sheaf-theoretic context*, for which we refer to [9], p. 316, Definition 8.1, along with p. 335 *ff.* So, we now have the following basic result.

Theorem 3.1 Let $(X, \mathcal{A}, \mathcal{P}, |\cdot|)$ be an ordered $\mathbb{R}$-algebraized space, endowed with an *absolute value*, and $\mathcal{E}$ the standard *free* $\mathcal{A}$*-module*, $\mathcal{A}^n$, of rank n on X.

Moreover let $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ be a *non-zero skew-symmetric* and *non-degenerate* $\mathcal{A}$-*bilinear sheaf morphism*. Then, there exists an $\mathcal{A}(U)$-*basis* of $\mathcal{A}^n(U)$, say,

$$s_1^U, \ldots, s_m^U, t_1^U, \ldots, t_m^U,$$

such that

$$\begin{array}{ll} n = 2m & \\ \omega(s_i^U, s_j^U) = 0 = \omega(t_i^U, t_j^U) & \text{for all } 1 \leq i, j \leq m \\ \omega(s_i^U, t_j^U) = \delta_{ij}^U & \text{for all } 1 \leq i, j \leq m. \end{array}$$

Proof. With no loss of generality, we assume that $U = X$. Therefore, since $\mathcal{A}^n \neq \{0\}$ (we already assumed that $\mathbb{C} \equiv \mathbb{C}_X \subseteq \mathcal{A}$), there exists an element

$$0 \neq s_1 \in \mathcal{A}^n(X) \cong \mathcal{A}(X)^n$$

(take e.g. an element from the *canonical basis* of (sections) of $\mathcal{A}^n(X) \cong \mathcal{A}(X)^n$, see [9], p. 123). Next, consider the "$\mathcal{A}(X)$-*line of* s_1", i.e.,

$$\mathcal{A}(X)[s_1] := \{\alpha s_1 \in \mathcal{A}^n(X) : \ \alpha \in \mathcal{A}(X)\},$$

which, by an obvious *abuse of notation*, we may still denote, for convenience, just, by

$$\mathcal{A}(s_1) \subseteq \mathcal{A}^n(X).$$

Now, it is also clear that there exists an element

$$0 \neq \bar{t}_1 \in \mathcal{A}^n(X) \setminus \mathcal{A}(s_1)$$

(just, take e.g. another element of the previously considered basis of $\mathcal{A}^n(X)$, *different from* s_1). Furthermore, due to the hypothesis concerning s_1, $\bar{t}_1$, and as well as, to that one for ω, one obtains that (see Lemma 3.1)

$$\omega(s_1, \bar{t}_1)(x) \neq 0_x \in \mathcal{A}_x,$$

for all $x \in X$. Hence, based also on our hypothesis for $\mathcal{A}$, that is, the existence of the sheaf morphism $|\cdot| : \mathcal{A} \longrightarrow \mathcal{A}^+ \equiv \mathcal{P}$, we also obtain that,

$$|\omega(s_1, \bar{t}_1)| > 0,$$

that is the section $|\omega(s_1, \bar{t}_1)| \in \mathcal{A}(X)$ is *strictly positive*; therefore, by assumption for $(X, \mathcal{A})$, see the inverse-positive-section condition above, it is also invertible in $\mathcal{A}(X)$. Hence taking further $t_1 := u^{-1}\bar{t}_1$, with $u \equiv |\omega(s_1, \bar{t}_1)| \in \mathcal{A}(X)$, one gets

$$|\omega(s_1, t_1)| = 1,$$

which implies that

$$\omega(s_1, t_1) = \pm 1 \in \mathcal{A}(X).$$

Now, let us consider

$$\mathrm{S}_1 := [s_1, t_1],$$

that is, the "*flag*" (alias, "$\mathcal{A}(X)$-*plane*"), defined by s_1 and t_1, in $\mathcal{A}^n(X)$, in effect, an $\mathcal{A}(X)$-*module, generated by* s_1*, and* t_1, along with its *"orthogonal complement"* in $\mathcal{A}^n(X)$, i.e.,

$$\mathrm{S}_1^\perp \equiv \mathrm{T}_1 := \{v \in \mathcal{A}^n(X) : \omega(v, z) = 0, \text{for all } z \in \mathrm{S}_1\}.$$

Now, we first remark that s_1, t_1 are also "*free generators*" of S_1, for, *if* $t_1 = \alpha s_1$, *then*

$$\pm 1 = \omega(s_1, t_1) = \omega(s_1, \alpha s_1) = \alpha \cdot \omega(s_1, s_1) = 0,$$

a *contradiction*. That is, $\{s_1, t_1\}$ yields actually an $\mathcal{A}(X)$-*basis of the flag* S_1. Furthermore, we prove that:

$$\begin{array}{ll} (i) & \mathrm{S}_1 \cap \mathrm{T}_1 = \{0\}, \text{ and} \\ (ii) & \mathrm{S}_1 + \mathrm{T}_1 = \mathcal{A}^n(X). \end{array}$$

Indeed, (i) if $z \equiv \alpha s_1 + \beta t_1 \in \mathrm{S}_1 \cap \mathrm{T}_1$, with $\alpha, \beta \in \mathcal{A}(X)$, one gets, by the very definition of S_1, T_1, and the fact that $\omega(s_1, t_1) = 1$, that,

$$\omega(z, s_1) = \beta = 0, \quad \text{and } \omega(z, t_1) = \alpha = 0,$$

that is, $z = 0$, *which proves* (i). On the other hand, (ii) *for every* $z \in \mathcal{A}^n(X)$, one has,

$$z = (-\omega(z, s_1)t_1 + \omega(z, t_1)s_1) + (z + \omega(z, s_1)t_1 - \omega(z, t_1)s_1),$$

with

$$-\omega(z, s_1)t_1 + \omega(z, t_1)s_1 \in \mathrm{S}_1,$$

and

$$z + \omega(z, s_1)t_1 - \omega(z, t_1)s_1 \in \mathrm{T}_1.$$

Thus,

$$\mathcal{A}^n(X) = \mathrm{S}_1 \oplus \mathrm{T}_1.$$

Now, in a manner similar to the manner we found the elements s_1, $t_1 \in \mathrm{S}_1$ with $\omega(s_1, t_1) = 1$, we conclude the existence of *sections* $s_2, t_2 \in \mathrm{T}_1 \setminus \{0\}$, *such that*

$$\omega(s_2, t_2) = 1 \in \mathcal{A}(X);$$

while we further consider the flag

$$\mathrm{S}_2 := [s_2, t_2],$$

along with

$$\mathrm{T}_2 \equiv \mathrm{S}_2^\perp := \{v \in \mathcal{A}^n(X) : \ \omega(v, z) = 0, \ z \in \mathrm{S}_2\}.$$

Yet, we still prove in a similar way, as before, that

$$\mathrm{T}_1 = \mathrm{S}_2 \oplus \mathrm{T}_2,$$

so that one obtains,

$$\mathcal{A}^n(X) = \mathrm{S}_1 \oplus \mathrm{S}_2 \oplus \mathrm{T}_2,$$

and so on. Now, the above process stops eventually, due to the *finite rank of* $\mathcal{A}^n(X)$, so that one finally obtains

$$\mathcal{A}^n(X) = \mathrm{S}_1 \oplus \mathrm{S}_2 \oplus \ldots \oplus \mathrm{S}_m$$

with the generators, s_i, t_i, of S_i $(1 \leq i \leq n)$ having the property that

$$\begin{aligned} &\omega(s_i, s_j) = 0 = \omega(t_i, t_j) \\ &\omega(s_i, t_j) = \delta_{ij}. \end{aligned}$$

Hence, the proof is finished. □

In the proof of the previous Theorem 3.1, one still essentially applies the following standard fact of the classical theory, which for convenience we also formulate, within the present context:

Lemma 3.1 Let $\mathcal{E}$ be a free $\mathcal{A}$-*module* of rank $n \in \mathbb{N}$ on a topological space X. Then, a family $\{s_i\}_{i \in I}$ of *global sections* of $\mathcal{E}$, i.e., $\{s_i\}_{i \in I} \subseteq \mathcal{E}(X)$, is $\mathcal{A}(X)$-*linearly independent* if, and only if, the relation

$$\sum_{i \in I} \alpha_i s_i = 0,$$

with $\{\alpha_i\}_{i \in I} \subseteq \mathcal{A}(X)$, having finite support, implies $\alpha_i = 0$, for any $i \in I$.

For the proof of Lemma 3.1, one can follow, for instance, the analogous argument in [4], Chap II; p. 25, remarks after Definition 10. Yet, for convenience, we recall that the term "*finite support*" for the family $\{\alpha_i\}_{i \in I} \subseteq \mathcal{A}(X)$, means that $\alpha_i \neq 0$, *only for finitely many indices* $i \in I$, *and the rest of the* α_i *being* 0; so that the *sum* used above acquires then a meaning.

When the skew-symmetric $\mathcal{A}$-bilinear sheaf morphism ω is not necessarily non-degenerate, then in place of Theorem 3.1, we have the theorem below. Let us first give the following definition:

Definition 3.3 Let $\omega \equiv (\omega^U) : \mathcal{A}^n \oplus \mathcal{A}^n \longrightarrow \mathcal{A}$ be an $\mathcal{A}$-bilinear morphism on the standard free $\mathcal{A}$-module $\mathcal{A}^n$. The *rank* of ω is the rank of the matrix (ω^U_{ij}), with $\omega^U_{ij} = \omega^U(\varepsilon^U_i, \varepsilon^U_j)$, where $\{\varepsilon^U_i\}_{1 \leq i \leq n}$ is the *Kronecker gauge* of $\mathcal{A}^n(U)$, and U is any *open subset* of X.

By the methods of Linear Algebra, one can easily show that the rank of ω is independent of the basis considered.

Note the notation $\omega \equiv (\omega^U)$ instead of the usual $\omega \equiv (\omega_U)$; the reason being that we want to avoid, in the sequel, too many sub-indices.

Theorem 3.2 Let $(X, \mathcal{A}, \mathcal{P}, |\cdot|)$ and $\mathcal{E}$ be as in Theorem 3.1. Let $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ be a skew-symmetric $\mathcal{A}$-bilinear sheaf morphism of rank r. Then, $r = 2m$, for some integer m, and for every $x \in X$ there are an open neighborhood $U \subseteq X$ of x and a basis

$$s_1^U, \ldots, s_n^U \in \mathcal{A}^n(U) = \mathcal{A}(U)^n$$

such that the matrix of $\omega^U \equiv \omega_U$ is

$$\begin{bmatrix} 0 & \mathrm{I}_m & 0 \\ -\mathrm{I}_m & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where ω_U is the component of ω relative to U.

Proof. Fix $x \in X$. Because of the continuity of ω_y, $y \in X$, and of the fact that the rank of ω is r, there exist an open neighborhood U_1 of x and $s_1^{U_1}, \bar{s}_{m+1}^{U_1} \in \mathcal{A}^n(U_1)$ such that

$$\omega^{U_1}(s_1^{U_1}, \bar{s}_{m+1}^{U_1})(y) \neq 0_y \in \mathcal{A}_y,$$

for all $y \in U_1$. Now, put $t_{1,m+1}^{U_1} := \omega^{U_1}(s_1^{U_1}, \bar{s}_{m+1}^{U_1}) \in \mathcal{A}(U_1)$. Based on our hypothesis for $\mathcal{A}$, we have that $u_{1,m+1} := |t_{1,m+1}^{U_1}| > 0$; therefore by the inverse-positive-section condition, $u_{1,m+1} \in \mathcal{A}^\bullet(U_1) = \mathcal{A}(U_1)^\bullet$. Hence, taking further $s_{m+1}^{U_1} = u_{1,m+1}^{-1}\bar{s}_{m+1}^{U_1}$, one gets

$$|\omega^{U_1}(s_1^{U_1}, s_{m+1}^{U_1})| = 1.$$

Assuming that $\omega^{U_1}(s_1^{U_1}, s_{m+1}^{U_1})| = 1_{U_1} =: 1$, the matrix of ω^{U_1} in the $\mathcal{A}(U_1)$-module $\mathrm{S}_1^{U_1} := [s_1^{U_1}, s_{m+1}^{U_1}]$, that is, the $\mathcal{A}(U_1)$-module spanned by $s_1^{U_1}$ and $s_{m+1}^{U_1}$, is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $(\mathrm{S}_1^{U_1})^\perp$ be the ω^{U_1}-orthogonal complement of $\mathrm{S}_1^{U_1}$ in $\mathcal{A}^n(U_1)$. As in the proof of Theorem 3.1, one shows that

$$\mathcal{A}^n(U_1) = \mathrm{S}_1^{U_1} \oplus (\mathrm{S}_1^{U_1})^\perp.$$

Now, we repeat the process on $(\mathrm{S}_1^{U_1})^\perp$ to get an open neighborhood $U_2 \subseteq X$ of x, and $s_2^{U_2}$ and $s_{m+2}^{U_2}$ such that $\omega^{U_2}(s_2^{U_2}, s_{m+2}^{U_2}) = 1$ and continue inductively.

Eventually, this process will stop because the rank of ω is finite. Certainly by taking $U = \cap_{i=1}^{m} U_i$, where $r = 2m$, one sees that ω^U has the stated matrix in the basis $\{s_1^U, \ldots, s_n^U\}$. □

Note that if we denote by $\{(s_i^U)^*\}_{1\leq i\leq m}$ the dual basis of $\{s_i^U\}_{1\leq i\leq m}$, then

$$\omega^U = \sum_{i=1}^{m} (s_i^U)^* \wedge (s_{m+i}^U)^*. \tag{15}$$

Corollary 3.1 Let $(X, \mathcal{A}, \mathcal{P})$ be an ordered $\mathbb{R}$-algebraized space such that $\mathcal{P}^* \subseteq \mathcal{A}^\bullet$. Let $\mathcal{E}$ be a *vector sheaf of rank* n on X, and $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ a *skew-symmetric* and *non-degenerate* $\mathcal{A}$*-bilinear morphism.* Then, given an *open* neighborhood U of $x \in X$ such that $\mathcal{E}|_U = \mathcal{A}^n|_U$, where $x \in X$ is arbitrary, there exists a *basis*, $s_1^U, \ldots, s_m^U, t_1^U, \ldots, t_m^U$, of $\mathcal{A}^n(U)$ such that

$$\begin{array}{ll} n = 2m & \\ \omega^U(s_i^U, s_j^U) = 0 = \omega^U(t_i^U, t_j^U) & \text{for all } 1 \leq i, j \leq m \\ \omega^U(s_i^U, t_j^U) = \delta_{ij}^U & \text{for all } 1 \leq i, j \leq m. \end{array}$$

The pair $(\mathcal{E}, \omega)$ is called a **locally free symplectic $\mathcal{A}$-module**, alias **symplectic vector sheaf**, of rank n, on X.

Proof. One proceeds in the same manner as for the proof of Theorem 3.1, the small nuance being that one works locally. □

Definition 3.4 Let $(X, \mathcal{A}, \mathcal{P})$ be an ordered $\mathbb{R}$-algebraized space, satisfying the inverse-positive-section condition. The non-degenerate skew-symmetric $\mathcal{A}$-bilinear morphism $\omega : \mathcal{A}^n \oplus \mathcal{A}^n \longrightarrow \mathcal{A}$ is called a **linear symplectic $\mathcal{A}$-structure** on the standard free $\mathcal{A}$-module $\mathcal{A}^n$, and the pair $(\mathcal{A}^n, \omega)$ is called a (**standard**) **free symplectic $\mathcal{A}$-module**. More generally, let $\mathcal{E}$ be an $\mathcal{A}$-module on X, and $\omega : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{A}$ a non-degenerate, skew-symmetric $\mathcal{A}$-bilinear morphism. Then, the pair $(\mathcal{E}, \omega)$ is called a **symplectic $\mathcal{A}$-module** on X.

With respect to the notation of Theorem 3.1, the basis, $s_1, \ldots, s_m, t_1, \ldots, t_m$, is called a **symplectic basis** of the standard symplectic free $\mathcal{A}$-module $(\mathcal{A}^n, \omega)$. It is clear that, with respect to a symplectic basis, the matrix representing ω is, as in the classical case, given by

$$J = \begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{bmatrix}. \tag{16}$$

Example 3.1 Let $\mathcal{E}$ be a vector sheaf of rank n on X, and consider the direct sum $\mathcal{E} \oplus \mathcal{E}^*$. The $\mathcal{A}$-bilinear morphism

$$\omega : (\mathcal{E} \oplus \mathcal{E}^*) \oplus (\mathcal{E} \oplus \mathcal{E}^*) \longrightarrow \mathcal{A},$$

defined by

$$\omega^U\left((s_1^U,\alpha_1^U),(s_2^U,\alpha_2^U)\right)=\alpha_2^U(s_1^U)-\alpha_1^U(s_2^U),$$

where U is a local gauge of $\mathcal{E}$, and $s_i^U\in\mathcal{E}(U)=\mathcal{A}^n(U)=\mathcal{A}(U)^n$ and $\alpha_i^U\in\mathcal{E}^*(U)=(\mathcal{A}^n)(U)=\mathcal{A}^n(U)$, is a symplectic $\mathcal{A}$-morphism. This example shows that any vector sheaf $\mathcal{F}$ of *even* rank admits a symplectic $\mathcal{A}$-structure, for one has

$$\mathcal{F}|_U=\mathcal{A}^{2n}|_U=\mathcal{A}^n|_U\oplus\mathcal{A}^n|_U=\mathcal{A}^n|_U\oplus(\mathcal{A}^n)^*|_U,$$

where U is a local gauge of $\mathcal{F}$. For the last equality of the previous line, see [[9], relation (3.14), p.122].

We now would like to show that a useful criterion for non-degeneracy is also available in this setting. To this effect, we suppose as usual that our *ordered algebraized space* $(X,\mathcal{A},\mathcal{P})$ is enriched with absolute value, see (14), and *square root*; the latter means a *morphism of (complete) presheaves*

$$\sqrt{\ }:\Gamma(\mathcal{P})\longrightarrow\Gamma(\mathcal{P}),\quad s\longmapsto\sqrt{s},\quad \text{for all } s\in\mathcal{P}(U),$$

with U running over all the open subsets of X. In addition, we assume that the pair $(\mathcal{A},\rho)$ is a *Riemann* $\mathcal{A}$-module on X, see [9], p. 320. We assume as well that $\overline{\rho}\equiv\rho^n$ (cf. [9], p. 324) is the extension of ρ to the standard free $\mathcal{A}$-module $\mathcal{A}^n$ on X.

Definition 3.5 Let $(X,\mathcal{A},\mathcal{P})$ and $\overline{\rho}$ be as above, and $\{s_i\}_{1\leq i\leq n}\subseteq\mathcal{A}^n(X)=\mathcal{A}(X)^n$ be a basis of $\mathcal{A}^n(X)$. Denoting by

$$\bigwedge^n(\mathcal{A}^n)^*$$

the n-th *exterior power* of the $\mathcal{A}$-module $(\mathcal{A}^n)^*$, see [9], p. 307, the *section*

$$\Omega=\sqrt{|\det(\overline{\rho}(s_i,s_j))|}s_1^*\wedge\ldots\wedge s_n^*\in\left(\Lambda^n(\mathcal{A}^n)^*\right)(X),\tag{17}$$

where $\{s_i^*\}_{1\leq 1\leq n}\subseteq(\mathcal{A}^n)^*(X)$ is the dual $\mathcal{A}$-basis of $\{s_i\}_{1\leq i\leq n}$, is called a **volume element of the $\mathcal{A}$-metric** $\overline{\rho}$. In the sequel, for the sake of brevity, the scaling factor $\sqrt{|\det(\overline{\rho}(s_1,s_j))|}$, above, will be denoted by $\sqrt{|\overline{\rho}(s)|}\equiv\sqrt{|\overline{\rho}|}$, where $s\equiv\{s_1,\ldots,s_n\}$.

It is clear that for an orthonormal gauge $\{s_i\}_{1\leq i\leq n}$ of $\mathcal{A}^n(X)$ (see [9], p. 340), relation (17) becomes

$$\Omega=s_1^*\wedge\ldots\wedge s_n^*.$$

Like in the classical case, we have:

Corollary 3.2 Let $\omega : \mathcal{A}^n \oplus \mathcal{A}^n \longrightarrow \mathcal{A}$ be a $\mathcal{A}$-bilinear and skew-symmetric $\mathcal{A}$-morphism on the standard free $\mathcal{A}$-module $\mathcal{A}^n$, where $(X, \mathcal{A})$ is an enriched (with square root and absolute value) ordered algebraized space and $(\mathcal{A}, \rho)$ a Riemannian $\mathcal{A}$-module. Then, ω is *non-degenerate* if and only if $n = 2m$, for some $m \in \mathbb{N}$, and $\omega^m = \omega \wedge \ldots \wedge \omega$ is a *volume element* on $\mathcal{A}^n$.

Note that
$$\omega^m \equiv (\omega^{U^m}) := (\omega^U \wedge \ldots \wedge \omega^U).$$
Therefore, that ω^m is a volume element on $\mathcal{A}^n$ means that $\omega^U \wedge \ldots \wedge \omega^U$ is a volume element on $\mathcal{A}^n(U)$, for every open $U \subseteq X$.

Proof. We refer to the proof in Abraham-Marsden [[1], p. 165] for detail.

Suppose that ω is non-degenrate. By Theorem 3.1, $n = 2m$, for some $m \in \mathbb{N}$. By virtue of Equation (15) and of induction, one sees that
$$\omega^{U^m} = m!(-1)^{[m/2]} s_1^{U^*} \wedge \ldots \wedge s_{2m}^{U^*},$$
where $[m/2]$ is the largest integer in $m/2$. Thus, assuming that $\sqrt{|\rho|} = m!(-1)^{[m/2]}$, ω^{U^m} is a volume. The converse is clear. □

Following Abraham-Marsden [[1], p. 167], we set that given a free symplectic $\mathcal{A}$-module $(\mathcal{A}^n, \omega)$, the volume element
$$\Omega_\omega = \frac{(-1)^{[m/2]}}{m!}\omega^m$$
defines an **orientation** on $\mathcal{A}^n$.

In all that precedes, by the $\mathcal{A}$-bilinear morphism $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$, we meant the map on sections of the corresponding $\mathcal{A}$-modules. So, using the presheaf $(\Gamma(U, \mathcal{A}^n), \sigma_V^U)$ of sections of the free $\mathcal{A}$-module $\mathcal{A}^n$ of Theorem 3.1, with U ranging over the topology $\mathcal{T}$ of $\mathcal{A}^n$, we adopt the following classical terminology. Let S be an $A(X)$-submodule of the $A(X)$-module $\Gamma(X, \mathcal{A}^n)$. Then,

- S is called *symplectic* if $\omega|_S$ is non-degenerate. For example, S_1, in the proof of Theorem 3.1, is symplectic.
- S is called *isotropic* if $\omega|_S \equiv 0$. For instance, the span of s_1, s_2, in Theorem 3.1 is isotropic.

Definition 3.6 Let $(\mathcal{E}, \omega)$ and $(\mathcal{E}', \omega')$ be symplectic $\mathcal{A}$-modules on the same topological space X. An $\mathcal{A}$-morphism $\varphi : \mathcal{E} \longrightarrow \mathcal{E}'$ is called **symplectic** if
$$\varphi^*\omega' := \omega' \circ (\varphi \times \varphi) = \omega, \tag{18}$$

that is, for any $s, t \in \mathcal{E}(U)$, $U \subseteq X$ open,

$$(\varphi^* \omega')(s,t) := \omega'(\varphi(s), \varphi(t)) = \omega(s,t). \tag{19}$$

A symplectic $\mathcal{A}$-isomorphism is called an **$\mathcal{A}$-symplectomorphism**. Symplectic $\mathcal{A}$-modules $(\mathcal{E}, \omega)$ and $(\mathcal{E}', \omega')$ are called $\mathcal{A}$-symplectomorphic if there is an $\mathcal{A}$-symplectomorphism φ between them.

Strictly speaking, Equations (19) and (18) should be written as follows:

$$(\varphi_U^* \omega'_U)(s,t) := \omega'_U(\varphi_U(s), \varphi_U(t)) = \omega_U(s,t),$$

and

$$\varphi_U^*(\omega'_U) := \omega'_U \circ (\varphi_U \times \varphi_U) = \omega_U,$$

respectively, where U varies over the topology of X.

It is clear that if $(\mathcal{E}, \omega)$ and $(\mathcal{E}', \omega')$ are symplectomorphic, then they are of the same rank, and their rank is an even positive integer. It is also clear that the set of symplectic $\mathcal{A}$-modules, defined over the same topological space, can be partitioned into equivalence classes. Furthermore, *a symplectic $\mathcal{A}$-morphism is* necessarily *injective*, since if $\varphi := (\varphi_V)_{X \supseteq V, open}$ and $\varphi_V(s) = 0$, where $s \in \Gamma(V, \mathcal{E})$, then necessarily $s = 0$, for ω is non-degenerate.

Lemma 3.2 Let $\mathcal{A}$ be a unital $\mathbb{C}$-algebra sheaf on a topological space X, and let

$$\mathcal{S}p(\mathcal{E}, \omega) \equiv \mathcal{S}p\ \mathcal{E}$$

be the sheaf on X, generated by the presheaf

$$U \longmapsto (\mathrm{S}p\ \mathcal{E})(U), \tag{20}$$

where U varies over the topology of X, such that for every open set $U \subseteq X$, $(\mathrm{S}p\ \mathcal{E})(U)$ is the group (under composition) of all $\mathcal{A}|_U$-symplectomorphisms

$$(\mathcal{E}|_U, \omega|_U \equiv \omega) \longrightarrow (\mathcal{E}|_U, \omega|_U \equiv \omega).$$

Then, the correspondence, given by (20), yields a complete presheaf of groups on X; so that one obtains

$$(\mathcal{S}p\ \mathcal{E})(U) = (\mathrm{S}p\ \mathcal{E})(U),$$

up to a group isomorphism, for every open set $U \subseteq X$. The sheaf $\mathcal{S}p\ \mathcal{E}$ is called the **symplectic group sheaf**, or even **group sheaf of symplectomorphisms** of $\mathcal{E}$ (in fact, of $(\mathcal{E}, \omega)$) on X.

Proof. We first show that for all open set $U \subseteq X$, $(\mathrm{S}p\ \mathcal{E})(U)$ is a group. In fact, let us consider the subset

$$\mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \subseteq \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$$

of all invertible elements of the $\mathcal{A}(U)$-module $\mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$. For $\mathcal{A}|_U$-morphisms φ, $\psi \in \mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$, we define

$$\varphi \circ \psi = (\varphi_V \circ \psi_V)_{U \supseteq V, open},$$

and

$$\varphi^{-1} = (\varphi_V^{-1})_{U \supseteq V, open}.$$

It is easy to see that under the above law of composition $\mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$ forms a group. Therefore, in order to show that $(\mathrm{S}p\ \mathcal{E})(U)$, where U is an open subset of X, is a group, we need only show that if $\varphi, \psi \in (\mathrm{S}p\ \mathcal{E})(U)$, then $\varphi \circ \psi$ and $\varphi^{-1} \in (\mathrm{S}p\ \mathcal{E})(U)$. To this end, let V be an open subset of U. Then, we have

$$(\varphi_V \circ \psi_V)^*(\omega) = (\psi_V^* \circ \varphi_V^*)(\omega) = \psi_V^*(\varphi_V^*(\omega)) = \psi_V^*(\omega) = \omega,$$

from which we deduce that $\varphi \circ \psi \in (\mathrm{S}p\ \mathcal{E})(U)$.

On the other hand, one also obtains

$$(\varphi_V^{-1})^*(\omega) = (\varphi_V^{-1})^*(\varphi_V^*(\omega)) = (\varphi_V \circ \varphi_V^{-1})^*(\omega) = \omega,$$

so that $\varphi^{-1} \in (\mathrm{S}p\ \mathcal{E})(U)$, as well.

Let us now show that (20) yields a complete presheaf of groups. It is easy to see that Correspondence (20), along with the obvious restriction maps, defines a presheaf of groups on X. Thus, we just prove that the presheaf of groups defined, on X, by (20) is complete.

Indeed, let U be an open subset of X and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open covering of U; let $\varphi, \psi \in (\mathrm{S}p\ \mathcal{E})(U)$ such that

$$\rho^U_{U_\alpha}(\varphi) \equiv \varphi_{U_\alpha} := \varphi_\alpha = \psi_\alpha =: \psi_{U_\alpha} \equiv \rho^U_{U_\alpha},$$

for all $\alpha \in I$, and where the $\rho^U_{U_\alpha}$ are the restriction maps characterizing the presheaf $((\mathrm{S}p\ \mathcal{E})(U), V)$. Since

$$(\mathrm{S}p\ \mathcal{E})(U) \subseteq \mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U), \quad \mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \subseteq \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U),$$

and the $\{\rho^U_{U_\alpha}\}_{\alpha \in I}$ are also the restriction maps making the diagram

$$U \longmapsto \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \tag{21}$$

into a presheaf, it follows that $\varphi = \psi$. So the presheaf on X, given by (20), satisfies Condition $(S1)$ of presheaves, see [9], p. 46.

For axiom $(S2)$, see [9], p. 47, let

$$(\varphi_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} (\mathrm{S}p\ \mathcal{E})(U_\alpha) \subseteq \prod_{\alpha\in I} \mathrm{H}om_{\mathcal{A}|_{U_\alpha}}(\mathcal{E}|_{U_\alpha}, \mathcal{E}|_{U_\alpha})$$

be such that

$$\rho^{U_\alpha}_{U_\alpha\cap U_\beta}(\varphi_\alpha) \equiv \varphi_\alpha|_{U_\alpha\cap U_\beta} = \varphi_\beta|_{U_\alpha\cap U_\beta} \equiv \rho^{U_\beta}_{U_\alpha\cap U_\beta}(\varphi_\beta)$$

for any $\alpha, \beta \in I$, with $U_\alpha \cap U_\beta \neq \emptyset$. Hence, since (21) yields a complete presheaf, there exists an element $\varphi \in \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$ such that one has

$$\varphi|_{U_\alpha} = \varphi_\alpha, \quad \alpha \in I.$$

It only remains to show that

$$\varphi^*\omega = \omega,$$

where $\omega \equiv \omega|_U$ is a symplectic structure on $\mathcal{E}|_U$.

To this end, we first observe that $\varphi^*\omega$, $\omega \in \mathrm{H}om_{\mathcal{A}|_U}((\mathcal{E}\times\mathcal{E})|_U, \mathcal{A}|_U)$, with

$$U \longmapsto \mathrm{H}om_{\mathcal{A}|_U}((\mathcal{E}\times\mathcal{E})|_U, \mathcal{A}|_U)$$

defining a complete presheaf of $\mathcal{A}$-modules on X, see [9], p. 134. But,

$$\varphi^*\omega|_{U_\alpha} = (\varphi|_{U_\alpha})^*\omega|_{U_\alpha} = \varphi^*_\alpha\omega|_{U_\alpha} = \omega|_{U_\alpha},$$

therefore

$$\varphi^*\omega = \omega,$$

as desired. □

On the other hand, the preceding notion of *symplectic sheaf of groups* can also be defined *through* an application of the *isomorphism* Γ, which is given in (12). Namely, since by (12), $\varphi \equiv (\varphi_U) : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E})$ is an A-symplectomorphism *if and only if* the corresponding $\mathcal{A}$-morphism is symplectomorphic, the symplectic sheaf of groups can be viewed as consisting of all $\mathcal{A}$-*symplectomorphisms*

$$\varphi : \mathcal{E} \longrightarrow \mathcal{E}.$$

Now, suppose that $\mathcal{E}$ is the *standard free* $\mathcal{A}$-*module* $\mathcal{A}^{2n}$ of rank $2n$, and let $\omega : \mathcal{A}^{2n}\oplus\mathcal{A}^{2n} \longrightarrow \mathcal{A}$ be a *skew-symmetric, non-degenerate* $\mathcal{A}$-*bilinear morphism.* Let $\{s_i\}_{1\leq i\leq 2n}$ be a *basis* of $\mathcal{A}^{2n}(X)$ such that *Theorem 3.1 holds*, and let $\varphi \in \mathcal{S}p\ \mathcal{A}^{2n}(X)$. Let's consider the *full matrix algebra sheaf* M_{2n} (see [9], p. 280) induced by the presheaf

$$U \longmapsto M_{2n}(\mathcal{A}(U)),$$

where $U \subseteq X$ is *open*, and the range, $M_{2n}(\mathcal{A}(U))$, of the preceding map consists of all $2n \times 2n$-matrices with entries in the $\mathbb{C}$-*algebra (unital and commutative)*

$$\mathcal{A}(U) \equiv \Gamma(U, \mathcal{A}).$$

Since

$$\mathcal{S}p\ \mathcal{A}^{2n}(X) \subseteq \mathcal{H}om_{\mathcal{A}}(\mathcal{A}^{2n}, \mathcal{A}^{2n})(X),$$

and by Statement 3.17, [9], p. 293,

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{A}^{2n}, \mathcal{A}^{2n})(X) = M_{2n}(\mathcal{A}(X)) = M_{2n}(\mathcal{A})(X),$$

it follows that if M is the *matrix representing* $\varphi \in \mathcal{S}p\ \mathcal{A}^{2n}(X)$ with respect to the basis $\{s_i\}_{1 \leq i \leq 2n}$ above, then Equation (18) becomes in matrix form

$$^{t}MJM = J, \tag{22}$$

where J is the matrix (16).

From (22), we deduce the following corollary.

Corollary 3.3 The *determinant* of an $\mathcal{A}$-*symplectomorphism*

$$\varphi : (\mathcal{A}^{2n}, \omega) \longrightarrow (\mathcal{A}^{2n}, \omega),$$

where $\mathcal{A}$ is a *unital and commutative* $\mathbb{C}$-*algebra sheaf* on a topological space X,is the *global identity section* $1 \in \Gamma(X, \mathcal{A}^{2n})$. More explicitly, given M as the matrix representing the $\mathcal{A}$-symplectomorphism φ, as in the paragraph preceding the corollary, one has

$$\overline{\partial et}(M) = 1 \in \Gamma(X, \mathcal{A}^{2n}). \tag{23}$$

We refer to [9], p. 294, for the definition of the determinant morphisms ∂et and $\overline{\partial et}$.

4 The Characteristic Polynomial Section

Let $\mathcal{A}$ and $\mathcal{B}$ be *sheaves of algebras* on a topological space $(X, \mathcal{T})$, and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a sheaf morphism such that, if

$$\overline{\varphi} \equiv (\overline{\varphi}_U)_{U \in \mathcal{T}} : \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B})$$

is the corresponding morphism between the associated complete presheaves of sections $\Gamma(\mathcal{A})$ and $\Gamma(\mathcal{B})$, then, for all $U \in \mathcal{T}$,

$$\overline{\varphi}_U(\mathcal{A}(U)) \subseteq C(\mathcal{B}(U)), \tag{24}$$

where $C(\mathcal{B}(U))$ stands for the center of the ring $\mathcal{B}(U)$, cf. [[7], p.121] . Explicitly, Equation (24) means that

$$\overline{\varphi}_U(s)t = t\overline{\varphi}_U(s)$$

for all $s \in \mathcal{A}(U)$ and $t \in \mathcal{B}(U)$. Now, given $s \in \mathcal{A}(U)$ and $t \in \mathcal{B}(U)$, with U an open set in X, the assignment

$$(s, t) \longmapsto \overline{\varphi}_U(s)t$$

makes $\mathcal{B}(U)$ into a module over $\mathcal{A}(U)$. What more is that $\mathcal{B}(U)$ is an algebra over $\mathcal{A}(U)$; in effect,

$$\overline{\varphi}_U(s + s')t = \overline{\varphi}_U(s)t + \overline{\varphi}_U(s')t$$

and

$$\overline{\varphi}_U(s)(t + t') = \overline{\varphi}_U(s)t + \overline{\varphi}_U(s)t'$$

for all $s, s' \in \mathcal{A}(U)$ and $t, t' \in \mathcal{B}(U)$. On the other hand, since the sheafification functor preserves algebraic structures, cf. [[9](1.54), p.101], $\mathcal{B}$ can be viewed as an $\mathcal{A}$-algebra sheaf. The $\mathcal{A}$-algebra sheaf $\mathcal{B}$ thus obtained is called an **$\mathcal{A}$-algebra sheaf with respect to the morphism** $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, above. More accurately, we may yet say that $\mathcal{B}$ is an **$\varphi(\mathcal{A})$-algebra sheaf**, with $\varphi(\mathcal{A}) \subseteq \mathcal{B}$, as above. Equivalently, by a **$\varphi(\mathcal{A})$-algebra sheaf** (or **$\varphi(\mathcal{A})$-algebra**, as a shorthand) , we shall always mean a *morphism* $\mathcal{A} \longrightarrow \mathcal{B}$ *of sheaves of algebras, as above.*

Now, let $\mathcal{A}$ be a sheaf of unital commutative $\mathbb{C}$-algebras, $\mathcal{E}$ an $\mathcal{A}$-module, and $\mathcal{R}$ a $\varphi(\mathcal{A})$-algebra. A **representation** of $\mathcal{R}$ in $\mathcal{E}$ is an $\mathcal{A}$-morphism

$$\Theta : \mathcal{R} \to \mathcal{E}nd_{\mathcal{A}}(\mathcal{E}),$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\;\Theta\;} & \mathcal{E}nd_{\mathcal{A}}(\mathcal{E}) \\ & \nwarrow \quad \nearrow & \\ & \mathcal{A} & \end{array}$$

commutative; the morphism $\mathcal{A} \longrightarrow \mathcal{E}nd_{\mathcal{A}}(\mathcal{E})$ in the above diagram is given by

$$a \longmapsto aI|_U = aI_U = (a_V I_V)_{U \supseteq V, open},$$

where $a \in \mathcal{A}(U)$ and $I_U : \mathcal{E}(U) \longrightarrow \mathcal{E}(U)$ with U open in X, denotes the identity $\mathcal{A}(U)$-morphism. Besides, for all open set V in U, $a_V \in \mathcal{A}(V)$, and $s \in \mathcal{E}(V)$,

$$(a_V I_V)(s) = a_V s \in \mathcal{E}(V).$$

We observe that for all open $U \subseteq X$, $\mathcal{E}(U)$ may be viewed as a *module* over $\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$. Indeed, let $f \equiv (f_V) \in \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$, where V runs over the open subsets of U, and $s \in \mathcal{E}(U)$. The action

$$(f, s) \longmapsto f_U(s)$$

defines a $\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$-module structure on $\mathcal{E}(U)$, as was to be shown. So, by means of the sheafification process, $\mathcal{E}$ may be viewed as an *$\mathcal{E}nd_{\mathcal{A}}\mathcal{E}$-module*. Furthermore, given a representation $\Theta : \mathcal{R} \longrightarrow \mathcal{E}nd_{\mathcal{A}}\mathcal{E}$ of a $\varphi(\mathcal{A})$-algebra sheaf $\mathcal{R}$ in $\mathcal{E}$, it turns out that $\mathcal{E}$ may be viewed as an $\mathcal{R}$-module, with the operation of $\mathcal{R}(U)$ on $\mathcal{E}(U)$ being given by

$$(s, e) \longmapsto \Theta_U(s)e \equiv \Theta_U(s)(e)$$

for $s \in \mathcal{R}(U)$ and $e \in \mathcal{E}(U)$. Like in [7], p.554, we write se instead of the more accurate notation $\Theta_U(s)(e)$.

Definition 4.1 Let $\mathcal{B}$ be a *sheaf of* $\mathbb{C}$*-algebras* over a topological space X, and $\mathcal{A}$ a subsheaf of $\mathcal{B}$. For all open set U in X, let

$$\mathcal{A}(U)[t]$$

denote the *ring of polynomials*, in the variable t, whose *coefficients* are the (*local*) *sections* of $\mathcal{A}$ on U. Similarly, let

$$\mathcal{A}(U)[s],$$

where $s \in \mathcal{B}(U)$, denote the subring of $\mathcal{B}(U)$ of all *polynomial values* $p(s)$, with $p \in \mathcal{A}(U)[t]$. A *local section* $s \in \mathcal{B}(U)$ is called **transcendental** over $\mathcal{A}(U)$ if the evaluation map

$$\mathcal{A}(U)[t] \longrightarrow \mathcal{A}(U)[s], \quad p \longmapsto p(s)$$

is an *isomorphism*.

Now, suppose that $\mathcal{A}$ is a *sheaf of unital commutative of* $\mathbb{C}$*-algebras* on the topological space X. As above, we let $\mathcal{A}(U)[t]$, where $U \subseteq X$ is open, be the polynomial ring. It is clear that the correspondence

$$U \longmapsto \mathcal{A}(U)[t], \tag{25}$$

where $U \subseteq X$ is *open*, yields a *complete presheaf of $\mathcal{A}$-modules on X*. So, (25) defines a *complete* $\Gamma(\mathcal{A})$*-presheaf on* X. The sheaf generated, on X, by the complete presheaf defined by (25) is called the **sheaf of germs of polynomials on** $\Gamma(\mathcal{A})$, and is denoted by

$$\mathcal{A}[t],$$

where t is a variable. It is easy to verify that the sheaf $\mathcal{A}[t]$ of germs of polynomials is an *$\mathcal{A}$-module* on X. Thus, based on [9], Proposition 11.1, p. 51, one obtains, for every open set $U \subseteq X$,

$$\mathcal{A}[t](U) = \mathcal{A}(U)[t],$$

up to an $\mathcal{A}(U)$-isomorphism.

Let $(X, \mathcal{A})$ be an ordered algebraized space, equipped with the *inverse-positive-section condition*, $\mathcal{E}$ a vector sheaf of rank n, and $\varphi \in \mathcal{E}nd_{\mathcal{A}}\mathcal{E}$. Let t be transcendental over $\mathcal{A}(U)$, with U open in X, and

$$\mathcal{A}(U)[t] \longrightarrow \mathcal{A}(U)[\varphi_U] \subseteq \mathrm{Hom}_{\mathcal{A}(U)}(\mathcal{E}(U), \mathcal{E}(U))$$

be a representation of the polynomial ring $\mathcal{A}(U)[t]$ in $\mathcal{E}(U)$. Like in [7], p. 561, we have for every *open* set $U \subseteq X$, a homomorphism

$$\mathcal{A}(U)[t] \longrightarrow \mathcal{A}(U)[\varphi_U],$$

which is obtained by *substituting* φ_U for t in *polynomials*. The $\mathcal{A}(U)$-algebra $\mathcal{A}(U)[\varphi_U]$ is the *subalgebra* of $\mathrm{End}_{\mathcal{A}(U)}\mathcal{E}(U)$, generated by φ_U, and is *commutative* because

$$\varphi_U^p \circ \varphi_U^q = \varphi_U^q \circ \varphi_U^p,$$

for all $p, q \in \mathbb{N}$. Thus, for all $s \in \mathcal{E}(U)$ and $f(t) \in \mathcal{A}(U)[t]$, where U is, as usual, an *open subset* of X, we put

$$f(t)s \equiv f(t)(s) := f(\varphi_U)(s) \equiv f(\varphi_U)s;$$

consequently $\mathcal{E}(U)$ turns out to be a *module* over $\mathcal{A}(U)[t]$. Let M_U be any $n \times n$ *matrix* in $\mathcal{A}(U)$ (for instance the matrix representing the $\mathcal{A}(U)$-endomorphism φ_U relative to a *canonical basis* $\{e_i^U\}$ of $\mathcal{E}(U)$, where U is a local gauge of the vector sheaf $\mathcal{E}$, and $e_i^U = \varepsilon_i^U \circ \varphi_i^U$, with φ^U being the $\mathcal{A}|_U$-isomorphism (2). The basis $\{e_i^U\}$ is called a *canonical gauge* of $\mathcal{E}(U)$.). We define the **characteristic polynomial section** $P_{M_U}(t)$ of M_U or of φ_U to be the *determinant*

$$\overline{\partial et}_U(tI_U - M_U) := \det{}_U(tI_U - M_U) \in \mathcal{A}(U)[t],$$

where I_U is the unit $n \times n$-matrix (here, $1 := 1_U$ is the (local) identity section over U). (We refer to [9], pp 294-298, for the sheaf-theoretic notation of the determinant morphism.) Next, we decree that the **characteristic polynomial of an endomorphism** $\varphi \in \mathcal{E}nd_{\mathcal{A}}\mathcal{E}$ is the endomorphism $P_\varphi(t) \in \mathcal{E}nd_{\mathcal{A}}\mathcal{A}$, given by

$$P_\varphi(t) \equiv (P_{\varphi_U}(t)) = (\det_U(tI_U - M_U)),$$

where M_U represents $\varphi_U \in \mathcal{E}nd_{\mathcal{A}(U)}\mathcal{E}(U)$ with respect to the local gauge $\{e_i^U\}$ of $\mathcal{E}(U)$. Let

$$M_x = M_U(x);$$

its characteristic polynomial is, as obviously expected, given by

$$\overline{\partial et}_U(tI_U - M_U)(x) := \overline{\partial et}_x(tI_x - M_x) = \det{}_x(tI_x - M_x) \in \mathcal{A}_x[t],$$

where $\mathcal{A}_x[t] := \lim_{\overrightarrow{U \ni x}} \mathcal{A}(U)[t]$.

The characteristic polynomial section will also be referred to simply as *characteristic polynomial*.

We further look at the following correspondence

$$U \longmapsto ChP(\mathcal{A}(U)), \tag{26}$$

where U ranges over the *open* subsets of X, while the range of (26) is the set of all *characteristic polynomials* $P_{M_U}(t)$ of $n \times n$*-matrices* M_U, whose *entries* are (*local*) *sections of* $\mathcal{A}$ *on* U. Now, the *presheaf of full matrix* $\mathbb{C}$*-algebras* on X,

$$U \longmapsto M_n(\mathcal{A}(U)), \tag{27}$$

where $U \subseteq X$ is *open*, and $M_n(\mathcal{A}(U))$ the (*full*) *algebra of* $n \times n$*-matrices*, with entries the (local) sections of $\mathcal{A}$ on U, is complete. Hence, (26) yields a complete presheaf on X; it is called *presheaf of characteristic polynomials on* X. For the restriction maps of the presheaf defined by (27), see [9], pp 280-281.

So, we denote by

$$\mathcal{C}h\mathcal{P}(\mathcal{A})$$

the *sheaf of modules on* X, generated by the previous presheaf.

Now before we proceed over to the version of the *Cayley-Hamilton theorem* in this setting, we signal in passing (One might work through all the details to their own satisfaction!) that all the fundamental classical properties of the *determinant morphism* are also valid in our context. One of the properties, useful for the scope of the present paper, follows after this: Let $(X, \mathcal{A}, \mathcal{P})$ be as usual an ordered $\mathbb{R}$-algebraized space satisfying the inverse-positive-section condition. Let $A = (s_{ij}) \in M_n(\mathcal{A}(X))$, and $\widetilde{A} = (t_{ij}) \in M_n(\mathcal{A}(X))$ such that

$$t_{ij} := (-1)^{i+j}\overline{\partial et}_X(A_{ji}) \equiv (-1)^{i+j}\det{}_X(A_{ji});$$

A_{ji} is the $(n-1) \times (n-1)$ matrix obtained from $A = (s_{ij})_{1 \leq i,j \leq n}$ by deleting the j-th row and i-th column.

Proposition 4.1 (***Laplace decomposition***)*, cf.* [*13*]. Let $\det_X(A) = s$, with $s \in \mathcal{A}^{\bullet}(X)$. Then, $A\widetilde{A} = \widetilde{A}A = sI$. Furthermore, $\det_X(A) \in \mathcal{A}^{\bullet}(X)$ if and only if $A \in M_n(\mathcal{A})^{\bullet}(X)$; consequently

$$A^{-1} = s^{-1}\widetilde{A}.$$

Proof. The proof goes along similar lines as the proof of Proposition 4.16 in [7]. □

Theorem 4.1 (**Cayley-Hamilton**) Let $\mathcal{E}$ be a *vector sheaf of rank* n on X, and $\varphi : \mathcal{E} \to \mathcal{E}$ an $\mathcal{A}$-morphism. Then,

$$P_{\varphi}(\varphi) \equiv (P_{\varphi_U}(\varphi_U)) = (0_U) \equiv 0.$$

Proof. Here as well, we base our proof on the proof of Theorem 3.1., [7], p.561. In fact, let U be an *open* subset of X such that $\mathcal{E}|_U = \mathcal{A}^n|_U$, and $\{e_i^U\}_{1\leq i\leq n}$ be the gauge on $\mathcal{E}|_U(U) = \mathcal{E}(U)$, corresponding the Kronecker gauge of . Then, since $\mathcal{E}(U)$ may be viewed as a module over $\mathcal{A}(U)[t]$, one has

$$te_j^U = \sum_{i=1}^{n} s_{ij}^U e_i^U,$$

where $1 \leq j \leq n$, and $(s_{ij}^U) \equiv M_U$ is the matrix representing φ_U with respect to $\{e_i^U\}$. Let $B_U(t) = tI_U - M_U$; then

$$\widetilde{B}_U(t)B_U(t) = P_{\varphi_U}(t)I_U,$$

and

$$\widetilde{B}_U(t)B_U(t)\begin{pmatrix} e_1^U \\ \vdots \\ e_n^U \end{pmatrix} = \begin{pmatrix} P_{\varphi_U}(t)e_1^U \\ \vdots \\ P_{\varphi_U}(t)e_n^U \end{pmatrix} = \begin{pmatrix} 0_U \\ \vdots \\ 0_U \end{pmatrix}$$

because

$$B(t)\begin{pmatrix} e_1^U \\ \vdots \\ e_n^U \end{pmatrix} = \begin{pmatrix} 0_U \\ \vdots \\ 0_U \end{pmatrix}.$$

It follows that $P_{\varphi_U}(t)(\mathcal{E}(U)) = \{0_U\}$, and therefore $P_{\varphi_U}(\varphi_U)(\mathcal{E}(U)) = \{0_U\}$. This implies that $P_{\varphi_U}(\varphi_U)(\mathcal{E}(U)) = \{0_U\}$, as was to be shown. □

Now, suppose that the pair $(X, \mathcal{A})$ is an ordered algebraized space, satisfying the *inverse-positive-section condition.* Moreover, as above, we suppose that $\mathcal{E}$ is a vector sheaf of rank n on X, and $\varphi : \mathcal{E} \longrightarrow \mathcal{E}$ an $\mathcal{A}$-endomorphism; and we let $s \in \mathcal{E}(U) \equiv \Gamma(U, \mathcal{E})$, where U is an open set in X. We further consider the *associated* endomorphism $\overline{\varphi} \equiv (\overline{\varphi}_U)_{U\in\mathcal{T}} : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E})$, where $\mathcal{T}$ is the assumed topology on X, and $\Gamma(\mathcal{E})$ is the presheaf of sections of $\mathcal{E}$. Now, we fix $U \in \mathcal{T}$ such that $\mathcal{E}|_U = \mathcal{A}^n|_U$. By an **eigenvector section**, or just **eigenvector**, of $\overline{\varphi}_U \in \mathrm{End}_{\mathcal{A}(U)}(\mathcal{E}(U))$, we mean a *nowhere-zero (local) section* $s \in \mathcal{E}(U)$, such that there exists a *section* $\lambda \in \mathcal{A}(U)$ for which we have

$$\overline{\varphi}_U(s) = \lambda s, \tag{28}$$

or equivalently

$$M_U s = \lambda s,$$

where M_U is the *matrix* representing $\overline{\varphi}_U$ with respect to the local *Kronecker gauge* on $\mathcal{E}(U)$. (A (local) section $s \in \mathcal{E}(U)$ is called a *nowhere-zero* section if $s_x \equiv s(x) \neq 0_x$ for all $x \in U$.) The *scalar section* λ, in Equation (28), is called an **eigenvalue section** or simply **eigenvalue** of the morphism $\overline{\varphi}_U$.

Next, we construct the following presheaf of sets, namely

$$U \longmapsto \mathrm{PV}(\mathcal{E}(U)), \tag{29}$$

where U varies over the *open subsets* of X, and the range of (29), i.e. $\mathrm{PV}(\mathcal{E}(U))$, is the *set of all eigenvectors* of $\overline{\varphi}_U$, with $\overline{\varphi}_U \in \mathrm{End}_{\mathcal{A}(U)}(\mathcal{E}(U))$.

Proposition 4.2 Let $\mathcal{E}$ be an $\mathcal{A}$-module on X. The presheaf $(\mathrm{PV}(\mathcal{E}(U)), \sigma^U_V)$, where for $s \in \mathrm{PV}(\mathcal{E}(U))$, $\sigma^U_V(s) \equiv s|_V$, is a *complete* presheaf.

Proof. Indeed, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of U, and let s, t be two elements of $\mathrm{PV}(\mathcal{E}(U))$ such that

$$\sigma^U_{U_\alpha}(s) \equiv s_\alpha = t_\alpha \equiv \sigma^U_{U_\alpha}(t), \quad \alpha \in I,$$

where the σ^U_V, $U \supseteq V =$ open, are the *restriction maps* of the aforementioned presheaf. Now, as before let $\Gamma(\mathcal{A}) \equiv (\Gamma(U, \mathcal{A}), \rho^U_V)$ be the presheaf of sections of the sheaf $\mathcal{A}$. Then, we have, assuming that $\lambda \in \mathcal{A}(U)$ is the eigenvalue associated with the eigenvector $s \in \mathrm{PV}(\mathcal{E}(U))$, that

$$\overline{\varphi}_{U_\alpha}(s_\alpha) = \sigma^U_{U_\alpha}(\overline{\varphi}_U(s)) = \sigma^U_{U_\alpha}(\lambda s) = \rho^U_{U_\alpha}(\lambda)\sigma^U_{U_\alpha}(s) = \rho^U_{U_\alpha}(\lambda)s_\alpha \equiv \lambda_\alpha s_\alpha.$$

Likewise,

$$\overline{\varphi}_{U_\alpha}(t_\alpha) = \mu_\alpha t_\alpha,$$

with $\mu \in \mathcal{A}(U)$ being the eigenvalue for the eigenvector t. As, by *hypothesis*, $s_\alpha = t_\alpha$, it follows that $\overline{\varphi}_{U_\alpha}(s_\alpha) = \overline{\varphi}_{U_\alpha}(t_\alpha)$; whence $\rho^U_{U_\alpha}(\lambda) \equiv \lambda_\alpha = \mu_\alpha \equiv \rho^U_{U_\alpha}(\mu)$. But $\Gamma(\mathcal{A})$ is a *complete* presheaf, therefore $\lambda = \mu$; so that $s = t$, as desired.

Now, let $(s_\alpha) \in \prod_\alpha \mathrm{PV}(\mathcal{E}(U_\alpha))$ such that for any $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ in $\mathcal{U}$, one has

$$\sigma^{U_\alpha}_{U_{\alpha\beta}}(s_\alpha) \equiv s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}} \equiv \sigma^{U_\alpha}_{U_{\alpha\beta}}(s_\beta). \tag{30}$$

The sequence (s_α) of *eigenvectors* gives rise to a sequence

$$(M_{n,\alpha}) \in \prod_\alpha M_n(\mathcal{E}(U_\alpha))$$

of $n \times n$-matrices whose entries are (local) *sections* of $\mathcal{E}$, and admitting the s_α as eigenvectors correspondingly. It is clear that for any α, $\beta \in I$ such that s_α, s_β fulfill (30), one has

$$M_n(\sigma^{U_\alpha}_{U_{\alpha\beta}})(s^{U_\alpha}_{ij}) := (\sigma^{U_\alpha}_{U_{\alpha\beta}}(s^{U_\alpha}_{ij})) = (\sigma^{U_\beta}_{U_{\alpha\beta}}(s^{U_\beta}_{ij})) =: M_n(\sigma^{U_\beta}_{U_{\alpha\beta}})(s^{U_\beta}_{ij}),$$

where s_α and s_β are eigenvectors of matrices $(s^{U_\alpha}_{ij}) \in M_n(\mathcal{E}(U_\alpha))$ and $(s^{U_\beta}_{ij}) \in M_n(\mathcal{E}(U_\beta))$, respectively. But the *presheaf*

$$(M_n(\mathcal{E}(U)), M_n(\sigma^U_V)),$$

cf. [9], p. 281, is a *complete* presheaf, therefore there exists a matrix $M \in M_n(\mathcal{E}(U))$ such that

$$M_n(\sigma^U_{U_\alpha})(M) = M_{n,\alpha}$$

for all $\alpha \in I$. Let $s \in \Gamma(U, \mathcal{E})$ such that $\sigma^U_{U_\alpha}(s) = s_\alpha$, $\alpha \in I$. It is easily seen that

$$\sigma^U_{U_\alpha}(Ms) = M_n(\sigma^U_{U_\alpha})(M)\sigma^U_{U_\alpha}(s) = M_{n,\alpha}s_\alpha = \lambda_\alpha s_\alpha,$$

$\alpha \in I$, which implies that

$$Ms = \lambda s,$$

where $\lambda \in \mathcal{A}(U)$ is derived from the λ_α, $\alpha \in I$, with $\lambda_\alpha \in \mathcal{A}(U_\alpha)$ and $\sigma^{U_\alpha}_{U_{\alpha\beta}}(\lambda_\alpha) = \sigma^{U_\beta}_{U_{\alpha\beta}}(\lambda_\beta)$. Hence, axiom (S2), see [9], p46, is satisfied. □

Definition 4.2 Let $\mathcal{A}$ be a unital commutative $\mathbb{C}$-algebra sheaf on a topological space X, and let $\mathcal{E}$ be an $\mathcal{A}$-module on X. We denote by

$$\mathcal{PV}(\mathcal{E})$$

the sheaf on X, generated by the presheaf defined by (29). We call it the **eigenvector sheaf** of $\mathcal{E}$ or **sheaf of germs of eigenvectors** of $\mathcal{E}$.

Proposition 4.3 Let $\mathcal{A}$ be a unital $\mathbb{C}$-algebra sheaf on a topological space X, $\omega : \mathcal{A}^{2n} \oplus \mathcal{A}^{2n} \longrightarrow \mathcal{A}$, $n \in \mathbb{N}$, a *symplectic structure* on $\mathcal{A}$, and $\varphi \in \mathcal{S}p\ \mathcal{A}^{2n}(X)$, cf. Lemma 3.2 for the definition of $\varphi \in \mathcal{S}p\ \mathcal{A}^{2n}(X)$. Moreover, let $\lambda \in \mathcal{A}^\bullet(X)$ be an *eigenvalue of* φ. Then, $\frac{1}{\lambda} \in \mathcal{A}(X)$ is an *eigenvalue of* φ too.

Proof. Let $\{s_i\}_{1\leq i\leq 2n}$ be a *basis* of $\mathcal{A}^{2n}(X)$ such that $(\omega_{ij})_{1\leq i,j\leq 2n} = J$, where J is given by (16) and $\omega_{ij} = \omega(s_i, s_j)$, and let M be the $2n \times 2n$-*matrix representing* the symplectomorphism φ with respect to the aforementioned basis.

Consider the *characteristic polynomial (section)* of M

$$P(t) = \det_X(M - tI),$$

where I is understood as the $2n \times 2n$ identity matrix, and $t \in \mathcal{A}(X)$ a *variable*. Then, by virtue of (22) and (23), we have

$$P(t) = t^{2n} P(\frac{1}{t}),$$

as is done in [1] and [2]. Thus,

$$P(\lambda) = \lambda^{2n} P(\frac{1}{\lambda}).$$

But $P(\lambda) = 0$ by Cayley-Hamilton theorem, and $\lambda \in \mathcal{A}(X) = \mathcal{A}^\bullet(X)$, so that $P(\frac{1}{\lambda}) = 0$. □

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