COMMON FIXED POINTS OF COMMUTATIVE SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we discuss characterizations of common fixed points of commutative semigroups of nonexpansive mappings. We next prove convergence theorems to a common fixed point. We finally discuss nonexpansive retractions onto the set of common fixed points. In our discussion, we may not assume the strict convexity of the Banach space.

1. INTRODUCTION

Let C be a closed convex subset of a Banach space E. A mapping T on C is called a *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Kirk [21] proved that $F(T)$ is nonempty in the case that C is weakly compact and has normal structure. See also [3, 4, 5, 14] and others. If C is weakly compact and E has the Opial property, then C has normal structure; see [15]. Thus, $F(T)$ is nonempty in the case that C is weakly compact and E has the Opial property.

Let $(S, +)$ be a *commutative semigroup*, i.e.,

(i) $s + t \in S$ for $s, t \in S$;
(ii) $(s + t) + u = s + (t + u)$ for $s, t, u \in S$; and
(iii) $s + t = t + s$ for $s, t \in S$.

Then a family $\{T(t) : t \in S\}$ of mappings on C is called a *commutative semigroup of nonexpansive mappings* on C (a *nonexpansive semigroup* on C, for short) if the following are satisfied:

(sg 1) for each $t \in S$, $T(t)$ is a nonexpansive mapping on C; and
(sg 2) $T(s + t) = T(s) \circ T(t)$ for all $s, t \in S$.

We put $F(\mathcal{S}) = \bigcap_{t \in S} F\big(T(t)\big)$. Common fixed point theorems for families of nonexpansive mappings are proved in [5, 7, 9] and others. The following is the corollary of the famous theorem proved by Bruck [7].

Theorem 1 (Bruck [7]). *Let S be a commutative semigroup and let $\{T(t) : t \in S\}$ be a nonexpansive semigroup on a weakly compact convex subset C of a Banach space E. Suppose that C has the fixed point property for nonexpansive mappings. Then $F(\mathcal{S})$ is a nonempty nonexpansive retract of C.*

We note that from this theorem, $F(\mathcal{S})$ is nonempty in the case that C is weakly compact and E has the Opial property.

Many convergence theorems for nonexpansive mappings and families of nonexpansive mappings have been studied; see [1, 2, 6, 11, 16, 22, 25, 29, 33, 40] and

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others. In these theorems, we assume the strict convexity of the Banach space E. In the results of [12, 18, 19, 36], we may not assume the strict convexity of E. Very recently, the author in [32] proved strong convergence theorems for families of nonexpansive mappings without the assumption of the strict convexity. See also [37].

In this paper, we extend the results in [32, 37] to commutative semigroups of nonexpansive mappings. We also discuss characterizations of common fixed points of nonexpansive semigroups and nonexpansive retractions onto the set of common fixed points. In our discussion, we may not assume the strict convexity of the Banach space.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. Let A be a subset of a set S. Then we define a function I_A from S into $\mathbb{R}$ by

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

Let E be a Banach space. We denote by E^* the dual of E. We recall that E is said to have the *Opial property* [27] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x_0, $\liminf_n \|x_n - x_0\| < \liminf_n \|x_n - x\|$ for all $x \in E$ with $x \neq x_0$. All Hilbert spaces, all finite dimensional Banach spaces and $\ell^p (1 \leq p < \infty)$ have the Opial property. A Banach space with a duality mapping which is weakly sequentially continuous also has the Opial property; see [15]. We know that every separable Banach space can be equivalently renormed so that it has the Opial property; see [10]. See also [23, 24, 28, 31] and others.

We recall that a closed convex subset C of a Banach space E has the fixed point property for nonexpansive mappings if for every bounded closed convex subset D of C and for every nonexpansive mapping on D has a fixed point. Every compact convex subset of any Banach space has the fixed point property for nonexpansive mappings. Also, does every weakly compact convex subset of a Banach space with the Opial property.

Let $(S, +)$ be a commutative semigroup. Then we can consider that S is a directed set with relation $\leq$ defined as follows: $s \leq t$ if and only if $s = t$ or there exists $u \in S$ such that $s + u = t$. We denote by $B(S)$ the Banach space consisting of all bounded functions from S into $\mathbb{R}$ with supremum norm. For $s \in S$, we define a mapping ℓ_s on $B(S)$ by

$$(\ell_s a)(t) = a(s + t)$$

for $a \in B(S)$ and $t \in S$. Let X be a linear subspace of $B(S)$ such that $I_S \in X$ and X is ℓ_s-invariant for all $s \in S$. We call that $\mu \in X^*$ is a *mean* on X if $\|\mu\| = \mu(I_S) = 1$. We know that μ is a mean on X if and only if

$$\inf_{t \in S} a(t) \leq \mu(a) \leq \sup_{t \in S} a(t)$$

for all $a \in X$; see [39] and others. We also know that if $a, b \in X$ satisfies $a(t) \leq b(t)$ for all $t \in S$, then $\mu(a) \leq \mu(b)$. Sometimes, we denote by $\mu_t\big(a(t)\big)$ the value $\mu(a)$. A mean μ on X is called *invariant* if

$$\mu_t\big(a(t)\big) = \mu_t\big(a(s + t)\big)$$

for all $a \in X$ and $s \in S$. We note that since S is commutative, there exists an invariant mean on X. Let $\{\mu_\alpha : \alpha \in D\}$ be a net of means on X. Then $\{\mu_\alpha\}$ is called *asymptotically invariant* [8] if

$$\lim_{\alpha \in D} \Big(\mu_\alpha(a) - (\mu_\alpha)_t\big(a(s+t)\big) \Big) = 0$$

for all $a \in X$ and $s \in S$. It is obvious that if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on X, then so is every subnet $\{\mu_{\alpha_\beta}\}$ of $\{\mu_\alpha\}$.

Let E be a Banach space and let C be a weakly compact convex subset of E. Let S be a commutative semigroup and let $\{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a linear subspace of $B(S)$ such that $I_S \in X$, X is ℓ_s-invariant for all $s \in S$, and $\Big(t \mapsto f\big(T(t)x\big)\Big) \in X$ for all $x \in C$ and $f \in E^*$. Let μ be a mean on X. Then we know that for each $x \in C$, there exists a unique element x_0 of C satisfying

$$\mu_t\Big(f\big(T(t)x\big)\Big) = f(x_0)$$

for all $f \in E^*$; see [17, 38]. Following Rodé [30], we denote such x_0 by $T_\mu x$. We also know that T_μ is a nonexpansive mapping on C.

We now prove the following, which are used in Section 3.

Lemma 1. *Let S be a commutative semigroup, and let $\{\alpha_t : t \in S\}$ be a real net. Then*

$$\liminf_{t \in S} \alpha_t = \liminf_{t \in S} \alpha_{s+t} \quad \text{and} \quad \limsup_{t \in S} \alpha_t = \limsup_{t \in S} \alpha_{s+t}$$

for $s \in S$.

Proof. Fix $\lambda \in \mathbb{R}$ with $\liminf_{t \in S} \alpha_t < \lambda$. For $t_1 \in S$, since $s + t_1 \in S$, there exists $t_2 \in S$ such that $t_2 \geq s + t_1$ and $\alpha_{t_2} < \lambda$. In the case of $t_2 = s + t_1$, we put $t_3 = t_1$. In the case that there exists $t_4 \in S$ such that $t_2 = s + t_1 + t_4$, we put $t_3 = t_1 + t_4$. In both cases, we have

$$t_3 \geq t_1 \quad \text{and} \quad \alpha_{s+t_3} = \alpha_{t_2} < \lambda.$$

Therefore $\liminf_{t \in S} \alpha_{s+t} \leq \lambda$. Since $\lambda \in \mathbb{R}$ is arbitrary, we obtain

$$\liminf_{t \in S} \alpha_t \geq \liminf_{t \in S} \alpha_{s+t}.$$

Fix $\lambda \in \mathbb{R}$ with $\liminf_{t \in S} \alpha_t > \lambda$. Then there exists $t_5 \in S$ such that $\alpha_t > \lambda$ for $t \geq t_5$. Since $t \geq t_5$ implies $s + t \geq t_5$, we have $\alpha_{s+t} > \lambda$ for $t \geq t_5$. Therefore $\liminf_{t \in S} \alpha_{s+t} \geq \lambda$. Since $\lambda \in \mathbb{R}$ is arbitrary, we obtain

$$\liminf_{t \in S} \alpha_t \leq \liminf_{t \in S} \alpha_{s+t}.$$

Hence $\liminf_{t \in S} \alpha_t = \liminf_{t \in S} \alpha_{s+t}$. We also have

$$\limsup_{t \in S} \alpha_t = -\liminf_{t \in S}(-\alpha_t) = -\liminf_{t \in S}(-\alpha_{s+t}) = \limsup_{t \in S} \alpha_{s+t}.$$

This completes the proof. □

Lemma 2. *Let S be a commutative semigroup, and let $\tilde{\mu}$ be a mean on $B(S)$. Let $A_1, A_2, A_3, \cdots, A_k$ be subsets of S. Put*

$$A = \bigcap_{j=1}^{k} A_j \quad \text{and} \quad \alpha = \sum_{j=1}^{k} \liminf_{s \in S} \tilde{\mu}_t\big(I_{A_j}(s+t)\big) - k + 1.$$

Suppose $\alpha > 0$. *Then*

$$\liminf_{s \in S} \tilde{\mu}_t\big(I_A(s+t)\big) \geq \alpha \quad \textit{and} \quad \{s_0 + t : t \in S\} \cap A \neq \varnothing$$

hold for all $s_0 \in S$.

Proof. It is obvious that $t \in A$ if and only if $\sum_{j=1}^k I_{A_j}(t) = k$, and $t \in S \setminus A$ if and only if $\sum_{j=1}^k I_{A_j}(t) \leq k-1$. Therefore we obtain

$$I_A(t) \geq \sum_{j=1}^k I_{A_j}(t) - k + 1$$

for all $t \in S$. Hence,

$$\begin{aligned}
\liminf_{s \in S} \tilde{\mu}_t\big(I_A(s+t)\big) &\geq \liminf_{s \in S} \tilde{\mu}_t \left(\sum_{j=1}^k I_{A_j}(s+t) - k + 1 \right) \\
&= \liminf_{s \in S} \left(\sum_{j=1}^k \tilde{\mu}_t\big(I_{A_j}(s+t)\big) - k + 1 \right) \\
&\geq \sum_{j=1}^k \liminf_{s \in S} \tilde{\mu}_t\big(I_{A_j}(s+t)\big) - k + 1 \\
&= \alpha > 0.
\end{aligned}$$

Therefore there exists $s_1 \in S$ such that

$$\inf_{s \geq s_1} \tilde{\mu}_t\big(I_A(s+t)\big) \geq \frac{\alpha}{2}.$$

We suppose that there exists $s_0 \in S$ such that $\{s_0 + t : t \in S\} \cap A = \varnothing$. Then $\{s_0 + s_1 + t : t \in S\} \cap A = \varnothing$ and hence $I_A(s_0 + s_1 + t) = 0$ for $t \in S$. Since $s_0 + s_1 \geq s_1$, we obtain

$$0 < \frac{\alpha}{2} \leq \tilde{\mu}_t\big(I_A(s_0 + s_1 + t)\big) = \tilde{\mu}(0) = 0.$$

This is a contradiction. This completes the proof. □

Lemma 3. *Let* S *be a commutative semigroup and let* μ *be an invariant mean on* $B(S)$. *Let* $A_1, A_2, A_3, \cdots, A_k$ *be subsets of* S. *Put*

$$A = \bigcap_{j=1}^k A_j \quad \textit{and} \quad \alpha = \sum_{j=1}^k \mu(I_{A_j}) - k + 1.$$

Suppose $\alpha > 0$. *Then* $\mu(I_A) \geq \alpha$ *holds and* $\{s_0 + t : t \in S\} \cap A \neq \varnothing$ *hold for all* $s_0 \in S$.

Proof. For every subset B of S, we have

$$\liminf_{s \in S} \mu_t\big(I_B(s+t)\big) = \liminf_{s \in S} \mu_t\big(I_B(t)\big) = \mu(I_B).$$

So, by Lemma 2, we obtain the desired result. □

3. Characterizations

In this section, we discuss the characterization of common fixed points.

We first prove the following, which plays an important role in this paper.

Theorem 2. *Let E be a Banach space and let C be a weakly compact convex subset of E. Let S be a commutative semigroup and let $\{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a linear subspace of $B(S)$ such that $I_S \in X$, X is ℓ_s-invariant for all $s \in S$, and $\Big(t \mapsto f\big(T(t)x\big)\Big) \in X$ for all $x \in C$ and $f \in E^*$. Let μ be an invariant mean on X. Suppose that $T_\mu z = z$ for some $z \in C$. Then there exist sequences $\{p_n\}$ and $\{q_n\}$ in S and $\{f_n\}$ in E^* such that*

$$\begin{aligned}
&p_{n+1} = p_n + q_n,\\
&\|T(p_n)z - z\| \geq \lambda - \frac{1}{3^{n+1}},\\
&f_\ell\big(T(p_n)z - z\big) \leq \frac{2^{\ell+1}}{3^{\ell+1}} \quad \textit{for } \ell = 1, 2, \cdots, n-1,\\
&\|f_n\| = 1 \quad \textit{and} \quad f_n\big(T(p_n)z - z\big) = \|T(p_n)z - z\|
\end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\lambda = \limsup_{t \in S} \|T(t)z - z\|.$$

Before proving it, we need some preliminaries. In the following lemmas and the proof of Theorem 2, we put

$$A(f, \varepsilon) = \{t \in S : f\big(T(t)z - z\big) \leq \varepsilon\}$$

for $f \in E^*$ and $\varepsilon > 0$, and

$$B(\varepsilon) = \{t \in S : \|T(t)z - z\| \geq \lambda - \varepsilon\}$$

for $\varepsilon > 0$. By the Hahn-Banach theorem, there exists an extension $\tilde{\mu}$ of μ such that the domain of $\tilde{\mu}$ is $B(S)$ and $\|\tilde{\mu}\| = \|\mu\| = 1$. It is obvious that such $\tilde{\mu}$ is a mean on $B(S)$.

Lemma 4. *For every $s \in S$, $\|T(s)z - z\| \leq \lambda$ holds.*

Proof. Fix $s \in S$ and $\varepsilon > 0$. Then by the definition of λ, there exists $t_0 \in S$ such that

$$\sup_{t \geq t_0} \|T(t)z - z\| \leq \lambda + \varepsilon.$$

Hence, for each $t \in S$, we have

$$\|T(t_0 + t)z - z\| \leq \lambda + \varepsilon$$

because $t_0 + t \geq t_0$. By the Hahn-Banach theorem, there exists $f \in E^*$ such that

$$\|f\| = 1 \quad \text{and} \quad f\big(T(s)z - z\big) = \|T(s)z - z\|.$$

For $t \in S$, we have

$$\begin{aligned}
\|T(s)z - z\| &= f\big(T(s)z - z\big)\\
&= f\big(T(s)z - T(s + t_0 + t)z\big) + f\big(T(s + t_0 + t)z - z\big)\\
&\leq \|f\|\ \|T(s)z - T(s + t_0 + t)z\| + f\big(T(s + t_0 + t)z - z\big)\\
&= \|T(s)z - T(s) \circ T(t_0 + t)z\| + f\big(T(s + t_0 + t)z - z\big)
\end{aligned}$$

$$\begin{aligned}&\le \|T(t_0+t)z-z\| + f\big(T(s+t_0+t)z-z\big)\\&\le \lambda+\varepsilon+f\big(T(s+t_0+t)z-z\big).\end{aligned}$$

Since μ is an invariant mean on X, we have

$$\begin{aligned}\|T(s)z-z\| &= \mu_t\big(\|T(s)z-z\|\big)\\&\le \mu_t\Big(\lambda+\varepsilon+f\big(T(s+t_0+t)z-z\big)\Big)\\&= \lambda+\varepsilon+\mu_t\Big(f\big(T(s+t_0+t)z\big)\Big)-f(z)\\&= \lambda+\varepsilon+\mu_t\Big(f\big(T(t)z\big)\Big)-f(z)\\&= \lambda+\varepsilon+f(T_\mu z)-f(z)\\&= \lambda+\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we have $\|T(s)z-z\|\le\lambda$. This completes the proof. □

Lemma 5. *Fix $s_0\in S$ and let $f\in E^*$ with*

$$\|f\|=1 \quad \textit{and} \quad f\big(T(s_0)z-z\big)=\|T(s_0)z-z\|.$$

Let δ be a positive real number satisfying $\|T(s_0)z-z\|\ge\lambda-\delta$. Then

$$\liminf_{s\in S}\tilde{\mu}_t\big(I_{A(f,\varepsilon)}(s+t)\big)\ge\frac{\varepsilon}{\varepsilon+\delta}$$

hold for all $\varepsilon>0$.

Proof. For $t\in S$, by Lemma 4, we have

$$\|T(s_0)z-T(s_0+t)z\|=\|T(s_0)z-T(s_0)\circ T(t)z\|\le\|T(t)z-z\|\le\lambda$$

and hence

$$\begin{aligned}f\big(T(s_0+t)z-z\big)&=f\big(T(s_0+t)z-T(s_0)z\big)+f\big(T(s_0)z-z\big)\\&\ge-\|f\|\,\|T(s_0+t)z-T(s_0)z\|+f\big(T(s_0)z-z\big)\\&=-\|T(s_0+t)z-T(s_0)z\|+\|T(s_0)z-z\|\\&\ge-\lambda+\lambda-\delta=-\delta.\end{aligned}$$

On the other hand, by the definition of $A(f,\varepsilon)$, $f\big(T(t)z-z\big)>\varepsilon$ for all $t\in S\setminus A(f,\varepsilon)$. Therefore we have

$$\begin{aligned}f\big(T(s_0+t)z-z\big)&\ge-\delta\, I_{A(f,\varepsilon)}(s_0+t)+\varepsilon\, I_{S\setminus A(f,\varepsilon)}(s_0+t)\\&=-\delta\, I_{A(f,\varepsilon)}(s_0+t)+\varepsilon\, I_S(s_0+t)-\varepsilon\, I_{A(f,\varepsilon)}(s_0+t)\\&=-(\delta+\varepsilon)\, I_{A(f,\varepsilon)}(s_0+t)+\varepsilon\end{aligned}$$

for $t\in S$. So, for $s\in S$, we have

$$\begin{aligned}0&=f(T_\mu z-z)\\&=\mu_t\Big(f\big(T(t)z-z\big)\Big)\\&=\mu_t\Big(f\big(T(s_0+s+t)z-z\big)\Big)\\&=\tilde{\mu}_t\Big(f\big(T(s_0+s+t)z-z\big)\Big)\end{aligned}$$

$$\begin{aligned}&\geq \tilde{\mu}_t\Big(-(\delta+\varepsilon)\, I_{A(f,\varepsilon)}(s_0+s+t)+\varepsilon\Big)\\&= -(\delta+\varepsilon)\, \tilde{\mu}_t\Big(I_{A(f,\varepsilon)}(s_0+s+t)\Big)+\varepsilon.\end{aligned}$$

Hence, we obtain

$$\tilde{\mu}_t\Big(I_{A(f,\varepsilon)}(s_0+s+t)\Big) \geq \frac{\varepsilon}{\varepsilon+\delta}$$

for all $s \in S$. So, by Lemma 1, we have

$$\liminf_{s\in S} \tilde{\mu}_t\Big(I_{A(f,\varepsilon)}(s+t)\Big) = \liminf_{s\in S} \tilde{\mu}_t\Big(I_{A(f,\varepsilon)}(s_0+s+t)\Big) \geq \frac{\varepsilon}{\varepsilon+\delta}.$$

This completes the proof. □

Lemma 6.

$$\liminf_{s\in S} \tilde{\mu}_t\big(I_{B(\varepsilon)}(s+t)\big) = 1$$

hold for all $\varepsilon > 0$.

Proof. We fix $\varepsilon > 0$ and $\eta \in \mathbb{R}$ with $1/2 < \eta < 1$ and put $\delta = \varepsilon(1-\eta)/(2\eta)$. We note that $0 < \delta < \varepsilon/2$. By the definition of λ, there exists $s_0 \in S$ such that $\|T(s_0)z - z\| \geq \lambda - \delta$. Fix $f \in E^*$ with $\|f\| = 1$ and $f\big(T(s_0)z - z\big) = \|T(s_0)z - z\|$. So, by Lemma 5, we have

$$\liminf_{s\in S} \tilde{\mu}_t\big(I_{A(f,\varepsilon/2)}(s+t)\big) \geq \frac{\varepsilon/2}{\varepsilon/2+\delta} = \eta.$$

For $t \in S$ with $s_0 + t \in A(f, \varepsilon/2)$, we have

$$\begin{aligned}\|T(t)z - z\| &\geq \|T(s_0)z - T(s_0)\circ T(t)z\|\\&= \|f\|\, \|T(s_0)z - T(s_0+t)z\|\\&\geq f\big(T(s_0)z - T(s_0+t)z\big)\\&= f\big(T(s_0)z - z\big) + f\big(z - T(s_0+t)z\big)\\&= \|T(s_0)z - z\| + f\big(z - T(s_0+t)z\big)\\&\geq \lambda - \delta - \frac{\varepsilon}{2}\\&\geq \lambda - \varepsilon\end{aligned}$$

and hence $t \in B(\varepsilon)$. Therefore $I_{B(\varepsilon)}(t) \geq I_{A(f,\varepsilon/2)}(s_0+t)$ for all $t \in S$. So, by Lemma 1, we obtain

$$\begin{aligned}\liminf_{s\in S} \tilde{\mu}_t\big(I_{B(\varepsilon)}(s+t)\big) &\geq \liminf_{s\in S} \tilde{\mu}_t\big(I_{A(f,\varepsilon/2)}(s_0+s+t)\big)\\&= \liminf_{s\in S} \tilde{\mu}_t\big(I_{A(f,\varepsilon/2)}(s+t)\big)\\&\geq \eta.\end{aligned}$$

Since η is arbitrary, we obtain the desired result. □

Proof of Theorem 2. By the definition of λ, there exists $p_1 \in S$ such that $\|T(p_1)z - z\| \geq \lambda - 1/3^2$. Take $f_1 \in E^*$ with $\|f_1\| = 1$ and $f_1\big(T(p_1)z - z\big) = \|T(p_1)z - z\|$. By Lemma 5, we have

$$\liminf_{s\in S} \tilde{\mu}_t\Big(I_{A\left(f_1,(2/3)^2\right)}(s+t)\Big) \geq \frac{2^2}{2^2+1}.$$

We now define inductively sequences $\{p_n\}$ in S and $\{f_n\}$ in E^*. Suppose $p_k \in S$ and $f_k \in E^*$ are known. Since

$$\begin{aligned}
&\liminf_{s\in S} \tilde{\mu}_t\big(I_{B(1/3^{k+2})}(s+t)\big) + \sum_{\ell=1}^{k} \liminf_{s\in S} \tilde{\mu}_t\big(I_{A\left(f_\ell,(2/3)^{\ell+1}\right)}(s+t)\big) - k \\
&\geq 1 + \sum_{\ell=1}^{k} \frac{2^{\ell+1}}{2^{\ell+1}+1} - k \\
&\geq 1 + \sum_{\ell=1}^{k} \frac{2^{\ell+1}-1}{2^{\ell+1}} - k = 1 + \sum_{\ell=1}^{k} \frac{-1}{2^{\ell+1}} \\
&> \frac{1}{2} > 0,
\end{aligned}$$

we have

$$\{p_k + t : t \in S\} \cap B(1/3^{k+2}) \cap \bigcap_{\ell=1}^{k} A\big(f_\ell, (2/3)^{\ell+1}\big) \neq \varnothing$$

by Lemma 2. So we can choose $p_{k+1} \in S$ such that $p_{k+1} = p_k + t$ for some $t \in S$,

$$\|T(p_{k+1})z - z\| \geq \lambda - \frac{1}{3^{k+2}}, \quad \text{and} \quad f_\ell\big(T(p_{k+1})z - z\big) \leq \frac{2^{\ell+1}}{3^{\ell+1}}$$

for $\ell = 1, 2, \cdots, k$. Take $f_{k+1} \in E^*$ with

$$\|f_{k+1}\| = 1 \quad \text{and} \quad f_{k+1}\big(T(p_{k+1})z - z\big) = \|T(p_{k+1})z - z\|.$$

Note that

$$\liminf_{s\in S} \tilde{\mu}_t\big(I_{A\left(f_{k+1},(2/3)^{k+2}\right)}(s+t)\big) \geq \frac{2^{k+2}}{2^{k+2}+1}$$

by Lemma 5. Hence we have defined $\{p_n\}$ and $\{f_n\}$. For each $n \in \mathbb{N}$, there exists $t \in S$ such that $p_{n+1} = p_n + t$. We put $q_n = t$. So we have defined a sequence $\{q_n\}$ in S. □

Now, we prove the following characterization.

Theorem 3. *Let C be a weakly compact convex subset of a Banach space E with the Opial property. Let S, $\{T(t) : t \in S\}$, X and μ be as in Theorem 2. Then $z \in C$ is a common fixed point of $\{T(t) : t \in S\}$ if and only if $T_\mu z = z$.*

Proof. We assume that z is a common fixed point of $\{T(t) : t \in S\}$. Then for $f \in E^*$, we have

$$f(T_\mu z) = \mu_t\ \Big(f\big(T(t)z\big)\Big) = \mu_t\ \Big(f(z)\Big) = f(z).$$

Hence we obtain $T_\mu z = z$. Conversely, we assume $T_\mu z = z$. By Theorem 2, there exist sequences $\{p_n\}$ and $\{q_n\}$ in S and $\{f_n\}$ in E^* satisfying the conclusion of Theorem 2. We put $\lambda = \limsup_{t\in S} \|T(t)z - z\|$. Since C is weakly compact, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{T(p_{n_k})z\}$ converges weakly to some point $u \in C$. If $n_k > \ell$, then

$$f_\ell\big(T(p_{n_k})z - z\big) \leq \frac{2^{\ell+1}}{3^{\ell+1}}.$$

So we obtain

$$f_\ell(u - z) \leq \frac{2^{\ell+1}}{3^{\ell+1}}$$

for all $\ell \in \mathbb{N}$. Since

$$\begin{aligned}\|T(p_\ell)z - u\| &= \|f_\ell\|\,\|T(p_\ell)z - u\| \\ &\geq f_\ell\bigl(T(p_\ell)z - u\bigr) \\ &= f_\ell\bigl(T(p_\ell)z - z\bigr) + f_\ell(z-u) \\ &= \|T(p_\ell)z - z\| + f_\ell(z-u) \\ &\geq \lambda - \frac{1}{3^{\ell+1}} - \frac{2^{\ell+1}}{3^{\ell+1}}\end{aligned}$$

for $\ell \in \mathbb{N}$, we have $\liminf_\ell \|T(p_\ell)z - u\| \geq \lambda$. By Lemma 4, we have

$$\begin{aligned}\liminf_{k\to\infty} \|T(p_{n_k})z - z\| &\leq \lambda \\ &\leq \liminf_{\ell\to\infty} \|T(p_\ell)z - u\| \\ &\leq \liminf_{k\to\infty} \|T(p_{n_k})z - u\|.\end{aligned}$$

By the Opial property of E, we obtain $z = u$. Using Lemma 4 again, for each $\ell \in \mathbb{N}$, we also have

$$\begin{aligned}&\liminf_{k\to\infty} \|T(p_{n_k})z - T(p_\ell)z\| \\ &= \liminf_{k\to\infty} \|T(p_\ell) \circ T(q_\ell + q_{\ell+1} + q_{\ell+2} + \cdots + q_{n_k-1})z - T(p_\ell)z\| \\ &\leq \liminf_{k\to\infty} \|T(q_\ell + q_{\ell+1} + q_{\ell+2} + \cdots + q_{n_k-1})z - z\| \\ &\leq \lambda.\end{aligned}$$

By the Opial property of E, we obtain $T(p_\ell)z = u = z$ for all $\ell \in \mathbb{N}$. Therefore $\lambda = 0$. By Lemma 4, we obtain $T(t)z = z$ for all $t \in S$. This completes the proof. □

Using Theorem 3, we obtain another characterization.

Theorem 4. *Let C be a weakly compact convex subset of a Banach space E with the Opial property. Let S, $\{T(t) : t \in S\}$ and X be as in Theorem 2. Let $\{\mu_\alpha : \alpha \in D\}$ be an asymptotically invariant net of means on X. Then $z \in C$ is a common fixed point of $\{T(t) : t \in S\}$ if and only if $\{T_{\mu_\alpha} z\}$ converges weakly to z.*

Proof. We assume that z is a common fixed point of $\{T(t) : t \in S\}$. As in the proof of Theorem 3, we obtain $T_{\mu_\alpha} z = z$ for all $\alpha \in D$. Therefore $\{T_{\mu_\alpha} z\}$ converges weakly to z. Conversely, we assume that $\{T_{\mu_\alpha} z\}$ converges weakly to z. Using Alaoglu's theorem, there exists a subnet $\{\mu_{\alpha_\beta} : \beta \in D'\}$ such that $\{\mu_{\alpha_\beta} : \beta \in D'\}$ converges weakly* to some point $\mu \in X^*$. We know that such μ is an invariant mean on X; see [8]. Since $\{T_{\mu_\alpha} z\}$ converges weakly to z, we have

$$\lim_{\beta\in D'} f(T_{\mu_{\alpha_\beta}} z) = \lim_{\alpha\in D} f(T_{\mu_\alpha} z) = f(z)$$

for $f \in E^*$. We also have

$$\begin{aligned}\lim_{\beta\in D'} f(T_{\mu_{\alpha_\beta}} z) &= \lim_{\beta\in D'} (\mu_{\alpha_\beta})_t \Bigl(f\bigl(T(t)z\bigr)\Bigr) \\ &= \mu_t \Bigl(f\bigl(T(t)z\bigr)\Bigr) \\ &= f(T_\mu z)\end{aligned}$$

for $f \in E^*$. Since $f(T_\mu z) = f(z)$ for all $f \in E^*$, we obtain $T_\mu z = z$. By Theorem 3, z is a common fixed point of $\{T(t) : t \in S\}$. □

Remark. In Theorems 3 and 4, we may replace "E has the Opial property" with the following condition: For each weakly convergent sequence $\{x_n\}$ in C with weak limit $x_0 \in C$, $\liminf_n \|x_n - x_0\| < \liminf_n \|x_n - x\|$ for all $x \in E$ with $x \neq x_0$. We remark that if C is a compact subset of any Banach space, then the above condition is satisfied even in the case that E does not have the Opial property; see the remark of Theorem 4 in [34].

From the above-mentioned remark, we obtain the following.

Theorem 5. *Let C be a compact convex subset of a Banach space E. Let S, $\{T(t) : t \in S\}$, X and μ be as in Theorem 2. Let $\{\mu_n\}$ be an asymptotically invariant sequence of means on X. Let $z \in C$. Then the following are equivalent:*

(i) *z is a common fixed point of $\{T(t) : t \in S\}$;*
(ii) *$T_\mu z = z$;*
(iii) *$\{T_{\mu_n} z\}$ converges strongly to z.*

4. Convergence Theorem

In [1], Atsushiba, Shioji and Takahashi proved convergence theorems with the assumption that the Banach space is strictly convex. In this section, we prove a convergence theorem to a common fixed point without the assumption of the strict convexity.

The following lemma concerns Krasnoselskii and Mann's type sequences [22, 26], and is proved in [32]. See also [13].

Lemma 7 ([32] Lemma 2)**.** *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n) z_n$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty} \Big(\|w_n - w_{n+k}\| - \|z_n - z_{n+k}\| \Big) \leq 0$$

for all $k \in \mathbb{N}$. Then $\liminf_n \|w_n - z_n\| = 0$.

Using Lemma 7, we obtain the following convergence theorem.

Theorem 6. *Let C be a compact convex subset of a Banach space E. Let S, $\{T(t) : t \in S\}$ and X be as in Theorem 2. Let $\{\mu_n\}$ be an asymptotically invariant sequence of means on X. Suppose that $\lim_n \|\mu_n - \mu_{n+1}\| = 0$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n T_{\mu_n} x_n + (1-\alpha_n) x_n$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a common fixed point z_0 of $\{T(t) : t \in S\}$.

Proof. For $n, k \in \mathbb{N}$, taking $f \in E^*$ with

$$\|f\| = 1 \quad \text{and} \quad f\big(T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\big) = \|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\|,$$

we have

$$\begin{aligned} &\|T_{\mu_n} x_n - T_{\mu_{n+k}} x_{n+k}\| - \|x_n - x_{n+k}\| \\ &\leq \|T_{\mu_n} x_n - T_{\mu_n} x_{n+k}\| + \|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\| - \|x_n - x_{n+k}\| \end{aligned}$$

$$
\begin{aligned}
&\le \|x_n - x_{n+k}\| + \|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\| - \|x_n - x_{n+k}\| \\
&= \|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\| \\
&= f\big(T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\big) \\
&= (\mu_n)_t \left(f\big(T(t)x_{n+k}\big)\right) - (\mu_{n+k})_t \left(f\big(T(t)x_{n+k}\big)\right) \\
&= (\mu_n - \mu_{n+k})_t \left(f\big(T(t)x_{n+k}\big)\right) \\
&\le \|\mu_n - \mu_{n+k}\| \sup_{t\in S} \left|f\big(T(t)x_{n+k}\big)\right| \\
&\le \|\mu_n - \mu_{n+k}\| \sup_{t\in S} \|f\| \left\|T(t)x_{n+k}\right\| \\
&\le \|\mu_n - \mu_{n+k}\| \sup_{x\in C} \|x\| \\
&\le \sum_{j=0}^{k-1} \|\mu_{n+j} - \mu_{n+j+1}\| \sup_{x\in C} \|x\|.
\end{aligned}
$$

Hence we obtain

$$\limsup_{n\to\infty} \Big(\|T_{\mu_n} x_n - T_{\mu_{n+k}} x_{n+k}\| - \|x_n - x_{n+k}\|\Big) \le 0$$

for all $k \in \mathbb{N}$. By Lemma 7, we obtain $\liminf_n \|T_{\mu_n} x_n - x_n\| = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \|T_{\mu_{n_k}} x_{n_k} - x_{n_k}\| = 0$$

and $\{x_{n_k}\}$ converges strongly to some point z_0. We have

$$
\begin{aligned}
&\limsup_{k\to\infty} \|T_{\mu_{n_k}} z_0 - z_0\| \\
&\le \limsup_{k\to\infty} \Big(\|T_{\mu_{n_k}} z_0 - T_{\mu_{n_k}} x_{n_k}\| + \|T_{\mu_{n_k}} x_{n_k} - x_{n_k}\| + \|x_{n_k} - z_0\|\Big) \\
&\le \limsup_{k\to\infty} \Big(2\,\|z_0 - x_{n_k}\| + \|T_{\mu_{n_k}} x_{n_k} - x_{n_k}\|\Big) = 0.
\end{aligned}
$$

Since $\{\mu_{n_k}\}$ is also an asymptotically invariant sequence of means on X, by Theorem 5, we have z_0 is a common fixed point of $\{T(t) : t \in S\}$. Since

$$
\begin{aligned}
\|x_{n+1} - z_0\| &\le \alpha_n \|T_{\mu_n} x_n - z_0\| + (1-\alpha_n)\|x_n - z_0\| \\
&= \alpha_n \|T_{\mu_n} x_n - T_{\mu_n} z_0\| + (1-\alpha_n)\|x_n - z_0\| \\
&\le \alpha_n \|x_n - z_0\| + (1-\alpha_n)\|x_n - z_0\| \\
&= \|x_n - z_0\|,
\end{aligned}
$$

we obtain $\lim_n \|x_n - z_0\| = \lim_k \|x_{n_k} - z_0\| = 0$. This completes the proof. □

Remark. The following lemma (Lemma 8) is a generalization of Lemma 7, which is useful and proved in [36]. Using Lemma 8, we can give the shorter proof of Theorem 6. However, we do not use Lemma 8 because Reference [36] is not yet published.

Lemma 8 ([36] Lemma 2)**.** *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n) z_n$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty} \Big(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|\Big) \le 0.$$

Then $\lim_n \|w_n - z_n\| = 0$.

5. Nonexpansive Retraction

In this section, we prove the existence theorems of the nonexpansive retraction onto the set of common fixed points.

Theorem 7. *Let S be a commutative semigroup and let $\{T(t) : t \in S\}$ be a nonexpansive semigroup on a weakly compact convex subset C of a Banach space E. Suppose that C has the fixed point property for nonexpansive mappings. Then there exists a nonexpansive retraction Q from C onto $F(\mathcal{S})$ satisfying $Q \circ T(t) = T(t) \circ Q = Q$ for all $t \in S$.*

Proof. By Theorem 1, there exists a nonexpansive retraction from C onto $F(\mathcal{S})$. Let μ be an invariant mean on $B(S)$ and put $Q = P \circ T_\mu$. Then Q is a nonexpansive mapping on C, because so are P and T_μ. For $x \in F(\mathcal{S})$, we have $T_\mu x = x \in F(\mathcal{S})$ and hence $Qx = x$. Therefore we have shown that Q is a nonexpansive retraction from C onto $F(\mathcal{S})$. So it is obvious that $T(t) \circ Q = Q$ for all $t \in S$. Fix $s \in S$ and $x \in C$. For $f \in E^*$, we have

$$\begin{aligned} f\big(T_\mu \circ T(s)x\big) &= \mu_t\Big(f\big(T(t) \circ T(s)x\big)\Big) = \mu_t\Big(f\big(T(t+s)x\big)\Big) \\ &= \mu_t\Big(f\big(T(t)x\big)\Big) = f(T_\mu x). \end{aligned}$$

Hence we have $T_\mu \circ T(s)x = T_\mu x$. Therefore we obtain $Q \circ T(s) = Q$ for all $s \in S$. This completes the proof. □

We give another nonexpansive retraction.

Theorem 8. *Let C be a weakly compact convex subset of a Banach space E with the Opial property. Let S, $\{T(t) : t \in S\}$, X and μ be as in Theorem 2. Define a mapping Q on C as*

$$Qx = \mathop{\textit{weak-lim}}_{n\to\infty} \left(\frac{1}{2}T_\mu + \frac{1}{2}I\right)^n \circ T_\mu x$$

for all $x \in C$, where I is the identity mapping on C. Then Q is a nonexpansive retraction from C onto $F(\mathcal{S})$ satisfying $Q \circ T(t) = T(t) \circ Q = Q$ for all $t \in S$. Further if a closed convex subset C' of C is $T(t)$-invariant for all $t \in S$, then C' is also Q-invariant.

Proof. Fix $x \in C$. Define a sequence $\{x_n\}$ in C by $x_1 = T_\mu x$ and $x_{n+1} = \frac{1}{2}T_\mu x_n + \frac{1}{2}x_n$ for $n \in \mathbb{N}$. Then by the result of Edelstein and O'Brien [12], $\{x_n\}$ converges weakly to a fixed point of T_μ. Since $x \in C$ is arbitrary, Q is well-defined and Qx is a fixed point of T_μ for $x \in C$. By Theorem 3, we have $Qx \in F(\mathcal{S})$ for all $x \in C$. For $x, y \in C$, we have

$$\begin{aligned} \|Qx - Qy\| &\le \liminf_{n\to\infty} \left\| \left(\frac{1}{2}T_\mu + \frac{1}{2}I\right)^n T_\mu x - \left(\frac{1}{2}T_\mu + \frac{1}{2}I\right)^n T_\mu y \right\| \\ &\le \liminf_{n\to\infty} \|T_\mu x - T_\mu y\| \\ &= \|T_\mu x - T_\mu y\| \le \|x - y\| \end{aligned}$$

and hence Q is nonexpansive. For each $x \in F(\mathcal{S})$, we have $T_\mu x = x$ and hence $Qx = x$. Therefore we have shown that Q is a nonexpansive retraction from C onto $F(\mathcal{S})$. So we also obtain that $T(t) \circ Q = Q$ for all $t \in S$. As in the proof of

Theorem 7, we have $T_\mu \circ T(t) = T_\mu$ for all $t \in S$. So, by the definition of Q, we obtain that $Q \circ T(t) = Q$ for all $t \in S$. We assume that a closed convex subset C' of C is $T(t)$-invariant for all $t \in S$. Then since C' is weakly compact and convex, we have that C' is T_μ-invariant. So, by the definition of Q, C' is also Q-invariant. □

From Ishikawa's convergence theorem [18], Theorem 5, and the proof of Theorem 8, we also obtain the following.

Theorem 9. *Let C be a compact convex subset of a Banach space E. Let S, $\{T(t) : t \in S\}$, X and μ be as in Theorem 2. Define a mapping Q on C as in Theorem 8. Then the conclusion of Theorem 8 holds.*

6. $S = \mathbb{N} \times \mathbb{N}$

Using the results in Sections 3, 4 and 5, we can prove many theorems. In this section, we state the deduced theorems in the case of $S = \mathbb{N} \times \mathbb{N}$. And in the next section, we state them in the case of $S = [0, \infty)$.

We first prove the following.

Lemma 9. *Put $S = \mathbb{N} \times \mathbb{N}$ and define a sequence $\{\mu_n\}$ of functions on $B(S)$ by*

$$\mu_n(a) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a(i,j)$$

for $n \in \mathbb{N}$ and $a \in B(S)$. Then $\{\mu_n\}$ is an asymptotically invariant sequence of means on $B(S)$ and satisfies $\lim_n \|\mu_n - \mu_{n+1}\| = 0$.

Proof. We know that $\{\mu_n\}$ is an asymptotically invariant sequence of means on $B(S)$; see [17, 30, 39]. For $n \in \mathbb{N}$ and $a \in B(S)$, we have

$$\begin{aligned}
&\left|\mu_n(a) - \mu_{n+1}(a)\right| \\
&= \left| \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a(i,j) - \frac{1}{(n+1)^2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a(i,j) \right| \\
&\le \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \sum_{i=1}^{n} \sum_{j=1}^{n} \left|a(i,j)\right| \\
&\quad + \frac{1}{(n+1)^2} \sum_{i=1}^{n} \left|a(i,n+1)\right| + \frac{1}{(n+1)^2} \sum_{j=1}^{n+1} \left|a(n+1,j)\right| \\
&\le \left(\frac{n^2}{n^2} - \frac{n^2}{(n+1)^2} + \frac{2n+1}{(n+1)^2} \right) \|a\|
\end{aligned}$$

and hence

$$\limsup_{n\to\infty} \|\mu_n - \mu_{n+1}\| \le \lim_{n\to\infty} \left(\frac{n^2}{n^2} - \frac{n^2}{(n+1)^2} + \frac{2n+1}{(n+1)^2} \right) = 0.$$

This completes the proof. □

The following can be proved easily.

Lemma 10. *Put $S = \mathbb{N} \times \mathbb{N}$ and define a sequence $\{\mu_n\}$ of functions on $B(S)$ as in Lemma 9. Let C be a weakly compact convex subset of a Banach space E and let T and U be nonexpansive mappings on C with $TU = UT$. Then*

$$T_{\mu_n} x = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} T^i U^j x$$

for all $n \in \mathbb{N}$ and $x \in C$.

Using Lemmas 9 and 10, we obtain the following.

Corollary 1 ([32])**.** *Let C be a compact convex subset of a Banach space E and let T and U be nonexpansive mappings on C with $TU = UT$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \frac{\alpha_n}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} T^i U^j x_n + (1 - \alpha_n) x_n$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a common fixed point z_0 of T and U.

Corollary 2 ([34])**.** *Let E be a Banach space with the Opial property and let C be a weakly compact convex subset of E. Let T and U be nonexpansive mappings on C with $TU = UT$. Then for $z \in C$, the following are equivalent:*

(i) *z is a common fixed point of T and U;*
(ii) *there exists a subnet of a sequence*

$$\left\{ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} T^i U^j z \right\}$$

in C converging weakly to z.

We also obtain the following new results.

Corollary 3. *Let C be a weakly compact convex subset of a Banach space E with the Opial property. Let T and U be nonexpansive mappings on C with $TU = UT$. Let μ be an invariant mean on $B(\mathbb{N} \times \mathbb{N})$. Then $z \in C$ is a common fixed point of T and U if and only if $T_\mu z = z$.*

Corollary 4. *Let C be a compact convex subset of a Banach space E. Let T and U be nonexpansive mappings on C with $TU = UT$. Let μ be an invariant mean on $B(\mathbb{N} \times \mathbb{N})$. Then $z \in C$ is a common fixed point of T and U if and only if $T_\mu z = z$.*

7. $S = [0, \infty)$

In this section, we state the deduced theorems in the case of $S = [0, \infty)$. As in Section 6, we prove the following.

Lemma 11. *Put $S = [0, \infty)$ and let X be the Banach space consisting of all bounded continuous functions from S into $\mathbb{R}$ with the supremum norm. Let $\{t_n\}$ be a sequence in $(0, \infty)$ satisfying $\lim_n t_n = \infty$. Define a sequence $\{\mu_n\}$ of functions on X by*

$$\mu_n(a) = \frac{1}{t_n} \int_0^{t_n} a(t)\, dt$$

for $n \in \mathbb{N}$ and $a \in X$. Then $\{\mu_n\}$ is an asymptotically invariant sequence of means on X. Further, if $\{t_n\}$ satisfies $\lim_n t_{n+1}/t_n = 1$, then $\lim_n \|\mu_n - \mu_{n+1}\| = 0$.

Proof. We know that $\{\mu_n\}$ is an asymptotically invariant sequence of means on X; see [17, 30, 39]. We assume that $\lim_n t_{n+1}/t_n = 1$. For $n \in \mathbb{N}$ and $a \in X$, putting $m = \min\{t_n, t_{n+1}\}$ and $M = \max\{t_n, t_{n+1}\}$, we have

$$\begin{aligned}
&\left|\mu_n(a) - \mu_{n+1}(a)\right| \\
&= \left|\frac{1}{t_n}\int_0^{t_n} a(t)\,dt - \frac{1}{t_{n+1}}\int_0^{t_{n+1}} a(t)\,dt\right| \\
&= \left|\frac{1}{m}\int_0^{m} a(t)\,dt - \frac{1}{M}\int_0^{M} a(t)\,dt\right| \\
&= \left|\left(\frac{1}{m} - \frac{1}{M}\right)\int_0^{m} a(t)\,dt - \frac{1}{M}\int_m^{M} a(t)\,dt\right| \\
&\le \left(\frac{1}{m} - \frac{1}{M}\right)\int_0^{m} \left|a(t)\right|\,dt + \frac{1}{M}\int_m^{M} \left|a(t)\right|\,dt \\
&\le \left(\frac{1}{m} - \frac{1}{M}\right)\int_0^{m} \|a\|\,dt + \frac{1}{M}\int_m^{M} \|a\|\,dt \\
&= \left(\frac{m}{m} - \frac{m}{M} + \frac{M-m}{M}\right)\|a\| \\
&= \left(2 - 2\frac{\min\{t_n, t_{n+1}\}}{\max\{t_n, t_{n+1}\}}\right)\|a\|
\end{aligned}$$

and hence

$$\limsup_{n\to\infty} \|\mu_n - \mu_{n+1}\| \le \lim_{n\to\infty}\left(2 - 2\frac{\min\{t_n, t_{n+1}\}}{\max\{t_n, t_{n+1}\}}\right) = 0.$$

This completes the proof. □

We recall that a family of mappings $\{T(t) : t \ge 0\}$ is called a *one-parameter strongly continuous semigroup of nonexpansive mappings* on C if the following are satisfied:

(i) for each $t \ge 0$, $T(t)$ is a nonexpansive mapping on C;
(ii) $T(s+t) = T(s) \circ T(t)$ for all $s, t \ge 0$;
(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is strongly continuous.

The following can be proved easily.

Lemma 12. *Let S, X, $\{t_n\}$ and $\{\mu_n\}$ be as in Lemma 11. Let C be a weakly compact convex subset of a Banach space E and let $\{T(t) : t \ge 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Then*

$$T_{\mu_n} x = \frac{1}{t_n}\int_0^{t_n} T(t)x\,dt$$

for all $n \in \mathbb{N}$ and $x \in C$.

Using Lemmas 11 and 12, we obtain the following.

Corollary 5 ([37])**.** *Let C be a compact convex subset of a Banach space E and let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \frac{\alpha_n}{t_n} \int_0^{t_n} T(s)x_n \, ds + (1 - \alpha_n)x_n$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$, and $\{t_n\}$ be a sequence in $(0, \infty)$ satisfying $\lim_n t_n = \infty$ and $\lim_n t_{n+1}/t_n = 1$. Then $\{x_n\}$ converges strongly to a common fixed point z_0 of $\{T(t) : t \geq 0\}$.

Corollary 6 ([35])**.** *Let E be a Banach space with the Opial property and let C be a weakly compact convex subset of E. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Then for $z \in C$, the following are equivalent:*

(i) *z is a common fixed point of $\{T(t) : t \geq 0\}$;*
(ii) *there exists a subnet of a net*

$$\left\{ \frac{1}{t} \int_0^t T(s)x \, ds \right\}$$

in C converging weakly to z.

We also obtain the following new results.

Corollary 7. *Let C be a weakly compact convex subset of a Banach space E with the Opial property. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Let X be as in Lemma 11 and let μ be an invariant mean on X. Then $z \in C$ is a common fixed point of $\{T(t) : t \geq 0\}$ if and only if $T_\mu z = z$.*

Corollary 8. *Let C be a compact convex subset of a Banach space E. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Let X be as in Lemma 11 and let μ be an invariant mean on X. Then $z \in C$ is a common fixed point of $\{T(t) : t \geq 0\}$ if and only if $T_\mu z = z$.*

8. Appendix

In this section, using the notion of universal nets, we give an invariant mean. We recall that a net $\{y_\alpha : \alpha \in D\}$ in a topological space Y is universal if for each subset A of Y, there exists $\alpha_0 \in D$ satisfying either of the following:

- $y_\alpha \in A$ for all $\alpha \in D$ with $\alpha \geq \alpha_0$; or
- $y_\alpha \in Y \setminus A$ for all $\alpha \in D$ with $\alpha \geq \alpha_0$.

For every net $\{y_\alpha : \alpha \in D\}$, by the Axiom of Choice, there exists a universal subnet $\{y_{\alpha_\beta} : \beta \in D'\}$ of $\{y_\alpha : \alpha \in D\}$. If f is a mapping from Y into a topological space Z and $\{y_\alpha : \alpha \in D\}$ is a universal net in Y, then $\{f(y_\alpha) : \alpha \in D\}$ is also a universal net in Z. If Y is compact, then a universal net $\{y_\alpha : \alpha \in D\}$ in Y always converges. See [20] and others for details.

Proposition 1. *Let S be a commutative semigroup, let X be a linear subspace of $B(S)$ such that $I_S \in X$ and X is ℓ_s-invariant for all $s \in S$. Let $\{\mu_\alpha : \alpha \in D\}$ be an asymptotically invariant net of means on X. Let $\{\mu_{\alpha_\beta} : \beta \in D'\}$ be a universal subnet of $\{\mu_\alpha : \alpha \in D\}$. Define a function μ from X into $\mathbb{R}$ by*

$$\mu(a) = \lim_{\beta \in D'} \mu_{\alpha_\beta}(a)$$

for all $a \in X$. Then μ is an invariant mean on X.

Proof. Since the net $\{\mu_{\alpha_\beta} : \beta \in D'\}$ is universal, μ is well-defined. Since $\{\mu_{\alpha_\beta} : \beta \in D'\}$ is also an asymptotically invariant net of means on X, μ is an invariant mean; see [8]. This completes the proof. □

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