INTERSECTIONS OF SHIFTED SETS

MAURO DI NASSO

Abstract. We consider shifts of a set $A \subseteq \mathbb{N}$ by elements from another set $B \subseteq \mathbb{N}$, and prove intersection properties according to the relative asymptotic size of A and B. A consequence of our main theorem is the following: If $A = \{a_n\}$ is such that $a_n = o(n^{k/k-1})$, then the k-recurrence set $R_k(A) = \{x \mid |A \cap (A+x)| \geq k\}$ contains the distance sets of arbitrarily large finite sets.

1. Introduction

It is a well-know fact that if a set of natural numbers A has positive upper asymptotic density, then its *set of distances*

$$\Delta(A) \ = \ \{a' - a \mid a', a \in A, a' > a\}$$

meets the set of distances $\Delta(X)$ of any infinite set X (see, *e.g.*, [1]). In consequence, $\Delta(A)$ is *syndetic*, that is there exists k such that $\Delta(A) \cap I \neq \emptyset$ for every interval I of length k. It is a relevant theme of research in combinatorial number theory to investigate properties of distance sets according to their "asymptotic size" (see, *e.g.*, [7, 8, 4, 2].)

The sets of distances are generalized by the *k-recurrence sets*, namely the sets of those numbers that are the common distance of at least k-many pairs:

$$R_k(A) \ = \ \{x \mid |A \cap (A+x)| \geq k\}\,.$$

Notice that $R_1(A) = \Delta(A)$. We now further generalize this notion.

Let $[A]^h = \{Z \subseteq A \mid |Z| = h\}$ denote the family of all finite subsets of A of cardinality h, namely the *h-tuples* of A.

Definition 1.1. For $k, h \in \mathbb{N}$ with $h > 1$, the *(h, k)-recurrence set* of A is the following set of h-tuples:

$$R_k^h(A) = \left\{\{t_1 < \ldots < t_h\} \in [\mathbb{N}]^h \,\middle|\, |(A+t_1) \cap \ldots \cap (A+t_h)| \geq k\right\}.$$

Note that a pair $\{t < t'\} \in R_k^2(A) \Leftrightarrow t' - t \in R_k(A)$, because trivially $|(A+t) \cap (A+t')| = |A \cap (A + (t' - t))|$.

2000 *Mathematics Subject Classification.* 05B10, 11B05, 11B37.

Key words and phrases. Asymptotic density, Delta-sets, k-Recurrence sets.

For sets of natural numbers, we write $A = \{a_n\}$ to mean that elements a_n of A are arranged in increasing order. We adopt the usual "little-O" notation, and for functions $f : \mathbb{N} \to \mathbb{R}$, we write $a_n = o(f(n))$ to mean that $\lim_{n\to\infty} a_n/f(n) = 0$.

Our main result is the following.

- **Theorem 2.3.** *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers, and let:*[1]
$$\liminf_{n,m\to\infty} \frac{a_n + b_m}{n\sqrt[k]{m}} \;=\; l\,.$$
If $l < \frac{1}{\sqrt[k]{h-1}}$ then $R_k^h(A) \cap [B]^h \neq \emptyset$; and if $l = 0$ then $R_k^h(A) \cap [B]^h$ is infinite for all h.

(Notice that when $k = 1$, for every infinite set A one has $R_1^h(A) \neq \emptyset$ for all h). As a consequence of the theorem above, the following intersection property is obtained.

- **Theorem 3.3.** *Let $k \geq 2$. If the infinite set $A = \{a_n\}$ is such that $a_n = o(n^{k/k-1})$ then $R_k(A)$ is a "finitely Delta-set", that is $\Delta(Z) \subseteq R_k(A)$ for arbitrarily large finite sets Z.*

(When $k = 1$, $R_1(A) = \Delta(A)$ is trivially a "finitely Delta-set".)

All proofs contained in this paper have been first obtained by working with the *hyperintegers* of nonstandard analysis. (Nonstandard integers seem to provide a convenient framework to investigate combinatorial properties of numbers which depend on density; see, *e.g.*, [5, 6, 3].) However, all used arguments in our original proof could be translated in terms of limits of subsequences in an (almost) straightforward manner, with the only inconvenience of a heavier notation. So, we eventually decided to keep to the usual language of elementary combinatorics.

2. The main theorem

The following finite combinatorial property will be instrumental for the proof of our main result.

Lemma 2.1. *Let $A = \{a_1 < \ldots < a_n\}$ and $B = \{b_1 < \ldots < b_m\}$ be finite sets of natural numbers. For every k there exists a subset $Z \subseteq B$ such that*

[1] By *limit inferior* of a double sequence $\langle c_{nm} \mid (n,m) \in \mathbb{N}\times\mathbb{N}\rangle$ we mean
$$\liminf_{n,m\to\infty} c_{nm} = \lim_{k\to\infty}\left(\inf_{n,m\geq k} c_{nm}\right).$$

(1) $|\bigcap_{z\in Z}(A+z)| \geq k$.

(2) $|Z| \geq L\cdot\left(\frac{n\sqrt[k]{m}}{a_n+b_m}\right)^k$ *where* $L = \prod_{i=1}^{k-1}\frac{1-\frac{i}{n}}{1-\frac{i}{a_n+b_m}}$.

Proof. For every $i \leq m$, let $A_i = A + b_i$ be the shift of A by b_i. Notice that $|A_i| = |A| = n$ and $A_i \subseteq I = [1, a_n + b_m]$ for all i. Then denote by $\vartheta_i : [\mathbb{N}]^k \to \{0,1\}$ the characteristic function of $[A_i]^k$, and for $H \in [\mathbb{N}]^k$ let

$$f(H) \;=\; \sum_{i=1}^{m}\vartheta_i(H).$$

Then:

$$\sum_{H\in[I]^k} f(H) \;=\; \sum_{i=1}^{m}\left(\sum_{H\in[I]^k}\vartheta_i(H)\right) \;=\; \sum_{i=1}^{m}|[A_i]^k| \;=\; \sum_{i=1}^{\nu}\binom{n}{k} \;=\; \nu\cdot\binom{n}{k}.$$

Since $|[I]^k| = \binom{a_n+b_m}{k}$, by the *pigeonhole principle* there exists $H_0 \in [I]^k$ such that

$$f(H_0) \;\geq\; \frac{\nu\cdot\binom{n}{k}}{\binom{a_n+b_m}{k}} \;=\; \nu\cdot\frac{n(n-1)(n-2)\cdots(n-(k-1))}{(a_n+b_m)(a_n+b_m-1)\cdots(a_n+b_m-(k-1))}$$
$$=\; \nu\cdot L\cdot\left(\frac{n}{a_n+b_m}\right)^k \;=\; L\cdot\left(\frac{n\sqrt[k]{m}}{a_n+b_m}\right)^k,$$

where L is the number defined in the statement of this lemma. Now consider the set $\Gamma = \{i \in [1,m] \mid H_0 \in [A_i]^k\}$. We have that

$$|\Gamma| \;=\; \sum_{i=1}^{m}\vartheta_i(H_0) \;=\; f(H_0) \;\geq\; L\cdot\left(\frac{n\sqrt[k]{m}}{a_n+b_m}\right)^k.$$

Now, $H_0 = \{h_1 < \ldots < h_k\} \in \bigcap_{i\in\Gamma}[A_i]^k \Rightarrow |\bigcap_{i\in\Gamma} A_i| \geq k$, and the set $Z = \{b_i \mid i \in \Gamma\}$ satisfies the thesis. □

We already noticed that $\{t < t'\} \in R_k^2(A)$ if and only if the distance $t' - t \in R_k(A)$. More generally, one has the property:

Proposition 2.2. *If* $Z \in R_k^h(A)$ *then its set of distances* $\Delta(Z) \subseteq R_k(A)$.

Proof. Let $Z = \{z_1 < \ldots < z_h\}$. By the hypothesis, one finds at least k-many elements $\xi_1 < \ldots < \xi_k$ in the intersection $(A+z_1)\cap\ldots\cap(A+z_h)$. This means that there exist elements $a_{ij} \in A$ for $i = 1,\ldots,k$ and $j = 1,\ldots,h$ such that

$$\xi_i = a_{i1} + z_1 \;=\; \ldots \;=\; a_{ij} + z_j \;=\; \ldots \;=\; a_{ij'} + z_{j'} \;=\; \ldots \;=\; a_{ih} + z_h.$$

So, for all $1 \leq j < j' \leq h$, we have that

$$a_{ij} = a_{ij'} + (z_{j'} - z_j) \in A \cap (A + (z_{j'} - z_j)).$$

Notice that $a_{ij} < a_{i'j}$ for $i < i'$, so $A \cap (A + (z_{j'} - z_j))$ contains at least k-many elements. We conclude that $z_{j'} - z_j \in R_k(A)$ for all $1 \leq j < j' \leq h$, *i.e.* $\Delta(Z) \subseteq R_k(A)$. □

We remark that the implication in the above proposition cannot be reversed when $h > 2$. *E.g.*, if $A = \{1, 2, 3, 5, 8\}$ and $F = \{1, 2, 4\}$ then $|A \cap (A + 1)| = |A \cap (A + 2)| = |A \cap (A + 3)| = 2$, and so $\Delta(F) = \{1, 2, 3\} \subseteq R_2(A)$. However $F \notin R_2^3(A)$ because $(A+1) \cap (A+2) \cap (A + 4) = \emptyset$.

We are finally ready to prove our main theorem.

Theorem 2.3. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers, and let*

$$\liminf_{n,m\to\infty} \frac{a_n + b_m}{n \sqrt[k]{m}} \ = \ l\,.$$

If $l < \frac{1}{\sqrt[k]{h-1}}$ then $R_k^h(A) \cap [B]^h \neq \emptyset$; and if $l = 0$ then $R_k^h(A) \cap [B]^h$ is infinite for all h.

Proof. Pick increasing functions $\sigma, \tau : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{a_{\sigma(n)} + b_{\tau(n)}}{\sigma(n) \sqrt[k]{\tau(n)}} \ = \ l\,.$$

For every n, apply Lemma 2.1 to the finite sets $A_n = \{a_1 < \ldots < a_{\sigma(n)}\}$ and $B_n = \{b_1 < \ldots < b_{\tau(n)}\}$, and get the existence of a subset $Z_n \subseteq B_n$ such that

(1) $\left|\bigcap_{z\in Z_n} (A_n + z)\right| \geq k$.

(2) $|Z_n| \geq L_n \cdot \left(\frac{\sigma(n) \sqrt[k]{\tau(n)}}{a_{\sigma(n)}+b_{\tau(n)}}\right)^k$ where $L_n = \prod_{i=1}^{k-1} \frac{1-\frac{i}{\sigma(n)}}{1-\frac{i}{a_{\sigma(n)}+b_{\tau(n)}}}$.

Since $\lim_{n\to\infty} L_n = 1$, we have that

$$\liminf_{n\to\infty} |Z_n| \ \geq \ \lim_{n\to\infty} L_n \cdot \left(\frac{\sigma(n) \sqrt[k]{\tau(n)}}{a_{\sigma(n)} + b_{\tau(n)}}\right)^k \ = \ 1 \cdot \left(\frac{1}{l}\right)^k \ > \ h - 1.$$

Let t be an index such that $|Z_t| > h - 1$, and pick $z_1 < \ldots < z_h \in Z_t$. Then:

$$\left|\bigcap_{i=1}^{h}(A + z_i)\right| \ \geq \ \left|\bigcap_{i=1}^{h}(A_t + z_i)\right| \ \geq \ \left|\bigcap_{z\in Z_t}(A_t + z)\right| \ \geq \ k.$$

As $Z_t \subset B$, we conclude that $\{z_1 < \ldots < z_h\} \in R_k^h(A) \cap [B]^h$.

Now let us turn to the case $l = 0$. Given $s > 1$, pick $j \leq s$ such that the set $T_j = \{\tau(n) \mid \tau(n) \equiv j \mod s\}$ is infinite, let $\xi, \zeta : \mathbb{N} \to \mathbb{N}$ be the increasing functions such that $T_j = \{\tau(\xi(n))\} = \{s \cdot \zeta(n) + j\}$, and let $B = \{b'_n\}$ be the set where $b'_n = b_{sn+j}$. Then for every $h > 1$:

$$\liminf_{n,m\to\infty} \frac{a_n + b'_m}{n \cdot \sqrt[k]{m}} \;\leq\; \lim_{n\to\infty} \frac{a_{\sigma(\xi(n))} + b'_{\zeta(n)}}{\sigma(\xi(n)) \cdot \sqrt[k]{\zeta(n)}} \;=$$

$$\lim_{n\to\infty} \frac{a_{\sigma(\xi(n))} + b_{\tau(\xi(n))}}{\sigma(\xi(n)) \cdot \sqrt[k]{\tau(\xi(n))}} \cdot \sqrt[k]{\frac{s \cdot \zeta(n) + j}{\zeta(n)}} \;=\; l \cdot \sqrt[k]{s} \;=\; 0 \;<\; \frac{1}{\sqrt[k]{h-1}}\,.$$

By what already proved above, we get the existence of an h-tuple

$$Z \;=\; \{z_1 < z_2 < \ldots < z_h\} \;\subseteq\; B'$$

such that $|\bigcap_{i=1}^h (A + z_i)| \geq k$. It is clear from the definition of B' that $\max Z \geq b'_h \geq sh + j > s$. Since s can be taken arbitrarily large, we conclude that $R_k^h(A) \cap [B]^h$ is infinite, as desired. □

Corollary 2.4. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers. If there exists a function $f : \mathbb{N} \to \mathbb{R}^+$ such that*

$$\limsup_{n\to\infty} \frac{a_n}{n \cdot f(n)} \;<\; \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{f(b_n)}{\sqrt[k]{n}} \;=\; 0\,,$$

then $R_k^h(A) \cap [B]^h$ is infinite for all h.

Proof. It directly follows from Theorem 2.3, since

$$\liminf_{n,m\to\infty} \frac{a_n + b_m}{n\sqrt[k]{m}} \;\leq\; \liminf_{m\to\infty} \frac{a_{b_m} + b_m}{b_m \cdot \sqrt[k]{m}} \;=\; \liminf_{m\to\infty} \frac{a_{b_m}}{b_m \cdot \sqrt[k]{m}} \;\leq$$

$$\leq\; \limsup_{m\to\infty} \frac{a_{b_m}}{b_m \cdot f(b_m)} \cdot \liminf_{m\to\infty} \frac{f(b_m)}{\sqrt[k]{m}} \;=\; 0\,.$$

□

An an example, we now see a property that also applies to all zero density sets having at least the same "asymptotic size" as the prime numbers.

Corollary 2.5. *Assume that the sets $A = \{a_n\}$ and $B = \{b_n\}$ satisfy the conditions $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$ and $\log b_n = o(n^{\varepsilon})$ for all $\varepsilon > 0$. Then for every h and k, there exist infinitely many h-tuples $\{\beta_1 < \ldots < \beta_h\} \subset B$ such that each distance $\beta_j - \beta_i$ equals the distance of k-many pairs of elements of A.*

Proof. By the hypothesis $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$ it follows that $a_n = o(n \log^2 n)$, and so the previous corollary applies with $f(n) = \log^2 n$. Clearly, every h-tuple $\{\beta_1 < \ldots < \beta_h\} \in R_k^h(A) \cap [B]^h$ satisfies the desired property. □

3. Finitely Δ-sets

Recall that a set $A \subseteq \mathbb{N}$ is called a *Delta-set* (or Δ*-set* for short) if $\Delta(X) \subseteq A$ for some infinite X. A basic result is the following: "If A has positive upper asymptotic density, then $\Delta(A) \cap \Delta(X) \neq \emptyset$ for all infinite sets X." (See, *e.g.*, [1].) Another relevant property is that Δ-sets are *partition regular*, *i.e.* the family $\mathcal{F}$ of Δ-sets satisfies the following property:

- If a set $A = A_1 \cup \ldots \cup A_r$ of $\mathcal{F}$ is partitioned into finitely many pieces, then at least one of the pieces A_i belongs to $\mathcal{F}$.

To see this, let an infinite set of distances $\Delta(X) = C_1 \cup \ldots \cup C_r$ be finitely partitioned, and consider the partition of the pairs $[X]^2 = D_1 \cup \ldots \cup D_r$ where $\{x < x'\} \in D_i \Leftrightarrow x' - x \in C_i$. By the infinite Ramsey Theorem, there exists an infinite $Y \subseteq X$ and an index i such that $[Y]^2 \subseteq D_i$, which means $\Delta(Y) \subseteq C_i$.

A convenient generalization of Δ-sets is the following.

Definition 3.1. A is a *finitely* Δ*-set* (or Δ_f*-set* for short) if it contains the distances of finite sets of arbitrarily large size, *i.e.*, if for every k there exists $|X| = k$ such that $\Delta(X) \subseteq A$.

Trivially every Δ-set is a Δ_f-set, but not conversely. For example, take any sequence $\{a_n\}$ such that $a_{n+1} > a_n \cdot n$, let $A_n = \{a_n \cdot i \mid i = 1, \ldots, n\}$, and consider the set $A = \bigcup_{n\in\mathbb{N}} A_n$. Notice that for every n, one has $\Delta(A_n) \subseteq A_n$, and hence A is a Δ_f-set. However A is not a Δ-set. Indeed, assume by contradiction that $\Delta(X) \subseteq A$ for some infinite $X = \{x_1 < x_2 < \ldots\}$; then $x_2 - x_1 = a_k \cdot i$ for some k and some $1 \leq i \leq k$. Pick a large enough m so that $x_m > x_2 + a_k \cdot k$. Then $x_m - x_1, x_m - x_2 \in \bigcup_{n>k} A_n$, and so $x_2 - x_1 = (x_m - x_1) - (x_m - x_2) \geq a_{k+1} > a_k \cdot k \geq x_2 - x_1$, a contradiction. We remark that there exist "large" sets that are not Δ_f-sets. For instance, consider the set O of odd numbers; it is readily seen that $\Delta(Z) \not\subseteq O$ whenever $|Z| \geq 3$.

The following property suggests the notion of Δ_f-set as combinatorially suitable.

Proposition 3.2. *The family of* Δ_f*-sets is partition regular.*

Proof. Let A be a Δ_f-set, and let $A = C_1 \cup \ldots \cup C_r$ be a finite partition. Given k, by the finite Ramsey theorem we can pick n large enough so that every r-partition of the pairs $[\{1,\ldots,n\}]^2$ admits a homogeneous set of size k. Now pick a set $X = \{x_1 < \ldots < x_n\}$ with n-many elements such that $\Delta(X) \subseteq A$, and consider the partition $[\{1,\ldots,n\}]^2 = D_1 \cup \ldots \cup D_r$ where $\{i < j\} \in D_t \Leftrightarrow x_j - x_i \in C_t$. Then there exists an index t_k and a set $H = \{h_1 < \ldots < h_k\}$ of cardinality k such that $[H]^2 \subseteq D_{t_k}$. This means that the set $Y = \{x_{h_1} < \ldots < x_{h_k}\}$ is such that $\Delta(Y) \subseteq C_{t_k}$. Since there are only finitely many pieces $C_1,\ldots,C_r$, there exists t such that $t_k = t$ for infinitely many k. In consequence, C_t is a Δ_f-set. □

As a straight consequence of Theorem 2.3, we can give a simple sufficient condition on the "asymptotic size" of a set A that guarantees the corresponding k-recurrence sets be finitely Δ-sets.

Theorem 3.3. *Let $k \geq 2$ and let the infinite set $A = \{a_n\}$ be such that $a_n = o(n^{k/k-1})$. Then $R_k(A)$ is a Δ_f-set.*

Proof. Let $B = \mathbb{N}$, so $b_m = m$. By taking $m = a_n$, we obtain that

$$\liminf_{n,m\to\infty} \frac{a_n + m}{n\sqrt[k]{m}} \;\leq\; \lim_{n\to\infty} \frac{a_n + a_n}{n\sqrt[k]{a_n}} \;=\; \lim_{n\to\infty} \left(2^{\frac{k}{k-1}} \cdot \frac{a_n}{n^{\frac{k}{k-1}}}\right)^{\frac{k-1}{k}} \;=\; 0\,.$$

Then Theorem 2.3 applies, and for every h we obtain the existence of a finite set Z of cardinality h such that $Z \in R_k^h(A) \cap [B]^h = R_k^h(A)$. But then, by Proposition 2.2, $\Delta(Z) \subseteq R_k(A)$. □

References

[1] V. Bergelson, *Ergodic Ramsey Theory - an update*, in "Ergodic Theory of $\mathbb{Z}^d$-actions", London Math. Soc. Lecture Notes Series **228** (1996), pp. 1–61.

[2] V. Bergelson, P. Erdös, N. Hindman and T. Łukzak, *Dense Difference Sets and their Combinatorial Structure*, in "The Mathematics of Paul Erdös, I" (R. Graham and J. Nešetřil, eds.), Springer (1997), pp. 165–175.

[3] M. Di Nasso, I. Goldbring, R. Jin, S. Leth, M. Lupini and K. Mahlburg, *Progress on a sumset conjecture by Erdös*, Canad. J. Math., to appear.

[4] P. Erdös, A. Sárkozy and V.T. Sós, *On additive properties of general sequences*, Discrete Math. **136** (1994), pp. 75–99.

[5] R. Jin, *Sumset phenomenon*, Proc. Amer. Math. Soc. **130** (2002), pp. 855–861.

[6] R. Jin, *Plünnecke's theorem for asymptotic densities*, Trans. Amer. Math. Soc. **363** (2011), pp 5059–5070.

[7] I.Z. Ruzsa, *On difference sets*, Studia Sci. Math. Hungar. **13** (1978), pp. 319–326.

[8] A. Sárközy, *On difference sets of sequences of integers*; part I: Acta Math. Hung. **31** (1978), pp. 125–149; part II: Ann. Univ. Sci. Budap., Sect. Math. **21** (1978), pp. 45–53; part III: Acta Math. Hung. **31** (1978), pp. 355–386.

Dipartimento di Matematica, Università di Pisa, Italy
E-mail address: dinasso@dm.unipi.it