

# PROPERLY DISCONTINUOUS ISOMETRIC ACTIONS ON THE UNIT SPHERE OF INFINITE DIMENSIONAL HILBERT SPACES

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ABSTRACT. We study the properly discontinuous and isometric actions on the unit sphere of infinite dimensional Hilbert spaces and we get some new examples of Hilbert manifold with constant positive sectional curvature. We prove some necessary conditions for a group to act isometrically and properly discontinuously, and in the case of finitely generated Abelian groups the necessary and sufficient conditions are given.

Key words: properly discontinuous, Hilbert manifold.

## 1. INTRODUCTION

A Hilbert manifold  $M$  is a manifold modeled on a Hilbert space  $\mathbb{H}$  and equipped with an inner product  $\langle \cdot, \cdot \rangle(p)$ , on each tangent space  $T_p M \cong \mathbb{H}$ , depending smoothly on  $p$  and defining on  $T_p M$  a norm equivalent to the original norm of  $\mathbb{H}$ . In what follows we assume that  $M$  is complete metric space with respect to the usual distance obtained from Riemannian metric  $\langle \cdot, \cdot \rangle$ . In infinite dimensional geometry most of the local results follow from general arguments analogous to those in the finite dimensional case (see [7] or [9]). The investigation of global properties, on the contrary, is harder than the finite dimensional case; for example the Hopf-Rinow Theorem is only generically satisfied on a complete Hilbert manifold (see [6]). An important application, due to Anderson ([1]), is the extension of the Bonnet Theorem about the estimate of the diameter of complete Hilbert manifolds with sectional curvature  $K \geq K_o > 0$ . In finite dimensional geometry the Bonnet Theorem implies also that the fundamental group must be finite, (see [5]), while in [1] doesn't appear any information about the fundamental group. As in the finite dimensional case there is a bijective correspondence between complete Hilbert manifolds, modeled on  $\mathbb{H}$ , with positive constant sectional curvature 1 and groups  $G$  acting freely, isometrically and properly discontinuously on the unit sphere  $\mathbf{S}(\mathbb{H} \times \mathbb{R})$  in  $\mathbb{H} \times \mathbb{R}$ .

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We recall that a group  $G$  acts *properly discontinuously* on a topological space  $X$  if it satisfies the following conditions:

- (1) if  $x, y \in X$ ,  $y \notin Gx$ , then  $x$  and  $y$  have neighborhoods  $U$  and  $V$  such that  $gU \cap V = \emptyset$  for every  $g \in G$ ;
- (2) for any  $x \in X$  the set  $G_x := \{g \in G : gx = x\}$  is finite;
- (3) for any  $x \in X$  there exists a neighborhood  $U$ , stable by  $G_x$ , such that  $gU \cap U = \emptyset$  for every  $g \in G - G_x$ ;

If  $G_x = e$  for every  $x \in X$  we say that  $G$  acts *freely*. Note that condition (1) implies that the quotient space  $X/G$  is Hausdorff. It is well known that if  $G$  acts properly discontinuously on  $X$ , then  $X/G$  is a manifold and the projection  $\pi : X \rightarrow X/G$  is a covering map.

In this article we study which groups act properly discontinuously on the unit sphere of infinite dimensional Hilbert space and we shall prove that the situation is quite different from finite dimensional case. The main result (Theorem 7) is that any group of the form

$$H \cong G \oplus \mathbb{Z}_{p_1}^{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{p_m}^{\alpha_m},$$

where  $G$  is a group without elements of finite order, acts isometrically and properly discontinuously on the unit sphere  $\mathbf{S}(l_2(G))$  of  $l_2(G)$  if and only if  $p_i$  are different primes. Moreover, we shall prove that a group  $H$  acts isometrically and properly discontinuously on the unit sphere of some separable Hilbert space then  $H$  acts, with the same properties, on the unit sphere of any Hilbert space. (Proposition 4). Some basic references for infinite dimensional geometry are [9] and [7].

## 2. PROPERLY DISCONTINUOUS ISOMETRIC ACTIONS ON THE INFINITE DIMENSIONAL UNIT SPHERE OF A HILBERT SPACE

Let  $f$  be a mapping from a set  $G$  into a real (complex) Banach space  $\mathbb{E}$ . We recall, briefly, the idea of unordered summation in terms of convergence of nets. The set  $\Lambda$  of finite subsets of  $G$  is directed by the inclusion relation  $\subseteq$ , and we can define a net by the equation

$$\eta : \Lambda \rightarrow \mathbb{E}, \quad \eta(F) = \sum_{g \in F} f(g).$$

We say that  $f$  is summable if the net  $\eta$  converges to some element, and this is possible if and only if there exists an element  $L_o \in \mathbb{E}$  with the following property: for every  $\epsilon > 0$  there exists  $J(\epsilon) \in \Lambda$  such that

$$\| L_o - \sum_{g \in J} f(g) \| < \epsilon, \text{ whenever } J(\epsilon) \subseteq J \in \Lambda.$$

It is well known, see [10], that

$$l_2(G) = \{x : G \rightarrow \mathbb{R} : \sum_{g \in G} [x(g)]^2 < \infty\},$$

is a Hilbert space with inner product  $\langle x, y \rangle = \sum_{g \in G} x(g)y(g)$  and a Hilbert basis is given by the functions  $e_h(g) = \delta_{hg}$ ,  $h, g \in G$ . When  $G$  is countable then  $l_2(G)$  is isometrically isomorphic to  $l_2$  and any Hilbert space is isometrically isomorphic to  $l_2(D)$ , where  $D$  is a set with the same cardinality of any Hilbert basis. Generally, we have the following properties:

- (1)  $N = \{g \in G : x(g) \neq 0\}$  is countable;
- (2)  $\sum_{g \in G} [x(g)]^2 = \sup\{\sum_{g \in F} [x(g)]^2 : F \in \Lambda\}$ .

The second property proves that for every bijective map  $\phi : G \rightarrow G$ , if  $x \in l_2(G)$  then  $x \circ \phi \in l_2(G)$  and  $\|x\| = \|x \circ \phi\|$ .

Now, we assume that  $G$  is a group and we will denote by  $R_g$  the right translation, i.e.  $h \rightarrow hg$ , that is a bijective map. It is easy to prove that the following application

$$\mu : G \times l_2(G) \rightarrow l_2(G) \quad \mu(g, x) = gx = x \circ R_g$$

is an isometric effective action of  $G$  into  $l_2(G)$ , i.e. if  $gx = x$  for every  $x$  then  $g = e$ , and, in particular, we have obtained an isometric action on the unit sphere  $\mathbf{S}(l_2(G))$  of  $l_2(G)$ . Summing up we have the following result.

**Proposition 1.** *Any group  $G$  is isomorphic to a subgroup of the unitary group of the Hilbert space  $l_2(G)$ .*

Now we shall prove the first step of our main result.

**Theorem 2.** *If  $G$  has no elements of finite order, then the action  $\mu$  on the unit sphere  $\mathbf{S}(l_2(G))$  is properly discontinuous. Moreover  $\mathbf{S}(l_2(G))/G$  has a complete Riemannian metric with constant sectional curvature 1.*

**Proof:** we will prove the above Theorem in three simple steps.

*Step 1: the action is free.*

If  $gx = x$  then  $g^n x = x$  and we have, for every  $h \in G$  and  $n \in \mathbb{Z}$ ,  $x(h) = x(hg^n)$ . The elements  $\{hg^n, n \in \mathbb{Z}\}$  are different because  $g$  has not finite order. Then there exists an element  $h \in G$  such that  $x(h) \neq 0$  and we have  $\sum_{n=1}^{\infty} [x(hg^n)]^2 = \sum_{n=1}^{\infty} [x(h)]^2 = \infty$ , which is absurd by property 2.

*Step 2: if the action is not properly discontinuous, there exists a sequence of different elements  $(g_n) \subseteq G$  and  $x \in \mathbf{S}(l_2(G))$  such that  $g_n x \rightarrow y$ , for some  $y$ .*

Since  $G$  acts isometrically, this is an easy consequence of the definition of properly discontinuous action.

*Step 3: there are no sequences  $(g_n) \subseteq G$  of different elements and  $x \in \mathbf{S}(l_2(G))$  such that  $g_n x$  converges to some  $y \in \mathbf{S}(l_2(G))$ .*

Assume that there exists a sequence  $g_n \subseteq G$  and  $x \in \mathbf{S}(l_2(G))$  such that  $g_n x \rightarrow y$ , for some  $y \in S(L_2(G))$ . Hence for every  $\alpha \in G$  we have

$$y(\alpha) = \langle y, e_\alpha \rangle = \lim_{n \rightarrow \infty} \langle g_n x, e_\alpha \rangle = \lim_{n \rightarrow \infty} x(\alpha g_n)$$

and by property 2 we conclude that  $y(\alpha) = 0$ , for every  $\alpha \in G$ , which is a contradiction.

Now, it is well known that  $G$  induces a metric on the manifold  $\mathbf{S}(l_2(G))/G$  such that it is an Hilbert manifold and the projection  $\pi : \mathbf{S}(l_2(G)) \rightarrow \mathbf{S}(l_2(G))/G$  is a local isometry and a covering map. Obviously the sectional curvature of  $\mathbf{S}(l_2(G))/G$  is constant and equal to 1, so to conclude the proof we shall prove that  $\mathbf{S}(l_2(G))/G$  is a complete Hilbert manifold. In the finite dimensional geometry we can prove it easily, because being geodesically complete at some point  $p$ , i.e. the  $\exp_p$  being defined on  $T_p M$ , implies completeness as metric space. Unfortunately, this fact doesn't hold in infinite dimensional geometry: Atkin, see [2], gave an example of a complete Hilbert manifold  $M$ , such that the exponential map is not surjective in some point  $p \in M$ . Take  $q \notin \exp_p(T_p M)$ ; clearly  $M - \{q\}$  is already geodesically complete in  $p$  but it is not complete as metric space. In this case we will use a simple criterion to resolve our problem.

We say that a continuous curve  $c : [a, b] \rightarrow M$  is *convergent* if there exists a discrete subset  $D \subset [a, b]$  such that

- $c : [a, b] - D \rightarrow M$  is differentiable;
- $\lim_{t \rightarrow b} \int_a^t \|\dot{c}(t)\| dt$  is finite.

*A Hilbert manifold is complete if and only if the trace of any convergent curve is relatively compact.*

If  $(M, g)$  is complete then it is easy to prove that any trace of convergent curve is relatively compact. Vice-versa, we assume that any trace of convergent curve is relatively compact. We shall prove that any Cauchy sequence has a convergent subsequence, which proves that  $M$  is complete as metric space. Let  $(x_n)$  be any Cauchy sequence. For any  $\epsilon = \frac{1}{2^k}$  there exists  $n(k)$  such that  $\forall n, m \geq n(k), \Rightarrow d(x_n, x_m) \leq \frac{1}{2^{k+1}}$ . Note that we shall assume  $n(k+1) \geq n(k) + 2$ . Let  $(a_n)$  be a sequence such that

$$\begin{cases} a_1 = 0; \\ a_n > 0, \quad n \geq 2; \\ \sum_1^\infty a_n = 1. \end{cases}$$

By definition of distance in a Hilbert manifold, for any  $k \in \mathbb{N}$  there exists a differential curve,  $\gamma_k : [\sum_1^k a_n, \sum_1^{k+1} a_n] \longrightarrow M$ , between  $x_{n(k)}$  and  $x_{n(k+1)}$  such that

$$L[\gamma_k] \leq d(x_{n(k)}, x_{n(k+1)}) + \frac{1}{2^{k+1}} \leq \frac{1}{2^k}.$$

We define  $\gamma : [0, 1) \longrightarrow M$ ,  $\gamma(t) = \gamma_k(t)$  if  $t \in [\sum_1^k a_n, \sum_1^{k+1} a_n]$ . It is easy to see that  $\gamma$  is a convergent curve so  $(x_n)$  has a convergent subsequence.

Now take a convergent curve  $\gamma; [a, b] \longrightarrow M$  in  $\mathbf{S}(l_2(G))/G$ . We can lift  $\gamma$  to a curve  $\bar{\gamma}$  in  $\mathbf{S}(l_2(G))$  that is a convergent curve because  $\pi$  is a local isometry. On the other hand  $\gamma([a, b]) \subseteq \pi(\overline{\bar{\gamma}([a, b])})$ , then the trace of  $\gamma$  is relative compact. Q.E.D.

Now, the infinite dimensional sphere is contractible, by Bessega Theorem [3], so we have proved the following result.

**Corollary 3.** *Let  $G$  be a group without elements of finite order. Then there exists a complete Hilbert manifold with positive constant sectional curvature with fundamental group  $G$ .*

The above result is different from the corresponding one in finite dimensions: by Bonnet Theorem the fundamental group of a complete finite dimensional manifold with constant positive curvature is finite. On the contrary, groups as  $\mathbb{Q}$  and  $\mathbb{Z}$  act properly discontinuously and isometrically on the unit sphere of  $l_2$ .

An interesting fact is that, when the model space is separable, then the cardinality of the fundamental group is at most countable. This is a trivial consequence of the lemma of Sierpinski [11], which says that every connected locally separable metric space is separable, and of the fact that the cardinality of the fiber of the universal covering map, is the cardinality of the fundamental group. In particular our Theorem is sharp, relative to the cardinality of the fundamental group, in a case of Hilbert manifold modeled on  $l_2$ .

Now, we shall study the case when the group  $G$  has elements of finite order and we shall investigate what are the necessary conditions for a linear map, with finite order, to have no fixed points except for the origin. In the finite dimensional case, the groups acting properly discontinuously, as isometric groups, on the unit sphere are classified [12] and the first question is: if a group acts on the finite dimensional sphere, does it act on any infinite dimensional Hilbert space? The following proposition gives a positive answer to this question.

**Proposition 4.** *If  $G$  acts isometrically and properly discontinuously on the unit sphere of  $\mathbb{R}^n$  or  $l_2$  then  $G$  acts with the same properties on the unit sphere of any infinite dimensional Hilbert space.*

**Proof:** first of all we shall prove that if  $G \subset \mathbf{O}(\mathbf{n})$ , then it acts on the unit sphere of  $l_2$  with the same properties. We define

$$T_g : l_2 \longrightarrow l_2, \quad T_g\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{\infty} g\left(\sum_{j=n(i-1)}^{ni-1} x_{j+1} e_{j+1}\right),$$

with the following identification:

$$\mathbb{R}^n \xrightarrow{T_i} l_2, \quad (x_1, \dots, x_n) \longrightarrow \sum_{j=1}^n x_j e_{n(i-1)+k-1}.$$

Clearly  $T_g$  is an isometry and it is easy to check that this defines an isometric action on  $\mathbf{S}(l_2)$ , that is properly discontinuous since  $G$  is finite. The general case is analogous. We know that any Hilbert space  $\mathbb{H}$  is isometrically isomorphic to  $l_2(D)$ , where  $D$  is a set that has the same cardinality of any Hilbert basis of  $\mathbb{H}$ , and

$$D = \bigcup_{i \in I} D_i, \quad D_i \cap D_j = \emptyset, \text{ if } i \neq j, \quad D_i \cong \mathbb{N}$$

(see [8] page 678). Let  $f \in l_2(D)$ . We define

$$f_i(x) = \begin{cases} f(x) & \text{if } x \in D_i; \\ 0 & \text{otherwise.} \end{cases}$$

Now,  $f \in l_2(D)$  and by property 2 we have that  $f_i \in l_2(D)$ . Another consequence of property 2, that is easy to check, is that for every  $\epsilon > 0$  there exist a  $J(\epsilon) \in \Lambda$  such that for every finite set  $J \subseteq D - J(\epsilon)$  we have  $\sum_{d \in J} [f(d)]^2 \leq \epsilon$ . Let  $\epsilon > 0$  and let  $J(\epsilon)$  be above. Let  $I_o := \{i \in I : D_i \cap J(\epsilon) \neq \emptyset\}$  and we compute, for every finite set  $I \supseteq I_o$ ,

$$\begin{aligned} \left\| \sum_{i \in I} f_i - f \right\|^2 &= \sup\{\sum_{g \in J} [f_i(g) - f(g)]^2, J \text{ finite}\} \\ &= \sup\{\sum_{g \in J - J(\epsilon)} [f(g)]^2, J \text{ finite}\} \\ &\leq \epsilon. \end{aligned}$$

Hence  $\sum_{i \in I} f_i$  converges to  $f$  and it is easy to check that  $\alpha f + \beta g = \sum_{i \in I} \alpha f_i + \beta g_i$  and  $f = \sum_{i \in I} f'_i$ , where  $f'_i = 0$  on the complement of  $D_i$ , so  $f_i = f'_i$ . Moreover, for every finite set  $J \subseteq I$  we have

$$\begin{aligned} \sum_{i \in J} \|f_i\|^2 &= \sum_{i \in J} \sup(\sum_{g_i \in J_i} [f_i(g_i)]^2) \quad J_i \subseteq D_i \text{ finite} \\ &= \sup(\sum_{g \in \cup J_i} [f(g)]^2) \end{aligned}$$

so  $\sum_{i \in I} \|f_i\|^2 = \|f\|^2$ . Now, if we define

$$T_g : l_2(D) \longrightarrow l_2(D), \quad T_g\left(\sum_{i \in I} f_i\right) = \sum_{i \in I} g f_i,$$

by the identifications  $l_2 \xrightarrow{T_i} l_2(D)$  as

$$f \longrightarrow \begin{cases} f \circ s_i(x) & \text{if } x \in D_i; \\ 0 & \text{otherwise,} \end{cases}$$

where  $s_i$  is a bijective map between  $D_i$  and  $\mathbb{N}$ , we have an isometric action that is free. If  $G$  does not act properly discontinuously, there will be a sequence  $g_n$  of distinct elements and an element  $x = \sum_{i \in I} x_i \neq 0$  such that

$$g_n x = \sum_{i \in I} g_n x_i \rightarrow y = \sum_{i \in I} y_i.$$

Hence  $g_n x_i \rightarrow y_i$ , that is a contradiction. Q.E.D.

Let  $\mathbb{H}$  be a Hilbert space and let  $T : \mathbb{H} \longrightarrow \mathbb{H}$  be a linear continuous map with finite order  $m$ , i. e.  $T^m = Id$  and  $m$  is the smallest integer with this property. Henceforth, when we say that a linear map has no fixed points, we will mean that the unique fixed point is the origin. If  $T$  has no fixed point, then it must satisfy the following equation

$$T^{m-1} + T^{m-2} + \cdots + T + Id = 0.$$

If  $m = 2$  then  $T = -Id$ . Hence, if a group  $H$  acts without fixed point then it has at most one element  $g$  of order two and  $g$  has to belong to the center of  $H$ . In particular, groups as  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ , the symmetric group  $S_3 = \langle \phi, \psi; \phi^2 = \psi^3 = 1, \phi\psi = \psi^2\phi \rangle$  and the dihedral group  $D_8 = \langle \sigma, \tau; \sigma^4 = \tau^2, \tau\sigma\tau = \sigma^3 \rangle$ , cannot act without fixed point.

We shall assume that  $\mathbb{H}$  is a complex Hilbert space. Otherwise we shall consider the complexification  $\mathbb{H} \otimes \mathbb{C}$  and  $T^{\mathbb{C}}$  and, clearly,  $T$  has fixed points if and only if  $T^{\mathbb{C}}$  does. Since  $T^m = Id$ , the minimal polynomial is well-defined and is given by  $p_T(z) = (z - \xi_1) \cdots (z - \xi_n)$ , where  $\xi_i$  are roots of unity. As in the finite dimensional case, we have the following decomposition

$$\mathbb{H} = \mathbb{H}_1 \oplus \cdots \oplus \mathbb{H}_n$$

where

- $\mathbb{H}_i = Ker(T - \xi_i Id)$ ;
- $p_{T|_{\mathbb{H}_i}}(z) = (z - \xi_i)$ .

Moreover,  $x = x_1 + \cdots + x_n$  is a fixed point of  $T^k$ ,  $1 \leq k \leq m$  if and only if  $T^k(x_i) = x_i$ . This simple remark proves the following result.

**Proposition 5.** *Let  $T : \mathbb{H} \rightarrow \mathbb{H}$  be a linear continuous map with finite order  $m$ . Then  $T, \dots, T^{m-1}$  have no fixed points if and only if*

$$\Phi_m(T) = 0$$

where  $\Phi_m(t)$  is the  $m$ -th cyclotomic polynomial.

Now, let  $T : \mathbb{H} \rightarrow \mathbb{H}$  be a linear continuous map of order  $n$  and let  $L : \mathbb{H} \rightarrow \mathbb{H}$  be a linear continuous map of order  $m$ . We suppose that  $L \circ T = T \circ L$  and there exists a prime  $p$  such that  $p|m$  and  $p|n$ . Let  $T_1 = T^{\frac{n}{p}}$ ,  $L_1 = L^{\frac{m}{p}}$ . Clearly the order of the last two linear operators is  $p$ , and  $L_1 \circ T_1 = T_1 \circ L_1$ . Relative to  $T_1$  we have

$$\mathbb{H} = \mathbb{H}_1 \oplus \dots \oplus \mathbb{H}_l,$$

and  $H_i$  are  $L_1$ -invariant. In particular, relative to  $L_1$  we have

$$\mathbb{H}_1 = V_1 \oplus \dots \oplus V_r.$$

Now it is easy to prove the following simple, but important, result.

**Proposition 6.** *Let  $G$  be a group that acts on  $\mathbb{H}$  without fixed points, except the origin, as linear group. Let  $g \in G$  be an element of finite order and let  $m \in C(g)$  (centralizer of  $g$ ) have finite order. Then, either  $g^s = m^k$ , for some  $k, s \in \mathbb{N}$  or  $MDC(o(m), o(g)) = 1$ .*

The above result proves that any group that has a subgroup of the form

$$\mathbb{Z}_n \oplus \mathbb{Z}_m$$

with  $MDC(n, m) \neq 1$ , does not act, as linear group, without fixed points.

Now, we shall construct explicitly an action of the group

$$\mathbb{Z}_{p_1^{\alpha_1}} \oplus \dots \oplus \mathbb{Z}_{p_m^{\alpha_m}}$$

with  $p_1 < \dots < p_m$  primes. Let  $\xi_1 \dots \xi_m$  be primitive  $p_i^{\alpha_i}$ -th roots of unity and  $\theta_i = \arg(\xi_i)$ . Generally for any  $\theta$  we define

$$T_\theta : \mathbb{H} \oplus \mathbb{H} \rightarrow \mathbb{H} \oplus \mathbb{H}, \quad T_\theta(x, y) = (\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y).$$

It is easy to check that

- (1)  $T_\theta$  is a surjective isometry;
- (2)  $\forall \theta, \alpha \in \mathbb{R}, T_\theta \circ T_\alpha = T_\alpha \circ T_\theta = T_{\theta+\alpha}$ ;
- (3)  $T_\theta$  has fixed points if and only if  $\theta = 2k\pi$ , with  $k \in \mathbb{Z}$ .
- (4)  $\prod_{i=1}^m \xi_i^{s_i}(x + iy) = \pi_1(\prod_{i=1}^m T_{\theta_i}^{s_i}(x, y)) + i(\pi_2(\prod_{i=1}^m T_{\theta_i}^{s_i}(x, y)))$ , where  $\pi_1$  and  $\pi_2$  are the natural projections.



We shall prove that the operators  $T_{\theta_i}$  represent  $\mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{\alpha_m}}$  as isometry subgroup of  $\mathbf{O}(\mathbb{H} \oplus \mathbb{H})$ . Suppose there exist  $s_1, \dots, s_m \in \mathbb{N}$  such that

$$T_{\theta_i}^{s_i} = \prod_{j \neq i} T_{\theta_j}^{s_j}.$$

Then by the property 4 we have

$$\xi_i^{s_i} = \prod_{j \neq i} \xi_j^{s_j}$$

which implies that  $s_1 = \dots = s_m = 0$ , because  $p_i$  are different primes. If

$$\prod_{i=1}^m T_{\theta_i}^{s_i}$$

has fixed points then we have, by property 4,

$$\prod_{i=1}^m \xi_i^{s_i} = 1$$

that implies, by the same argument as above,  $s_1 = \dots = s_m = 0$ . Now we shall prove the principal result.

**Theorem 7.** *Let  $H \cong G \oplus \mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{\alpha_m}}$  be a group where  $G$  is a group without elements of finite order. Then  $H$  acts isometrically and properly discontinuously on the unit sphere  $\mathbf{S}(l_2(G))$  of  $l_2(G)$  if and only if  $p_i$  are different primes. Furthermore,  $\mathbf{S}(l_2(G))/H$  has a complete Riemannian metric with constant sectional curvature 1.*

**Proof:** the forward implication has already been verified in Proposition 6. For the reverse, we shall prove that  $H$  acts isometrically on the Hilbert space  $l_2(G) \oplus l_2(G)$  that is isometric to  $l_2(G)$ . Any element  $g \in G$  acts on  $l_2(G) \oplus l_2(G)$  as  $g \oplus g$ , and any  $g$  acts as in Theorem 2. It is easy to check that this action has the same properties of the action of  $G$  into  $l_2(G)$ . The elements of finite order, that we will indicated as  $A$ , acts as above. The proof is divided in three steps.

$$(1) G \cap A = e.$$

If  $\prod_{i=1}^n T_{\theta_i}^{s_i} = T_{\theta} = g_o$ , where  $\theta = \theta_1 s_1 + \cdots + \theta_m s_m$ , then

$$\begin{aligned} g_o(e_e, e_e) &= (e_{g_o^{-1}}, e_{g_o^{-1}}) \\ &= (e_e(\cos \theta - \sin \theta), e_e(\cos \theta + \sin \theta)). \end{aligned}$$

Then  $g_o = e$  and  $\theta = 2\pi k$ ;

$$(2) \forall g \in G \text{ e } \forall a \in A, ag = ga.$$

We shall prove that  $T_{\theta}$  commutes with  $g$ , for every  $g \in G$  and every  $\theta \in \mathbb{R}$ .

$$\begin{aligned}
& g(T_\theta((\sum_{h \in G} x_h e_h, \sum_{h \in G} y_h e_h))) = \\
& g(\sum_{h \in G} (x_h \cos \theta - y_h \sin \theta) e_h, \sum_{h \in G} (x_h \sin \theta + y_h \cos \theta) e_h) = \\
& (\sum_{h \in G} (x_h \cos \theta - y_h \sin \theta) e_{hg^{-1}}, \sum_{h \in G} (x_h \sin \theta + y_h \cos \theta) e_{hg^{-1}}) = \\
& T_\theta((\sum_{h \in G} x_h e_{hg^{-1}}, \sum_{h \in G} y_h e_{hg^{-1}})) = \\
& T_\theta(g(\sum_{h \in G} x_h e_h, \sum_{h \in G} y_h e_h)).
\end{aligned}$$

(3)  $ag$  has no fixed points and the action is properly discontinuous.

If

$$ag\left(\sum_{h \in G} x_h e_h, \sum_{h \in G} y_h e_h\right) = \left(\sum_{h \in G} x_h e_h, \sum_{h \in G} y_h e_h\right)$$

then

$$\sum_{h \in G} (x_h \cos \theta - y_h \sin \theta) e_{hg^{-1}} = \sum_{h \in G} x_h e_h$$

and

$$\sum_{h \in G} (x_h \cos \theta + y_h \sin \theta) e_{hg^{-1}} = \sum_{h \in G} y_h e_h.$$

Hence, for every  $\alpha \in G$  we have

$$x_\alpha \cos \theta + y_\alpha \cos \theta = y_{\alpha g^{-1}}; \quad (1)$$

$$x_\alpha \cos \theta - y_\alpha \cos \theta = x_{\alpha g^{-1}}. \quad (2)$$

Now, there exists an  $\alpha \in G$  such that  $x_\alpha \neq 0$  or  $y_\alpha \neq 0$ . Then, it is easy to check by (1) and (2), that for every  $n \in \mathbb{N}$  we have

$$|x_{\alpha g^{-n}}|^2 + |y_{\alpha g^{-n}}|^2 = |x_\alpha|^2 + |y_\alpha|^2.$$

Hence

$$\sum_{n=1}^{\infty} |x_{\alpha g^{-n}}|^2 + \sum_{n=1}^{\infty} |y_{\alpha g^{-n}}|^2 = \infty$$

which is a contradiction, because the above series have to converge. If the action is not properly discontinuous there exists a sequence of distinct elements  $g_n a_n$  and an element  $x \neq 0$ , such that  $g_n a_n x$  converge to some element. The group  $A$  is finite, so we shall consider a subsequence such that  $a_{n(k)} = a_o$ . Then we have obtained a contradiction because  $G$  acts properly discontinuously on the unit sphere. Now, we conclude our proof as in Theorem 1. Q.E.D.

A trivial application of main Theorem and Proposition 4 is the following result that gives a necessary and sufficient conditions when the group  $G$  is a finitely generated Abelian group.

**Corollary 8.** *Every finitely generated Abelian group  $G$  acts isometrically and properly discontinuously on the unit sphere of any infinite dimensional Hilbert space if and only if  $G$  has a torsion of the form*

$$\mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{\alpha_m}},$$

where  $p_i$  are distinct primes. In particular, there exists a complete Hilbert manifold, possibly modeled on  $l_2$ , with positive constant sectional curvature, whose fundamental group is  $G$ .

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