On Thompson's group T and algebraic K-theory

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Abstract

Using a theorem of Lück-Reich-Rognes-Varisco, we show that the Whitehead group of Thompson's group T is infinitely generated, even when tensored with the rationals. To this end we describe the structure of the centralizers and normalizers of the finite cyclic subgroups of T, via a direct geometric approach based on rotation numbers. This also leads to an explicit computation of the source of the Farrell-Jones assembly map for the rationalized higher algebraic K-theory of the integral group ring of T.

1 Introduction and statement of results

Thompson's groups F and T are well-known groups having both type F_∞ and infinite cohomological dimension. Recall that F and T can be defined as the groups of orientation-preserving dyadic piecewise-linear homeomorphisms of the closed unit interval $I = [0, 1]$ and of the circle $S^1 = \mathbb{R}/\mathbb{Z}$; see Section 2 (and [CFP96] for a comprehensive introduction).

Essentially nothing has been known about the algebraic K-theory of these groups. Here we show that the Whitehead group of T is infinitely generated, even when tensored with the rationals. More precisely, our main theorem is the following.

Theorem 1.1. *The Farrell-Jones assembly map in algebraic K-theory induces an injective homomorphism*

$$\operatorname*{colim}_{k\in\mathbb{N}} Wh(C_k) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow Wh(T) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{1.2}$$

and in particular $Wh(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an infinite dimensional $\mathbb{Q}$-vector space.

On the left-hand side of (1.2) $C_k = \mathbb{Z}/k\mathbb{Z}$ denotes the finite cyclic group of order k. It is well known that $Wh(C_k) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$ for all $k \notin \{1, 2, 3, 4, 6\}$; see for example [Oli88, top of page 6]. The colimit in (1.2) is taken over the poset $\mathbb{N}$ with respect to the divisibility relation, and the homomorphisms

$Wh(C_k) \to Wh(C_\ell)$ induced by $C_k = \mathbb{Z}/k\mathbb{Z} \to C_\ell = \mathbb{Z}/\ell\mathbb{Z}$, $1 \mapsto \frac{\ell}{k}$ whenever $k \mid \ell$. The map in (1.2) is induced by identifying C_k with the cyclic subgroup $\langle \gamma_k \rangle$ of T generated by the pseudo-rotation of order k from Example 2.1; see the proof of Corollary 3.11.

Theorem 1.1 is a direct application to T of the paper [LRRV15]. That work and its applicability here are discussed in Section 3, where we also obtain results about the higher algebraic K-theory groups of the integral group ring of T. The ingredients about T needed for this application are summarized in the following theorem.

Theorem 1.3. *Every finite subgroup of T is cyclic, and for every integer $k \geq 0$ there is exactly one conjugacy class in T of cyclic subgroups of order k. Moreover, for every finite cyclic subgroup C of T, the centralizer $Z_T C$ and the normalizer $N_T C$ of C in T are equal, and there is a short exact sequence*

$$1 \longrightarrow C \longrightarrow Z_T C \longrightarrow T \longrightarrow 1\,. \tag{1.4}$$

We proved Theorem 1.3 in 2007 and only lately became aware that essentially the same result (but without the observation about normalizers, which is important for our application) appeared in Matucci's 2008 thesis [Mat08, Theorem 7.1.5], and was subsequently generalized in [MPN13, MPMN14]. The full details of our proof of Theorem 1.3 are given in Section 2.

The group T is of type F_∞ by a theorem of Brown and Geoghegan; see for example [Bro87, Remark 2 on page 56], where this is shown to follow immediately from the same result for the group F [BG84]. Thus the short exact sequence (1.4) of Theorem 1.3 has the following corollary (see for example [Geo08, Section 7.2]).

Corollary 1.5. *For every finite cyclic subgroup C of T the centralizer $Z_T C$ of C in T is of type F_∞. Moreover, for every $s \in \mathbb{N}$, the abelian group $H_s(BZ_T C;\mathbb{Z})$ is finitely generated, and $H_s(BZ_T C;\mathbb{Q}) \cong H_s(BT;\mathbb{Q})$.*

The rational homology groups $H_s(BT;\mathbb{Q})$ of T (and hence also of $Z_T C$) are completely known thanks to a theorem of Ghys and Sergiescu. In fact, in [GS87, Corollaire C on pages 187–188] it is proved that $H^1(T;\mathbb{Z}) = 0$, $H^2(T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ with natural generators α and χ, and $H^*(T;\mathbb{Q}) \cong \mathbb{Q}[\alpha, \chi]$.

2 Thompson's group T and centralizers of finite subgroups

In this section we recall the definition of Thompson's groups F and T, and then prove Theorem 1.3; see Theorem 2.3 and Corollary 2.6 below.

We say that an interval $I \subset \mathbb{R}$ is *dyadic* if its endpoints are dyadic rationals. If I and J are closed dyadic intervals, a homeomorphism $f\colon I \to J$

is called *dyadic piecewise linear*, or *DPL* for short, if f is piecewise linear, the breakpoints occur at dyadic rational points, and the slopes are integer powers of 2. Notice that the inverse of a DPL homeomorphism is again DPL. *Thomspon's group* F is defined as the group of orientation-preserving DPL homeomorphisms of $[0,1]$.

We define an $\mathbb{R}$-*space* to be a pair (X,p) where X is a topological space and $p\colon \mathbb{R} \to X$ is a covering map. In other words, X is a connected 1-dimensional manifold together with a chosen universal covering map p. We consider every $\mathbb{R}$-space to be oriented via p. The primary example is of course $X = S^1 = \mathbb{R}/\mathbb{Z}$ together with the usual universal covering map $u\colon \mathbb{R} \to S^1$.

Let (X,p) and (Y,q) be $\mathbb{R}$-spaces, and let $f\colon X \to Y$ be a map. We say that f is *locally DPL* (short for local dyadic piecewise linear homeomorphism) if for every $x \in X$ there exist closed dyadic intervals I, J in $\mathbb{R}$ such that:

- $p_{|I}$ and $q_{|J}$ are embeddings;
- x belongs to the interior of $p(I)$ and $f(x)$ belongs to the interior of $q(J)$;
- f induces a homeomorphism $f_{|p(I)}\colon p(I) \to q(J)$;
- and the composition

  $$I \xrightarrow{p_|} p(I) \xrightarrow{f_|} q(J) \xrightarrow{q_|^{-1}} J$$

  is a DPL homeomorphism.

If (X,p) is an $\mathbb{R}$-space, then we define $H(X,p) = H(X)$ to be the group of all orientation-preserving homeomorphisms of X, and $T(X,p)$ to be the subgroup of $H(X,p)$ consisting of those orientation-preserving homeomorphisms that are locally DPL. *Thompson's group* T is defined as $T = T(S^1,u)$. Similarly we write $H = H(S^1,u)$. Thompson's group F can then be identified with the subgroup of T fixing a base point.

Example 2.1 (pseudo-rotations)**.** The following elements of T play an important role in our work. Given $q \geq 2$ we denote by $\gamma_q \in T$ be the pseudo-rotation of order q, i.e., the element of T (called C_{q-2} in [CFP96, pages 236–237]) that cyclically permutes the images of the q intervals

$$\left[0, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, \tfrac{3}{4}\right], \ldots, \left[1 - \tfrac{1}{2^j}, 1 - \tfrac{1}{2^{j+1}}\right], \ldots, \left[1 - \tfrac{1}{2^{q-1}}, 1\right] \tag{2.2}$$

and is affine on each of them.

The main result in this section is the following.

Theorem 2.3. *Let C be a finite subgroup of H. Then C is cyclic, the centralizer $Z_H C$ and the normalizer $N_H C$ of C in H are equal, and there is a short exact sequence*

$$1 \longrightarrow C \longrightarrow Z_H C \longrightarrow H \longrightarrow 1\,.$$

Moreover, if $C < T$, then $Z_T C = N_T C$, and there is a short exact sequence

$$1 \longrightarrow C \longrightarrow Z_T C \longrightarrow T \longrightarrow 1\,.$$

The proof of Theorem 2.3 uses Poincaré rotation numbers. We now recall their definition and basic properties, and we refer to [KH95, Chapter 11] for more details and proofs.

Given $h \in H$, choose a lift $\overline{h}\colon \mathbb{R} \to \mathbb{R}$ such that $u\overline{h} = hu$, and choose a point $x \in \mathbb{R}$. Define

$$\rho(h) = \mathbb{Z} + \lim_{n\to\infty} \frac{\overline{h}^n(x) - x}{n} \in \mathbb{R}/\mathbb{Z}\,.$$

Then $\rho(h) \in \mathbb{R}/\mathbb{Z}$ is independent of the choices of $\overline{h}$ and x (see [KH95, Proposition 11.1.1]), and it is called the *rotation number* of h.

Proposition 2.4. *Let $h, g \in H$ and let m be an integer.*

- *(i) If $h(x) = x+\vartheta \mod \mathbb{Z}$, i.e., if h is a rotation by $2\pi\vartheta$, then $\rho(h) = \vartheta$. In particular, $\rho(\mathrm{id}_{S^1}) = 0$.*
- *(ii) $\rho(h^m) = m\rho(h)$.*
- *(iii) $\rho(hgh^{-1}) = \rho(g)$.*
- *(iv) If $\rho(h) = 0$, then h has a fixed point.*
- *(v) If $h \neq \mathrm{id}_{S^1}$ has finite order, then $\rho(h) \in \mathbb{Q}/\mathbb{Z}$ and $\rho(h) \neq 0$. Let $\rho(h) = \frac{p}{q}$ with $(p, q) = 1$ and $0 < p < q$. Then the order of h is q; for every $x \in S^1$ the ordering of $\{x, h(x), h^2(x), \ldots, h^{q-1}(x)\}$ in S^1 is the same as that of $\{0, \frac{p}{q}, \frac{2p}{q}, \frac{(q-1)p}{q}\}$; and h is conjugate to the rotation by $2\pi\frac{p}{q}$.*

Proof. Statements (i) and (ii) follow immediately from the definition, whereas (iii) and (iv) are proved in [KH95, Propositions 11.1.3 and 11.1.4].

(v) Let $h \neq \mathrm{id}_{S^1}$ have finite order. From [KH95, Proposition 11.1.1] we have that $\rho(h) \in \mathbb{Q}/\mathbb{Z}$. From (iv) and Lemma 2.5 below we conclude that $\rho(h) \neq 0$. So let $\rho(h) = \frac{p}{q}$ with $(p, q) = 1$ and $0 < p < q$. Suppose that the order of h is m. Then, using (i) and (ii), $0 = \rho(\mathrm{id}_{S^1}) = \rho(h^m) = m\rho(h) = m\frac{p}{q}$, and therefore $q|m$ since $(p, q) = 1$. On the other hand $\rho(h^q) = q\rho(h) = q\frac{p}{q} = 0 \in \mathbb{R}/\mathbb{Z}$, and therefore from (iv) and Lemma 2.5 we conclude that $h^m = \mathrm{id}_{S^1}$ and hence $m|q$. So the order of h is q. The last statements then follow from [KH95, Proposition 11.2.1]. □

Lemma 2.5. *If $h \in H$ has finite order and has a fixed point, then $h = \mathrm{id}_{S^1}$.*

Proof. If h has a fixed point, then h induces an orientation-preserving homeomorphism of a closed interval. Since the group of orientation-preserving homeomorphisms of a closed interval is torsion-free, if h also has finite order then $h = \mathrm{id}_{S^1}$. □

Corollary 2.6. *Any two cyclic subgroups of H (respectively, of T) with the same order are conjugate in H (respectively, in T).*

Proof. Let C be a cyclic subgroup of H with order q. Proposition 2.4(ii) implies that C has a unique generator g with rotation number $\frac{1}{q}$, and (v) implies that g is conjugate in H to the rotation by $\frac{2\pi}{q}$, and therefore the corollary is true for H. So assume that $g \in T$. By Proposition 2.4(v), the ordering of $\mathcal{O} = \{0, g(0), g^2(0), \dots, g^{q-1}(0)\}$ in S^1 is the same as that of $\{0, \frac{p}{q}, \frac{2p}{q}, \frac{(q-1)p}{q}\}$, and so each $g^k(0)$ is a dyadic rational. Think of $\mathcal{O}$ as a dyadic subdivision of S^1. Then there is a finer dyadic subdivision $\mathcal{O}'$ such that g is affine on each segment of $\mathcal{O}'$. Now let $\gamma_q \in T$ be the pseudo-rotation of order q from Example 2.1. Define $\mathcal{O}''$ to be the dyadic subdivision of (2.2) corresponding to $\mathcal{O}'$, so that γ_q is also affine on each segment of $\mathcal{O}''$, and define $h \in T$ to be the locally DPL homeomorphism that maps each segment of $\mathcal{O}'$ affinely onto the corresponding segment of $\mathcal{O}''$. Then $hgh^{-1} = \gamma_q$, and therefore the corollary is also true for T. □

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let C be a finite subgroup of H. Assume that $C \neq 1$, otherwise there is nothing to prove. Define $S^1_0 = C\backslash S^1$ to be the quotient, and denote by $q\colon S^1 \to S^1_0$ the quotient map.

We first show that C is cyclic. By Lemma 2.5, if $g \in C$ has a fixed point, then $g = \mathrm{id}_{S^1}$. It follows that $q\colon S^1 \to S^1_0$ is a covering map and that S^1_0, being a closed 1-dimensional manifold, is homeomorphic to S^1, and therefore C is cyclic by covering space theory.

Notice that S^1_0 together with the composition qu is an $\mathbb{R}$-space. We abbreviate $H_0 = H(S^1_0, qu)$ and $T_0 = T(S^1_0, qu)$.

Fix a generator g of C. By Proposition 2.4(v), we know that $\rho(g) = \frac{p}{q} \neq 0$, with $(p,q) = 1$ and $0 < p < q$, and q is the order of C. Let s and t be such that $sp + tq = 1$. Then $g^s(1)$ is the element in the orbit of $1 \in S^1$ coming directly after 1 in the cyclic order. Let ℓ be the length in S^1 of $[1, g^s(1)]$. Then S^1_0 can be identified with $\mathbb{R}/\ell\mathbb{Z}$, and multiplication by $\frac{1}{\ell}$ induces a homeomorphism $f\colon S^1_0 \to S^1$. It follows that conjugation by f yields an isomorphism $H_0 \cong H$. Moreover, if $g \in T$, then ℓ is a dyadic rational and therefore the homeomorphism f is locally DPL, so conjugation by f restricts to an isomorphism $T_0 \cong T$.

To prove that $N_H C = Z_H C$, let $h \in N_H C$ be given. Then $hgh^{-1} = g^m$ for some integer m. By Proposition 2.4(ii)-(iii) we see that $\rho(g) = m\rho(g)$, and by Proposition 2.4(iv) and Lemma 2.5 we see that $\rho(g) \neq 0$. Therefore $m = 1$ and so $h \in Z_H C$. Since obviously $Z_H C \leq N_H C$, we conclude that $N_H C = Z_H C$.

Since $Z_H C$ acts on the quotient $S^1_0 = C\backslash S^1$, we get a group homomorphism $\pi\colon Z_H C \to H_0$. We are going to show next that there is a short exact sequence

$$1 \longrightarrow C \longrightarrow Z_H C \xrightarrow{\pi} H_0 \longrightarrow 1\,, \tag{2.7}$$

i.e., that $\ker \pi = C$ and that π is surjective. Since $H_0 \cong H$, as observed above, this will prove the first part of the theorem.

To show that $\ker \pi = C$, let $h \in \ker \pi$ be given. Then for any $x \in S^1$, $h(x) = g^{m(x)}(x)$ for some integer $m(x)$. By continuity and since S^1 is connected, it follows that $m(x)$ is constant, i.e., that $h \in C$. Since obviously $C \leq \ker \pi$, we conclude that $\ker \pi = C$.

To show that π is surjective, let $h_0 \in H_0$ be given. Choose a basepoint $x_0 \in S^1_0$ and define $y_0 = h_0(x_0)$. Since h_0 is freely homotopic to $\mathrm{id}_{S^1_0}$, choose such a homotopy and let α be the track of this homotopy at x_0; α is then a path from x_0 to y_0. It follows that given any loop ω at x_0, ω is homotopic to $\alpha \cdot (h_0\omega) \cdot \alpha^{-1}$ relative to x_0. Now choose $x \in S^1$ such that $q(x) = x_0$, and let $\widetilde{\alpha}$ be the lift of α starting at x. Define $y = \widetilde{\alpha}(1)$ and $y_0 = q(y) = \alpha(1)$.

We want to show that h_0 lifts to a homeomorphism $h \in H$ such that:

$$qh = h_0 q \quad \text{and} \quad h(x) = y\,. \tag{2.8}$$

We first show that $h_0 q\colon S^1 \to S^1_0$ lifts to a map $h\colon S^1 \to S^1$ satisfying (2.8). It follows then easily that $h \in H$.

By covering space theory, it is enough to show that if γ is any loop in S^1 at x then there is a loop σ in S^1 at y such that $h_0 q\gamma$ is homotopic to $q\sigma$ relative to y_0. Given γ, since $q\gamma$ is homotopic to $\alpha \cdot (h_0 q\gamma) \cdot \alpha^{-1}$ relative to x_0 and $q\gamma$ lifts to a loop at x, it follows that $\alpha \cdot (h_0 q\gamma) \cdot \alpha^{-1}$ lifts to a loop τ at x. Let σ be the lift of $h_0 q\gamma$ at y. We claim that σ is a loop. Indeed, if $\sigma(1) = g^m y$ for some integer m, then $\tau(1) = g^m x$. Hence $m = 0$ and σ is a loop, as claimed. Therefore h_0 lifts to an $h \in H$ satisfying (2.8).

It only remains to show that h commutes with g, i.e., $h \in Z_H(C)$. Let $x \in S^1$. Let $\rho(g) = \frac{p}{q}$ with $(p,q) = 1$. By Proposition 2.4(v) we know that the cyclic order of $Cx = \big\{x, gx, g^2x, \ldots, g^{q-1}x\big\}$ in S^1 is the same as that of $\big\{0, \frac{p}{q}, \frac{2p}{q}, \frac{(q-1)p}{q}\big\}$. Since h preserves cyclic order, it follows that the cyclic orders of

$$\begin{aligned} Ch(x) &= \big\{h(x), gh(x), g^2h(x), \ldots, g^{q-1}h(x)\big\}\,,\\ h\big(Cx\big) &= \big\{h(x), h(gx), h(g^2x), \ldots, h(g^{q-1}x)\big\}\,,\\ \text{and} &\quad \big\{0, \tfrac{p}{q}, \tfrac{2p}{q}, \tfrac{(q-1)p}{q}\big\} \end{aligned}$$

are all the same. Since $qh = h_0 q$, h sends orbits to orbits, and therefore we have $Ch(x) = h\big(Cx\big)$, from which it follows that $gh(x) = h(gx)$.

Finally, assume that $C < T$. Since $Z_T C = T \cap Z_H C$ and $N_T C = T \cap N_H C$, it is now clear that $Z_T C = N_T C$. For all $h \in H$, since membership in $T(X,p)$ is a local property, $h \in T = T(S^1, u)$ if and only if $\pi(h) \in T_0 = T(S^1_0, qu)$, therefore (2.7) induces a short exact sequence

$$1 \longrightarrow C \longrightarrow Z_T C \xrightarrow{\pi_|} T_0 \longrightarrow 1\,. \tag{2.9}$$

But as observed above, $T_0 \cong T$, and so the theorem is proved. $\square$

3 Assembly maps and algebraic K-theory of T

In this last section we review assembly maps and isomorphism conjectures in algebraic K-theory, focusing on the rationalized case and referring the reader to [LR05, Lüc10] for comprehensive surveys. Then we explain the main results of [LRRV15] and how they imply Theorem 1.1 as well as a generalization to higher algebraic K-theory.

Let G be a discrete group. The algebraic K-theory groups $K_n(\mathbb{Z}G)$ of the integral group ring of G play a central role in geometric topology, in particular in the classification of high-dimensional manifolds and their automorphisms. Arguably the most important K-theoretic invariant is the *Whitehead group* $Wh(G)$, which classifies high-dimensional h-cobordisms, and which is defined as the quotient of $K_1(\mathbb{Z}G) = \big(\bigcup_{k\in\mathbb{N}} GL_k(\mathbb{Z}G)\big)_{\mathrm{ab}}$ by the image of the 1-by-1 invertible matrices $(\pm g)$, $g \in G$. The following conjecture is one of the most well-known and consequential open problems in this area.

Conjecture 3.1. *If G is torsion-free, then $Wh(G) = 0$. If G has torsion, then the inclusions of finite subgroups H of G induce an injective group homomorphism*

$$\operatorname*{colim}_{H\in \mathtt{Sub}_{\mathcal{F}\mathrm{in}}G} Wh(H) \longrightarrow Wh(G). \tag{3.2}$$

The colimit in (3.2) is taken over the finite subgroup category $\mathtt{Sub}_{\mathcal{F}\mathrm{in}}G$, whose objects are the finite subgroups H of G and whose morphisms are defined as follows. Given subgroups H and K of G, let $\mathrm{conhom}_G(H, K)$ be the set all group homomorphisms $H \to K$ given by conjugation by an element of G. The group $\mathrm{inn}(K)$ of inner automorphisms of K acts on $\mathrm{conhom}_G(H, K)$ on the left by post-composition. The set of morphisms in $\mathtt{Sub}_{\mathcal{F}\mathrm{in}}G$ from H to K is then defined as the quotient $\mathrm{inn}(K)\backslash \mathrm{conhom}_G(H, K)$. For example, in the special case when G is abelian, then $\mathtt{Sub}_{\mathcal{F}\mathrm{in}}G$ is just the poset of finite subgroups of G ordered by inclusion. Equivalently, the colimit in (3.2) could be taken over the restricted orbit category $\mathtt{Or}_{\mathcal{F}\mathrm{in}}G$, which has as objects the homogeneous G-sets G/H for any finite subgroup H of G, and as morphisms the G-equivariant maps. The relation between $\mathtt{Sub}_{\mathcal{F}\mathrm{in}}G$ and $\mathtt{Or}_{\mathcal{F}\mathrm{in}}G$ and the equivalence of the two approaches is explained, for example, in [LRV03, page 152, Lemma 3.11].

Conjecture 3.1 is known to be true for all Gromov hyperbolic groups [BLR08] and all CAT(0)-groups [BL12], for example. One of the most interesting open cases of Conjecture 3.1 is Thompson's group F: is $Wh(F) = 0$? Our main result, Theorem 1.1, is that for Thompson's group T the map (3.2) is injective when tensored with the rational numbers $\mathbb{Q}$.

Before explaining this, we want to discuss how Conjecture 3.1 is a special case of the more general Farrell-Jones Conjecture in algebraic K-theory. This conjecture asserts that certain *assembly maps* are isomorphisms. The targets of the assembly maps are the algebraic K-theory groups $K_n(\mathbb{Z}G)$ that we are interested in. The sources are other groups that are easier to compute and homological in nature, and that only depend on the algebraic K-theory of relatively "small" subgroups of G. The construction of these assembly maps is rather technical, and we will not explain it here—see e.g. [LR05, LRRV15] for details. But the picture simplifies after rationalizing, i.e., after tensoring with $\mathbb{Q}$, and we are going to focus on it now.

The *rationalized classical assembly map* for $K_n(\mathbb{Z}G)$, $n \in \mathbb{Z}$, is a homomorphism

$$\bigoplus_{\substack{s,t\geq 0\\ s+t=n}} H_s(BG;\mathbb{Q}) \underset{\mathbb{Q}}{\otimes} (K_t(\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \longrightarrow K_n(\mathbb{Z}G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\,. \tag{3.3}$$

The *rationalized Farrell-Jones assembly map* for $K_n(\mathbb{Z}G)$, $n \in \mathbb{Z}$, is a homomorphism

$$\bigoplus_{C\in(\mathcal{F}\mathrm{in}\mathcal{C}\mathrm{yc})} \bigoplus_{\substack{s\geq 0,t\geq -1\\ s+t=n}} H_s(BZ_GC;\mathbb{Q}) \underset{\mathbb{Q}[W_GC]}{\otimes} \Theta_C(K_t(\mathbb{Z}C) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \longrightarrow K_n(\mathbb{Z}G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\,. \tag{3.4}$$

Here $(\mathcal{F}\mathrm{in}\mathcal{C}\mathrm{yc})$ denotes the set of conjugacy classes of finite cyclic subgroups in G, Z_GC denotes the centralizer in G of C, W_GC denotes the quotient N_GC/Z_GC of the normalizer modulo the centralizer, and $\Theta_C(K_t(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q})$ is a direct summand of $K_t(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$ naturally isomorphic to

$$\operatorname{coker}\left(\bigoplus_{D\subsetneq C} K_t(\mathbb{Z}D) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow K_t(\mathbb{Z}C) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\right).$$

The dimensions of the $\mathbb{Q}$-vector spaces $\Theta_C(K_n(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q})$ can be explicitly computed; see [Pat14, Theorem on page 9].

Moreover, the summand in the source of (3.4) corresponding to $C = 1$ is the same as the source of (3.3), since $K_{-1}(\mathbb{Z}) = 0$. Therefore, if G is torsion-free, then the classical and the Farrell-Jones assembly maps are the same.

Conjecture 3.5 (Rationalized Farrell-Jones Conjecture)**.** *For any group G and for any $n \in \mathbb{Z}$, the rationalized Farrell-Jones assembly map* (3.4) *is an isomorphism. In particular, if G is torsion-free, then the map* (3.3) *is an isomorphism for any $n \in \mathbb{Z}$.*

Conjecture 3.5, even in its much stronger integral version that we are not discussing here, is known to be true for all Gromov hyperbolic groups [BLR08] and all CAT(0)-groups [BL12, Weg12], for example.

Theorem 1.3 and Corollary 1.5 have the following immediate consequence.

Corollary 3.6. *The source of the rationalized Farrell-Jones assembly map for Thompson's group T is isomorphic to*

$$\bigoplus_{k\geq 0}\bigoplus_{\substack{s\geq 0, t\geq -1\\ s+t=n}} H_s(BT;\mathbb{Q}) \underset{\mathbb{Q}}{\otimes} \Theta_{C_k}(K_t(\mathbb{Z}C_k) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \tag{3.7}$$

for any $n \in \mathbb{Z}$. In particular, if the Farrell-Jones conjecture is true for T, then $K_n(\mathbb{Z}T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to (3.7) *for any $n \in \mathbb{Z}$.*

As we already remarked, thanks to theorems of Ghys-Sergiescu and Patronas, the dimension over $\mathbb{Q}$ of each individual summand in (3.7) is explicitly computable.

Now we recall a famous result about the injectivity of the rationalized classical assembly map.

Theorem 3.8 (Bökstedt-Hsiang-Madsen [BHM93])**.** *Let G be any group, not necessarily torsion-free. Assume that for every $s \in \mathbb{N}$ the abelian group $H_s(BG;\mathbb{Z})$ is finitely generated. Then for every $n \in \mathbb{N}$ the rationalized classical assembly map* (3.3) *is injective.*

In particular, Theorem 3.8 applies to Thompson's group F, since any group of type F_∞ satisfies the assumption above. However, this injectivity result produces no information about $Wh(G)$.

In [LRRV15], Theorem 3.8 is generalized to the Farrell-Jones assembly map, yielding also information about $Wh(G)$.

Theorem 3.9 ([LRRV15, Main Theorem 1.11])**.** *Let G be any group. Assume that for every finite cyclic subgroup C of G the following conditions hold:*

(i) for every $s \in \mathbb{N}$ the abelian group $H_s(BZ_GC;\mathbb{Z})$ is finitely generated;

(ii) let k be the order of C and let ζ_k be any primitive k^{th} root of unity; for every $t \in \mathbb{N}$ the natural homomorphism

$$K_t(\mathbb{Z}[\zeta_k]) \longrightarrow \prod_{p\in\mathbb{P}} K_t\Big(\mathbb{Z}_p \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\zeta_k];\mathbb{Z}_p\Big)$$

is injective after tensoring with $\mathbb{Q}$, where $\mathbb{P}$ denotes the set of all primes and $\mathbb{Z}_p$ denotes the ring of p-adic integers for $p \in \mathbb{P}$.

Then the restriction of the rationalized Farrell-Jones assembly map (3.4) *to the summands where $t \neq -1$ induces an injective homomorphism*

$$\bigoplus_{C\in(\mathcal{F}\text{in}\mathcal{C}\text{yc})}\bigoplus_{\substack{s,t\geq 0\\ s+t=n}} H_s(BZ_GC;\mathbb{Q}) \underset{\mathbb{Q}[W_GC]}{\otimes} \Theta_C(K_t(\mathbb{Z}C) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \longrightarrow K_n(\mathbb{Z}G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

for every $n \geq 0$.

Corollary 3.10 ([LRRV15, Theorem 1.16]). *Assume that a group G satisfies assumption (i) of Theorem 3.9. Then there is an injective homomorphism*

$$\operatorname*{colim}_{H \in \mathtt{Sub}_{\mathcal{F}\mathrm{in}} G} Wh(H) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow Wh(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\,,$$

i.e., the second part of Conjecture 3.1 is true rationally for G.

Some remarks are in order about assumption (ii) of Theorem 3.9. First of all, (ii) is true for all k if $t = 0, 1$, and for all t if $k = 1$. This explains why the assumption is absent from Theorem 3.8 and Corollary 3.10. Moreover, assumption (ii) is conjecturally always true, in the sense that it is automatically satisfied if a weak version of the Leopoldt-Schneider conjecture holds for cyclotomic fields; see [LRRV15, Section 2] for details.

Now Theorem 1.3 and its Corollaries 1.5 and 3.6, combined with Theorem 3.9 and Corollary 3.10, immediately imply our main result; cf. Theorem 1.1.

Corollary 3.11. *Conjecture 3.1 is true rationally for Thompson's group T, and there is an injective homomorphism*

$$\operatorname*{colim}_{k \in \mathbb{N}} Wh(C_k) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow Wh(T) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\,.$$

In particular, $Wh(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an infinite dimensional $\mathbb{Q}$-vector space. Moreover, if assumption (ii) of Theorem 3.9 holds for all $k, t \in \mathbb{N}$, then there is an injective homomorphism

$$\bigoplus_{k \geq 0} \bigoplus_{\substack{s,t \geq 0 \\ s+t=n}} H_s(BT; \mathbb{Q}) \underset{\mathbb{Q}}{\otimes} \Theta_{C_k}(K_t(\mathbb{Z}C_k) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \longrightarrow K_n(\mathbb{Z}T) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

for all $n \in \mathbb{N}$.

Proof. The only step that remains to be explained is the identification

$$\operatorname*{colim}_{k \in \mathbb{N}} Wh(C_k) \cong \operatorname*{colim}_{H \in \mathtt{Sub}_{\mathcal{F}\mathrm{in}} T} Wh(H)\,, \tag{3.12}$$

where on the left-hand side we have the colimit described right after Theorem 1.1. Recall that all finite subgroups of T are cyclic. Suppose that C and D are finite subgroups of T of orders k and ℓ, respectively, and assume that $k \mid \ell$. Then there is exactly one subgroup C' of D of order k, and C and C' are conjugate in T by Corollary 2.6. As explained in the proof of that Corollary, C has a unique generator with rotation number $\frac{1}{k}$, and the same is true for C'. Since rotation numbers are preserved by conjugation by Proposition 2.4(iii), we conclude that there is exactly one morphism in $\mathtt{Sub}_{\mathcal{F}\mathrm{in}} T$ from C to D. Now, identifying C_k with the cyclic subgroup $\langle \gamma_k \rangle$ of T generated by the pseudo-rotation of order k from Example 2.1, the isomorphism (3.12) follows by cofinality. □

References

[BL12] Arthur Bartels and Wolfgang Lück, *The Borel conjecture for hyperbolic and* CAT(0)*-groups*, Ann. of Math. (2) **175** (2012), no. 2, 631–689, DOI 10.4007/annals.2012.175.2.5. MR2993750 ↑7, 8

[BLR08] Arthur Bartels, Wolfgang Lück, and Holger Reich, *The K-theoretic Farrell-Jones conjecture for hyperbolic groups*, Invent. Math. **172** (2008), no. 1, 29–70, DOI 10.1007/s00222-007-0093-7. MR2385666 ↑7, 8

[BHM93] Marcel Bökstedt, Wu Chung Hsiang, and Ib Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), no. 3, 465–539, DOI 10.1007/BF01231296. MR1202133 ↑9

[Bro87] Kenneth S. Brown, *Finiteness properties of groups*, J. Pure Appl. Algebra **44** (1987), no. 1-3, 45–75, DOI 10.1016/0022-4049(87)90015-6. Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985). MR885095 ↑2

[BG84] Kenneth S. Brown and Ross Geoghegan, *An infinite-dimensional torsion-free FP_∞ group*, Invent. Math. **77** (1984), no. 2, 367–381, DOI 10.1007/BF01388451. MR752825 ↑2

[CFP96] James W. Cannon, William J. Floyd, and Walter R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) **42** (1996), no. 3-4, 215–256. MR1426438 ↑1, 3

[Geo08] Ross Geoghegan, *Topological methods in group theory*, Graduate Texts in Mathematics, vol. 243, Springer, New York, 2008. 10.1007/978-0-387-74614-2. MR2365352 ↑2

[GS87] Étienne Ghys and Vlad Sergiescu, *Sur un groupe remarquable de difféomorphismes du cercle*, Comment. Math. Helv. **62** (1987), no. 2, 185–239, DOI 10.1007/BF02564445. MR896095 ↑2

[KH95] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995. MR1326374 ↑4

[Lüc10] Wolfgang Lück, *K- and L-theory of group rings*, Proceedings of the International Congress of Mathematicians, Vol. II, Hindustan Book Agency, New Delhi, 2010, pp. 1071–1098. MR2827832 ↑7

[LR05] Wolfgang Lück and Holger Reich, *The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory*, Handbook of K-theory, Vol. 2, Springer, Berlin, 2005, pp. 703–842, DOI 10.1007/978-3-540-27855-9_15, (to appear in print). MR2181833 ↑7, 8

[LRRV15] Wolfgang Lück, Holger Reich, John Rognes, and Marco Varisco, *Algebraic K-theory of group rings and the cyclotomic trace map* (2015), preprint, available at `http://arxiv.org/abs/1504.03674v1`. ↑2, 7, 8, 9, 10

[LRV03] Wolfgang Lück, Holger Reich, and Marco Varisco, *Commuting homotopy limits and smash products*, K-Theory **30** (2003), no. 2, 137–165, DOI 10.1023/B:KTHE.0000018387.87156.c4. MR2064237 ↑7

[MPMN14] Conchita Martínez-Pérez, Francesco Matucci, and Brita E.A. Nucinkis, *Cohomological finiteness conditions and centralisers in generalisations of Thompson's group V* (2014), preprint, available at `http://arxiv.org/abs/1309.7858v2`. ↑2

[MPN13] Conchita Martínez-Pérez and Brita E.A. Nucinkis, *Bredon cohomological finiteness conditions for generalisations of Thompson groups*, Groups Geom. Dyn. **7** (2013), no. 4, 931–959, DOI 10.4171/GGD/211. MR3134031 ↑2

[Mat08] Francesco Matucci, *Algorithms and classification in groups of piecewise-linear homeomorphisms*, Ph.D. thesis, Cornell University, 2008, `http://arxiv.org/abs/0807.2871v1`. ↑2

[Oli88] Robert Oliver, *Whitehead groups of finite groups*, London Mathematical Society Lecture Note Series, vol. 132, Cambridge University Press, Cambridge, 1988. MR933091 ↑1

[Pat14] Dimitrios Patronas, *The Artin defect in algebraic K-theory*, Ph.D. thesis, Freie Universität Berlin, 2014. ↑8

[Weg12] Christian Wegner, *The K-theoretic Farrell-Jones conjecture for CAT(0)-groups*, Proc. Amer. Math. Soc. **140** (2012), no. 3, 779–793, DOI 10.1090/S0002-9939-2011-11150-X. MR2869063 ↑8