DUKE-CGTP-99-06

hep-th/9909108



# Discrete Torsion and Gerbes I

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In this technical note we give a purely geometric understanding of discrete torsion, as an analogue of orbifold Wilson lines for two-form tensor field potentials. In order to introduce discrete torsion in this context, we describe gerbes and the description of certain type II supergravity tensor field potentials as connections on gerbes. Discrete torsion then naturally appears in describing the action of the orbifold group on (1-)gerbes, just as orbifold Wilson lines appear in describing the action of the orbifold group on the gauge bundle. Our results are not restricted to trivial gerbes – in other words, our description of discrete torsion applies equally well to compactifications with nontrivial $H$-field strengths. We also describe a specific program for rigorously deriving analogues of discrete torsion for many of the other type II tensor fields, and we are able to make specific conjectures for the results.

September 1999

# Contents

# 1 Introduction

Historically discrete torsion has been a rather mysterious aspect of string theory. Discrete torsion was originally discovered [1] as an ambiguity in the choice of phases to assign to twisted sectors of string orbifold partition functions. Although other work has been done on the subject (see, for example, [2, 3, 4, 5, 6]), no work done to date has succeeded in giving any sort of genuinely deep understanding of discrete torsion. In fact, discrete torsion has sometimes been referred to has an inherently stringy degree of freedom, without any geometric analogue.

In this paper we shall give a purely geometric understanding of discrete torsion. Specifically, we describe discrete torsion as a precise analogue of orbifold Wilson lines, for 2-form fields rather than vector fields. Put another way, we shall argue that discrete torsion can be understood as "orbifold Wilson surfaces."

Our description of discrete torsion hinges on a deeper understanding of type II $B$-fields than is common in the literature. More specifically, just as vector potentials (gauge fields) are understood as connections on bundles, we describe $B$-fields as connections on (1-)gerbes. Although gerbes seem to be well-known in some circles, their usefulness does not seem to be widely appreciated. We shall review a recent description of transition functions for gerbes, given in [7, 8], which provides a simplified language in which to discuss gerbes, and then shall discuss gerbes themselves (in the language of stacks) in detail in [9]. As accessible accounts of gerbes which provide the level of detail we need do not seem to exist, we provide such an overview in [9]. In a later paper [10] we shall provide a simplified way of understanding orbifold group actions on $B$ fields, and shall also derive additional physical manifestations of discrete torsion from the ideas presented here.

Let us take a moment to give some general explanation of our ideas. In defining an orbifold of a physical theory, the orbifold group $\Gamma$ must define a symmetry of the theory. Specifying the action of the orbifold group on the underlying topological space is, however, not sufficient when bundles or other objects (such as gerbes) are present – specifying an action of the orbifold group $\Gamma$ on a space does not uniquely specify an action of $\Gamma$ on a bundle. Put another way, for any given action of $\Gamma$ on the base space, there can be multiple inequivalent actions of $\Gamma$ on a bundle or a gerbe. This fact is usually glossed over in descriptions of orbifolds.

What is the physical meaning of this ambiguity in the lifting of the action of $\Gamma$ to a bundle? Specifying a specific action of $\Gamma$ on a bundle with connection implicitly defines an orbifold Wilson line. In other words, orbifold Wilson lines are precisely choices of actions of $\Gamma$ on a bundle with connection. We shall show that, similarly, discrete torsion is a choice of an action of $\Gamma$ on a (1-)gerbe with connection.

Technically, specifying a lift of $\Gamma$ to a bundle, or a gerbe, is known as specifying an "equivariant structure" on the bundle or gerbe. Thus, in the paper we shall often speak of classifying equivariant structures, which means classifying lifts of $\Gamma$.

Our results are not restricted to trivial gerbes – in other words, our description of discrete torsion applies equally well to compactifications with nontrivial $H$-field strengths. Also, we do not make any assumptions concerning the nature of $\Gamma$ – it does not matter whether $\Gamma$ is abelian or nonabelian. It also does not matter whether $\Gamma$ acts freely on the underlying topological space – in our description, freely-acting orbifolds are understood in precisely the same way as orbifolds with fixed points.

We are also able to describe analogues of discrete torsion for the type IIA RR 1-form and the IIB RR 2-form fields. In addition, our approach makes it clear that there should exist analogues of discrete torsion for the other tensor fields appearing in supergravity theories. We describe a specific program for rigorously deriving analogues of discrete torsion for many of the other type II tensor field potentials (specifically, those which can be understood in terms of gerbes), and conjecture the results – that analogues of discrete torsion for $p$-form fields are measured by $H^p(\Gamma, U(1))$, where $\Gamma$ is the orbifold group.

We begin in section 2 by reviewing orbifold Wilson lines in language that will easily generalize. More specifically, we describe orbifold Wilson lines as an ambiguity in lifting the action of an orbifold group to a bundle with connection. In section 3 we give a basic discussion of $n$-gerbes, describing how they can be used to understand many of the tensor field potentials appearing in supergravity theories. This discussion is necessary because we shall describe discrete torsion as a precise analogue of orbifold $U(1)$ Wilson lines for 1-gerbes. In section 4 we outline precisely how one can derive discrete torsion as an ambiguity in specifying the action of an orbifold group on a 1-gerbe with connection, in other words, as an analogue of orbifold Wilson lines. We do not give a rigorous derivation of discrete torsion in this paper; see instead [9]. Finally, we include an appendix on group cohomology, which is used a great deal in this paper and may not be familiar to the reader.

In this paper we concentrate on developing some degree of physical intuition for our results on discrete torsion, and give simplified (and rather loose) derivations. A rigorous derivation of our results on 1-gerbes, together with a detailed description of 1-gerbes, is provided in a separate paper [9].

We should remark that our purpose in writing this paper and [9] was primarily conceptual, rather than computational, in nature. In these papers we give a new conceptual understanding of discrete torsion. Along the way we provide some fringe benefits, such as an understanding of orbifold Wilson lines for nontrivial bundles, a description of discrete torsion in backgrounds with nontrivial torsion[1], and a thorough pedagogical review of 1-gerbes in terms of sheaves of categories. However, we do not provide significant new computational methods.

[1]Torsion in the sense of, nontrivial 3-form curvature, as opposed to the mathematical senses.

We should mention that there are a few issues concerning discrete torsion which we shall not address in either this paper or [9]. First, we shall only discuss discrete torsion for orbifold singularities. One might wonder if discrete torsion, or some close analogue, can be defined for non-orbifold singularities, such as conifold singularities; we shall not address this matter here. Second, we shall not attempt to discuss how turning on discrete torsion alters the moduli space structure, i.e., how discrete torsion changes the allowed resolutions of singularities. See instead [2] for a preliminary discussion of this matter. These matters will be discussed in [10].

We should also mention that in an earlier version of this paper, we made a slightly stronger claim than appears here. Namely, in earlier versions of this paper we claimed that the difference between any two equivariant structures on a 1-gerbe with connection is defined by an element of $H^2(\Gamma, U(1))$. However, we have since corrected a minor error in [9]. Here, we only claim that the difference between equivariant structures is an element of a group which includes $H^2(\Gamma, U(1))$ – in other words, there are additional degrees of freedom which we missed previously. These additional degrees of freedom will be discussed in more detail in [10].

# 2 Orbifold Wilson lines

We shall begin by making a close examination of principal bundles and heterotic string orbifolds, in order to carefully review the notion of an "orbifold Wilson line." In the next section, we describe gerbes, which provide the generalization of line bundles required to describe higher rank tensor potentials in supergravity theories. Once we have given a basic description of gerbes, we shall describe how the notion of orbifold Wilson line generalizes to the case of gerbes, and in the process, recover discrete torsion (and its analogues for the other tensor potentials appearing in supergravity theories) as a precise analogue of an orbifold Wilson line for a gerbe.

Let $X$ be a smooth manifold, and $\Gamma$ a discrete group acting by diffeomorphisms on $X$. In this section we shall discuss how to extend the action of $\Gamma$ to a bundle on $X$ and to a connection on the bundle. We shall explicitly recover, for example, the orbifold Wilson lines that crop up in toroidal heterotic orbifolds. In the next section we shall generalize the same methods to describe discrete torsion as an analogue of orbifold Wilson lines for higher-degree gerbes.

## 2.1 Basics of orbifold Wilson lines

Before we begin discussing orbifold Wilson lines in mathematical detail, we shall take a moment to discuss them in generality. We should point out that in this paper we will always assume that gauge bundles in question are abelian; that is, any principal $G$-bundle appearing implicitly or explicitly has abelian $G$.

In constructing heterotic orbifolds, people often mention orbifold Wilson lines. What are they? In constructing a heterotic toroidal orbifold, one can combine the action of $\Gamma$ on $X$ with a gauge transformation, so as to create Wilson lines on the quotient space. The action on the gauge bundle defines the lift of $\Gamma$ to the bundle. Such Wilson lines are typically called orbifold Wilson lines.

The simple description above works precisely in the case that the bundle being orbifolded is trivial. In this case, there is a canonical (in fact, trivial) lift that leaves the fibers invariant, and any other lift can be described as combining a gauge transformation with the action of $\Gamma$ on the base. In general, when the bundle is nontrivial, there is no canonical lift, and so one has to work harder. A specification of a lift of $\Gamma$ to a bundle is known technically as a choice of "equivariant structure" on the bundle, and so to derive orbifold Wilson lines in the general context we will speak of classifying equivariant structures. We shall study equivariant structures in much more detail in the next subsection. In the remainder of this subsection, we shall attempt to give some intuition for the relevant ideas.

Consider for simplicity the special case of an orbifold group $\Gamma$ acting freely (without fixed points) on a space $X$. How precisely do we describe a Wilson line on the quotient space? Let $x \in X$, and pick some path from $x$ to $g \cdot x$ for some $g \in \Gamma$. In essence, a Wilson loop on the quotient space $X/\Gamma$ is the composition of the (nonclosed) Wilson loop along this path from $x$ to $g \cdot x$ with a gauge transformation describing the action of $g \in \Gamma$ on the corresponding principal bundle. It should be clear from this description that equivariant structures on a bundle are encoding information about Wilson lines on the quotient space, among other things. For this reason, choices of equivariant structures are often called orbifold Wilson lines.

How should orbifold Wilson lines be classified? Again, for simplicity assume $\Gamma$ acts freely on $X$. We shall examine how flat connections on the quotient space are related to flat connections on the cover, in order to shed some light. (In later sections we shall not assume that bundles under consideration admit flat connections; we make this assumption here in order to perform an enlightening calculation.) First, recall that for any $G$, the moduli space of flat $G$-connections on $X/\Gamma$, for abelian $G$, is given by

$$\mathrm{Hom}\left(\pi_1(X/\Gamma), G\right)/G$$

where $G$ acts by conjugation. For abelian $G$, conjugation acts trivially, and so the moduli space of flat $G$-connections on $X/\Gamma$ is simply

$$\mathrm{Hom}\left(\pi_1(X/\Gamma), G\right)$$

Thus, in order to study orbifold Wilson lines on $X$, we need to understand how $\pi_1(X/\Gamma)$ is related to $\pi_1(X)$. Assuming $\Gamma$ is discrete and $X$ is connected, then from the long exact sequence for homotopy[2] we find the short exact sequence

$$0 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/\Gamma) \longrightarrow \pi_0(\Gamma) \longrightarrow 0$$

so $\pi_1(X/\Gamma)$ is an extension of $\pi_0(\Gamma) \cong \Gamma$ by $\pi_1(X)$. As $\pi_1(X/\Gamma)$ receives a contribution from $\Gamma$, we see that orbifolding enhances the space of possible Wilson lines by $\mathrm{Hom}(\Gamma, G)$, roughly speaking. More precisely, we have the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(\Gamma, G) \longrightarrow \mathrm{Hom}\left(\pi_1(X/\Gamma), G\right) \longrightarrow \mathrm{Hom}\left(\pi_1(X), G\right) \longrightarrow \cdots$$

For example, for the special case $G = U(1)$, we have the short exact sequence[3]

$$0 \longrightarrow H^1(\Gamma, U(1)) \longrightarrow \mathrm{Hom}\left(\pi_1(X/\Gamma), U(1)\right) \longrightarrow \mathrm{Hom}\left(\pi_1(X), U(1)\right) \longrightarrow 0$$

where $H^1(\Gamma, U(1))$ denotes group cohomology of $\Gamma$ with trivial action on the coefficients $U(1)$. Thus, we see explicitly that for $\Gamma$ discrete and freely-acting, flat $U(1)$-connections on the quotient space pick up a contribution from the group cohomology group $H^1(\Gamma, U(1))$, which we can identify with orbifold $U(1)$ Wilson lines.

The results of the discussion above are important and bear repeating. We just argued that, for $\Gamma$ discrete and freely-acting, flat $U(1)$ connections on the quotient get a contribution from $H^1(\Gamma, U(1))$. We shall argue in later sections that for general abelian $G$ and general discrete $\Gamma$ (not necessarily freely-acting), orbifold $G$ Wilson lines are classified by $H^1(\Gamma, G)$, where $H^1(\Gamma, G)$ denotes group cohomology of $\Gamma$, with coefficients[4] $G$. In later sections we shall also not make any assumptions concerning the nature of the bundle – we shall not assume the bundle in question admits flat connections. We shall rigorously derive the classification of orbifold Wilson lines as a classification of equivariant structures on principal bundles with connection. When we classify equivariant structures on gerbes with connection[5], we shall recover a classification which includes $H^2(\Gamma, U(1))$.

At this point we shall take a moment to clarify an issue that may have been puzzling the reader. We claimed in the introduction that we would describe discrete torsion in terms

[2] Applied to the principal $\Gamma$ bundle

$$\Gamma \longrightarrow X \longrightarrow X/\Gamma$$

whose existence follows from the fact that $\Gamma$ acts freely.

[3] Using the fact that $U(1) = \mathbf{R}/\mathbf{Z}$ is an injective $\mathbf{Z}$-module [11, section I.7].

[4] Technically, we are also assuming that the action of $\Gamma$ on the coefficients $G$ is trivial. We shall make this assumption on group cohomology throughout this paper.

[5] And band $C^\infty(U(1))$, technically.

of orbifold Wilson lines for $B$ fields. However, discrete torsion is measured in terms of group cohomology, whereas (for flat connections) Wilson lines are given by $\mathrm{Hom}(\pi_1, G)/G$. However, for the special case $G = U(1)$,

$$\mathrm{Hom}\left(\pi_1, G\right)/G \; = \; H^1(\Gamma, U(1))$$

where $H^1(\Gamma, U(1))$ is group cohomology. It should now be clear to the reader that the usual classification of discrete torsion – given by $H^2(\Gamma, U(1))$ – is quite similar to this formal classification of orbifold $U(1)$ Wilson lines – given by $H^1(\Gamma, U(1))$. In particular, the reader should now be less surprised that orbifold Wilson lines and discrete torsion are related.

One issue we have glossed over so far concerns "fake" Wilson lines, which we shall now take a moment to discuss. Consider for example the orbifold $\mathbf{C}^2/\mathbf{Z}_2$. This space is simply-connected, yet the usual prescriptions for orbifold Wilson lines tell us that there is a physical degree of freedom (given by $\mathrm{Hom}(\mathbf{Z}_2, G)$) which we would usually associate with Wilson lines. Such degrees of freedom are often referred to as fake Wilson lines [12].

This degree of freedom is in fact physical – not some unphysical artifact. In the next few sections we shall see mathematically that one will recover degrees of freedom measured by

$$H^1(\Gamma, G) \; = \; \mathrm{Hom}(\Gamma, G)$$

for $\Gamma$-orbifolds of spaces with $G$-bundles (with $G$ abelian), regardless of whether or not $\Gamma$ is freely acting.

How precisely should fake Wilson lines be interpreted on the quotient space? It can be shown [17, chapter 14] that if one quotients the total spaces of bundles, using equivariant structures defining fake Wilson lines, then the resulting object over the quotient space is not a fiber bundle. (For example, a $\mathbf{Z}_2$ orbifold of a rank $n$ complex vector bundle over $\mathbf{C}^2$ with nontrivial orbifold Wilson lines is not a fiber bundle over the quotient space $\mathbf{C}^2/\mathbf{Z}_2$ – one gets an object whose fiber over most points is $\mathbf{C}^n$, appropriate for a rank $n$ vector bundle, but whose fiber over the singularity is $\mathbf{C}^n/\mathbf{Z}_2$.) We have not pursued this question in depth, but we do have a strong suspicion. In the case that $X/\Gamma$ is an algebraic variety[6], it is possible to construct (reflexive) sheaves which are closely related to, but not quite the same as, bundles. For example, on $\mathbf{C}^2/\mathbf{Z}_2$, in addition to line bundles there are also reflexive rank 1 sheaves. We find it very tempting to conjecture that these reflexive rank 1 sheaves correspond to quotients of equivariant line bundles on $\mathbf{C}^2$ with nontrivial fake Wilson lines, and that more generally, fake Wilson lines on quotient spaces that are algebraic varieties have an interpretation in terms of reflexive sheaves which are not locally free. Moreover, isomorphism classes of reflexive sheaves on affine spaces $\mathbf{C}^2/\Gamma$ are classified in the same way as orbifold Wilson lines [18], a fact that forms one corner of the celebrated McKay correspondence. We shall have nothing further to say on this matter in this paper.

[6] Technically, a Noetherian normal variety.

Before we move on to discuss lifts $\tilde{\Gamma}$ of $\Gamma$ acting on line bundles with connection, we shall discuss some amusing technical points regarding equivariant bundles. One natural question to ask is the following: given some equivariant structure on a bundle $P$, how can one compute the characteristic classes of the quotient bundle $P/\Gamma$?

The basic idea is to construct a principle $G$-bundle on $E\Gamma \times_\Gamma X$, such that the projection to $X/\Gamma$ yields the quotient bundle. We shall not work out the details here; see instead [13], where this program is pursued in detail. In principle one could follow the same program for the equivariant gerbes we shall construct in later sections, and discuss their equivariant characteristic classes. However, we shall not pursue this direction in this paper.

In passing we should also note that on rare occasions, equivariant bundles and equivariant bundles with connection have been discussed in the physics literature in terms of "V-bundles" [14, 15, 16]. The language of V-bundles is rather different from the language we shall use in this paper to describe orbifold Wilson lines, though it is technically equivalent.

## 2.2 Equivariant bundles

Let $P$ be a principal $G$-bundle on $X$ for some abelian Lie group $G$ (e.g., $G = U(1)^n$ for some positive $n$). Given the action of $\Gamma$ on $X$, we would like to study lifts of the action of $\Gamma$ on $X$ to the total space of $P$.

What precisely is a lift of the action of $\Gamma$? Let $\pi : P \to X$ denote the projection, then a lift of the action of an element $g \in \Gamma$ is a diffeomorphism $\tilde{g} : P \to P$ such that $\tilde{g}$ is a morphism of principal $G$-bundles. The statement that $\tilde{g}$ is a morphism of bundles means precisely that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{g}} & P \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{g} & X \end{array} \tag{1}$$

The statement that $\tilde{g}$ is a morphism of principal $G$-bundles, not merely a morphism of bundles, means that, in addition to the commutativity of the diagram above, the action of $G$ on the total space must commute with $\tilde{g}$. In other words, $\tilde{g}(h \cdot p) = h \cdot \tilde{g}(p)$ for all $h \in G$ and $p \in P$. Furthermore, in order to lift the action of the entire group $\Gamma$ and not just its elements, we shall impose the constraint that

$$\widetilde{g_1 \cdot g_2} \;=\; \tilde{g}_1 \circ \tilde{g}_2 \tag{2}$$

for all $g_1, g_2 \in \Gamma$.

The constraint given in equation (2) is quite important; one is not always guaranteed of finding lifts that satisfy (2). As an example[7], we shall examine the nontrivial $\mathbf{Z}_2$ bundle over

[7] We would like to thank A. Knutson for pointing out this example to us.

$S^1$. Consider the group $\Gamma = \mathbf{Z}_2$ that acts on the base $S^1$ as a half-rotation of the circle. On the total space of the nontrivial $\mathbf{Z}_2$ bundle, essentially a 720° object, $\Gamma$ must act by rotation by either $+180°$ or $-180°$, in order to cover the action of $\Gamma$ on the base $S^1$ (i.e., in order for diagram (1) to commute). Unfortunately, neither such action on the total space of the bundle squares to the identity, and so equation (2) can not be satisfied in this case.

As the example just given demonstrates, although trivial bundles admit lifts of orbifold group actions, not all nontrivial bundles admit lifts of orbifold group actions. Rather than digress to explain conditions for the existence of lifts, we shall simply assume lifts exist in all examples considered in this paper. (We shall make a similar assumption when discussing equivariant gerbes.)

A lift of the action of $\Gamma$ to a bundle is also called a choice of ($\Gamma$-)equivariant structure on the bundle.

In passing, we should also mention that instead of speaking of lifts, we could equivalently work in terms of pullbacks. Loosely speaking, in terms of pullbacks, a bundle $P$ is "almost equivariantizable" with respect to the action of $\Gamma$ if, for all $g \in \Gamma$, $g^* P \cong P$. As above, not all bundles will necessarily be equivariant with respect to some given $\Gamma$, but we shall not discuss relevant constraints in this paper. More precisely, an equivariant bundle $P$ is defined to be a bundle with a choice of equivariant structure, which can be defined as a specific set of isomorphisms $\psi_g : g^* P \stackrel{\sim}{\longrightarrow} P$ for all $g \in \Gamma$, subject to the obvious analogue of equation (2).

It is easy to check that the definitions of equivariant structures in terms of lifts and in terms of pullbacks are equivalent. For completeness, we shall outline the argument here. Let $\psi_g : g^* P \stackrel{\sim}{\longrightarrow} P$ define an equivariant structure (in terms of pullbacks) on a principal bundle $P$. Then, we can define a lift $\tilde{g}$ of $g \in \Gamma$ as, $\tilde{g} = \psi_{g^{-1}} \circ (g^{-1})^*$. The reader can easily check that $\tilde{g}$ does indeed define a lift of $g$, as defined above. Conversely, given a lift $\tilde{g}$ of $g \in \Gamma$, we can define an equivariant structure (in terms of pullbacks) $\psi_g : g^* P \stackrel{\sim}{\longrightarrow} P$ by, $\psi_g^{-1} = g^* \circ \tilde{g}$. It is easy to check that the two constructions given here are inverses of one another.

How many possible lifts of the action of a given $g \in \Gamma$ exist? Given any one lift, we can certainly make any other lift by composing the action of the lift with a gauge transformation. More precisely, given a set of lifts $\{\tilde{g} | g \in \Gamma\}$, we can construct a new set of lifts $\{\tilde{g}'\}$ by composing each $\tilde{g}$ with a gauge transformation $\phi_g : X \rightarrow G$ such that $\phi_{g_2}(x) \cdot \phi_{g_1}(g_2^{-1} x) = \phi_{g_2 g_1}(x)$ for all $x \in X$. Moreover, any two lifts differ by a set of such gauge transformations. We can rephrase this by saying that any two lifts of the action of $\Gamma$ to $P$ differ by an element of $\mathrm{Hom}(\Gamma, C^\infty(G)) = H^1(\Gamma, C^\infty(G))$.

Now, from our knowledge of orbifold Wilson lines, we will eventually want (equivalence classes of) lifts to be classified by $H^1(\Gamma, G) = \mathrm{Hom}(\Gamma, G)$, but above we only have $H^1(\Gamma, C^\infty(G))$. What have we forgotten?

So far we have only studied how to extend the action of $\Gamma$ to the total space of a bundle.

We have not yet spoken about further requiring the action of $\Gamma$ to preserve the connection on the bundle. This requirement will place additional constraints on the lifts. When we are done, we will see that by considering lifts of the action of $\Gamma$ to line bundles with connection, instead of just line bundles, we will recover the classification $H^1(\Gamma, G)$, as desired.

For more information on equivariant bundles, see for example [17].

## 2.3 Equivariant bundles with connection

In the previous subsection we described the action of the group $\Gamma$ on principal $G$-bundles, for $G$ an abelian Lie group. In this section we shall extend this discussion to include consideration of a connection on a bundle. We shall argue that equivalence classes of lifts of the action of $\Gamma$ to pairs (line bundle, connection) are classified by the group cohomology group $H^1(\Gamma, G)$. (More precisely, we shall find a non-canonical correspondence between equivariant structures on principal $G$-bundles with connection and elements of $H^1(\Gamma, G)$. In special cases, such as trivial principal $G$-bundles, there will be a canonical correspondence.) For more information on connection-preserving lifts, see[8] [13] and [23, section 1.13].

Before going on, we shall take a moment to very briefly review connections on principal bundles and what it means for a lift to preserve a connection. One way to think of a connection on a bundle is as a set of gauge fields $A_\mu$, one for each element of a good cover of the base. However, there is a slightly more elegant description which we shall use instead [22, section Vbis.A]. If $P$ is the total space of a principal $G$-bundle on $X$, then a connection on $P$ is a map $TP \rightarrow \text{Lie } G$, or a (Lie $G$)-valued 1-form on $P$, satisfying certain properties we shall not describe here (see instead, for example, [21, section 11.4] or [22, section V bis A1]). Given an open set $U \subset X$ such that $P|U$ is trivial, let $s : U \rightarrow P$ be a section, and let $\alpha$ denote the connection on $P$, i.e., the corresponding (Lie $G$)-valued 1-form, then we can recover a gauge field on the base $X$ as $s^*\alpha$. Any two distinct sections $s_1, s_2 : U \rightarrow P$ define gauge fields differing by a gauge transformation, i.e., $s_1^*\alpha = s_2^*\alpha - (dg)g^{-1}$. If $\phi : X \rightarrow G$ defines a gauge transformation, then it acts on the connection $\alpha$ as ([22, section V bis, problem 1], [23, section 1.10])

$$\alpha(p) \;\rightarrow\; \phi(\pi(p))\,\alpha(p)\,\phi^{-1}(\pi(p)) \;-\; d\ln(\phi \circ \pi)(p)$$

for $p \in P$ and $\pi : P \rightarrow X$ the projection. Clearly, a gauge transformation $\phi : X \rightarrow G$ will

[8] There is also related information in [19, sections 2.4, 2.5] and [20, section V.2]. These references analyze a distinct but related problem; their discussion might at first confuse the reader. Specifically, instead of considering representations of $\Gamma$ in the group of all connection-preserving lifts of diffeomorphisms of the base, they study the space of connection-preserving lifts itself, and argue that it is a central extension of the group of bundle-with-connection-preserving diffeomorphisms of the base by $U(1)$, for principal $U(1)$-bundles. The reader might be then tempted to argue that lifts of $\Gamma$ should be classified by $H^2(\Gamma, U(1))$, but this is not quite correct. In particular, when viewing the set of all connection-preserving lifts as a central extension, the elements that project to $\Gamma$ will not, in general, form a representation of $\Gamma$, i.e., will not satisfy equation (2).

preserve the connection (not just up to gauge equivalence) if and only if $\phi$ is a constant map.

How does a morphism of principal bundles act on a connection? Let $\tau : P_1 \rightarrow P_2$ denote a morphism of principal bundles, then if $\alpha_2$ is a connection on $P_2$, $\tau^* \alpha_2$ (defined in the obvious way) is a connection on $P_1$. More relevantly to the problem under discussion, if $g \in \Gamma$ and $\tilde{g}$ denotes the lift of $g$, then we shall say that $\tilde{g}$ preserves the connection $\alpha$ if $\tilde{g}^* \alpha = \alpha$, not just up to gauge transformation.

In order to get a well-defined connection living over the quotient $X/\Gamma$, we shall demand that the lift of the action of $\Gamma$ preserves the connection itself, not just its gauge-equivalence class. (If this were not the case, then we would not be able to immediately write down a well-defined connection over the quotient space.) Phrased another way, a lift of the action of $\Gamma$ on $X$ will yield a well-defined connection on the quotient precisely if it can be deformed by an element of $H^1(\Gamma, C^\infty(G))$ so that it preserves the connection itself, not just its gauge-equivalence class. Phrased another way still, if we merely demanded that the lift of $\Gamma$ preserve only the gauge-equivalence class of the connection, then naively spoke of the gauge-equivalence class descending to the quotient, we would not be guaranteed of finding any representatives of the quotiented gauge-equivalence class.

Necessary and sufficient conditions for a lift of $\Gamma$ to preserve the connection are known and easy to describe [23]. In fact, the action of $g \in \Gamma$ on the base $X$ is liftable to a map of bundle with connection if and only if the action of $g$ preserves the values[9] of Wilson loops on the base [23, prop. 1.12.2]. (Note that even for bundles with a non-flat connection – nontrivial bundles – one can still define Wilson loops – however, only in the special case of a flat connection will the value of a Wilson loop depend only on the homotopy class of the loop.) The reader should not be surprised by this result, as this fact is often implicitly claimed in the literature on toroidal heterotic orbifolds, for example.

Now, how many lifts of $\Gamma$ preserve the connection itself? Let $\{\tilde{g}\}$ denote a lift of $\{g \in \Gamma\}$ that preserves the connection itself. We can compose $\{\tilde{g}\}$ with an element of $H^1(\Gamma, C^\infty(G))$ to get another lift, but only the constant elements, namely those in $H^1(\Gamma, G) \subset H^1(\Gamma, C^\infty(G))$, will act so as to preserve the connection itself. Thus, $H^1(\Gamma, G)$ acts on the set of connection-preserving lifts of $\Gamma$, and it should be clear this action is both transitive and free.

Note that in the very special case that the equivariant bundle on $X$ is trivial, then there is (distinguished) trivial lift, and so there is a canonical correspondence between elements of $H^1(\Gamma, G)$ and connection-preserving lifts. For more general bundles, there is no such distinguished lift.

As this result is important, we shall repeat it. We have just shown that ($\Gamma$-)equivariant structures on $G$-bundles with connection can be (noncanonically) identified with elements of

[9]Strictly speaking, preserves the values of the Wilson loops up to conjugation; however, for bundles with abelian structure group, conjugation acts trivially.

$H^1(\Gamma, G)$. In special cases, such as trivial bundles, there is a canonical identification.

In passing, note that our analysis of equivariant structures on bundles with connection did not assume $\Gamma$ was freely-acting or that $\Gamma$ be abelian: our analysis applies equally well to cases in which $\Gamma$ has fixed points on the base space, as well as cases in which $\Gamma$ is nonabelian.

## 2.4 Example: heterotic orbifolds

As a more explicit example, let us consider how to define a toroidal heterotic orbifold. Here we have some principal $G$ bundle (for some $G$) over the torus, which for simplicity we shall assume to be trivial[10]. We shall also assume the connection on the bundle on the torus is not merely flat, but actually trivial. In these special circumstances, any diffeomorphism of the base lifts to an action on the bundle.

Now, to define a lift of an action of $\Gamma$ on the torus to the total space of the principal bundle is trivial. Since the total space of a trivial principal bundle is just $X \times G$, clearly we can lift the action of $\Gamma$ to the total space by defining it to be trivial on the fiber $G$. (More generally, for a nontrivial bundle, demanding that the group $\Gamma$ lift to an action on the total space of the bundle is not trivial. Depending upon both $X$ and the bundle in question, there are often obstructions.) Given any one such lift, we can find all other possible lifts simply by composing the trivial lift with a gauge transformation.

In order to get a well-defined connection on the quotient space, however, there are some constraints on the possible lifts. First, note that in these special circumstances, we can describe any lift as the composition of the trivial lift with a gauge transformation. For any $g \in \Gamma$, let $\phi_g : X \rightarrow G$ denote the corresponding gauge transformation. Then in order to preserve the connection itself, $\phi_g$ must be constant, in other words, $\phi_g = \epsilon(g)$ for some $\epsilon : \Gamma \rightarrow G$. These $\epsilon(g)$ define the usual orbifold Wilson lines.

## 2.5 Discussion in terms of Čech cohomology

Eventually in this paper we will work through arguments closely analogous to those above to derive analogues of orbifold Wilson lines for gerbes. In order to do this properly is somewhat difficult and time-consuming – (1-)gerbes are properly described in terms of sheaves of categories, and their full analysis can be somewhat lengthy. In order to give some general intuition for the results at the level of Hitchin's [7, 8] discussion of gerbes, we will eventually give a rather loose derivation of the results in terms of Čech cohomology. (A rigorous

[10] Although the bundle has a flat connection, it need not be topologically trivial or even trivializable – this is a stronger constraint than necessary, which we are introducing in order to keep this warm-up example simple.

derivation will appear in [9].)

As a warm-up for that eventual discussion, in this section we shall very briefly describe how one can re-derive orbifold Wilson lines while working at the level of Čech cohomology, i.e., at the level of transition functions for bundles. We feel that such an approach is philosophically somewhat flawed – the transition functions of a bundle do not really capture all information about the bundle. For example, gauge transformations of a bundle are completely invisible at the level of transition functions. Thus, we do not find the notion of defining an equivariant structure on a bundle by putting an equivariant structure on its transition functions to be completely satisfying. Thus, when we study equivariant gerbes, we shall not limit ourselves to only discussing equivariant structures on gerbe transition functions, but shall also discuss equivariant structures on the gerbes themselves.

Experts will note that in this subsection we implicitly make some assumptions regarding the behavior of bundle trivializations under the action of the orbifold group. As our purpose in this subsection is not to give a rigorous derivation but merely to perform an enlightening calculation, we shall gloss over such issues.

Let $P$ be a principal $G$-bundle on a manifold $X$, and let $\Gamma$ be a group acting on $X$ by diffeomorphisms. Let $\{U_\alpha\}$ be a "good invariant" cover of $X$, by which we means that each $U_\alpha$ is invariant under $\Gamma$, and each $U_\alpha$ is a disjoint union of contractible open sets. For example, one can often get good invariant covers of a space $X$ from good covers of the quotient $X/\Gamma$. Note that a good invariant cover is not a good cover, in general. We shall assume good invariant covers exist in this subsection, though it is not clear that this need be true in general. (Again, our purpose in this subsection is to present enlightening calculations and plausibility arguments, not completely rigorous proofs.)

Let $h_{\alpha\beta}$ denote transition functions for the bundle $P$. Assume $P$ has an equivariant structure, which at the level of transition functions means that for all $g \in \Gamma$ there exist functions $\nu^g_\alpha : U_\alpha \rightarrow G$ such that

$$g^* h_{\alpha\beta} \, ( = \, h_{\alpha\beta} \circ g \, ) \; = \; \nu^g_\alpha \, h_{\alpha\beta} \, (\nu^g_\beta)^{-1} \tag{3}$$

The functions $\nu^g_\alpha$ are local trivialization realizations of an isomorphism of principal $G$-bundles $\nu^g : P \stackrel{\sim}{\longrightarrow} g^* P$ for each $g \in \Gamma$. It should be clear that $\nu^g = (\psi_g)^{-1}$ where the $\psi_g$ were defined in the section on equivariant bundles.

The $\nu^g$ partially specify an equivariant structure on $P$, but we also need a little more information. In particular, we must also demand that for $g_1, g_2 \in \Gamma$,

$$\nu^{g_2}_\alpha \, g_2^* \nu^{g_1}_\alpha \; = \; \nu^{g_1 g_2}_\alpha \tag{4}$$

Note that this is the appropriate Čech version of equation (2).

Now, suppose $\nu^g_\alpha$ and $\overline{\nu}^g_\alpha$ define two distinct equivariant structures on $P$, with respect to the same group $\Gamma$. Define $\phi^g_\alpha : U_\alpha \rightarrow U(1)$ by,

$$\phi^g_\alpha \equiv \frac{\nu^g_\alpha}{\overline{\nu}^g_\alpha}$$

From the fact that both $\nu^g_\alpha$ and $\overline{\nu}^g_\alpha$ must satisfy equation (3), we can immediately derive the fact that

$$\phi^g_\alpha = \phi^g_\beta$$

on $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and so the $\phi^g_\alpha$ define a function $\phi^g : X \rightarrow U(1)$. This is a gauge transformation describing the difference between two equivariant structures. It is almost, but not quite, the same as the gauge transformation $\phi_g$ described in the section on equivariant bundles.

From equation (4), we see that the $\phi^g$ must obey

$$\phi^{g_2} \cdot g_2^* \phi^{g_1} = \phi^{g_1 g_2}$$

Thus, any two equivariant structures on $P$ differ by an element of $H^1(\Gamma, C^\infty(G))$, as described in the section on equivariant bundles.

We shall now recover the fact that equivariant structures on bundles with connection differ by elements of $H^1(\Gamma, G)$, for abelian $G$.

Let $A^\alpha$ be a (Lie $G$)-valued one-form on the open set $U_\alpha$, defining a connection on $P$. In other words, on overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$,

$$A^\alpha - A^\beta = d \ln h_{\alpha\beta}$$

Under the action of $g \in \Gamma$, since

$$g^* h_{\alpha\beta} = \nu^g_\alpha \, h_{\alpha\beta} \, (\nu^g_\beta)^{-1}$$

we know that

$$g^* A^\alpha = A^\alpha + d \ln \nu^g_\alpha$$

From this we see that if $\nu^g_\alpha$ and $\overline{\nu}^g_\alpha$ define two distinct equivariant structures on the transition functions, then we must have $\nu^g_\alpha / \overline{\nu}^g_\alpha$ be constant, in order for $g^* A^\alpha$ to be well-defined. Thus, we recover the fact that $\phi^g$ is constant, and so the subset of $H^1(\Gamma, C^\infty(G))$ that describes equivariant bundles with connection is given by $H^1(\Gamma, G)$, as claimed.

In essence, we have been using a form of equivariant Čech cohomology. The mathematics literature seems to contain multiple[11] versions of equivariant Čech cohomology, unfortunately none of them are quite adequate for our eventual needs (i.e., none of them correspond to the precise way we set up equivariant structures on gerbes), and so we shall not speak about them further.

[11] One version of equivariant Čech cohomology is described in [24, chapitre V]. Another version is described in [25, section 2] and [26, section 5].

# 3 $n$-Gerbes

Discrete torsion has long been associated with the two-form $B$-fields of supergravity theories. The reader should not be surprised, therefore, that a deep understanding of discrete torsion hinges on a deep understanding of the two-form $B$ fields. We shall argue that $B$ fields should be understood as connections on 1-gerbes, and that discrete torsion arises when lifting the action of an orbifold group to a 1-gerbe with connection.

Why might one want a new mathematical object to describe $B$-fields in type II string theory? The reason is a dilemma that has no doubt puzzled for many years. The torsion[12] $H$ is defined to be $H = dB$. So long as $H$ is taken to be cohomologically trivial, this is certainly a sensible definition. Unfortunately, in general one often wants to speak of $H$ which is not a cohomologically trivial element of $H^3(\mathbf{R})$. In such a case, the relation $H = dB$ can only hold locally. (This point has been made previously in, for example, [27].)

We shall see shortly that such $H$ can be understood globally as a connection on a 1-gerbe. More generally, many of the tensor field potentials appearing in type II string theories naively appear to have a natural and obvious interpretation in terms of connections on $n$-gerbes, though for the sake of simplifying the discussion we will usually only discuss the two-form tensor field in examples.

In passing, we should also mention that although some tensor field potentials may be understood in terms of gerbes, it is not clear that all tensor field potentials have such an understanding. One notable exception is the $B$-field of heterotic string theory. Recall that one has the anomaly cancellation constraint

$$dH \,\propto\, \mathrm{Tr}\, R \wedge R \,-\, \mathrm{Tr}\, F \wedge F$$

If the heterotic $B$ field were a 1-gerbe connection, then we shall see that the curvature $H$ would be a closed form, whereas that is certainly not the case here in general. Moreover, many other tensor field potentials have nontrivial interactions and "mixed" gauge transformations, and it is not at all clear whether these phenomena can be understood in terms of gerbes. As a result, one should be somewhat careful about blindly identifying all tensor field potentials with connections on gerbes. These issues should not arise for the comparatively simpler cases of type II 2-form potentials (with other background fields turned off), which is the primary case of interest for us in this paper.

We should also mention a slight technical caveat. We shall only be discussing gerbes with "band" $C^\infty(U(1))$ [9], which means, in less technical language, that there exist more general gerbes than those discussed in this section. For example, some theories contain multiple coupled tensor multiplets (for one example, see [28]), which would be described in terms of connections on gerbes with "band" $C^\infty(U(1)^N)$. We shall not discuss gerbes with general bands in this paper; see instead [9].

[12] No relation to the various mathematical concepts of torsion.

In this section we will give a description of gerbes and connections on gerbes, due to [7, 8]. We shall begin by defining gerbes themselves, then afterwards we shall describe connections on gerbes. In the next section we will discuss the analogue of "orbifold Wilson lines" for gerbes.

### 3.1 Description in terms of cohomology

We shall begin by describing characteristic classes of abstract objects called "$n$-gerbes," following the discussion in [7, 8]. These characteristic classes, which for $n$-gerbes on a space $X$, are elements of the sheaf cohomology group $H^{n+1}(X, C^\infty(U(1)))$. This is closely analogous to describing a line bundle in terms of Chern classes. More intrinsic definitions of gerbes are given in the next section and in [9]. Gerbes themselves take considerably longer to define; by first describing their characteristic classes, we hope to give the reader some basic intuitions for these objects.

In passing we should comment on our usage of the terminology "$n$-gerbe." We are following the simplified conventions of [7, 8]. In general, an $n$-gerbe should, morally, be understood in terms of sheaves of multicategories. Unfortunately, $n$-categories for $n > 2$ are not well understood at present. As a result, although 1-gerbes and, to a slightly lesser extent, 2-gerbes are well understood, higher degree gerbes are not on as firm a footing. It seems reasonably clear that such objects should exist, however, and one can certainly describe many properties that a general $n$-gerbe should possess in terms of characteristic classes (as in this section) and Deligne cohomology. Thus, we shall often speak (loosely) of general $n$-gerbes, though for $n > 2$ the reader should probably take such remarks with a small grain of salt.

A couple of paragraphs ago we mentioned that the characteristic classes of gerbes could be understood in terms of sheaf cohomology, and more specifically that the characteristic classes of possible $n$-gerbes on a space $X$ live in $H^{n+1}(X, C^\infty(U(1)))$. For those readers not acquainted with sheaf cohomology, we can express this somewhat more simply (and loosely) in terms of Čech cocycles with respect to some fixed open cover. Let $U_\alpha$ be a "reasonably nice[13]" open cover of $X$. Then an element of $H^{n+1}(X, C^\infty(U(1)))$ is essentially defined by a set of smooth functions $h_{\alpha_0 \cdots \alpha_{n+1}} : U_{\alpha_0 \cdots \alpha_{n+1}} \rightarrow U(1)$, one for each overlap $U_{\alpha_0 \cdots \alpha_{n+1}} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_{n+1}}$, subject to the constraint

$$(\delta h)_{\alpha_0 \cdots \alpha_{n+2}} \; = \; 1 \tag{5}$$

where $\delta h$ is defined by

$$(\delta h)_{\alpha_0 \cdots \alpha_{n+2}} \; \equiv \; \prod_{i=0}^{n+2} h^{(-)^i}_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_{n+2}}$$

[13]Specifically, a good open cover.

on the intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_{n+2}}$. Two such sets of functions $h_{\alpha_0 \cdots \alpha_{n+1}}$, $h'_{\alpha_0 \cdots \alpha_{n+1}}$ are identified with the same element of $H^{n+2}(X, C^\infty(U(1)))$ if and only if

$$h_{\alpha_0 \cdots \alpha_{n+1}} \; = \; h'_{\alpha_0 \cdots \alpha_{n+1}} \prod_{i=0}^{n+1} f^{(-)^i}_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_{n+1}} \tag{6}$$

for some functions $f_{\alpha_0 \cdots \alpha_n} : U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \rightarrow U(1)$.

As a special case, let us see how this duplicates line bundles. In the classification of $n$-gerbes implicit in the description of characteristic classes above, it should be clear that line bundles are very special examples of $n$-gerbes, specifically, 0-gerbes. A 0-gerbe is specified by an element of $H^1(X, C^\infty(U(1)))$, that is, a set of smooth functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow U(1)$, such that

$$h_{\beta\gamma} h^{-1}_{\alpha\gamma} h_{\alpha\beta} \; = \; 1$$

on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ (this is the specialization of equation (5)). The reader should immediately recognize this as defining transition functions for a smooth $U(1)$ bundle on $X$. Equation (5) is precisely the statement that transition functions agree on triple overlaps. Moreover, two $U(1)$ line bundles are equivalent if and only if their transition functions $h_{\alpha\beta}$, $h'_{\alpha\beta}$ are related by [29, chapter 5.2]

$$h_{\alpha\beta} \; = \; h'_{\alpha\beta} f_\alpha / f_\beta$$

for some set of functions $f_\alpha : U_\alpha \rightarrow U(1)$. The reader should immediately recognize this as the specialization of equation (6).

Although the sheaf cohomology group $H^1(X, C^\infty(U(1)))$ precisely describes (equivalence classes of) transition functions for 0-gerbes (smooth principal $U(1)$ bundles), the same is not true for higher degree gerbes – an element of sheaf cohomology for a higher degree gerbe does not define a set of transition functions. (We shall study transition functions for gerbes in the next subsection.)

We can rewrite these characteristic classes of $n$-gerbes in a format that is more accessible to calculation. Using the short exact sequence of ($C^\infty$) sheaves

$$0 \rightarrow C^\infty(\mathbf{Z}) \cong \mathbf{Z} \rightarrow C^\infty(\mathbf{R}) \rightarrow C^\infty(U(1)) \rightarrow 0$$

one can immediately prove, from the associated long exact sequence, that for $n \geq 0$,

$$H^{n+1}(X, C^\infty(U(1))) \; \cong \; H^{n+2}(X, \mathbf{Z})$$

As a special case, this implies that 0-gerbes are classified by elements of $H^2(X, \mathbf{Z})$, and indeed it is a standard fact that $C^\infty$ line bundles are classified by their first Chern class.

In general, any two trivializations of a trivializable $n$-gerbe, that is, one described by a cohomologically trivial $(n+1)$-cocycle, differ by an $(n-1)$-gerbe. This should be clear from the description above – any cohomologically trivial $(n+1)$-cocycle is a coboundary of some $n$-cochain, and any two such cochains differ by an $n$-cocycle, defining an $(n-1)$-gerbe.

Before going on, we should mention that in the remainder of this paper (as well as [9]) we shall usually abbreviate "1-gerbes" to simply "gerbes." Unfortunately, on rare occasion we shall also use "gerbes" as shorthand for $n$-gerbes. The usage should be clear from context.

## 3.2 Description in terms of transition functions

In the previous section we described $n$-gerbes in terms of sheaf cohomology, which is precisely analogous to describing line bundles in terms of Chern classes. Traditionally gerbes are typically defined in terms of sheaves of multicategories, as we shall do in [9]. In this section, we shall give a simplified account, due to [7, 8], which amounts to describing gerbes in terms of transition functions. In [9] we shall review sheaves of categories and the description of 1-gerbes in such language, in addition to giving a geometric first-principles derivation of discrete torsion.

Before grappling with transition functions for $n$-gerbes, we shall begin with a description of transition functions for 1-gerbes. Let $\{U_\alpha\}$ be a good cover of $X$, then we can define a 1-gerbe on $X$ in terms of two pieces of data:

1. A principal $U(1)$ bundle $\mathcal{L}_{\alpha\beta}$ over each $U_{\alpha\beta} = U_\alpha \cap U_\beta$ (subject to the convention $\mathcal{L}_{\beta\alpha} = \mathcal{L}_{\alpha\beta}^{-1}$), such that $\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\alpha}$ is trivializable on $U_{\alpha\beta\gamma}$
2. An explicit trivialization $\theta_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \rightarrow U(1)$ of $\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\alpha}$ on $U_{\alpha\beta\gamma}$

Then, $\theta$ naturally defines a Čech 2-cochain, and it should be clear that $\delta\theta = 1$, i.e., the canonical section of the canonically trivial bundle defined by tensoring the obvious twelve factors of principal $U(1)$ bundles.

Thus, $\theta$ defines a 2-cocycle, and it should be clear that this 2-cocycle is the same 2-cocycle defining the 1-gerbe in the description in the previous subsection.

We should take a moment to clarify the precise relationship between the construction above and 1-gerbes defined in terms of sheaves of categories. In the description of gerbes in terms of sheaves of categories, one can define transition functions for the gerbe with respect to a local trivialization, in precise analogy to transition functions for bundles. However, for 1-gerbes the objects one associates to overlaps of open sets are not maps into the group, but rather line bundles, subject to a constraint on triple overlaps. Put more directly, the description given in the paragraphs above precisely describes transition functions for a 1-gerbe. The corresponding element of sheaf cohomology is merely a characteristic class, classifying isomorphism classes.

Thus, the description of 1-gerbes given so far in this section is technically a description of transition functions for 1-gerbes. The reader may well wonder what is a 1-gerbe; the answer is, a special kind of sheaf of categories. Sheaves of categories and related concepts have been banished to [9], but we shall give a very quick flavor of the construction here.

Sometimes one speaks of "objects" of the 1-gerbe. These are line bundles $\mathcal{L}_\alpha$ over open sets $U_\alpha$, such that $\mathcal{L}_{\alpha\beta} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta^{-1}$. Objects exist locally on $X$, but in general will not exist globally (unless the 1-gerbe is trivializable, meaning the associated Čech 2-cocycle is trivial in cohomology).

In more formal treatments of gerbes, one often associates sheaves of categories with gerbes[14]. In this description, the "objects" mentioned above are precisely objects of a category associated to some open set on $X$. We shall not go into a detailed description of gerbes as sheaves of categories in this section; see instead [9].

Now that we have discussed 1-gerbes, how are $n$-gerbes defined? In principle an analogous description should hold true – transition functions for an $n$-gerbe should consist of associating an $(n-1)$-gerbe to each overlap, subject to constraints at triple overlaps. Although we are well-acquainted with more intrinsic definitions of 1-gerbes [9], we have not worked through higher $n$-gerbes in comparable detail, and so we hesitate to say much more concerning transition functions for higher order gerbes. We hope to return to this elsewhere [30].

## 3.3 Connections on gerbes

Now that we have stated the definition of an $n$-gerbe, we shall define a connection on an $n$-gerbe, which is a straightforward generalization of the notion of connection on a $C^\infty$ line bundle.

For simplicity, fix some good open cover $U_\alpha$ of $X$. A connection on an $n$-gerbe is defined by a choice of $H \in \Omega^{n+2}(X)$ such that $dH = 0$ (a closed $(n+2)$-form on $X$), and a choice of $(B_\alpha) \in C^1(\Omega^{n+1})$, that is, a choice of smooth $(n+1)$-form on $U_\alpha$ for each $\alpha$, such that on each $U_\alpha$, $H|_{U_\alpha} = dB_\alpha$, and such that on overlaps $U_\alpha \cap U_\beta$, $B_\alpha - B_\beta = dA_{\alpha\beta}$, where $A_{\alpha\beta}$ is a smooth $n$-form on $U_{\alpha\beta}$. In general there will more more terms, of lower-degree-forms, filling out an entire cocycle in the Čech-de Rham complex.

To be complete, we need to specify how the forms on various open sets are related by the transition functions for the $n$-gerbe. For simplicity, consider a 1-gerbe. Here, we have a globally-defined 3-form $H$, a family of 2-forms $B_\alpha$, one for each open set $U_\alpha$, a family of 1-

[14] More precisely, there is a standard method to associate sheaves of 1-categories and 2-categories to 1-gerbes and 2-gerbes, respectively. The higher-degree gerbes outlined in [7, 8] presumably correspond to sheaves of higher-degree multicategories, however the precise definitions required have not been worked out, to our knowledge.

forms $A_{\alpha\beta}$, one for each intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Recall transition functions for a 1-gerbe consist of line bundles $\mathcal{L}_{\alpha\beta}$ associated to each $U_{\alpha\beta}$; the 1-forms $A_{\alpha\beta}$ are precisely connections[15] on the $U(1)$ bundles $\mathcal{L}_{\alpha\beta}$. If $\theta_{\alpha\beta\gamma}$ denotes the specified trivialization of $\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\alpha}$ on $U_{\alpha\beta\gamma}$, then we have

$$A_{\alpha\beta} \,+\, A_{\beta\gamma} \,+\, A_{\gamma\alpha} \;=\; d \ln \theta_{\alpha\beta\gamma}$$

Then, as per the description above,

$$B_\alpha \,-\, B_\beta \;=\; dA_{\alpha\beta}$$

on overlaps $U_{\alpha\beta}$, and

$$H|_{U_\alpha} \;=\; dB_\alpha$$

In principle, similar remarks hold for more general $n$-gerbes.

The reader should immediately recognize that a connection on a 1-gerbe is precisely the same thing as a type II string theory $B$-field. (The point that $B$ fields and the relation $H = dB$ should really only be understood locally has been made previously in the physics literature, albeit not usually in terms of connections on gerbes; see for example [27].) This relationship seems to be well understood in certain parts of the field; we repeat it here simply to make this note more self-contained. In general, the reader should recognize that tensor field potentials appearing in type II supergravities often look like connections on gerbes.

The reader should also notice that a connection on a 0-gerbe precisely coincides with the usual notion of connection on a smooth line bundle. To make this more clear, change notation as follows: change $H$ to $F$, and change $B$ to $A$. For a connection on a smooth line bundle, locally we have the relation $F = dA$, but globally this does not hold if $F$ is a nonzero element of $H^2(X, \mathbf{R})$.

In the special case that $F$ descends from an element of $H^2(X, \mathbf{Z})$ via the natural map $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{R})$, then there exists a $C^\infty$ line bundle whose first Chern class is represented by $F$. (In particular, Kähler forms can be interpreted as the first Chern classes of (holomorphic) line bundles precisely when the Kähler form lies in the image of $H^2(X, \mathbf{Z})$ in $H^2(X, \mathbf{R})$.) Analogously, for an $n$-gerbe, when the curvature $H$ descends from an element of $H^{n+2}(X, \mathbf{Z})$, then there exists an $n$-gerbe whose characteristic class is defined by $H$.

In fact, we have been slightly sloppy about certain details. Suppose that $H^{n+2}(X, \mathbf{Z})$ contains torsion[16], that is, elements of finite order, then if an $n$-gerbe-connection defines an $n$-gerbe, it does not do so uniquely – one will get several $n$-gerbes, each of which has a characteristic class that descends to the same element of $H^{n+2}(X, \mathbf{R})$. Are these extra degrees of freedom physically relevant – in other words, must there be an actual gerbe underlying these connections?

[15] Note we are implicitly using the fact that the $\{U_\alpha\}$ form a good cover, so each $U_{\alpha\beta}$ is contractible.

[16] In the mathematical sense.

It is easy to see that an actual gerbe must underlie such connections. The point is that torsion elements of $H^{n+2}(X, \mathbf{Z})$ contain physically relevant information, as was noted in, for example, [4].

Given that $n$-gerbes can be loosely interpreted as one generalization of line bundles, the reader may wonder if there is some gerbe-analogue of the holonomy of a flat $U(1)$ connection. Indeed, it is possible to define the holonomy of a flat $n$-gerbe-connection, though we shall not do so here. Such holonomies have been observed in physics previously; one example is [28].

As mentioned earlier, gerbes are often described in terms of sheaves of categories. There is a corresponding notion of connection in such a description, which we have summarized in [9] and can also be found in [19, section 5.3].

## 3.4 Gauge transformations of gerbes

For principal $U(1)$-bundles there is a well-defined notion of gauge transformation: a gauge transformation is defined by a map $f : X \to U(1)$. What is the analogue for $n$-gerbes?

We shall begin by describing gauge transformations for 1-gerbes. It can be shown that the analogue of a gauge transformation for a 1-gerbe is given by a principal $U(1)$-bundle, and a gauge transformation of a 1-gerbe with connection is defined by a principal $U(1)$-bundle with connection. (Strictly speaking, an equivalence class of principal $U(1)$-bundles, but we shall defer such technicalities to later discussions.) A rigorous derivation of this fact and related material is given in [9]. We shall describe the implications of this fact for connections on 1-gerbes, and for transition functions.

Intuitively, how does a principal $U(1)$-bundle act on a 1-gerbe? Very roughly, the general idea is that given a bundle with connection $A$, under a gauge transformation the $B$ field will transform as $B \mapsto B + dA$. (At the same level, we can see that only equivalence classes of bundles with connection are relevant. If $A$ and $A'$ differ by a gauge transformation (of bundles), then $dA = dA'$, and so they define the same action on $B$.) In terms of sheaves of categories, a 1-gerbe is locally a category of all principal $U(1)$-bundles, so given any one object in that category, we can tensor with a principal $U(1)$-bundle to recover another object. This is essentially the action, in somewhat loose language.

To properly describe how a principal $U(1)$-bundle acts on a 1-gerbe requires understanding 1-gerbes in terms of sheaves of categories. The reader might well ask, however, how a bundle acts on the transition functions for a 1-gerbe? We described transition functions for 1-gerbes by associating principal $U(1)$ bundles to intersections $U_{\alpha\beta}$, together with an explicit trivialization of Čech coboundaries. The reader should (correctly) guess that a gauge transformation of a 1-gerbe, at the level of transition functions, should be a gauge transformation of the bundle on each coordinate overlap, such that the gauge transformations preserve the trivializations on triple intersections. In other words, a gauge transformation of a 1-gerbe should be a set of maps $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow U(1)$, subject to the condition that $\delta g = 1$. Put more simply still, a gauge transformation of a 1-gerbe is precisely a principal $U(1)$-bundle. Note that as expected by analogy with bundles, the transition functions are invariant (the bundles on coordinate overlaps are unchanged by gauge transformations). Note that by analogy with bundles, one should expect a gauge transformation to leave transition functions invariant – and indeed, our 1-gerbe gauge transformation does leave the transition functions invariant, as a gauge transformation of each bundle is an automorphism of the bundle.

How does a gauge transformation of a 1-gerbe act on a connection on the 1-gerbe? Principal bundles have well-defined actions on gerbes; a unique specification of the action of a principal $U(1)$-bundle, call it $P$, on a gerbe connection is equivalent to a choice of connection on $P$. Let $\{h_{\alpha\beta} : U_{\alpha\beta} \rightarrow U(1)\}$ be transition functions for $P$, and $\{A^\alpha\}$ a set of 1-forms on elements of the cover $\{U_\alpha\}$ defining a connection on $P$. Let $(B^\alpha, A^{\alpha\beta}, g_{\alpha\beta\gamma})$ be a set of data defining a connection on a 1-gerbe. Then under the action of $P$, this data transforms as follows:

$$
\begin{aligned}
B^\alpha &\mapsto B'^\alpha \equiv B^\alpha + dA^\alpha \\
A^{\alpha\beta} &\mapsto A'^{\alpha\beta} \equiv A^{\alpha\beta} + d\ln h_{\alpha\beta} \\
g_{\alpha\beta\gamma} &\mapsto g_{\alpha\beta\gamma} + \delta h \\
&\quad\;\; = g_{\alpha\beta\gamma}
\end{aligned}
$$

More generally, the reader should correctly guess that a gauge transformation of an $n$-gerbe is an $(n-1)$-gerbe. We shall not attempt to justify this statement here, however.

## 3.5 Gerbes versus K theory

It has recently been claimed that the Ramond-Ramond tensor field potentials of type II theories can be understood in terms of K theory, so we feel we should take a moment to justify our assumption of a gerbe description of certain fields.

In this paper, when we discuss discrete torsion, we have in mind the NS two-form field potential of type II theories, and we implicitly assume that the other tensor field potentials have vanishing vacuum expectation value, to assure that the curvature of the $B$ field is a closed form. In these circumstances, the $B$ field is well-described as a connection on a gerbe.

However, in general terms the basic ideas presented in this paper should also hold more generally. At some level, the point of this paper is that in any theory containing fields with gauge invariances, specifying the orbifold group action on the base space does not suffice to define the action of the orbifold group on the theory, because the orbifold group action can always be combined with gauge transformations. To properly understand possible equivariant structures (equivalently, orbifold group actions) on tensor field potentials not described as gerbes involves certain technical distinctions, but the basic point is the same.

## 3.6 Why gerbes?

So far we have presented gerbes as being a natural mathematical structure for which many of the tensor field potentials of supergravities can be understood as connections. A doubting Thomas might argue, are gerbes really necessary? After all, in supergravity theories, we only see the tensor fields themselves; why not only speak of tensor fields on coordinate charts, and forget about more abstract underlying structures?

We shall answer this question with another question: why bundles? Whenever one sees a vector field with the usual gauge invariances, it is commonplace to associate it with some bundle. One can ask, why? In supergravity and gauge theories containing vector fields, one does not see a bundle, only a set of vector fields on coordinate charts. Bundles (formulated as topological spaces) describe auxiliary spaces – fibers – which are fibered over spacetime, but these auxiliary structures are neither seen nor detected in physics. There are no extra dimensions in the theory corresponding to the fiber of a fiber bundle, so why work with bundles at all? Since using bundles means invoking physically meaningless auxiliary structures, why not just ignore bundles and only work with vector fields on coordinate patches?

Part of the reason people speak of bundles and not just vector fields on coordinate patches is that bundles give an insightful, elegant way of thinking about vector fields on coordinate patches. For example, recent discussions of brane charges in terms of K-theory [31, 32] would have been far more obscure if the notion of bundles was not commonly accepted.

Similarly, the notion of a gerbe gives an insightful and elegant structure in which to understand many of the tensor field potentials appearing in supergravity theories. In principle, one could understand tensor fields without thinking about gerbes, in the same way that one can understand vector fields without thinking about bundles. However, just as bundles give an insightful and useful way to think about vector fields, so gerbes give an insightful and useful way to think about tensor fields.

A slightly more subtle question that could be asked is the following. In [9], we shall describe 1-gerbes in terms of sheaves of categories; however, this description is not unique – gerbes can be described in several different ways. If one should work with gerbes, which description is relevant?

A closely analogous problem arises in dealing with bundles. A bundle has multiple descriptions – as a topological space, or a special kind of sheaf, for example. Connections on bundles can be described in terms of vector fields, or, in special circumstances, as holomorphic structures on certain topological spaces. The correct description one should use varies depending upon the application and one's personal taste. Similarly, which description of gerbes is relevant varies depending upon both the application and personal inclination.

# 4 Discrete torsion

In defining orbifolds, it is well-known that the Riemannian space being orbifolded must have a well-defined action of the orbifold group $\Gamma$. However, after our discussion of gerbes, the reader should suspect that something has been omitted from standard discussions of orbifolds in string theory. Namely, if we understand some of the tensor fields appearing in type II supergravities in terms of connections on gerbes, then we must also specify the precise $\Gamma$-action on the gerbes. This action need not be trivial, and (to our knowledge) has been completely neglected in previous discussions of type II orbifolds.

We shall find that the set of actions of an orbifold group $\Gamma$ on a 1-gerbe is naturally acted upon by a group which includes $H^2(\Gamma, U(1))$, just as the set of orbifold group actions on a principal $U(1)$-bundle with connection is naturally acted upon by $H^1(\Gamma, U(1))$. In special cases, such as trivial gerbes for example, there will be a canonical orbifold group action, and in such cases we can identify the set of orbifold group actions is identified with a group. The possible equivariant structures (meaning, the possible orbifold group actions) correspond to analogues of orbifold Wilson lines for $B$-fields, in the same way that equivariant structures on a principal $U(1)$-bundle with connection correspond to orbifold Wilson lines.

It is natural to speculate that the action of an orbifold group $\Gamma$ on an $n$-gerbe is described by a group including $H^{n+1}(\Gamma, U(1))$, in a fashion analogous to the above. This can be checked for 2-gerbes in the same fashion as for 1-gerbes described in this paper, and we are presently studying this issue [30]. For gerbes of higher degree, a precise understanding in terms of sheaves of multicategories is not yet known, and so one can only make somewhat more limited remarks [30].

## 4.1 Basics of discrete torsion

Discrete torsion was originally discovered as an ambiguity in the choice of phases of different twisted sector contributions to partition functions of orbifold string theories. The possible inequivalent choices of phases are counted by elements of the group cohomology group[17] $H^2(\Gamma, U(1))$, where $\Gamma$ is the orbifold group. Since its discovery, discrete torsion has been considered a rather mysterious quantity.

[17] Where the action of the group $\Gamma$ on the coefficients $U(1)$ is trivial.

Our description of discrete torsion essentially boils down to the observation that discrete torsion is the analogue of orbifold Wilson lines for 2-form-fields rather than vectors. Recall orbifold Wilson lines could be described as a (discrete) ambiguity in lifting the orbifold action on a space to a bundle with connection. A similar ambiguity arises in lifting orbifold actions to gerbes with connection, and this ambiguity is partially measured by $H^2(\Gamma, U(1))$ [9]. More precisely, in general the set of lifts of orbifold actions (in more technical language, the set of equivariant structures on a (1-)gerbe with connection) is (noncanonically) isomorphic to a group which includes $H^2(\Gamma, U(1))$, viewed as a set rather than a group. In special cases, such as trivial gerbes, there exists a canonical isomorphism.

Just as for bundles, not all nontrivial gerbes admit lifts of orbifold group actions. We shall not attempt to study conditions under which a nontrivial gerbe admits such a lift; rather, we shall simply assume that lifts exist in all examples in this paper.

How does this description of discrete torsion as an analogue of orbifold Wilson lines mesh with the original definition in terms of distinct phases associated to twisted sectors of string partition functions? Recall there are factors

$$\exp\left( \int \phi^* B \right) \tag{7}$$

in the partition function, contributing the holonomy of the $B$-field. (We have used $\phi$ to denote the embedding map $\phi : \Sigma \rightarrow X$ of the worldsheet into spacetime.) Because of these holonomy factors, distinct lifts of the orbifold action to the 1-gerbe with connection (i.e., distinct equivariant structures on the 1-gerbe with connection) yield distinct phases in the twisted sectors of orbifold partition functions – we recover the original description of discrete torsion [1].

Put another way, we do not just derive some set of discrete degrees of freedom that happen coincidentally to also be measured by $H^2(\Gamma, U(1))$; the discrete degrees of freedom we recover necessarily describe discrete torsion. The passage from lifts of orbifold actions to phase factors is provided by the partition function factors (7).

In passing, we should mention that the phase factors (7) were the original reason that discrete torsion, viewed as a set of phases of twisted sector contributions to partition functions, was associated with $B$-fields at all [1]. In some sense, our description of discrete torsion is a natural outgrowth of some of the original ideas in [1].

In passing, we should also briefly speak to discrete torsion on D-branes as discussed in [5, 6]. In those references, D-branes on orbifolds with discrete torsion were argued to be described by specifying a projective representation of the orbifold group on the bundle on the worldvolume. We believe (although we have not checked in total detail) that this can be derived from our description of discrete torsion, using the interconnection between $B$-field backgrounds and bundles on worldvolumes of D-branes, as recently described in [35, section 6].

It is one thing to classify possible lifts of orbifold actions to gerbes with connection; it is quite another to describe precisely the characteristic classes and holonomies of the resulting gerbe on the quotient space. In principle, both could be determined as for orbifold Wilson lines: for a gerbe on a space $X$, with orbifold action $\Gamma$, construct a gerbe on the space $E\Gamma \times_{\Gamma} X$, such that the projection to $X/\Gamma$ yields the quotient gerbe, analogously to the program pursued in [13]. We shall not pursue this program here.

Suppose the (discrete) orbifold group $\Gamma$ acting on $X$ acts freely, i.e., without fixed points. In section 2.1, we studied moduli spaces of flat connections on quotient spaces, in order to gain insight into orbifold Wilson lines. In particular, we argued that (for bundles admitting flat connections) orbifold $U(1)$ Wilson lines could be understood directly in terms of extra elements of $\mathrm{Hom}(\pi_1, U(1))$ in the quotient space, for $\Gamma$ a freely-acting discrete group. In other words, in this case orbifold Wilson lines were precisely Wilson lines on the quotient space. We shall perform analogous calculations here. (In the next few paragraphs we shall implicitly only consider flat $n$-gerbes – but only for the purposes of performing illuminating calculations. We do not make such assumptions elsewhere.)

For gerbes, the interpretation is slightly more obscure. First, note that in the case $\Gamma$ acts freely, we have a principal $\Gamma$ bundle $\Gamma \rightarrow X \rightarrow X/\Gamma$, so we can apply the long exact sequence for homotopy to find that $\pi_n(X) = \pi_n(X/\Gamma)$ for $n > 1$ and $\Gamma$ discrete. In other words, although the fundamental group of $X/\Gamma$ received a contribution from $\Gamma$, the higher homotopy groups of $X/\Gamma$ are identical to those of $X$. Thus, the higher-dimensional analogues of orbifold Wilson lines (for flat $n$-gerbes) can not correspond to extra elements of homotopy groups. We shall find, rather, that the higher-dimensional analogues correspond to extra elements of

$$\mathrm{Hom}_{\mathbf{Z}} \left( H_n(X/\Gamma), U(1) \right)$$

We shall now describe the homology of the quotient $X/\Gamma$. One way to compute the homology of the quotient space $X/\Gamma$ is as the limit of the Cartan-Leray spectral sequence [36, section VII.7]

$$E^2_{p,q} \; = \; H_p(\Gamma, H_q(X)) \tag{8}$$

Note that the group homology appearing in the definition above has the property that, in general, $\Gamma$ acts nontrivially[18] on the coefficients $H_q(X)$, even if $\Gamma$ acts freely on $X$.

In special cases, the homology of $X/\Gamma$ can be computed more directly. For example, for any path-connected space $Y$, for any $n > 1$ such that $\pi_r(Y) = 0$ for all $1 < r < n$, we have that [37, theorem II] the following sequence is exact:

$$0 \longrightarrow \pi_n(Y) \longrightarrow H_n(Y) \longrightarrow H_n(\pi_1(Y), \mathbf{Z}) \longrightarrow 0$$

[18] An example should make this clear. Let $X$ be the disjoint union of 2 identical disks, and let $\Gamma$ be a $\mathbf{Z}_2$ exchanging the two disks. Then $H_0(X) = \mathbf{Z}^2$, and $\Gamma$ exchanges the two $\mathbf{Z}$ factors, i.e., $\Gamma$ acts nontrivially on $H_0(X)$.

where the group homology $H_n(\pi_1(Y), \mathbf{Z})$ is defined by the group $\pi_1(Y)$ having trivial action on the coefficients $\mathbf{Z}$. Using the results above, we find that for path-connected $X$ such that $\pi_r(X) = 0$ for $1 < r < n$ for some $n > 1$, the following sequence is exact:

$$0 \longrightarrow \pi_n(X) \longrightarrow H_n(X/\Gamma) \longrightarrow H_n(\pi_1(X/\Gamma), \mathbf{Z}) \longrightarrow 0$$

In the special case that $\pi_1(X) = 0$, we can rewrite this as

$$0 \longrightarrow \pi_n(X) \longrightarrow H_n(X/\Gamma) \longrightarrow H_n(\Gamma, \mathbf{Z}) \longrightarrow 0$$

Moreover, using the Hurewicz theorem, applying the functor $\mathrm{Hom}_{\mathbf{Z}}(-, U(1))$, and using a relevant universal coefficient theorem, we can rewrite the short exact sequence above as[19]

$$0 \longrightarrow H^n(\Gamma, U(1)) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\left(H_n(X/\Gamma), U(1)\right) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\left(H_n(X), U(1)\right) \longrightarrow 0$$

(Technically we are also assuming that $X/\Gamma$ is a path-connected space.)

From the calculation above we can extract two important lessons. First, for $n = 2$, we see that (in special cases), the holonomy of a $B$-field (of a flat 1-gerbe) on the quotient $X/\Gamma$, as measured by $\mathrm{Hom}(H_2(X/\Gamma), U(1))$, differs from the possible holonomies on the covering space by $H^2(\Gamma, U(1))$, and so we can understand discrete torsion in such cases as being precisely the extra contribution to $\mathrm{Hom}(H_2(X), U(1))$ on the quotient. (More generally, the precise relationship between group cohomology and holonomies of $B$-fields is described by the spectral sequence (8).)

The second, and more basic, lesson we can extract from the calculation above is that it is quite reasonable to believe that there exist analogues of discrete torsion and orbifold Wilson lines for the higher-ranking tensor fields appearing in supergravity theories, and that those analogues of discrete torsion should be measured by higher-degree group cohomology $H^n(\Gamma, U(1))$. We shall return to this point later. In this paper, we only derive[20] discrete torsion for $B$-fields, and in so doing find $H^2(\Gamma, U(1))$. However, our general methods should apply equally well to higher-ranking tensor fields, and it is extremely tempting to conjecture that the analogue of discrete torsion for an $n$-gerbe is measured by $H^{n+1}(\Gamma, U(1))$.

Before we go on to outline how discrete torsion can be derived, we shall mention that in this paper, when speaking of an $n$-gerbe on a space $X$, we shall assume that $X$ has $(\mathbf{R})$ dimension at least $n$.

[19] This result has been independently derived, using other methods, by P. Aspinwall.

[20] Our derivation in [9] is not restricted to flat 1-gerbes; the restriction to flat 1-gerbes in the previous few paragraphs was for purposes of making illustrative calculations only. We should also mention that our derivation in [9] does not assume $\Gamma$ is freely acting, or that it is abelian – our derivation holds equally well regardless.

## 4.2 Equivariant gerbes

In this section we shall try to give some intuitive understanding of the classification of equivariant structures on 1-gerbes, that is, the classification of lifts of the orbifold action to 1-gerbes. More precisely, we shall study what equivariant structures on 1-gerbes mean at the level of transition functions for 1-gerbes. We shall not be able to rigorously derive results on equivariant gerbes in this fashion – such derivations are instead given in [9]. However, we hope that this approach should give the reader some intuitive understanding of our results, without requiring them to gain a detailed understanding of 1-gerbes in terms of stacks.

Let $\mathcal{C}$ denote a 1-gerbe on a space $X$, and let $\Gamma$ denote a group acting on $X$ by homeomorphisms. Let $\{U_\alpha\}$ be a "good invariant" cover of $X$ – namely, a cover such that each $U_\alpha$ is invariant under $\Gamma$ and each $U_\alpha$ is a disjoint union of contractible open sets. (For example, we can often obtain such a cover as the inverse image of a good cover on the quotient $X/\Gamma$.) Note that a good invariant cover is not usually a good cover.

In order to define $\mathcal{C}$ at the level of transition functions for the cover $\{U_\alpha\}$, recall we need to specify a line bundle $\mathcal{L}_{\alpha\beta}$ on each overlap $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta$, and an explicit trivialization $\theta_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \rightarrow U(1)$ of $\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\alpha}$ on $U_{\alpha\beta\gamma}$.

Now, let us describe how one defines an equivariant structure on the 1-gerbe $\mathcal{C}$ at the level of transition functions. First, we need $g^* \mathcal{L}_{\alpha\beta} \cong \mathcal{L}_{\alpha\beta}$ for all $g \in \Gamma$. Let $\psi^g_{\alpha\beta} : \mathcal{L}_{\alpha\beta} \xrightarrow{\sim} g^* \mathcal{L}_{\alpha\beta}$ denote a specific choice of isomorphism. Since $\{U_\alpha\}$ is a good invariant cover of $X$, we can represent each $\psi^g_{\alpha\beta}$ by a function $\nu^g_{\alpha\beta} : U_{\alpha\beta} \rightarrow U(1)$.

Note that the $\theta_{\alpha\beta\gamma}$ necessarily now obey

$$g^* \theta_{\alpha\beta\gamma} \, ( = \, \theta_{\alpha\beta\gamma} \circ g \, ) \; = \; \theta_{\alpha\beta\gamma} \, \nu^g_{\alpha\beta} \, \nu^g_{\beta\gamma} \, \nu^g_{\gamma\alpha} \tag{9}$$

Before going on, we should pause to derive an implication of equation (9). Let $\nu^g_{\alpha\beta}$ and $\overline{\nu}^g_{\alpha\beta}$ denote a pair of maps (partially) defining equivariant structures on $\mathcal{C}$. Define

$$\gamma^g_{\alpha\beta} \; \equiv \; \frac{\nu^g_{\alpha\beta}}{\overline{\nu}^g_{\alpha\beta}}$$

then the $\gamma^g_{\alpha\beta}$ satisfy

$$\gamma^g_{\alpha\beta} \, \gamma^g_{\beta\gamma} \, \gamma^g_{\gamma\alpha} \; = \; 1$$

for all $g \in \Gamma$, and so define transition functions for a bundle on $X$ we shall denote $T_g$. Thus, even though we have not finished describing equivariant structures on the 1-gerbe $\mathcal{C}$ at the level of transition functions, we can already derive the fact that any two equivariant structures will differ by, among other things, a set of principal $U(1)$-bundles $T_g$, one for each $g \in \Gamma$.

Before we can claim to have defined an equivariant structure on the transition functions for $\mathcal{C}$, we need to fill in a few more details. In particular, how do the $\nu$ behave under composition of actions of elements of $\Gamma$? We shall demand that for any pair $g_1, g_2 \in \Gamma$,

$$(\nu^{g_2}_{\alpha\beta})\, g_2^*(\nu^{g_1}_{\alpha\beta}) \;=\; (\nu^{g_1 g_2}_{\alpha\beta})\, h(g_1, g_2)_\alpha \, h(g_1, g_2)^{-1}_\beta \tag{10}$$

for some functions $h(g_1, g_2)_\alpha : U_\alpha \rightarrow U(1)$. We shall also demand that the functions $h(g_1, g_2)_\alpha$ satisfy

$$h(g_1, g_2)_\alpha \, h(g_1 g_2, g_3)_\alpha \;=\; h(g_2, g_3)_\alpha \, h(g_1, g_2 g_3)_\alpha \tag{11}$$

These constraints probably seem relatively unnatural to the reader. In our discussion of equivariant gerbes in terms of stacks, we shall show how these constraints (or, rather, their more complete versions for stacks) are quite natural.

We can attempt to rewrite equations (10) and (11) somewhat more invariantly in terms of the line bundles $\mathcal{L}_{\alpha\beta}$ on overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Recall that $\nu^g_{\alpha\beta}$ is the local trivialization representation of the bundle morphism $\psi^g_{\alpha\beta} : \mathcal{L}_{\alpha\beta} \rightarrow g^* \mathcal{L}_{\alpha\beta}$, then equation (10) states that the two bundle morphisms

$$(g_2^* \psi^{g_1}_{\alpha\beta}) \circ \psi^{g_2}_{\alpha\beta} : \; \mathcal{L}_{\alpha\beta} \; \longrightarrow \; (g_1 g_2)^* \mathcal{L}_{\alpha\beta}$$

and

$$\psi^{g_1 g_2}_{\alpha\beta} : \; \mathcal{L}_{\alpha\beta} \; \longrightarrow \; (g_1 g_2)^* \mathcal{L}_{\alpha\beta}$$

are related by a gauge transformation on $(g_1 g_2)^* \mathcal{L}_{\alpha\beta}$ defined by $h(g_1, g_2)_\alpha h(g_1, g_2)^{-1}_\beta$, i.e.,

$$(g_2^* \psi^{g_1}_{\alpha\beta}) \circ \psi^{g_2}_{\alpha\beta} \;=\; \kappa \circ \psi^{g_1 g_2}_{\alpha\beta}$$

where $\kappa : (g_1 g_2)^* \mathcal{L}_{\alpha\beta} \rightarrow (g_1 g_2)^* \mathcal{L}_{\alpha\beta}$ is the gauge transformation defined by the function $h(g_1, g_2)_\alpha h(g_1, g_2)^{-1}_\beta$ on $U_{\alpha\beta}$.

Given two distinct equivariant structures on the same transition functions, labelled by $\nu, \overline{\nu}$ and $h, \overline{h}$, if we define functions

$$\omega(g_1, g_2)_\alpha \;\equiv\; \frac{h(g_1, g_2)_\alpha}{\overline{h}(g_1, g_2)_\alpha}$$

then from equation (10) we have the relation

$$(\gamma^{g_2}_{\alpha\beta})\, g_2^*(\gamma^{g_1}_{\alpha\beta}) \;=\; (\gamma^{g_1 g_2}_{\alpha\beta})\, \omega(g_1, g_2)_\alpha \, \omega(g_1, g_2)^{-1}_\beta \tag{12}$$

The functions $\omega(g_1, g_2)_\alpha$ define local trivialization realizations of isomorphisms of principal $U(1)$-bundles. We denote these bundle isomorphisms by $\omega_{g_1, g_2}$, and so we can rewrite equation (12) more invariantly as the definition of $\omega_{g_1, g_2}$:

$$\omega_{g_1, g_2} : \; T_{g_1 g_2} \; \stackrel{\sim}{\longrightarrow} \; T_{g_2} \cdot g_2^* T_{g_1}$$

Furthermore, from equation (11) we see that the bundles $T_g$ and isomorphisms $\omega_{g_1,g_2}$ are further related by

$$\begin{array}{ccc} T_{g_1g_2g_3} & \stackrel{\omega_{g_1g_2,g_3}}{\longrightarrow} & T_{g_3}\cdot g_3^* T_{g_1g_2} \\ \omega_{g_1,g_2g_3} \downarrow & & \downarrow \omega_{g_1,g_2} \\ T_{g_2g_3}\cdot (g_2g_3)^* T_{g_1} & \stackrel{\omega_{g_2,g_3}}{\longrightarrow} & T_{g_3}\cdot g_3^*(T_{g_2}\cdot g_2^* T_{g_1}) \end{array} \tag{13}$$

So far we have argued that the difference between two equivariant structures on a 1-gerbe is determined by the data $(T_g, \omega_{g_1,g_2})$, where the $\omega$ are required to make diagram (13) commute. However, it should also be said that the bundles $T_g$ are only determined up to isomorphism. Given a set of principal bundle isomorphisms $\kappa_g : T_g \to T'_g$, we can replace the data $(T_g, \omega_{g_1,g_2})$ by the data $\left(T'_g, \kappa_{g_1g_2}\circ\omega_{g_1,g_2}\circ(\kappa_{g_2}\cdot g_2^*\kappa_{g_1})^{-1}\right)$ and describe the same difference between equivariant structures.

## 4.3 Equivariant gerbes with connection

To properly derive the classification of equivariant gerbes with connection at the level of transition functions is rather messy and not very illuminating, so instead we shall settle for outlining the main points. (A complete derivation, in terms of gerbes as stacks, can be found in [9], and a complete derivation at the level of transition functions, as well as related information, will appear in [10].)

In the previous section, we argued that any two equivariant structures on a (1-)gerbe differ by a set of principal $U(1)$ bundles $T_g$ ($g \in \Gamma$), together with appropriate bundle isomorphisms $\omega_{g_1,g_2}$, such that diagram (13) commutes, modulo isomorphisms of bundles.

A gauge transformation of a gerbe with connection is defined by a principal $U(1)$-bundle with connection, so the reader should not be surprised to hear that the difference between two equivariant structures on a 1-gerbe with connection is defined by bundles $T_g$ with connection, together with connection-preserving maps $\omega_{g_1,g_2}$ such that diagram (13) commutes. Also, the connections on the bundles $T_g$ are constrained to be flat.

Note that this structure is closely analogous to the discussion of orbifold $U(1)$ Wilson lines. In both cases, we find that the difference between two equivariant structures is determined by a set of gauge transformations, such that the gauge transformation associated to the product $g_1g_2$ is isomorphic to the product of the gauge transformations associated to $g_1$ and $g_2$. The constraint for bundles that the gauge transformations be constant becomes the present constraint that the gerbe gauge transformations defined by bundles with connection, must have a flat connection.

Just as before, the bundles $T_g$ are only defined up to equivalence. We can replace any of the bundles $T_g$ with connection with an isomorphic bundle with connection (changing $\omega_{g_1,g_2}$ appropriately), and describe the same difference between equivariant structures.

Where does $H^2(\Gamma, U(1))$ appear in this structure? Assume for simplicity that $X$ is connected. Take all the bundles $T_g$ to be topologically trivial, with gauge-trivial connections. Then, by replacing these bundles with isomorphic bundles, we can assume without loss of generality that each bundle $T_g$ is the canonical trivial bundle, with identically zero connection. The morphisms $\omega_{g_1,g_2}$ are now gauge transformations of the canonical trivial bundle, and since they must preserve the connection, they must be constant maps. From commutivity of diagram (13), it is clear that any set of $\omega_{g_1,g_2}$ defines a cocycle representative of an element of $H^2(\Gamma, U(1))$ (with trivial action of $\Gamma$ on the coefficients $U(1)$), in the inhomogeneous representation. Now, we still can act on any of the $T_g$ by constant gauge transformations to get isomorphic equivariant structures, and it is easy to see that these define group coboundaries.

Now, in general not every set of data $(T_g, \omega_{g_1,g_2})$ corresponds to topologically-trivial $T_g$ with gauge-trivial connection – the $T_g$ are only constrained to be flat, so it is not difficult to find new degrees of freedom that do not correspond to elements of $H^2(\Gamma, U(1))$. We shall discuss these further in [10].

In special cases, such as trivial gerbes, there exist canonical trivial equivariant structures, and so elements of the group $H^2(\Gamma, U(1))$ can be identified with (some of) the equivariant structures. More generally, the identification of (some) equivariant structures with $H^2(\Gamma, U(1))$ is not canonical[21].

## 4.4 Analogues of discrete torsion

So far in this paper we have outlined how orbifold Wilson lines and discrete torsion both appear as an ambiguity in the choice of orbifold group action on some tensor field potential. Although we have only concerned ourselves with vector fields and NS-NS $B$ fields, in principle analogous ambiguity exists for every tensor field potential appearing in string theory.

Put another way, in general whenever one has a theory containing fields with gauge invariances, specifying an orbifold group action on the base space does not suffice to define the orbifold group action on the fields of the theory, as one can combine any orbifold group action with gauge transformations.

For example, the other RR tensor field potentials of type II theories should also have analogues of discrete torsion, given as the ambiguity in the choice of orbifold group action on the fields. It has recently been pointed out that these fields should be understood in terms of K-theory, so given some Cheeger-Simons-type description of K-theory, one should be able to calculate the analogues of discrete torsion for these other fields.

What might analogues of discrete torsion be for other tensor field potentials? There is an obvious conjecture. For vector fields, we found that the set of equivariant structures is

[21] Technically, in general the set of equivariant structures on a gerbe with connection is merely a torsor.

a torsor under the group $H^1(\Gamma, U(1))$. For $B$ fields, we found that the set of equivariant structures is a torsor under a group which includes $H^2(\Gamma, U(1))$. Therefore, for a rank $p$ tensor field potential, it is tempting to conjecture that the set of equivariant structures is a torsor under some group which includes $H^p(\Gamma, U(1))$. We are presently studying this matter [30].

The reader might well ask how such degrees of freedom could be seen in perturbative string theory. Orbifold Wilson lines and discrete torsion both crop up unavoidably; but how could one turn on analogues for RR fields? The answer surely lies in the description of RR field backgrounds in perturbative string theory. Judging from the results in, for example, [38, 39, 40, 41, 42], it seems reasonable to assume that one can understand Ramond-Ramond backgrounds in conformal field theory after coupling to the superconformal ghosts, so in principle analogues of discrete torsion for RR fields in conformal field theory might emerge when considering orbifolds of such backgrounds. Unfortunately, it might also be true that the RR analogues of discrete torsion are simply not visible in string perturbation theory.

It is quite possible that there may also be certain analogues of modular invariance conditions for these analogues of discrete torsion. We have only discussed gerbes in isolation, whereas in type II theories, the gerbes interact with one another (and so cannot really be understood as gerbes). It is quite conceivable that, in order for any given orbifold to define a symmetry of the full physical theory, there are nontrivial constraints among analogues of discrete torsion for various gerbes. We have nothing particularly concrete to say on this matter, though we hope to return to it in [30].

It is not clear, however, whether all analogues of modular invariance conditions can be described in this fashion. For example, in [43] it was argued that there existed a constraint on orbifold Wilson lines associated to the IIA RR 1-form, arising nonperturbatively. (We are referring to the so-called "black hole level matching" of that reference.) Unfortunately we are not able to address the existence and interpretation of such constraints.

# 5 Conclusions

In this paper we have given a geometric description of discrete torsion, as a precise analogue of orbifold Wilson lines. Put another way, we have described discrete torsion as "orbifold Wilson surfaces." After giving a mathematically precise discussion of orbifold Wilson lines, we outlined how the classification of orbifold Wilson lines (as equivariant structures on bundles with connection) could be extended to discrete torsion (as equivariant structures on 1-gerbes with connection). Although we outlined how this result on discrete torsion was proven, we have deferred a rigorous examination to [9].

# 6 Acknowledgements

We would like to thank P. Aspinwall, D. Freed, A. Knutson, D. Morrison, and R. Plesser for useful conversations, and E. Silverstein for pointing out reference [13]. We would also like to thank J. Stasheff for extremely extensive comments on the first version of this paper, and we would like to apologize for taking so long to implement his suggestions.

# A Review of group cohomology

For a complete technical overview of group cohomology, the standard reference is [36]. For much shorter and more accessible accounts, we recommend [45, section IV.4] and [46].

Let $G$ and $M$ be groups, $M$ abelian, with a (possibly trivial) action of $G$ on $M$ by group automorphisms. We shall assume that the action of $G$ commutes with the group operation of $M$ on itself.

Define $C^n(G, M)$ to be the set of all maps

$$\epsilon : G \times \cdots \times G = G^{n+1} \rightarrow M$$

such that $\epsilon(gg_0, gg_1, \cdots, gg_n) = g\epsilon(g_0, g_1, \cdots, g_n)$ for all $g, g_i \in G$. (This representation of the cochains is known as a homogeneous representation, because of the obvious analogy with projective spaces.)

Define a coboundary operator $\delta : C^n(G, M) \rightarrow C^{n+1}(G, M)$ by

$$(\delta\epsilon)(g_0, \cdots, g_{n+1}) \;=\; \sum_{k=0}^{n+1} (-)^k \epsilon(g_0, \cdots, \hat{g_k}, \cdots, g_{n+1})$$

Note that $\delta^2\epsilon = 1$.

Define $Z^n(G, M)$ to be the set of cocycles, that is, $\epsilon \in \ker \delta \subset C^n(G, M)$. Define $B^n(G, M)$ to be the set of coboundaries, that is, $\epsilon \in \text{im } \delta \subset C^n(G, M)$. Then define the group cohomology to be $H^n(G, M) = Z^n(G, M)/B^n(G, M)$.

There is an alternative presentation of group cohomology, which can be defined as follows. Given a cochain $\epsilon \in C^n(G, M)$, which is to say, a map $G^{n+1} \rightarrow M$, define a map $\tilde{\epsilon} : G^n \rightarrow M$ as,

$$\tilde{\epsilon}(g_1, g_2, \cdots, g_n) \;=\; \epsilon(e, g_1, g_1g_2, g_1g_2g_3, \cdots, g_1g_2\cdots g_n)$$

This is known as an inhomogeneous representation, that is, these are called inhomogeneous cochains. It is then easy to demonstrate that

$$(\delta\tilde{\epsilon})(g_1, g_2, \cdots, g_{n+1}) \;=\; g_1\tilde{\epsilon}(g_2, \cdots, g_{n+1})$$

$$+\sum_{k=1}^{n}(-)^k \tilde{\epsilon}(g_1, g_2, \cdots, g_k g_{k+1}, \cdots, g_{n+1})$$
$$+(-)^{n+1}\tilde{\epsilon}(g_1, g_2, \cdots, g_n)$$

In the group cohomology appearing in this paper, and to our knowledge in the physics literature to date[22], we always assume that the action of the group on the coefficients is trivial.

When the action of $G$ on $M$ is assumed trivial, if $\epsilon : G \to M$ is a homogeneous 0-cochain, then it is easy to check that $\epsilon$ is constant. From the definitions of coboundaries for homogeneous and inhomogeneous cochains, it is easy to derive that the associated inhomogeneous 0-cochain $\tilde{\epsilon}$ must always be the identity of $M$. To repeat, if $\tilde{\epsilon}$ is an inhomogeneous 0-cochain, then $\tilde{\epsilon} = 1 \in M$.

As a consequence, for trivial action of $G$ on $M$, we have that $H^1(G, M) = Z^1(G, M)$, that is, $H^1(G, M)$ is precisely the set of group homomorphisms $G \to M$.

For group 2-cochains (defined with trivial group action on the coefficients), there is a gauge choice that is often used. From manipulating the group 2-cocycle condition (in inhomogeneous representation), it is easy to check that $\tilde{\epsilon}(1, g) = \tilde{\epsilon}(g, 1) = \tilde{\epsilon}(1, 1)$ for any $g$. For convenience, one often sets $\tilde{\epsilon}(1, 1) = 1$ (just pick a group coboundary conveniently). Then, in this gauge, $\tilde{\epsilon}(1, g) = \tilde{\epsilon}(g, 1) = 1$ for all $g$. One is still free to add any group coboundary in this gauge, modulo the constraint that if $\mu(g)$ defines a group coboundary, one needs $\mu(1) = 1$ in order to stay in the gauge.

In passing, we shall mention that more formally, for any $G$-module $M$, we can define group cohomology as

$$H^n(G, M) \equiv \mathrm{Ext}^n_{\mathbf{Z}[G]}(\mathbf{Z}, M)$$

where $\mathbf{Z}[G]$ is the free $\mathbf{Z}$-module generated by the elements of $G$. In other words, any element of $\mathbf{Z}[G]$ can be written uniquely in the form

$$\sum_{g \in G} a(g) g$$

where $a(g) \in \mathbf{Z}$. This definition of group cohomology does not make any assumptions concerning the nature of the $G$-action on $M$.

In addition to group cohomology, one can also define group homology in a very similar manner, though we shall not do so here. For the case of group homology and cohomology defined by groups with trivial actions on the coefficients, there exist precise analogues of the usual universal coefficient theorems for homology and cohomology [36, exercise III.1.3]. There is also a Künneth formula [36, section V.5].

[22]For example, experts should note that it is this latter, inhomogeneous form, restricted to the special case that the action of $G$ on $M$ is trivial, which appears in [46].

For reference, we shall now list some commonly used group homology and cohomology groups. First, the homology groups $H_i(\mathbf{Z}_n, \mathbf{Z})$, where the group $\mathbf{Z}_n$ acts trivially on the coefficients $\mathbf{Z}$, are given by

$$H_i(\mathbf{Z}_n, \mathbf{Z}) \;=\; \begin{cases} \mathbf{Z} & i = 0 \\ \mathbf{Z}_n & i \text{ odd} \\ 0 & i \text{ even, } i > 0 \end{cases}$$

The cohomology groups $H^i(\mathbf{Z}_n, U(1))$, where the group $\mathbf{Z}_n$ acts trivially on the coefficients $U(1)$, are given by

$$H^i(\mathbf{Z}_n, U(1)) \;=\; \begin{cases} U(1) & i = 0 \\ \mathbf{Z}_n & i \text{ odd} \\ 0 & i \text{ even, } i > 0 \end{cases}$$

From the Künneth formula [36, section V.5], we find that the homology groups $H_i(\mathbf{Z}_n \times \mathbf{Z}_m, \mathbf{Z})$, where the group acts trivially on the coefficients $\mathbf{Z}$, are given by

$$H_i(\mathbf{Z}_n \times \mathbf{Z}_m, \mathbf{Z}) \;=\; \begin{cases} \mathbf{Z} & i = 0 \\ \mathbf{Z}_n \oplus \mathbf{Z}_m \oplus \bigoplus_{(i-1)/2} \mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}_n, \mathbf{Z}_m) & i \text{ odd} \\ \bigoplus_{i/2} (\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m) & i \text{ even, } i > 0 \end{cases}$$

In other words,

$$\begin{aligned} H_0(\mathbf{Z}_n \times \mathbf{Z}_m, \mathbf{Z}) &= \mathbf{Z} \\ H_1(\mathbf{Z}_n \times \mathbf{Z}_m, \mathbf{Z}) &= \mathbf{Z}_n \oplus \mathbf{Z}_m \\ H_2(\mathbf{Z}_n \times \mathbf{Z}_m, \mathbf{Z}) &= (\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m) \\ H_3(\mathbf{Z}_n \times \mathbf{Z}_m, \mathbf{Z}) &= \mathbf{Z}_n \oplus \mathbf{Z}_m \oplus \mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}_n, \mathbf{Z}_m) \\ H_4(\mathbf{Z}_n \times \mathbf{Z}_m, \mathbf{Z}) &= (\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m) \oplus (\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m) \end{aligned}$$

and so forth.

Using the identities

$$\begin{aligned} \mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m &= \mathbf{Z}_{gcd(n,m)} \\ \mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}_n, \mathbf{Z}_m) &= \mathbf{Z}_{gcd(n,m)} \end{aligned}$$

and the appropriate universal coefficient theorem, one can compute the cohomology groups $H^i(\mathbf{Z}_n \times \mathbf{Z}_m, U(1))$, where the group $\mathbf{Z}_n \times \mathbf{Z}_m$ is assumed to act trivially on the coefficients $U(1)$:

$$H^i(\mathbf{Z}_n \times \mathbf{Z}_m, U(1)) \;=\; \begin{cases} U(1) & i = 0 \\ \mathbf{Z}_n \oplus \mathbf{Z}_m \oplus \bigoplus_{(i-1)/2} \mathbf{Z}_{gcd(n,m)} & i \text{ odd} \\ \bigoplus_{i/2} \mathbf{Z}_{gcd(n,m)} & i \text{ even, } i > 0 \end{cases}$$

In other words,

$$
\begin{aligned}
H^0(\mathbf{Z}_n \times \mathbf{Z}_m, U(1)) &= U(1) \\
H^1(\mathbf{Z}_n \times \mathbf{Z}_m, U(1)) &= \mathbf{Z}_n \oplus \mathbf{Z}_m \\
H^2(\mathbf{Z}_n \times \mathbf{Z}_m, U(1)) &= \mathbf{Z}_{gcd(n,m)} \\
H^3(\mathbf{Z}_n \times \mathbf{Z}_m, U(1)) &= \mathbf{Z}_n \oplus \mathbf{Z}_m \oplus \mathbf{Z}_{gcd(n,m)} \\
H^4(\mathbf{Z}_n \times \mathbf{Z}_m, U(1)) &= \mathbf{Z}_{gcd(n,m)} \oplus \mathbf{Z}_{gcd(n,m)}
\end{aligned}
$$

and so forth. Note that we have used the notation $\times$ and $\oplus$ in this subsection interchangeably.

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