On Grothendieck-Serre conjecture concerning principal G-bundles over regular semi-local domains containing a finite field: I

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Abstract

In three preprints [Pan2], [Pan3] and the present one we prove Grothendieck-Serre's conjecture concerning principal G-bundles over regular semi-local domains R containing **a finite field** (here G is a reductive group scheme). The present preprint contains main geometric presentation theorems which are necessary for that. The preprint [Pan2] contains reduction of the Grothendieck-Serre's conjecture to the case of a simple simply-connected group scheme. The preprint [Pan3] contains a proof of Grothendieck-Serre's conjecture for regular semi-local domains R containing a finite field. One of the main result of the present preprint is Theorem 1.1.

The Grothendieck–Serre conjecture for the case of regular semi-local domains containing **an infinite field** is proven in [FP]. *Thus the conjecture holds for regular semi-local domains containing a field.*

We use results on Bertini theorems from [Poo] and [ChPoo] to get an appropriative elementary fibration (Proposition 2.3). The present preprint is inspired by [PSV].

1 Introduction

Recall that an R-group scheme G is called reductive (respectively, semi-simple or simple), if it is affine and smooth as an R-scheme and if, moreover, for each ring homomorphism $s : R \to \Omega(s)$ to an algebraically closed field $\Omega(s)$, its scalar extension $G_{\Omega(s)}$ is a connected reductive (respectively, semi-simple or simple) algebraic group over $\Omega(s)$. The class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes. This notion of a simple R-group scheme coincides with the notion of a simple semi-simple R-group scheme from Demazure—Grothendieck [SGA3, Exp. XIX, Defn. 2.7 and Exp. XXIV, 5.3]. *Throughout the paper R denotes an integral domain and G denotes a semi-simple R-group scheme, unless explicitly stated otherwise. All commutative rings that we consider are assumed to be Noetherian.*

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A semi-simple R-group scheme G is called *simply connected* (respectively, *adjoint*), provided that for an inclusion $s : R \hookrightarrow \Omega(s)$ of R into an algebraically closed field $\Omega(s)$ the scalar extension $G_{\Omega(s)}$ is a simply connected (respectively, adjoint) $\Omega(s)$-group scheme. This definition coincides with the one from [SGA3, Exp. XXII. Defn. 4.3.3].

A well-known conjecture due to J.-P. Serre and A. Grothendieck [Se, Remarque, p.31], [Gr1, Remarque 3, p.26-27], and [Gr2, Remarque 1.11.a] asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme G over R the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion of R into K, has trivial kernel.

In three preprints [Pan2], [Pan3] an the present one we prove this conjecture for regular semi-local domains containing a finite field. For such a domain containing an infinite field the conjecture is proved in [FP]. *Thus the conjecture holds for regular semi-local domains containing a field.*

The preprint [Pan3] contains a proof of the conjecture for regular semi-local domains containing a finite field. The general plan of the proof of the conjecture realized in [Pan3] is this. Using D.Popescu theorem the case of an arbitrary regular local domain containing a finite field is reduced to the case of regular local domains of the form as in Theorem 1.2. Next, Theorem 1.2 is used to reduce the case of an arbitrary reductive group scheme to the case of semi-simple simply connected group schemes. The latter case is easily reduced to the case of simple simply connected group schemes. Finally, using Theorem 1.1 the case of simple simply connected group schemes is proved in [Pan3]. This latter case is proved using all the technical tools developed in [FP]. Since we work over finite fields results from [Poo] and [ChPoo] are heavily used.

Theorem 1.1. *Let k be a finite field. Let $\mathcal{O}$ be the semi-local ring of finitely many* **closed points** *on a k-smooth irreducible affine k-variety X and let K be its field of fractions. Let G be a simple simply connected group scheme over $\mathcal{O}$. Let G be a* **semi-simple** *simply connected group scheme over $\mathcal{O}$. Let $\mathcal{G}$ be a principal G-bundle over $\mathcal{O}$ which is trivial over K. Then there exists a principal G-bundle $\mathcal{G}_t$ over $\mathcal{O}[t]$ and a monic polynomial $h(t) \in \mathcal{O}[t]$ such that*

(i) the G-bundle $\mathcal{G}_t$ is trivial over $\mathcal{O}[t]_h$,

(ii) the evaluation of $\mathcal{G}_t$ at $t = 0$ coincides with the original G-bundle $\mathcal{G}$.

Clearly, this Theorem looks similarly to the Theorem 1.2 from [PSV]. However the proof of Theorem 1.1 is much more involved since the base field is finite.

Theorem 1.2. *Let k be* **a finite field**. *Assume that for any irreducible k-smooth affine variety X and any finite family of its* **closed points** *$x_1, x_2, \dots, x_n$ and the semi-local k-algebra $\mathcal{O} := \mathcal{O}_{X,x_1,x_2,\dots,x_n}$ and all semi-simple simply connected reductive $\mathcal{O}$-group schemes H the pointed set map*

$$H^1_{\text{ét}}(\mathcal{O}, H) \to H^1_{\text{ét}}(k(X), H),$$

induced by the inclusion of $\mathcal{O}$ into its fraction field $k(X)$, has trivial kernel.

Then for any regular semi-local domain $\mathcal{O}$ *of the form* $\mathcal{O}_{X,x_1,x_2,\dots,x_n}$ *above and any reductive* $\mathcal{O}$*-group scheme* G *the pointed set map*

$$H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion of $\mathcal{O}$ *into its fraction field* K*, has trivial kernel.*

Theorem 1.1 is one of the main result of the present preprint. Theorem 1.2 is one of the main result of the preprint [Pan2].

The preprint is organized as follows. In Section 2 elementary fibrations are discussed. In Section 3 the concept of a nice triple is recalled and Theorems 3.3 and 3.4 are formulated. In Section 4 Theorem 3.4 is proved. In Section 5 Theorem 3.3 is proved. In Section 6 a basic nice triple is constructed. In Section 7 Theorem 1.1 is proved.

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2 Elementary fibrations

In this Section we modify a result of M. Artin from [A] concerning existence of nice neighborhoods. The following notion is a modification of the one introduced by Artin in [A, Exp. XI, Déf. 3.1].

Definition 2.1. *An elementary fibration is a morphism of schemes* $p : X \to S$ *which can be included in a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{j} & \overline{X} & \xleftarrow{i} & Y \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \overline{p}} & \swarrow{\scriptstyle q} & \\ & & S & & \end{array} \qquad (1)$$

of morphisms satisfying the following conditions:

(i) j *is an open immersion dense at each fibre of* $\overline{p}$*, and* $X = \overline{X} - Y$*;*

(ii) $\overline{p}$ *is smooth projective all of whose fibres are geometrically irreducible of dimension one;*

(iii) q *is finite étale all of whose fibres are non-empty.*

Remark 2.2. Clearly, an elementary fibration is an almost elementary fibration in the sense of [PSV, Defn.2.1].

Using repeatedly [Poo, Thm.1.3] and [ChPoo, Thm.1.1] and modifying Artin's arguments [A, Exp. XI,Prop. 3.3], one can prove the following result, which is a slight extension of Artin's result [A, Exp. XI,Prop. 3.3].

Proposition 2.3. *Let k be* **a finite field**, *X be a smooth* **geometrically** *irreducible* **affine** *variety over k, $x_1, x_2, \ldots, x_n \in X$ be a family of* **closed** *points. Then there exists a Zariski open neighborhood X^0 of the family $\{x_1, x_2, \ldots, x_n\}$ and* **an elementary fibration** *$p : X^0 \to S$, where S is an open sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$.*

If, moreover, Z is a closed co-dimension one subvariety in X, then one can choose X^0 and p in such a way that $p|_{Z \bigcap X^0} : Z \bigcap X^0 \to S$ is finite surjective.

The following result is proved in [PSV, Prop.2.4].

Proposition 2.4. *Let $p : X \to S$ be an elementary fibration. If S is a regular semi-local irreducible scheme, then there exists a commutative diagram of S-schemes*

$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & \overline{X} & \xleftarrow{\ i\ } & Y \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \overline{\pi}} & & \Big\downarrow \\ \mathbf{A}^1 \times S & \xrightarrow{\ \mathrm{in}\ } & \mathbf{P}^1 \times S & \xleftarrow{\ i\ } & \{\infty\} \times S \end{array} \qquad (2)$$

such that the left hand side square is Cartesian. Here j and i are the same as in Definition 2.1, while $\mathrm{pr}_S \circ \pi = p$, where pr_S is the projection $\mathbf{A}^1 \times S \to S$.

In particular, $\pi : X \to \mathbf{A}^1 \times S$ is a finite surjective morphism of S-schemes, where X and $\mathbf{A}^1 \times S$ are regarded as S-schemes via the morphism p and the projection pr_S, respectively.

3 Nice triples

In the present section we introduce and study certain collections of geometric data and their morphisms. The concept of a *nice triple* is very similar to that of a *standard triple* introduced by Voevodsky [Vo, Defn. 4.1], and was in fact inspired by the latter notion. Let k be **a finite field**, let X/k be a smooth geometrically irreducible **affine** variety, and let $x_1, x_2, \ldots, x_n \in X$ be **a family of closed points**. Further, let $\mathcal{O} = \mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}}$ be the corresponding geometric semi-local ring.

Definition 3.1. *Let $U := \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}})$. A* nice triple *over U consists of the following data:*

(i) *a smooth morphism $q_U : \mathcal{X} \to U$, where $\mathcal{X}$ is an irreducible scheme,*

(ii) *an element $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,*

(iii) *a section Δ of the morphism q_U,*

subject to the following conditions:

(a) *each irreducible component of each fibre of the morphism q_U has dimension one,*

(b) *the module $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finite as a $\Gamma(U, \mathcal{O}_U) = \mathcal{O}$-module,*

(c) *there exists a finite surjective U-morphism* $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$,

(d) $\Delta^*(f) \neq 0 \in \Gamma(U, \mathcal{O}_U)$.

Definition 3.2. *A* morphism *between two nice triples over U*

$$(q' : \mathcal{X}' \to U, f', \Delta') \to (q : \mathcal{X} \to U, f, \Delta)$$

is an étale morphism of U-schemes $\theta : \mathcal{X}' \to \mathcal{X}$ such that

(1) $q'_U = q_U \circ \theta$,

(2) $f' = \theta^*(f) \cdot h'$ *for an element* $h' \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,

(3) $\Delta = \theta \circ \Delta'$.

Two observations are in order here.

• Item (2) implies in particular that $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\theta^*(f) \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a finite $\mathcal{O}$-module.

• It should be emphasized that no conditions are imposed on the interrelation of Π' and Π.

Let us state two crucial results which will be used in our main construction. Their proofs are given in Sections 5 and 4 respectively. If U as in Definition 3.1 the for any U-scheme V and any closed point $u \in U$ set $V_u = u \times_U V$. For a finite set A denote $\sharp A$ the cardinality of A.

Theorem 3.3. *Let U be as in Definition 3.1. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Let $G_{\mathcal{X}}$ be a semi-simple $\mathcal{X}$-group scheme, and let $G_U := \Delta^*(G_{\mathcal{X}})$. Finally, let G_{const} be the pull-back of G_U to $\mathcal{X}$. Then there exists a morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ of nice triples* **over** *U satisfying the following conditions*

(1) *there is an isomorphism $\Phi : \theta^*(G_{\text{const}}) \to \theta^*(G_{\mathcal{X}})$ of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = \text{id}_{G_U}$,*

(2) *for the closed sub-scheme $\mathcal{Z}'$ of $\mathcal{X}'$ defined by $\{f' = 0\}$ and any closed point $u \in U$ the point $\Delta'(u) \in \mathcal{Z}'_u$ is the only $k(u)$-rational point of $\mathcal{Z}'_u$,*

(3) *for any closed point $u \in U$ and any integer $r \geq 1$ and for $\mathcal{Z}'$ as in* (2) one has

$$\sharp\{z \in \mathcal{Z}'_u | \deg[z : u] = r\} \leq \ \sharp\{x \in \mathbf{A}^1_u | \deg[z : u] = r\}$$

Theorem 3.4. *Let U be as in Definition 3.1. Let $(\mathcal{X}', f', \Delta')$ be a nice triple over U, such that f'* **vanishes at every closed point of** *$\Delta'(U)$. Let $\mathcal{Z}'$ be the closed sub-scheme of $\mathcal{X}'$ defined by $\{f' = 0\}$. Assume that $\mathcal{Z}'$ satisfies the conditions* (2) *and* (3) *from Theorem 3.3. Then there exist a distinguished finite surjective morphism*

$$\sigma : \mathcal{X}' \to \mathbf{A}^1 \times U$$

of U-schemes, a monic polinomial $h \in Ker[\mathcal{O}[t] \xrightarrow{\bar{\ } \circ \sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')]$ and an element $g \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ which enjoys the following properties:

(a) *the morphism* $\sigma_g = \sigma|_{\mathcal{X}'_g} : \mathcal{X}'_g \to \mathbf{A}^1 \times U$ *is étale,*

(b) *data* $(\mathcal{O}[t], \sigma_g^* : \mathcal{O}[t] \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g, h)$ *satisfies the hypotheses of [C-T/O, Prop.2.6], i.e.* $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$ *is a finitely generated as the* $\mathcal{O}[t]$*-algebra, the element* $(\sigma_g)^*(h)$ *is not a zero-divisor in* $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$ *and* $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$,

(c) $(\Delta'(U) \cup \mathcal{Z}') \subset \mathcal{X}'_g$,

(d) $\mathcal{X}'_{gh} \subseteq \mathcal{X}'_{gf'}$.

Remark 3.5. The item (b) of this theorem shows that the cartesian square

$$\begin{array}{ccc} \mathcal{X}'_{gh} & \xrightarrow{\text{inc}} & \mathcal{X}'_g \\ {\scriptstyle \sigma_{gh}}\downarrow & & \downarrow{\scriptstyle \sigma_g} \\ (\mathbf{A}^1 \times U)_h & \xrightarrow{\text{inc}} & \mathbf{A}^1 \times U \end{array} \qquad (3)$$

can be used to glue principal G-bundles. The items (a) and (b) show that the square (16) is an elementary **distinguished** square in the category of smooth U-schemes in the sense of [MV, Defn.3.1.3]. The item (d) guaranties that a principal G-bundle on $\mathcal{X}'$, which is trivial being restricted to $\mathcal{X}'_{f'}$ is trivial being restricted to $\mathcal{X}'_{gh}$.

4 Proof of Theorem 3.4

The nearest aim is to prove Theorem 3.4. The following theorem is a step to do that.

Theorem 4.1. *Let* U *be as in Definition 3.1. Let* $(q'_U : \mathcal{X}' \to U, f', \Delta')$ *be a nice triple over* U*, such that* f' **vanishes at every closed point of** $\Delta'(U)$*. Let* $\mathcal{Z}'$ *be the closed sub-scheme of* $\mathcal{X}'$ *defined by* $\{f' = 0\}$*. Assume that* $\mathcal{Z}'$ *satisfies the conditions* (2) *and* (3) *from Theorem 3.3. Then there exists a distinguished finite surjective morphism*

$$\sigma : \mathcal{X}' \to \mathbf{A}^1 \times U$$

of U*-schemes which enjoys the following properties:*

(a) *the morphism* $\sigma|_{\mathcal{Z}'} : \mathcal{Z}' \to \mathbf{A}^1 \times U$ *is a closed embedding;*

(b) σ *is étale in a neighborhood of* $\mathcal{Z}' \cup \Delta'(U)$*;*

(c) $\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \coprod \mathcal{Z}''$ *scheme theoretically and* $\mathcal{Z}'' \cap \Delta'(U) = \emptyset$*;*

(d) $\sigma^{-1}(\{0\} \times U) = \Delta'(U) \coprod \mathcal{D}$ *scheme theoretically and* $\mathcal{D} \cap \mathcal{Z}' = \emptyset$*;*

(e) *for* $\mathcal{D}_1 := \sigma^{-1}(\{1\} \times U)$ *one has* $\mathcal{D}_1 \cap \mathcal{Z}' = \emptyset$*.*

(f) *there is a monic polinomial* $h \in \mathcal{O}[t]$ *such that* $(h) = Ker[\mathcal{O}[t] \xrightarrow{\bar{\ }\circ\sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')]$*.*

Sketch of the proof of Theorem 4.1. For any closed point $u \in U$ and any U-scheme V let $V_u = u \times_U V$ be the fibre of the scheme V over the point u.

Step (i). For any closed point $u \in U$ and any point $z \in \mathcal{Z}'_u$ there is a closed embedding $z^{(2)} \hookrightarrow \mathbf{A}^1_u$, where $z^{(2)} := Spec(\Gamma(\mathcal{X}'_u, \mathcal{O}_{\mathcal{X}'_u})/\mathfrak{m}_z^2)$ for the maximal ideal $\mathfrak{m}_z \subset \Gamma(\mathcal{X}'_u, \mathcal{O}_{\mathcal{X}'_u})$ of the point z regarded as a point of $\mathcal{X}'$. This holds, since the $k(u)$-scheme $\mathcal{X}'_u$ is equidimensional of dimension one, affine and $k(u)$-smooth.

Step (ii). For any closed point $u \in U$ there is a closed embedding $i_u : \coprod_{z \in \mathcal{Z}'_u} z^{(2)} \hookrightarrow \mathbf{A}^1_u$ of the $k(u)$-schemes. To see this apply Step (i) and use that $\mathcal{Z}'$ satisfies the condition (3) from Theorem 3.3.

Step(iii) is to introduce some notation. Since $(\mathcal{X}', f', \Delta')$ is a nice triple over U there is a finite surjective morphism $\mathcal{X}' \xrightarrow{\Pi} \mathbf{A}^1 \times U$ of the U-schemes. Take the composite $\mathcal{X}' \xrightarrow{\Pi} \mathbf{A}^1 \times U \hookrightarrow \mathbf{P}^1 \times U$ morphism and denote by $\bar{\mathcal{X}}'$ the normalization of $\mathbf{P}^1 \times U$ in the fraction field $k(\mathcal{X}')$ of the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. The normalization of $\mathbf{A}^1 \times U$ in $k(\mathcal{X}')$ coincides with the scheme $\mathcal{X}'$, since $\mathcal{X}'$ is a regular scheme. So, we have a Cartesian diagram of U-schemes

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\text{inc}} & \bar{\mathcal{X}}' \\ \Pi \downarrow & & \downarrow \bar{\Pi} \\ \mathbf{A}^1 \times U & \xrightarrow{\text{inc}} & \mathbf{P}^1 \times U, \end{array} \qquad (4)$$

in which the horizontal arrows are open embedding.

Let $\mathcal{X}'_\infty$ be the Cartie-divisor $(\bar{\Pi})^{-1}(\infty \times U)$ in $\bar{\mathcal{X}}'$. Let $\mathcal{L} := \mathcal{O}_{\bar{\mathcal{X}}'}(\mathcal{X}'_\infty)$ be the corresponding invertible sheaf and let $s_0 \in \Gamma(\bar{\mathcal{X}}', \mathcal{L})$ be its section vanishing exactly on $\mathcal{X}'_\infty$. One has a Cartesian square of U-schemes

$$\begin{array}{ccc} \mathcal{X}'_{\infty,u} & \xrightarrow{j_\infty} & \mathcal{X}'_\infty \\ in_u \downarrow & & \downarrow in \\ \bar{\mathcal{X}}'_u & \xrightarrow{j} & \bar{\mathcal{X}}', \end{array} \qquad (5)$$

which shows that the closed embedding $in_u : \mathcal{X}'_{\infty,u} \hookrightarrow \bar{\mathcal{X}}'_u$ is a Cartie-divisor on $\bar{\mathcal{X}}'_u$. Set $\mathcal{L}_u = j^*(\mathcal{L})$ and $s_{0,u} = s_0|_{\bar{\mathcal{X}}'_u} \in \Gamma(\bar{\mathcal{X}}'_u, \mathcal{L}_u)$.

Step (iv). There exists an integer $n > 0$ and a section $s_{1,u} \in \Gamma(\bar{\mathcal{X}}'_u, \mathcal{L}_u^{\otimes n})$ which has no zeros on $\mathcal{X}'_{\infty,u}$ and such that the morphism

$$[s^n_{0,u} : s_{1,u}] : \bar{\mathcal{X}}'_u \to \mathbf{P}^1_u$$

has the following two properties

(a) the morphism $\sigma_u = s_{1,u}/s^n_{0,u} : \mathcal{X}'_u \to \mathbf{A}^1_u$ is finite surjective,

(b) $\sigma_u|_{\coprod_{z \in \mathcal{Z}'_u} z^{(2)}} = i_u : \coprod_{z \in \mathcal{Z}'_u} z^{(2)} \hookrightarrow \mathbf{A}^1_u$, where i_u is from the step (ii); in particular, σ_u is étale at every point $z \in \mathcal{Z}'_u$.

Step (v). There exists a section $s_1 \in \Gamma(\bar{\mathcal{X}}', \mathcal{L}^{\otimes n})$ such that for any closed point $u \in U$ one has $s_1|_{\mathcal{X}'_u} = s_{1,u}$.

Step (vi). If s_1 is such as in the step (v), then the morphism

$$\sigma = (s_1/s_0^n, pr_U) : \mathcal{X}' \to \mathbf{A}^1 \times U$$

is finite and surjective.

We are ready now to check step by step all the statements of the Theorem.

The assertion (b). Since the schemes $\mathcal{X}'$ and $\mathbf{A}^1 \times U$ are regular and the morphism σ is finite and surjective, the morphism σ is flat by a theorem of Grothendieck.

So, to check that σ is étale at a closed point $z \in \mathcal{Z}'$ it suffices to check that for the point $u = q'_U(z)$ the morphism $\sigma_u : \mathcal{X}'_u \to \mathbf{A}^1_u$ is étale at the point z. The latter does hold by the step (iv), item (b). Whence σ is étale at all the closed points of $\mathcal{Z}'$. By the hypotheses of the Theorem the set of closed points of $\Delta'(U)$ is contained in the set of the closed points of $\mathcal{Z}'$. Whence σ is étale also at all the closed points of $\Delta'(U)$. The schemes $\Delta'(U)$ and $\mathcal{Z}'$ are both semi-local. Thus, σ is étale in a neighborhood of $\mathcal{Z}' \cup \Delta'(U)$.

The assertion (a). For any closed point $u \in U$ and **any point** $z \in \mathcal{Z}'_u$ the $k(u)$-algebra homomorphism $\sigma^*_u : k(u)[t] \to k(u)[\mathcal{X}'_u]$ is étale at the maximal ideal $\mathfrak{m}_z$ of the $k(u)$-algebra $k(u)[\mathcal{X}'_u]$ and the composite map $k(u)[t] \xrightarrow{\sigma^*_u} k(u)[\mathcal{X}'_u] \to k(u)[\mathcal{X}'_u]/\mathfrak{m}_z$ is an epimorphism. Thus, for any integer $r > 0$ the $k(u)$-algebra homomorphism $k(u)[t] \to k(u)[\mathcal{X}'_u]/\mathfrak{m}^r_z$ is an epimorphism. The local ring $\mathcal{O}_{\mathcal{Z}'_u,z}$ of the scheme $\mathcal{Z}'_u$ at the point z is of the form $k(u)[\mathcal{X}'_u]/\mathfrak{m}^s_z$ for an integer s. Thus, the $k(u)$-algebra homomorphism

$$k(u)[t] \xrightarrow{\sigma^*_u} k(u)[\mathcal{X}'_u] \to \mathcal{O}_{\mathcal{Z}'_u,z}$$

is surjective. Since $\sigma_u|_{\coprod_{z\in\mathcal{Z}'_u} z^{(2)}} = i_u$ and i_u is a closed embedding one concludes that the $k(u)$-algebra homomorphism

$$k(u)[t] \to \prod_{z/u} O_{\mathcal{Z}'_u,z} = \Gamma(\mathcal{Z}'_u, \mathcal{O}_{\mathcal{Z}'_u})$$

is surjective. Let $\mathbf{u} = \coprod Spec(k(u))$ regarded as the closed sub-scheme of U, where u runs over all closed points of U. Then, for the scheme $\mathcal{Z}'_{\mathbf{u}} = \mathbf{u} \times_U \mathcal{Z}'$ the $k[\mathbf{u}]$-algebra homomorphism

$$k[\mathbf{u}][t] \to \Gamma(\mathcal{Z}'_{\mathbf{u}}, \mathcal{O}_{\mathcal{Z}'_{\mathbf{u}}}) \qquad (6)$$

is surjective.

Since $(\mathcal{X}', f', \Delta')$ is a nice triple over U, the $\mathcal{O}$-module $\Gamma(\mathcal{Z}', \mathcal{O}_{\mathcal{Z}'})$ is finite. Thus, the $k[\mathbf{u}]$-module $\Gamma(\mathcal{Z}'_{\mathbf{u}}, \mathcal{O}_{\mathcal{Z}'_{\mathbf{u}}})$ is finite. Now the surjectivity of the $k[\mathbf{u}]$-algebra homomorphism (6) and the Nakayama lemma show that the $\mathcal{O}$-algebra homomorphism $\mathcal{O}[t] \to \Gamma(\mathcal{Z}', \mathcal{O}_{\mathcal{Z}'})$ is surjective. Thus, $\sigma|_{\mathcal{Z}'} : \mathcal{Z}' \to \mathbf{A}^1 \times U$ is a closed embedding.

The assertion (e). The morphism Δ' is a section of the structure morphism $q'_U : \mathcal{X}' \to U$ and the morphism σ is a morphism of the U-schemes. Hence the composite morphism $t_0 := \sigma \circ \Delta'$ is a section of the projection $pr_U : \mathbf{A}^1 \times U \to U$. This section is defined by an element $a \in \mathcal{O}$. There is another section t_1 of the projection pr_U defined by the element $1 - a \in \mathcal{O}$. Making an affine change of coordinates on $\mathbf{A}^1_U$ we may and will assume that $t_0(U) = \{0\} \times U$ and $t_1(U) = \{1\} \times U$.

Since $\mathcal{D}_1$ and $\mathcal{Z}'$ are semi-local, to prove the assertion (e) it suffices to check that $\mathcal{D}_1$ and $\mathcal{Z}'$ have no common closed points. Let $z \in \mathcal{D}_1 \cap \mathcal{Z}'$ be a common closed point. Then $\sigma(z) \in \{1\} \times U$. Let $u = q'_U(z)$. We already know that $\sigma|_{\mathcal{Z}'}$ is a closed embedding. Thus $deg[z : u] = deg[\sigma(z) : u] = 1$. The U-scheme $\mathcal{Z}'$ satisfies the conditions (2) of Theorem 3.3. Thus, $z = \Delta'(u)$. In this case $\sigma(z) \in \{0\} \times U$. But $\sigma(z) \in \{1\} \times U$. This is a contradiction. Whence $\mathcal{D}_1 \cap \mathcal{Z}' = \emptyset$.

The assertion (c). The finite morphism σ is étale in a neighborhood of $\mathcal{Z}'$ by the item (b) of the Theorem. By the item (a) of the Theorem $\sigma|_{\mathcal{Z}'}$ is a closed embedding. Thus, the morphism $\sigma^{-1}(\sigma(\mathcal{Z}')) \to \sigma(\mathcal{Z}')$ of affine schemes is finite and there is an affine open sub-scheme V of the scheme $\sigma^{-1}(\sigma(\mathcal{Z}'))$ such that the morphism $V \to \sigma(\mathcal{Z}')$ is étale. Since $\sigma|_{\mathcal{Z}'}$ is a closed embedding there is a unique section s of the morphism $\sigma^{-1}(\sigma(\mathcal{Z}')) \to \sigma(\mathcal{Z}')$ with the image $\mathcal{Z}'$ and this image is contained in V. By [OP1, Lemma 5.3] the scheme $\sigma^{-1}(\sigma(\mathcal{Z}'))$ has the form $\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \coprod \mathcal{Z}''$.
By a similar reasoning the scheme $\sigma^{-1}(\{0\} \times U)$ has the form $\Delta'(U) \coprod \mathcal{D}$. All the closed points of $\Delta'(U)$ are closed points of $\mathcal{Z}'$ and $\mathcal{Z}' \cap \mathcal{Z}'' = \emptyset$. Thus, $\Delta'(U) \cap \mathcal{Z}'' = \emptyset$.

The assertion (d). It remains to show that $\mathcal{D} \cap \mathcal{Z}' = \emptyset$. It suffices to check that $\mathcal{D}$ and $\mathcal{Z}'$ have no common closed points. Let $z \in \mathcal{D} \cap \mathcal{Z}'$ be a common closed point. Then $\sigma(z) \in \{0\} \times U$. Let $u = q'_U(z)$. We already know that $\sigma|_{\mathcal{Z}'}$ is a closed embedding. Thus $deg[z : u] = deg[\sigma(z) : u] = 1$. The U-scheme $\mathcal{Z}'$ satisfies the conditions (2) of Theorem 3.3. Thus, $z = \Delta'(u) \in \Delta'(U)$. So, $z \in \Delta'(U) \cap \mathcal{D}$. But as we already know $\Delta'(U) \cap \mathcal{D} = \emptyset$. This contradiction shows that $\mathcal{D}$ and $\mathcal{Z}'$ have no common closed points. Thus, $\mathcal{D} \cap \mathcal{Z}' = \emptyset$.

The assertion (f). Recall that $\mathcal{X}'$ is affine irreducible and regular. So, the principal ideal (f') has the form $\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2} \cdots \mathfrak{p}_n^{r_n}$, where $\mathfrak{p}_i$'s are hight one prime ideals in $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Let $\mathcal{Z}'_i$ be the closed subscheme in $\mathcal{X}'$ defined by the ideal $\mathfrak{p}_i$. Let $\mathfrak{q}_i = \mathcal{O}[t] \cap \mathfrak{p}_i$. The morphism $\sigma|_{\mathcal{Z}'} : \mathcal{Z}' \to \mathbf{A}^1 \times U$ is a closed embedding by the item (a) of Theorem 4.1. This yields that $\sigma|_{\mathcal{Z}'_i} : \mathcal{Z}'_i \to \mathbf{A}^1 \times U$ is a closed embedding too. Thus $\mathfrak{p}_i$ is a hight one prime ideal in $\mathcal{O}[t]$. So, it is a principal prime ideal. Since $\mathcal{Z}'$ is finite over U the scheme $\mathcal{Z}'_i$ is finite over U too. Hence the principal prime ideal $\mathfrak{p}_i$ is of the form (h_i) for a unique monic polinomial $h_i \in \mathcal{O}[t]$.

Set $h = h_1^{r_1} h_2^{r_2} \ldots h_n^{r_n}$. Clearly, $h \in Ker[\mathcal{O}[t] \xrightarrow{\bar{\ } \circ \sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')]$. Since the map $\mathcal{O}[t] \xrightarrow{\bar{\ } \circ \sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')$ is surjective, to prove the assertion (f) it suffices to check that the surjective $\mathcal{O}$-module homomorphism

$$\bar{\ } \circ \sigma^* : \mathcal{O}[t]/(h) \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')$$

is an isomorphism. Both sides are finitely generated projective $\mathcal{O}$-modules. It remains to check that both sides have the same rank as the $\mathcal{O}$-modules. For that it suffices to know that $\mathcal{O}[t]/(h_i)$ and $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\mathfrak{p}_i$ are of the same rank as the $\mathcal{O}$-modules. This is the case since they are isomorphic $\mathcal{O}$-modules. Indeed, the composite map

$$\mathcal{O}[t] \xrightarrow{\bar{\ } \circ \sigma^*} \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f') \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\mathfrak{p}_i$$

is an $\mathcal{O}$-algebra epimorphism and the kernel of this epimorphism is the ideal $\mathfrak{q}_i = (h_i)$.

Whence the assertion (f) and whence the Theorem.

□

Proof of Theorem 3.4. We need to find σ, h and g which enjoy the properties (a) to (d) from the Theorem. For that we will use notation from Theorem 4.1.

Take σ *as in Theorem* 4.1. Since $\mathcal{X}'$ is a regular affine irreducible and $\sigma : \mathcal{X}' \to \mathbf{A}^1_U$ is finite surjective the induced $\mathcal{O}$-algebra homomorphism $\sigma^* : \mathcal{O}[t] \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a monomorphism. We will regard below the $\mathcal{O}$-algebra $\mathcal{O}[t]$ as a subalgebra via σ^*.

Take $h \in \mathcal{O}[t]$ *as in the item* (f) *of Theorem* 4.1.

Let $I(\mathcal{Z}'') \subseteq \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ be the ideal defining the closed subscheme $\mathcal{Z}''$ of the scheme $\mathcal{X}'$. Using the items (b) and (c) of Theorem 4.1 find an element $g \in I(\mathcal{Z}'')$ such that
(1) $(f') + (g) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,
(2) $Ker((\Delta')^*) + (g) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,
(3) $\sigma_g = \sigma|_{\mathcal{X}'_g} : \mathcal{X}'_g \to \mathbf{A}^1_U$ is étale.
With this choice of σ, h and g complete the proof of Theorem 3.4. The assertions (a) and (c) hold by our choice of g. The assertion (d) holds, since $\sigma^*(h) \in (f')$. It remains to prove the assertion (b). The morphism σ is finite. Hence the $\mathcal{O}[t]$-algebra $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$ is finitely generated. The scheme $\mathcal{X}'$ is regular and irreducible. Thus, the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a domain. The homomorphism σ^* is injective. Hence, the element h is not zero and is not a zero divisor in $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$.

It remains to check that $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$. Firstly, by the choice of h and by the item (a) of Theorem 4.1 one has $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f')$. Secondly, by the property (1) of the element g one has $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f') = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / f'\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g$. Finally, by the items (c) and (a) of Theorem 4.1 one has

$$\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f') \ \times \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/I(\mathcal{Z}'') = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(h). \tag{7}$$

Localizing both sides of (7) in g one gets an equality

$$\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / f'\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g,$$

hence

$$\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/(f') = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / f'\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g / h\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g.$$

Whence the Theorem.

□

5 Proof of Theorem 3.3

Proposition 5.1. *Let* S *be a regular semi-local irreducible scheme. Assume that all the closed points of* S *have* **finite residue fields**. *Let* G_1, G_2 *be two* **semi-simple** S*-group schemes which are twisted forms of each other. Further, let* $T \subset S$ *be a closed sub-scheme of* S *and* $\varphi : G_1|_T \to G_2|_T$ *be an* S*-group scheme isomorphism. Then there exists a finite étale morphism* $S' \xrightarrow{\pi} S$ *together with its section* $\delta : T \to S'$ *over* T *and an* S'*-group scheme isomorphism* $\Phi : \pi^* G_1 \to \pi^* G_2$ *such that* $\delta^*(\Phi) = \varphi$.

Proof. The proof literally repeats the proof of [PSV, Prop.5.1] except exactly one reference, which is the reference to [OP2, Lemma 7.2]. That reference one has to replace with the reference to the following

Lemma 5.2. *Let $S = Spec(R)$ be a regular semi-local scheme such that* **the residue field at any of its closed point is finite**. *Let T be a closed subscheme of S. Let $\bar{X}$ be a closed subscheme of $\mathbb{P}^d_S = Proj(S[X_0, \dots, X_d])$ and $X = \bar{X} \cap \mathbf{A}^d_S$, where $\mathbb{A}^d_S$ is the affine space defined by $X_0 \neq 0$. Let $X_\infty = \bar{X} \setminus X$ be the intersection of $\bar{X}$ with the hyperplane at infinity $X_0 = 0$. Assume that over T there exists a section $\delta : T \to X$ of the canonical projection $X \to S$. Assume further that*

(1) *X is smooth and equidimensional over S, of relative dimension r;*

(2) *For every closed point $s \in S$ the closed fibres of X_∞ and X satisfy*

$$dim(X_\infty(s)) < dim(X(s)) = r \ .$$

Then there exists a closed subscheme $\tilde{S}$ of X which is finite étale over S and contains $\delta(T)$.

The proof of the lemma is given below and repeats literally the proof of [OP2, Lemma 7.2]. The only difference is that we refer below to a Poonen's article [Poo] on Bertini theorems over finite fields rather than to Artin's result.

Since S is semilocal, after a linear change of coordinates we may assume that δ maps T into the closed subscheme of $\mathbf{P}^d_T$ defined by $X_1 = \cdots = X_d = 0$. For each closed fibre $\mathbf{P}^d_s$ of $\mathbf{P}^d_S$ using repeatedly [Poo, Thm.1.2], we can choose a family of **homogeneous** polynomials $H_1(s), \dots, H_r(s)$ (in general of increasing degrees) such that the subscheme $Y(s)$ of $\mathbf{P}^d_S(s)$ defined by the equations

$$H_1(s) = 0 \, , \ \dots \ , \, H_r(s) = 0$$

intersects $X(s)$ transversally, contains the point $[1 : 0 : \cdots : 0]$ and avoids $X_\infty(s)$. By the chinese remainders' theorem there exists a common lift $H_i \in R[X_0, \dots, X_d]$ of all polynomials $H_i(s)$, $s \in \mathrm{Max}(R)$. We may choose this common lift H_i such that $H_i(1, 0, \dots, 0) = 0$. Let Y be the closed subscheme of $\mathbf{P}^d_S$ defined by

$$H_1 = 0 \, , \dots , H_r = 0 \ .$$

We claim that the subscheme $\tilde{S} = Y \cap X$ has the required properties. Note first that $X \cap Y$ is finite over S. In fact, $X \cap Y = \bar{X} \cap Y$, which is projective over S and such that every closed fibre (hence every fibre) is finite. Since the closed fibres of $X \cap Y$ are finite étale over the closed points of S, to show that $X \cap Y$ is finite étale over S it only remains to show that it is flat over S. Noting that $X \cap Y$ is defined in every closed fibre by a regular sequence of equations and localizing at each closed point of S, we see that flatness follows from [OP2, Lemma 7.3].

□

Let k be **a finite field**. Let U be as in Definition 3.1. Let S' be an irreducible regular semi-local scheme over k and $p : S' \to U$ be a k-morphism. Let $T' \subset S'$ be a closed sub-scheme of S' such that the restriction $p|_{T'} : T' \to U$ is an isomorphism. We will assume below that $dim(T') < dim(S')$, where dim is the Krull dimension. For any closed point $u \in U$ and any U-scheme V let $V_u = u \times_U V$ be the fibre of the scheme V over the point u. For a finite set A denote $\sharp A$ the cardinality of A.

Lemma 5.3. *Assume that all the closed points of S' have* **finite residue fields**. *Then there exists a finite étale morphism $\rho : S'' \to S'$ (with an irreducible scheme S'') and a section $\delta' : T' \to S''$ of ρ over T' such that the following holds*

(1) *for any closed point $u \in U$ let $u' \in T'$ be a unique point such that $p(u') = u$, then the point $\delta'(u') \in S''_u$ is the only $k(u)$-rational point of S''_u,*

(2) *for any closed point $u \in U$ and any integer $r \geq 1$ one has*

$$\sharp\{z \in S''_u | deg[z : u] = r\} \leq \ \sharp\{x \in \mathbf{A}^1_u | deg[z : u] = r\}$$

Proof of Theorem 3.3. We can start by almost literally repeating arguments from the proof of [OP1, Lemma 8.1], which involve the following purely geometric lemma [OP1, Lemma 8.2].

For reader's convenience below we state that Lemma adapting notation to the ones of Section 3. Namely, let U be as in Definition 3.1 and let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Further, let $G_{\mathcal{X}}$ be a simple simply-connected $\mathcal{X}$-group scheme, $G_U := \Delta^*(G_{\mathcal{X}})$, and let G_{const} be the pull-back of G_U to $\mathcal{X}$. Finally, by the definition of a nice triple there exists a finite surjective morphism $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$ of U-schemes.

Lemma 5.4. *Let $\mathcal{Y}$ be a closed nonempty sub-scheme of $\mathcal{X}$, finite over U. Let $\mathcal{V}$ be an open subset of $\mathcal{X}$ containing $\Pi^{-1}(\Pi(\mathcal{Y}))$. There exists an open set $\mathcal{W} \subseteq \mathcal{V}$ still containing $q_U^{-1}(q_U(\mathcal{Y}))$ and endowed with a finite surjective morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$ (in general $\neq \Pi$).*

Let $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$ be the above finite surjective U-morphism. The following diagram summarises the situation:

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & & \downarrow & & \\ \mathcal{X} - \mathcal{Z} & \hookrightarrow & \mathcal{X} & \xrightarrow{\Pi} & \mathbf{A}^1 \times U \\ & & {\scriptstyle\Delta}\uparrow\downarrow{\scriptstyle q_U} & & \\ & & U & & \end{array}$$

Here $\mathcal{Z}$ is the closed sub-scheme defined by the equation $f = 0$. By assumption, $\mathcal{Z}$ is finite over U. Let $\mathcal{Y} = \Pi^{-1}(\Pi(\mathcal{Z} \cup \Delta(U)))$. Since $\mathcal{Z}$ and $\Delta(U)$ are both finite over U and since Π is a finite morphism of U-schemes, $\mathcal{Y}$ is also finite over U. Denote by $y_1, \ldots, y_m$ its closed points and let $S = \mathrm{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \ldots, y_m})$. Set $T = \Delta(U) \subseteq S$. Further, let $G_U = \Delta^*(G_{\mathcal{X}})$ be as in the hypotheses of Theorem 3.3 and let G_{const} be the pull-back of G_U to $\mathcal{X}$. Finally, let $\varphi : G_{\mathrm{const}}|_T \to G_{\mathcal{X}}|_T$ be the canonical isomorphism. Recall that by assumption $\mathcal{X}$ is U-smooth and irreducible, and thus S is regular and irreducible.

By Proposition 5.1 there exists a finite étale covering $\theta_0 : S' \to S$, a section $\delta : T \to S'$ of θ_0 over T and an isomorphism

$$\Phi_0 : \theta_0^*(G_{\mathrm{const},S}) \to \theta_0^*(G_{\mathcal{X}}|_S)$$

such that $\delta^*\Phi_0 = \varphi$. **Replacing S' with a connected component of S' which contains $T' := \delta(T) = \delta(\Delta(U))$ we may and will assume that S' is irreducible.**

Let $p = q_U \circ \theta_0 : S' \to U$. By Lemma 5.3 there exists a finite étale morphism $\rho : S'' \to S'$ (with an irreducible scheme S'') and a section $\delta' : T' \to S''$ of ρ over T' such that the properties (1) and (2) from Lemma 5.3 holds. Set $\delta'' = \delta' \circ \delta : T \to S''$ and $\theta_0'' = \theta_0 \circ \rho : S'' \to U$. We are also given the S''-group scheme isomorphism

$$\rho^*(\Phi_0) : (\theta_0'')^*(G_{\mathrm{const},S}) \to (\theta_0'')^*(G_{\mathcal{X}}|_S)$$

We can extend these data to a neighborhood $\mathcal{V}$ of $\{y_1, \dots, y_n\}$ and get the diagram

$$\begin{array}{ccccccc} & & S'' & \hookrightarrow & \mathcal{V}'' & & \\ & \stackrel{\delta''}{\nearrow} & \downarrow{\scriptstyle \theta_0''} & & \downarrow{\scriptstyle \theta} & & \\ T & \hookrightarrow & S & \hookrightarrow & \mathcal{V} & \hookrightarrow & \mathcal{X} \end{array} \tag{8}$$

where $\theta : \mathcal{V}'' \to \mathcal{V}$ finite étale, and an isomorphism $\Phi : \theta^*(G_{\mathrm{const}}) \to \theta^*(G_{\mathcal{X}})$.

Since T isomorphically projects onto U, it is still closed viewed as a sub-scheme of $\mathcal{V}$. Note that since $\mathcal{Y}$ is semi-local and $\mathcal{V}$ contains all of its closed points, $\mathcal{V}$ contains $\Pi^{-1}(\Pi(\mathcal{Y})) = \mathcal{Y}$. By Lemma 5.4 there exists an open subset $\mathcal{W} \subseteq \mathcal{V}$ containing $\mathcal{Y}$ and endowed with a finite surjective U-morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$.

Let $\mathcal{X}' = \theta^{-1}(\mathcal{W})$, $f' = \theta^*(f)$, $q_U' = q_U \circ \theta$, and let $\Delta' : U \to \mathcal{X}'$ be the section of q_U' obtained as the composition of δ'' with Δ. We claim that the triple $(\mathcal{X}', f', \Delta')$ is a nice triple over U. Let us verify this. Firstly, the structure morphism $q_U' : \mathcal{X}' \to U$ coincides with the composition

$$\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X} \xrightarrow{q_U} U.$$

Thus, it is smooth. The element f' belongs to the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$, the morphism Δ' is a section of q_U'. Each component of each fibre of the morphism q_U has dimension one, the morphism $\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X}$ is étale. Thus, each component of each fibre of the morphism q_U' is also of dimension one. Since $\{f = 0\} \subset \mathcal{W}$ and $\theta : \mathcal{X}' \to \mathcal{W}$ is finite, $\{f' = 0\}$ is finite over $\{f = 0\}$ and hence also over U. In other words, the $\mathcal{O}$-module $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/f' \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is finite. The morphism $\theta : \mathcal{X}' \to \mathcal{W}$ is finite and surjective. We have constructed above in Lemma 5.4 the finite surjective morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$. It follows that $\Pi^* \circ \theta : \mathcal{X}' \to \mathbf{A}^1 \times U$ is finite and surjective.

Clearly, the étale morphism $\theta : \mathcal{X}' \to \mathcal{X}$ is a morphism between the nice triples, with $h' = 1$.

Denote the restriction of Φ to $\mathcal{X}'$ simply by Φ. The equality $(\Delta')^*\Phi = \mathrm{id}_{G_U}$ holds by the very construction of the isomorphism Φ. Theorem follows.

All the closed points of the sub-scheme $\{f = 0\} \subset \mathcal{X}$ are in S. The morphism θ is finite and $\theta^{-1}(S) = S''$. Thus all the closed points of the sub-scheme $\{f' = 0\} \subset \mathcal{X}'$ are in S''. Now the properties (1) and (2) of the U-scheme S'' show that the assertions (2) and (3) of Theorem 3.3 do hold for the closed sub-scheme $\mathcal{Z}'$ of $\mathcal{X}'$ defined by $\{f' = 0\}$.

Theorem 3.3 follows.

□

6 A basic nice triple

Let k be **a finite field**. Fix a smooth geometrically irreducible affine k-scheme X, and a finite family of **closed** points $x_1, x_2, \dots, x_n$ on X, and a non-zero function $\mathrm{f} \in k[X]$, which vanishes at each of x_i's for $i = 1, 2, \dots, n$. Let $\mathcal{O} = \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ be the semi-local ring of the family $x_1, x_2, \dots, x_n$ on X, $U = Spec(\mathcal{O})$ and $\mathrm{can} : U \hookrightarrow X$ the canonical inclusion of schemes. The definition of a nice triple over U is given in 3.1. The main aim of the present section is to prove the following

Proposition 6.1. *One can shrink X such that $x_1, x_2, \dots, x_n$ are still in X and X is affine, and then to construct a nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over U and an essentially smooth morphism $q_X : \mathcal{X} \to X$ such that $q_X \circ \Delta = can$, $f = q_X^*(\mathrm{f})$ and the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.*

Proof. By Proposition 2.3 there exist a Zariski open neighborhood X^0 of the family $\{x_1, x_2, \dots, x_n\}$ and an almost elementary fibration $p : X^0 \to S$, where S is an open sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$, such that

$$p|_{\{\mathrm{f}=0\}\cap X^0} : \{\mathrm{f} = 0\} \cap X^0 \to S$$

is finite surjective. Let $s_i = p(x_i) \in S$, for each $1 \le i \le n$. Shrinking S, we may assume that S is *affine* and still contains the family $\{s_1, s_2, \dots, s_n\}$. Clearly, in this case $p^{-1}(S) \subseteq X^0$ contains the family $\{x_1, x_2, \dots, x_n\}$. We replace X by $p^{-1}(S)$ and f by its restriction to this new X.

In this way we get an almost elementary fibration $p : X \to S$ such that

$$\{x_1, \dots, x_n\} \subset \{\mathrm{f} = 0\} \subset X,$$

S is an open affine sub-scheme in the projective space $\mathbf{P}^{\dim X - 1}$, and the restriction $p|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to S$ of p to the vanishing locus of f is a finite surjective morphism. In other words, $k[X]/(\mathrm{f})$ is finite as a $k[S]$-module.

As an open affine sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$ the scheme S is regular. By Proposition 2.4 one can shrink S in such a way that S is still affine, contains the family $\{s_1, s_2, \dots, s_n\}$ and there exists a finite surjective morphism

$$\pi : X \to \mathbf{A}^1 \times S$$

such that $p = \mathrm{pr}_S \circ \pi$. Clearly, in this case $p^{-1}(S) \subseteq X$ contains the family $\{x_1, x_2, \ldots, x_n\}$. We replace X by $p^{-1}(S)$ and f by its restriction to this new X.

In this way we get an almost elementary fibration $p : X \to S$ such that

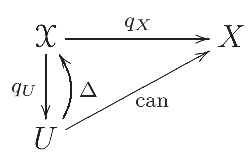
$$\{x_1, \ldots, x_n\} \subset \{\mathrm{f} = 0\} \subset X,$$

S is an open affine sub-scheme in the projective space $\mathbf{P}^{\dim X - 1}$, and the restriction $p|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to S$ is a finite surjective morphism. Eventually we conclude that there exists a finite surjective morphism $\pi : X \to \mathbf{A}^1 \times S$ such that $p = \mathrm{pr}_S \circ \pi$.

Let $p_U = p \circ \mathrm{can} : U \to S$, **where** $U = Spec(\mathcal{O})$ **and** $can : U \hookrightarrow X$ **are as above**. Further, we consider the fibre product

$$\mathcal{X} := U \times_S X.$$

Then the canonical projections $q_U : \mathcal{X} \to U$ and $q_X : \mathcal{X} \to X$ and the diagonal morphism $\Delta : U \to \mathcal{X}$ can be included in the following diagram



(9)

where

$$q_X \circ \Delta = \mathrm{can} \tag{10}$$

and

$$q_U \circ \Delta = \mathrm{id}_U\,. \tag{11}$$

Note that q_U is *a smooth morphism with geometrically irreducible fibres of dimension one*. Indeed, observe that q_U is a base change via p_U of the morphism p which has the desired properties. Note that $\mathcal{X}$ is irreducible. Indeed, U is irreducible and the fibre of q_U over the generic point of U is irreducible.

Taking the base change via p_U of the finite surjective morphism $\pi : X \to \mathbf{A}^1 \times S$, we get *a finite surjective morphism*

$$\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$$

such that $q_U = \mathrm{pr}_U \circ \Pi$, where $pr_U : \mathbf{A}^1 \times U \to U$ is the natural projection.

Set $f := q_X^*(\mathrm{f})$. The $\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}}$-module $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finite, since the $k[S]$-module $k[X]/\mathrm{f} \cdot k[X]$ is finite.

Now the data

$$(q_U : \mathcal{X} \to U, f, \Delta) \tag{12}$$

form an example of a *nice triple* as in Definition 3.1. Moreover, we have

Claim 6.2. *The schemes* $\Delta(U)$ *and* $\{f = 0\}$ *are both semi-local and the set of closed points of* $\Delta(U)$ *is contained in the set of closed points of* $\{f = 0\}$.

This holds since the set $\{x_1, x_2, \ldots, x_n\}$ is contained in the vanishing locus of the function f. **The nice triple (12) together with the essentially smooth morphism** q_X **are the required one. Whence the proposition.** □

7 Proof of Theorem 1.1

The main result of this Section is Corollary 7.2 (= Theorem 1.1).

Fix a k-smooth irreducible affine k-scheme X, a finite family of points $x_1, x_2, \dots, x_n$ on X, and set $\mathcal{O} := \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ and $U := \mathrm{Spec}(\mathcal{O})$. Further, consider a simple simply connected U-group scheme G and a principal G-bundle P over $\mathcal{O}$ which is trivial over K for the field of fractions K of $\mathcal{O}$. We may and will assume that for certain $0 \neq \mathrm{f} \in \mathcal{O}$ the principal G-bundle P is trivial over $\mathcal{O}_{\mathrm{f}}$.

Shrinking X if necessary, we may secure the following properties

(i) The points $x_1, x_2, \dots, x_n$ are still in X and X is affine.

(ii) The group scheme G is defined over X and it is a simple group scheme. We will often denote this X-group scheme by G_X and write G_U for the original G.

(iii) The principal G_U-bundle P is the restriction to U of a principal G_X-bundle P_X over X and $\mathrm{f} \in k[X]$. We often will write P_U for the original principal G_U-bundle P over U.

(iv) The restriction P_{f} of the bundle P_X to the principal open subset X_{f} is trivial and f vanishes at each x_i's.

If we shrink X further such that the property (i) is secured, then we automatically secure the properties (ii) to (iv). For any such X we will write $can : U \hookrightarrow X$ for the canonical embedding.

After substituting k by its algebraic closure $\tilde{k}$ in $k[X]$, we can assume that X is a $\tilde{k}$-smooth geometrically irreducible affine $\tilde{k}$-scheme. To simplify the notation, we will continue to denote this new $\tilde{k}$ by k.

In particular, we are given now the smooth geometrically irreducible affine k-scheme X, the finite family of points $x_1, x_2, \dots, x_n$ on X, and the non-zero function $\mathrm{f} \in k[X]$ vanishing at each point x_i. **We may shrink X further securing the property (i) and construct the nice triple $(q_U : \mathcal{X} \to U, \Delta, f)$ over U and the essentially smooth morphism $q_X : \mathcal{X} \to U$ as in Proposition 6.1. Since the property (i) is secured the properties (ii) to (iv) are secured too. Consider the $\mathcal{X}$-group scheme $G_{\mathcal{X}} := (q_X)^*(G_X)$. Note that the U-group scheme $\Delta^*(G_{\mathcal{X}})$ coincides with G_U from the item (ii) since $can = q_X \circ \Delta$ by Proposition 6.1. Consider one more $\mathcal{X}$-group scheme, namely**

$$G_{const} := (q_U)^*(\Delta^*(G_{\mathcal{X}})) = (q_U)^*(G_U).$$

By Theorem 3.3 there exists a morphism of nice triples

$$\theta : (q'_U : \mathcal{X}' \to U, f', \Delta') \to (q_U : \mathcal{X} \to U, f, \Delta)$$

and an isomorphism

$$\Phi : \theta^*(G_{\mathrm{const}}) \to \theta^*(G_{\mathcal{X}}) =: G_{\mathcal{X}'} \tag{13}$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = \mathrm{id}_{G_U}$ and such that the closed sub-scheme $\mathcal{Z}' := \{f' = 0\}$ satisfies the conditions (2) and (3) from Theorem 3.3. Set

$$q'_X = q_X \circ \theta : \mathcal{X}' \to X. \tag{14}$$

Recall that

$$q'_U = q_U \circ \theta : \mathcal{X}' \to U, \tag{15}$$

since θ is a morphism of nice triples.

Note that, since by Claim 6.2 f vanishes on all closed points of $\Delta(U)$, and θ is a morphism of nice triples, f' vanishes on all closed points of $\Delta'(U)$ as well. Therefore, the nice triple $(q'_U : \mathcal{X}' \to U, f', \Delta' : U \to \mathcal{X}')$ is subject to Theorem 3.4.

By Theorem 3.4 there exists a finite surjective morphism $\sigma : \mathcal{X}' \to \mathbf{A}^1 \times U$ of U-schemes, a monic polinomial $h \in ker(\sigma^*)$ and an element $g \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ which enjoys the properties (a) to (d) from that Theorem. The item (b) of Theorem 3.4 shows that the cartesian square

$$\begin{array}{ccc} \mathcal{X}'_{gh} & \xrightarrow{\text{inc}} & \mathcal{X}'_g \\ {\scriptstyle \sigma_{gh}}\downarrow & & \downarrow{\scriptstyle \sigma_g} \\ (\mathbf{A}^1 \times U)_h & \xrightarrow{\text{inc}} & \mathbf{A}^1 \times U \end{array} \tag{16}$$

can be used to glue principal G-bundles.

Below, we use this to construct principal G_U-bundles over $\mathbf{A}^1 \times U$ out of the following initial data: a principal G_U-bundle over $\mathcal{X}'_g$, the trivial principal G_U-bundle over $(\mathbf{A}^1 \times U)_h$, and a principal G_U-bundle isomorphism of their pull-backs to $\mathcal{X}'_{gh}$.

Consider $(q'_X)^*(P_X)$ as a principal $(q'_U)^*(G_U) = \theta^*(G_{\text{const}})$-bundle via the isomorphism Φ. Recall that P_X is trivial as a principal G_X-bundle over X_{f}. Therefore, $(q'_X)^*(P_X)$ is trivial as a principal $\theta^*(G_{\mathcal{X}})$-bundle over $\mathcal{X}'_{f'}$. So, $(q'_X)^*(P_X)$ is trivial over $\mathcal{X}'_{f'}$, when regarded as a principal $(q'_U)^*(G_U) = \theta^*(G_{\text{const}})$-bundle via the isomorphism Φ.

Thus, regarded as a principal G_U-bundle, the bundle $(q'_X)^*(P_X)$ over $\mathcal{X}'$ becomes trivial over $\mathcal{X}'_{f'}$, and a fortiori over $(\mathcal{X}')_{gh}$. Indeed, $(\mathcal{X}')_{gh} \subseteq (\mathcal{X}')_{gf'}$ by the item (d) of Theorem 3.4. Take the trivial G_U-bundle over $(\mathbf{A}^1 \times U)_h$ and an isomorphism

$$\psi : G_U \times_U \mathcal{X}'_{gh} \to (q'_X)^*(P_X)|_{\mathcal{X}'_{gh}} \tag{17}$$

of the principal G_U-bundles. By item (2) of Theorem 3.4 the triple

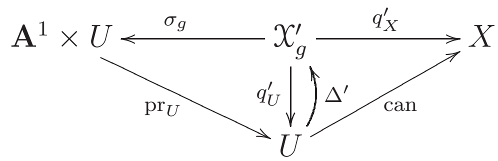
$$(\mathcal{O}[t], \sigma^*_g : \mathcal{O}[t] \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})_g, h)$$

satisfies the hypotheses of [C-T/O, Prop.2.6.(iv)]. **The latter statement implies that one can find** *a principal G_U-bundle $\mathcal{G}_t$ over* $\mathbf{A}^1 \times U$ such that

(1) $\mathcal{G}_t|_{[(\mathbf{A}^1 \times U)_h]} = G_U \times_U [(\mathbf{A}^1 \times U)_h]$,

(2) there is an isomorphism $\varphi : \sigma^*_g(\mathcal{G}_t) \to (q'_X)^*(P_X)|_{\mathcal{X}'_g}$ of the principal G_U-bundles, where $(q'_X)^*(P_X)$ is regarded as a principal G_U-bundle via the $\mathcal{X}'$-group scheme isomorphism Φ from (13).

Finally, form the following diagram



(18)

This diagram is well-defined, since by Item (c) of Theorem 3.4 the image of the morphism Δ' lands in $\mathcal{X}'_{g'}$.

Theorem 7.1. *The principal G_U-bundle $\mathcal{G}_t$ over $\mathbf{A}^1 \times U$, the monic polynomial $h \in \mathcal{O}[t]$, the diagram* (18)*, and the isomorphism Φ from* (13) *constructed above, satisfy the following conditions* (1*)–(6*).

(1)* $q'_U = \mathrm{pr}_U \circ \sigma_g$,

(2)* σ_g *is étale,*

(3)* $q'_U \circ \Delta' = \mathrm{id}_U$,

(4)* $q'_X \circ \Delta' = \mathrm{can}$,

(5) the restriction of $\mathcal{G}_t$ to $(\mathbf{A}^1 \times U)_h$ is a trivial G_U-bundle,*

(6) $\sigma_g^*(\mathcal{G}_t)$ and $(q'_X)^*(P_X)$ are isomorphic as G_U-bundles over $\mathcal{X}'_g$. Here $(q'_X)^*(P_X)$ is regarded as a principal G_U-bundle via the group scheme isomorphism Φ from* (13).

Proof. By the choice of σ it is an U-scheme morphism, which proves (1*). By the choice of $\mathcal{X}'_g \hookrightarrow \mathcal{X}'$ in Theorem 3.4, the morphism σ is étale on this sub-scheme, hence one gets (2*). Property (3*) holds for Δ' since $(q'_X : \mathcal{X}' \to U, f', \Delta')$ is a nice triple and, in particular, Δ' is a section of q'_U. Property (4*) can be established as follows:

$$q'_X \circ \Delta' = (q_X \circ \theta) \circ \Delta' = q_X \circ \Delta = \mathrm{can}.$$

The first equality here holds by the definition of q'_X, the second one holds since θ is a morphism of nice triples; the third one follows from equality (10). Property (5*) is just Property (1) in the above construction of $\mathcal{G}_t$. Property (6*) is precisely Property (2) in the construction of $\mathcal{G}_t$. ☐

The composition

$$s := \sigma_g \circ \Delta' : U \to \mathbf{A}^1 \times U$$

is a section of the projection pr_U by the properties (1*) and (3*). Recall that G_U over U is the original group scheme G introduced in the very beginning of this Section. Since U is semi-local, we may assume that *s is the zero section of the projection $\mathbf{A}^1_U \to U$.*

Corollary 7.2 (=Theorem 1.1). *The principal G_U-bundle $\mathcal{G}_t$ over $\mathbf{A}^1_U$ and the monic polynomial $h \in \mathcal{O}[t]$ are subject to the following conditions*

(i) the restriction of $\mathcal{G}_t$ to $(\mathbf{A}^1 \times U)_h$ is a trivial principal G_U-bundle,

(ii) the restriction of $\mathcal{G}_t$ to $\{0\} \times U$ is the original G_U-bundle P_U.

Proof. The property (i) is just the property (5*) above. Now by (6*) the G_U-bundles

$$\mathcal{G}_t|_{\{0\}\times U} = s^*(\mathcal{G}_t) = (\Delta')^*(\sigma_g^*(\mathcal{G}_t)) \text{ and } (\Delta')^*(q'_X)^*(P_X) = \operatorname{can}^*(P_X)$$

are isomorphic, since $\Delta'^*(\Phi) = \mathrm{id}_{G_U}$. It remains to recall that the principal G_U-bundle $\operatorname{can}^*(P_X)$ is the original G_U-bundle P_U by the choice of P_X. Whence the Corollary.

□

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