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MAXIMAL QUANTUM MECHANICAL SYMMETRY: PROJECTIVE REPRESENTATIONS OF THE INHOMOGENOUS SYMPLECTIC GROUP

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Abstract. A symmetry in quantum mechanics is described by the projective representations of a Lie symmetry group that transforms between physical quantum states such that the square of the modulus of the states is invariant. The Heisenberg commutation relations, that are fundamental to quantum mechanics, must be valid in all of these physical states. This paper shows that the maximal quantum symmetry group, whose projective representations preserve the Heisenberg commutation relations in this manner, is the inhomogeneous symplectic group. The projective representations are equivalent to the unitary representations of the central extension of the inhomogeneous symplectic group. This centrally extended group is the semidirect product of the cover of the symplectic group and the Weyl-Heisenberg group. Its unitary irreducible representations are computed explicitly using the Mackey representation theorems for semidirect product groups.

1. Introduction

The Heisenberg commutation relations are

$$\left[\widehat{P}_i, \widehat{Q}_j\right] = i\hbar\delta_{i,j}\mathbf{1}, \tag{1}$$

where $i, j, ... = 1, ..., n$. The hermitian operators $\widehat{P}_i$ and $\widehat{Q}_j$ represent quantum mechanical momentum and position observables acting on states $|\psi\rangle$ that are elements of a Hilbert space $\mathbf{H}$ for which $\mathbf{1}$ is the unit operator. (We will use natural units in which $\hbar = 1$ throughout the paper.) These relations are fundamental to quantum mechanics in its original formulation.

Weyl [1] established that these relations are the Hermitian representation of the algebra of a Lie group $\mathcal{H}(n)$ that we now call the Weyl-Heisenberg group. The Weyl-Heisenberg Lie group is a semidirect product [2] of two abelian groups[1]

$$\mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1), \tag{2}$$

where $\mathcal{A}(m)$ is the abelian Lie group isomorphic to the reals under addition, $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$. Therefore, it has an underlying manifold diffeomorphic to $\mathbb{R}^{2n+1}$ and is

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[1]In our notation for a semidirect product $\mathcal{G} \simeq \mathcal{K}\otimes_s\mathcal{N}$, $\mathcal{N}$ is the normal subgroup (see Definition 1 in Appendix A). Also, $\mathcal{A} \simeq \mathcal{B}$ is the notation for a group isomorphism.

simply connected. In a global coordinate system $p, q \in \mathbb{R}^n$, $\iota \in \mathbb{R}$, the group product and inverse of the Weyl-Heisenberg group may be written

$$\Upsilon(p', q', \iota')\Upsilon(p, q, \iota) = \Upsilon(p' + p, q' + q, \iota + \iota' + \frac{1}{2}\left(p' \cdot q - q' \cdot p\right)), \tag{3}$$

$$\Upsilon(p, q, \iota)^{-1} = \Upsilon(-p, -q, -\iota). \tag{4}$$

The identity element is $e = \Upsilon(0, 0, 0)$. Its Lie algebra is given by

$$[P_i, Q_i] = \delta_{i,j} I, \quad [P_i, I] = 0, [Q_i, I] = 0. \tag{5}$$

The faithful unitary irreducible representations ξ of the Weyl-Heisenberg group may be written as

$$\psi'(x) = \left(\xi(\Upsilon(p, q, \iota))\psi\right)(x) = e^{i\lambda(\iota + x \cdot p - \frac{1}{2} p \cdot q)}\psi(x - q) \tag{6}$$

where $p, q, x \in \mathbb{R}^n$, $\iota \in \mathbb{R}$. $\lambda \in \mathbb{R}\backslash\{0\}$ label the irreducible representations and $\psi(x) = \langle x|\psi\rangle \in \mathbf{H}^\xi \simeq \boldsymbol{L}^2(\mathbb{R}^n, \mathbb{C})$. We label the Hilbert space with the unitary representation ξ as this Hilbert space, on which the unitary representation ξ acts, is determined by the unitary irreducible representation and is not given *a priori*.

The Stone-von Neumann theorem [3], [4] establishes that (6) defines the complete set of faithful irreducible representations of the Weyl-Heisenberg group. This theorem is not constructive; it does not give a prescription to obtain these representations but only establishes that they are a complete set of faithful irreducible representations. However, as the Weyl-Heisenberg group has the form of the semidirect product given in (2), the unitary irreducible representations (6) can also be directly calculated using the Mackey theorems as these theorems are constructive. This is reviewed in Section 3.1.

The position and momentum operators in (1) are given by the faithful[2] hermitian representation ξ' of the Weyl-Heisenberg algebra. (The prime designates the lift of the unitary representation ξ of the group to the algebra, $\xi' = T_e\xi$.)

$$\widehat{P}_i = \xi'(P_i), \widehat{P}_i = \xi'(P_i), \widehat{I} = \xi'(I). \tag{7}$$

These operators also act on the Hilbert space $\mathbf{H}^\xi \simeq \boldsymbol{L}^2(\mathbb{R}^n, \mathbb{C})$. As the representation is a homomorphism, its lift preserves the Lie bracket,

$$\upsilon'([P_i, Q_i]) = \left[\upsilon'(P_i), \upsilon'(Q_i)\right]) = \quad i \quad \delta_{i,j}\upsilon'(I) = i \quad \lambda\delta_{i,j}\mathbf{1}. \tag{8}$$

The i appears simply because we are using hermitian rather than anti-hermitian operators[3]. Schur's lemma states that the representation of the central generators are a multiple of the identity for irreducible representations and so $\widehat{I} = \lambda\mathbf{1}$ where $\lambda \in \mathbb{R}\backslash\{0\}$. With $\lambda = 1$, these are the Heisenberg commutation relations given in (1).

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[2]There are also degenerate representations corresponding to the homomorphism $\pi : \mathcal{H}(n) \to \mathcal{A}(2n)$ for which $\lambda = 0$ (See Appendix A, Theorem 4). These representations of the abelian group are not discussed further here.

[3]In some neighborhood, the group element g is given in terms of an element X of the algebra by $g = e^X$. Then for a unitary representation υ of the group, the representation $\upsilon'$of the algebra is Hermitian (rather than non-Hermitian only if we insert an i, $\upsilon(g) = e^{i\upsilon'(X)}$. This follows as $\upsilon(g)^{-1} = \upsilon(g)^\dagger$ implies $-i\upsilon'(X) = (i\upsilon'(X))^\dagger$ and hence $\upsilon'(X) = \upsilon'(X)^\dagger$.

1.1. **Symmetry of Physical States.** A basic assumption of quantum mechanics is that the Heisenberg commutation relations (1) are valid when acting on any physical state. Physically observable probabilities are given by the square of the modulus of the states. Therefore, physical states in quantum mechanics are rays Ψ that are equivalence classes of states $|\psi\rangle$ in the Hilbert space that are equal up to a phase [5], [6],

$$\Psi = [|\psi\rangle], \left|\widetilde{\psi}\right\rangle \simeq |\psi\rangle \qquad \left|\widetilde{\psi}\right\rangle = e^{i\theta}|\psi\rangle. \tag{9}$$

The square of the modulus is the same for any representative state in the ray,

$$P(\alpha \to \beta) = |(\Psi_\beta, \Psi_\alpha)|^2 = |\left\langle \widetilde{\psi}_\beta | \widetilde{\psi}_\alpha \right\rangle|^2 = |\langle \psi_\beta | \psi_\alpha \rangle|^2. \tag{10}$$

Symmetry transformations between physical states (i.e. rays Ψ) are given by operators U that leave invariant the square of modulus,

$$|(U\Psi_\beta, U\Psi_\alpha)|^2 = |(\Psi_\beta, \Psi_\alpha)|^2. \tag{11}$$

These transformations U are the representation of a group in the space $U(\mathrm{H})$ of linear or anti-linear operators on $\mathbf{H}$

$$\varrho : \mathcal{G} \to U(\mathrm{H}) : g \mapsto U = \varrho(g). \tag{12}$$

This operator also acts on any representative in the equivalence class of states that defines the ray,

$$\Psi' = U\Psi, \quad \left|\psi'\right\rangle = U|\psi\rangle. \tag{13}$$

Theorem 2 in Appendix A states that any representation of a Lie group [7], [8] that leaves invariant the square of the modulus is always equivalent to a linear unitary or anti-linear, anti-unitary operator mapping the Hilbert space $\mathbf{H}$ into itself. Furthermore, if the Lie group is connected[4], it is always equivalent to a linear unitary operator.

The representations ϱ are referred to as projective representations. If $\mathcal{G}$ is a connected Lie group, the fundamental Theorem 3 states that these projective representations are equivalent to the ordinary unitary representations υ of the central extension $\breve{\mathcal{G}}$ of $\mathcal{G}$.

We seek the maximal group with projective representations that preserve the Heisenberg commutation relations. As the Heisenberg commutation relations are a faithful unitary representation of the Lie algebra of the Weyl-Heisenberg group, the group we seek must be a subgroup of the automorphism group of the Weyl-Heisenberg algebra. As the Weyl-Heisenberg group is simply connected, the automorphism group of the algebra is equivalent to the automorphism group $\mathcal{A}ut_{\mathcal{H}(n)}$ of the Weyl-Heisenberg group itself.

Under the action of elements $g \in \mathcal{A}ut_{\mathcal{H}(n)}$, the elements of the algebra transform to a new basis

$$P'_i = gP_ig^{-1}, Q'_i = gQ_ig^{-1}, I' = gIg^{-1}. \tag{14}$$

such that the form of the Lie algebra is preserved,

$$\left[P'_i, Q'_i\right] = \delta_{i,j} I'. \tag{15}$$

[4]In this paper, a connected group is abbreviation for a group for which every element is connected by a continuous path to the identity element.

The element I' is central and as I spans the center of the algebra, we must have $I' = dI$ with $d \in \mathbb{R}\backslash\{0\}$. Furthermore, the elements of the automorphism group that preserves the center of the algebra,

$$I' = gIg^{-1} = I, \tag{16}$$

defines a subgroup.

The group inner automorphisms of a group is isomorphic to the group itself. The full group of automorphisms always contains the group of inner automorphisms as a normal subgroup. For the case of the Weyl-Heisenberg group, this means that the Weyl-Heisenberg group is a normal subgroup of its automorphism group, $\mathcal{H}(n) \subset \mathcal{A}ut_{\mathcal{H}(n)}$.

The projective representations of $\mathcal{A}ut_{\mathcal{H}(n)}$ are equivalent to the unitary representations υ of its central extension $\widetilde{\mathcal{A}ut}_{\mathcal{H}(n)}$ acting on a Hilbert space $\mathbf{H}^{\upsilon}$. If we restrict υ to the normal subgroup $\mathcal{H}(n)$ of inner automorphisms, these are the unitary representations of the Weyl-Heisenberg group, $\upsilon|_{\mathcal{H}(n)} = \xi$. Therefore, the Hilbert space $\mathbf{H}^{\xi}$ is an invariant subspace of $\mathbf{H}^{\upsilon}$. The generators of the Weyl-Heisenberg group transform under the action of elements $U = \upsilon(g)$, $g \in \mathcal{A}ut_{\mathcal{H}(n)}$ as

$$\begin{aligned}
\widehat{P}'_i &= \upsilon'(P'_i) = \upsilon'(gP_ig^{-1}) = \upsilon(g)\xi'(P_i)\upsilon(g)^{-1} = U\widehat{P}_iU^{-1}, \\
\widehat{Q}'_i &= \upsilon'(Q'_i) = \upsilon'(gQ_ig^{-1}) = \upsilon(g)\xi'(Q_i)\upsilon(g)^{-1} = U\widehat{Q}_iU^{-1}, \\
\widehat{I}' &= \upsilon'(I') = \upsilon'(gIg^{-1}) = \upsilon(g)\xi'(I)\upsilon(g)^{-1} = U\widehat{I}U^{-1},
\end{aligned} \tag{17}$$

For the faithful representation υ, the commutation relations for the transformed generators are, using (15),

$$\left[\widehat{P}'_i, \widehat{Q}'_i\right] = i \;\; \delta_{i,j}\widehat{I}' = i\lambda'\delta_{i,j}\mathbf{1} \tag{18}$$

where $\widehat{I}' = d\widehat{I}$ and so $\lambda' = d\lambda$. Now, as we have noted, the λ label the faithful irreducible representations of the Weyl-Heisenberg group. Furthermore, the physical cases corresponds to the choice $\lambda = 1$. This must also be true for the transformed operators and therefore $\lambda' = 1$ and so $\widehat{I}' = \widehat{I}$ with $d = 1$. That is, the projective representation of the symmetry group of the Heisenberg commutation relations leaves the representation of the center of the Weyl-Heisenberg group invariant. As the representation is faithful, the symmetry group also must leave the central generator of the Weyl-Heisenberg algebra invariant, $I' = I$.

Therefore, the maximal group of symmetries of the Heisenberg commutation relations are the projective representation of the subgroup of the automorphism group of the Weyl-Heisenberg group that leaves the central generator I invariant.

The problem that this paper addresses is to determine the explicitly this symmetry group and its projective representations. We will show that the automorphism group of the Weyl-Heisenberg group is [2]

$$\mathcal{A}ut_{\mathcal{H}(n)} \simeq \mathcal{D} \otimes_s \mathcal{H}\overline{\mathcal{S}p}(2n), \tag{19}$$

where

$$\mathcal{H}\overline{\mathcal{S}p}(2n) \simeq \overline{\mathcal{S}p}(2n) \otimes_s \mathcal{H}(n), \qquad \mathcal{D} \simeq (\mathbb{R}\backslash\{0\}, \times), \tag{20}$$

where $\mathcal{D}$ is the reals excluding $\{0\}$ viewed as a group under multiplication, $\mathcal{D} \simeq (\mathbb{R}\backslash\{0\}, \times)$. We will show that the subgroup of the automorphism group that leaves

the central generator I invariant is

$$\mathcal{H}\overline{\mathcal{S}p}(2n). \tag{21}$$

The group $\mathcal{H}\overline{\mathcal{S}p}(2n)$ is connected and is the central extension of the Inhomogeneous group, $\mathcal{H}\overline{\mathcal{S}p}(2n) \simeq \mathcal{I}\breve{\mathcal{S}p}(2n)$ that is defined by the short exact sequence

$$e \to \mathbb{Z} \otimes \mathcal{A}(1) \to \mathcal{H}\overline{\mathcal{S}p}(2n) \to \mathcal{I}\mathcal{S}p(2n) \to e. \tag{22}$$

$\mathbb{Z}$ is the center of $\overline{\mathcal{S}p}(2n)$ and $\mathcal{A}(1)$ is the center of $\mathcal{H}(n)$. $\mathcal{I}\mathcal{S}p(2n)$ is the inhomogeneous symplectic group familiar from classical Hamiltonian mechanics,

$$\mathcal{I}\mathcal{S}p(2n) \equiv \mathcal{S}p(2n) \otimes_s \mathcal{A}(2n). \tag{23}$$

To establish the above results we start by reviewing the Weyl-Heisenberg group. We then derive its automorphism group and the subgroup that leaves the center of the Weyl-Heisenberg group invariant. This is the maximal symmetry group. The projective representations of this symmetry group are equivalent to the unitary representations of its central extension. We use the Mackey theorems to compute the unitary irreducible representations of the symmetry group from first principles. (As the symmetry group contains the Weyl-Heisenberg group as normal subgroup, this first requires the computation of the faithful unitary irreducible representations of the Weyl-Heisenberg group itself using the Mackey theorems.) We will enumerate and comment on the degenerate cases.

2. The symmetry group

In this section, we review basic properties of the Weyl-Heisenberg group and determine its automorphism group. We then determine the subgroup leaving the center of the Weyl-Heisenberg group invariant and study certain of its properties.

2.1. **The Weyl-Heisenberg group.** The Weyl-Heisenberg Lie group is defined to be the semi-direct product of two abelian groups of the form given in (2). We first verify that these group product (3) and inverse (4) relations result in the semidirect product of this form. First, the group product and inverse (3-4) enable us to identify the abelian subgroups

$$\Upsilon(0,q,\iota) \in \mathcal{A}(n+1), \Upsilon(p,0,0) \in \mathcal{A}(n). \tag{24}$$

where again $p, q \in \mathbb{R}^n$ and $\iota \in \mathbb{R}$. These subgroups satisfy the group product and inverse relations

$$\Upsilon(0,q',\iota')\Upsilon(0,q,\iota) = \Upsilon(0,q'+q,\iota+\iota'), \quad \Upsilon(0,q,\iota)^{-1} = \Upsilon(0,-q,-\iota) \tag{25}$$

$$\Upsilon(p',0,0)\Upsilon(p,0,0) = \Upsilon(p'+p,0,0), \Upsilon(p,0,0)^{-1} = \Upsilon(-p,0,0). \tag{26}$$

Additional abelian subgroups are likewise given by

$$\Upsilon(p,0,\iota) \in \mathcal{A}(n+1), \quad \Upsilon(0,q,0) \in \mathcal{A}(n+1) \tag{27}$$

We calculate the inner automorphisms of the group using (3-4) to be[5]

$$\begin{aligned} \varsigma_{\Upsilon(p',q',\iota')}\Upsilon(p,q,\iota) &= \Upsilon(p',q',\iota')\Upsilon(p,q,\iota)\Upsilon(p',q',\iota')^{-1} \\ &= \Upsilon(p,q,\iota+p'q-q'\cdot p). \end{aligned} \tag{28}$$

[5]We always use ς to define the similarity map $\varsigma_g h \equiv ghg^{-1}$ in what follows.

In particular, note that for each of the choices of the subgroups

$$\varsigma_{\Upsilon(p',q',\iota')}\Upsilon(0,q,\iota) = \Upsilon(0,q,\iota + p'q), \tag{29}$$

$$\varsigma_{\Upsilon(p',q',\iota')}\Upsilon(p,0,\iota) = \Upsilon(p,0,\iota - q'\cdot p). \tag{30}$$

This means that both of the $\mathcal{A}(n+1)$ subgroups given in (24), (27) are normal subgroups. Another special case of (3) is

$$\varsigma_{\Upsilon(0,0,\iota')}\Upsilon(p,q,\iota) = \Upsilon(p,q,\iota). \tag{31}$$

and therefore the elements $\Upsilon(0,0,\iota')$ commute with all elements of the group. Furthermore, these are the only elements that commute with all other elements of the group. Therefore the $\mathcal{A}(1)$ group that is defined by the elements $\Upsilon(0,0,\iota)$ is the center of the group, $\mathcal{Z} \simeq \mathcal{A}(1)$.

The final step to verify that the group relations defined by (3-4) results in the Weyl-Heisenberg group having the structure of a semidirect product given in (2). We have already established that there are two choices for the $\mathcal{A}(n)$ subgroup and $\mathcal{A}(n+1)$ normal subgroup. It is clear in both cases that

$$\mathcal{A}(n) \cap \mathcal{A}(n+1) = \boldsymbol{e}, \tag{32}$$

as the identity $\Upsilon(0,0,0)$ is the only element in both groups for both cases. It remains to show that $\mathcal{A}(n+1)\mathcal{A}(n) \simeq \mathcal{H}(n)$. Using the group product (3), for each of the cases (24), (27), this is

$$\Upsilon(0,q,\iota)\Upsilon(p,0,0) = \Upsilon(p,q,\iota - \frac{1}{2}q\cdot p), \tag{33}$$

$$\Upsilon(p,0,\iota)\Upsilon(0,q,0) = \Upsilon(p,q,\iota + \frac{1}{2}p\cdot q). \tag{34}$$

The map

$$\varphi^{\pm} : \mathcal{H}(n) \to \mathcal{H}(n) : \Upsilon(p,q,\iota) \mapsto \Upsilon^{\pm}(p,q,\iota^{\pm}) = \Upsilon(p,q,\iota \mp \frac{1}{2}p\cdot q) \tag{35}$$

is a homomorphism that is onto and the kernel is trivial. Therefore, the map $\varphi^{\pm}$ is an isomorphism and the Weyl-Heisenberg group has the semidirect product structure given in (2) for either of the choices of abelian subgroup given by (24), (27).

The Weyl-Heisenberg Lie group is a matrix group and may be realized by the $2n+2$ dimensional square matrices

$$\Upsilon(p,q,\iota) = \begin{pmatrix} 1_n & 0 & 0 & p \\ 0 & 1_n & 1 & q \\ q^{\mathrm{t}} & -p^{\mathrm{t}} & 1 & 2\iota \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

1_m denotes the unit matrix in m dimensions and the t superscript denotes the transpose. The group multiplication and inverse (3-4) are realized by matrix multiplication and inverse.

The Lie algebra of the Weyl-Heisenberg group may be computed from this matrix realization. The coordinates are nonsingular at the origin and therefore, choosing the unpolarized form, the generators are given by

$$Q_i = \frac{\partial}{\partial p^i}\Upsilon(p,q,\iota)|_e, P_i = \frac{\partial}{\partial q^i}\Upsilon(p,q,\iota)|_e, I = \frac{\partial}{\partial \iota}\Upsilon(p,q,\iota)|_e. \tag{37}$$

A general element of the algebra is then

$$W = p^i Q_i + q^i P_i + \iota I. \tag{38}$$

The nonzero commutation relations are, as expected,

$$[P_i, Q_i] = \delta_{i,j} I \tag{39}$$

where I is a central generator.

It is convenient to also introduce the notation that combines the $p, q \in \mathbb{R}^n$ into a single $2n$ tuple $z = (p, q)$, $z \in \mathbb{R}^{2n}$. Then the group product and inverse are

$$\Upsilon(z', \iota')\Upsilon(z, \iota) = \Upsilon(z' + z, \iota + \iota - \frac{1}{2} z'^{\mathrm{t}} \zeta z), \quad \Upsilon(z, \iota)^{-1} = \Upsilon(-z, -\iota) \tag{40}$$

and the unpolarized matrix realization is

$$\Upsilon(z, \iota) = \begin{pmatrix} 1_{2n} & 0 & z \\ -\ z^{\mathrm{t}}\zeta & 1 & 2\iota \\ 0 & 0 & 1 \end{pmatrix}, \quad \zeta = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \tag{41}$$

The Lie algebra has general element

$$W(z, \iota) = z^\alpha Z_\alpha + \iota I, \tag{42}$$

$\alpha, \beta, ... = 1, ...2n$ where the matrix form of the algebra is

$$W(z, \iota) = \begin{pmatrix} 0 & 0 & z \\ -\ z^{\mathrm{t}}\zeta & 0 & 2\iota \\ 0 & 0 & 0 \end{pmatrix}, \tag{43}$$

The generators satisfy the nonzero commutation relations

$$[Z_\alpha, Z_\beta] = \zeta_{\alpha,\beta} I. \tag{44}$$

2.2. **The automorphism group of the Weyl-Heisenberg group.** The automorphism group of a group $\mathcal{G}$ is the maximal group for which $\mathcal{G}$ is a normal subgroup. We have established in the previous section that the Weyl-Heisenberg group is a simply connected matrix group and this enables us to prove the following theorem.

Theorem 1. *The automorphism group of the Weyl-Heisenberg group $\mathcal{H}(n)$ is*

$$\mathcal{A}ut_{\mathcal{H}(n)} \simeq \mathcal{D} \otimes_s \overline{\mathcal{S}p}(2n) \otimes_s \mathcal{H}(n). \tag{45}$$

$\mathcal{H}(n)$ is the Weyl-Heisenberg group, $\overline{\mathcal{S}p}(2n)$ is the cover of the real symplectic group that leaves invariant a real skew symmetric form and $\mathcal{D}$ is the reals excluding zero viewed as a group under multiplication $\mathcal{D} \simeq (\mathbb{R}\backslash\{0\}, \times)$.

As the Weyl-Heisenberg group is simply connected, Theorem 7 states that the automorphism group of its algebra and group are equivalent. We can therefore establish the result by determining the maximal group for which its elements Ω satisfy

$$\varsigma_\Omega W = \Omega W \Omega^{-1} = W'. \tag{46}$$

W, W' are general elements of the algebra of the Weyl-Heisenberg group (42). The most general transformation between a primed and unprimed basis is

$$Z'_\alpha = a_\alpha^\beta Z_\beta + x_\alpha I, \quad I' = c^\alpha Z_\alpha + dI. \tag{47}$$

The commutator $[Z'_\alpha, I'] = 0$ requires $c^\alpha = 0$ so that $I' = dI$ with $d \in \mathbb{R}\backslash\{0\}$. Next,

$$\begin{aligned} \zeta_{\alpha,\beta} I' = [Z'_\alpha, Z'_\beta] &= \left[a^\kappa_\alpha Z_\kappa + x_\alpha, a^\kappa_\beta Z_\kappa + x_\beta\right] \\ &= \tfrac{1}{d} a^\delta_\alpha a^\gamma_\beta \zeta_{\delta,\gamma} I'. \end{aligned} \tag{48}$$

This has the solution $a^\beta_\alpha = \delta\Sigma^\beta_\alpha$ and $d = \delta^2$. Therefore, for $W(z,\iota) = z^\kappa Z_\kappa + \iota I$ we have

$$W(z',\iota') = \varsigma_\Omega W(z,\iota) = z'^\kappa Z_\kappa + \iota' I \tag{49}$$

with[2]

$$z' = \delta\Sigma z, \iota' = \iota\delta^2 + z \cdot x. \tag{50}$$

To determine the group with elements Ω that satisfies (46, 50), we can use the matrix realization of the algebra given in (43). As Ω is nonsingular, (46) is equivalent to

$$\Omega W(z,\iota) = W(z',\iota')\Omega. \tag{51}$$

where Ω is a $2n+2$ dimensional square matrix. We can write Ω in terms of the submatrices

$$\Omega = \begin{pmatrix} a & c & z \\ f & d & j \\ g & h & \epsilon \end{pmatrix} \tag{52}$$

where $j, d, r, h, e \in \mathbb{R}$, $c, w, f, g \in \mathbb{R}^{2n}$ and a is a $2n$ dimensional square submatrices and then solve (51) to obtain

$$\Omega(\delta,\Sigma,z,\iota) = \begin{pmatrix} \delta\Sigma & 0 & z \\ -\delta z^{\mathrm{t}}\zeta\Sigma & \delta^2 & 2\iota \\ 0 & 0 & 1 \end{pmatrix} \tag{53}$$

where $z \in \mathbb{R}^{2n}$, $\delta \in \mathcal{D} \equiv \mathbb{R}\backslash\{0\}$, $\iota \in \mathbb{R}$ and $\Sigma^{\mathrm{t}}\zeta\Sigma = \zeta$ and so $\Sigma \in \mathcal{S}p(2n)$.

Direct matrix multiplication shows that the elements $\Omega(\delta,\Sigma,w,r)$ define a group that we call $\mathrm{aut}_{\mathcal{H}(n)}$ with product and inverse

$$\begin{aligned} \Omega(\delta'',\Sigma'',z'',\iota'') &= \Omega(\delta',\Sigma',z',\iota')\Omega(\delta,\Sigma,z,\iota) \\ &= \Omega(\delta'\delta, \Sigma'\Sigma, z' + \delta'\Sigma' z, \iota' + \delta'^2\iota - \tfrac{1}{2}\delta' z'^t\zeta\Sigma' z), \end{aligned} \tag{54}$$

$$\Omega(\delta,\Sigma,z,\iota)^{-1} = \Omega(\delta^{-1},\Sigma^{-1},-\delta^{-1}\Sigma^{-1}z,-\delta^{-2}\iota), \tag{55}$$

where the identity element is $e = \{1, 1_{2n}, 0, 0\}$. From these relations, we can explicitly compute that automorphisms of the algebra given in (46)

$$W(z',\iota') = \varsigma_{\Omega(\delta,\Sigma,z'',\iota'')} W(z,\iota) \tag{56}$$

where

$$z' = \delta\Sigma z, \iota' = \iota\delta^2 - \delta z''^{\mathrm{t}}\zeta\Sigma \cdot z. \tag{57}$$

Comparing with the general expression given in (50), they are equivalent where we identify $x = \delta z''^{\mathrm{t}}\zeta\Sigma$. As $\det\Sigma \neq 0 \text{and} \delta \neq 0$, there is a bijection between values of x and z''.

Using these relations, the next step is to show that the group $\mathrm{aut}_{\mathcal{H}(n)}$ has the form of a semidirect product[6]

$$\mathrm{aut}_{\mathcal{H}(n)} \simeq (\mathcal{D} \otimes \mathcal{S}p(2n)) \otimes_s \mathcal{H}(n). \tag{58}$$

First, using the group product and inverse (54-55), we can establish that $\mathcal{D}$, $\mathcal{S}p(2n)$ and $\mathcal{H}(n)$ are subgroups of $\mathrm{aut}_{\mathcal{H}(n)}$ with elements

$$\begin{array}{l} \Omega(\delta, 1_{2n}, 0, 0) \in \mathcal{D}, \\ \Omega(1, \Sigma, 0, 0) \simeq \Sigma \in \mathcal{S}p(2n) \\ \Omega(1, 1_{2n}, z, \iota) = \Upsilon(z, \iota) \in \mathcal{H}(n) \end{array} \tag{59}$$

The direct product $\mathcal{D} \otimes \mathcal{S}p(2n)$ is immediately established from the special case of the group multiplication (54)

$$\begin{aligned} \Omega(\delta, \Sigma, 0, 0) &= \Omega(\delta, 1_{2n}, 0, 0)\Omega(1, \Sigma, 0, 0) \\ &= \Omega(1, \Sigma, 0, 0)\Omega(\delta, 1_{2n}, 0, 0), \end{aligned} \tag{60}$$

The semidirect product in (58) is established by first noting that

$$(\mathcal{D} \otimes \mathcal{S}p(2n)) \cap \mathcal{H}(n) \simeq \{\Omega(\delta, \Sigma, 0, 0)\} \cap \{\Omega(1, 1_{2n}, z, \iota)\} \simeq e, \tag{61}$$

Then, using the group product (54),

$$\Omega(1, 1_{2n}, z, \iota)\Omega(\delta, \Sigma, 0, 0) = \Omega(\delta, \Sigma, z, \iota). \tag{62}$$

Direct computation using (54-55) shows that the Weyl-Heisenberg subgroup $\mathcal{H}(n)$ is a normal subgroup with the automorphisms given by

$$\varsigma_{\Omega(\delta', \Sigma', z', \iota')} \Upsilon(z, \iota) = \Upsilon(\delta' \Sigma' z, {\delta'}^2 \iota - \delta' {z'}^{\mathrm{t}} \zeta \Sigma' z). \tag{63}$$

This establishes that $\mathrm{aut}_{\mathcal{H}(n)}$ has the semidirect product form given in (58). The right associative property of the semidirect product allows this to be written as

$$\begin{aligned} \mathrm{aut}_{\mathcal{H}(n)} &\simeq (\mathcal{D} \otimes \mathcal{S}p(2n)) \otimes_s \mathcal{H}(n) \\ &\simeq \mathcal{D} \otimes_s \mathcal{H}\mathcal{S}p(2n) \end{aligned} \tag{64}$$

where $\mathcal{H}\mathcal{S}p(2n)$ is a semidirect product of the form

$$\mathcal{H}\mathcal{S}p(2n) \simeq \mathcal{S}p(2n) \otimes_s \mathcal{H}(n) \tag{65}$$

This the local characterization of the automorphism group. It remains to consider any global topological properties that could result in a larger group that behaves the same locally.

The group $\mathcal{D}$ may be written as the direct product $\mathcal{D} \simeq \mathbb{Z}_2 \otimes \mathcal{D}^+$ where $\mathcal{D}^+ \simeq (\mathbb{R}^+, \times)$ is the positive reals considered as a group under multiplication. $\mathbb{Z}_2$ is the discrete group with two elements $\{\pm 1\}$. $\mathcal{D}^+$ is simply connected but $\mathcal{D}$ has two components, $\mathcal{D}/\mathcal{D}^+ \simeq \mathbb{Z}_2$. Therefore, the connected component of the group is

$$\mathrm{aut}^c_{\mathcal{H}(n)} \simeq \mathcal{D}^+ \otimes_s \mathcal{S}p(2n) \otimes_s \mathcal{H}(n). \tag{66}$$

$\mathcal{H}(n)$ and $\mathcal{D}^+$ are simply connected and $\mathcal{S}p(2n)$ is connected with fundamental group $\mathbb{Z}$. Its simply connected universal cover is denoted $\overline{\mathcal{S}p}(2n)$ with

$$\pi : \overline{\mathcal{S}p}(2n) \to \mathcal{S}p(2n) : \overline{\Sigma} \to \Sigma = \pi(\overline{\Sigma}), \ker \pi \simeq \mathbb{Z}. \tag{67}$$

Therefore, by the universal covering theorem,

$$\mathcal{A}ut^c_{\mathcal{H}(n)} \simeq \overline{\mathrm{aut}}^c_{\mathcal{H}(n)} \simeq \mathcal{D}^+ \otimes_s \overline{\mathcal{S}p}(2n) \otimes_s \mathcal{H}(n), \tag{68}$$

[6]The definition of a semidirect product is reviewed in Definition 1 in Appendix A.

is well defined and unique with the following group product and inverse

$$\begin{aligned}\Omega(\delta'',\Sigma'',z'',\iota'') &= \Omega(\delta',\overline{\Sigma}',z',\iota')\Omega(\delta,\overline{\Sigma},z,\iota)\\ &= \Omega(\delta'\delta,\overline{\Sigma}'\overline{\Sigma},z'+\delta'\Sigma' z,\iota'+\delta'^2\iota-\tfrac{1}{2}\delta' z'^t\zeta\Sigma' z),\end{aligned} \tag{69}$$

$$\Omega(\delta,\Sigma,z,\iota)^{-1} = \Omega(\delta^{-1},\overline{\Sigma}^{-1},-\delta^{-1}\Sigma^{-1}z,-\delta^{-2}\iota). \tag{70}$$

where $z\in\mathbb{R}^{2n}$, $\delta\in\mathcal{D}^+$, $\iota\in\mathbb{R}$ and $\overline{\Sigma}\in\overline{\mathcal{S}p}(2n)$. Note that in these expressions $\Sigma=\pi(\overline{\Sigma})$. The expression for automorphisms of the Weyl-Heisenberg subgroup remains the same as given in (63).

The cover of a disconnected group may be defined to be the central extension of the group with a discrete central group. The problem is, that in general, this does not give a unique cover and so this must be checked on a case by case basis. This is discussed in Appendix C where we show that

$$\mathcal{A}ut_{\mathcal{H}(n)} \simeq \mathrm{aut}_{\mathcal{H}(n)} \simeq \mathcal{D}\otimes_s \overline{\mathcal{S}p}(2n)\otimes_s \mathcal{H}(n) \tag{71}$$

is unique and well defined. It has the group product and inverse given in (69-70) where now $\delta\in\mathcal{D}$. Again, the automorphisms of the Weyl-Heisenberg subgroup remains the same as given in (63).

The group $\mathcal{A}ut_{\mathcal{H}(n)}$ is the largest group that the topological properties admit that is homomorphic to $\mathrm{aut}_{\mathcal{H}(n)}$ and therefore we completed the proof of Theorem 1.

2.3. **Subgroup of automorphism group with invariant center.** The action of the automorphism group on the algebra is given in (57). Invariance of the central element requires $\delta=1$ which is the unit element for $\mathcal{D}^+$. Thus the maximal symmetry group that leaves the center of the Weyl-Heisenberg algebra invariant is $\mathcal{H}\overline{\mathcal{S}p}(2n)$.

As given in (22), the central extension of $\mathcal{HS}p(2n)$ is equivalent to the central extension of the inhomogeneous symplectic group familiar from classical mechanics.

$$\mathcal{H}\breve{\mathcal{S}p}(2n) \simeq \mathcal{H}\overline{\mathcal{S}p}(2n) \simeq \mathcal{I}\breve{\mathcal{S}p}(2n) \tag{72}$$

This is a very remarkable fact. The central extension of the $\mathcal{A}(2n)$ is generally $n(2n-1)$ dimensional. However, because it is a subgroup of $\mathcal{I}\breve{\mathcal{S}p}(2n)$, the Lie algebra relations with the symplectic group constrain the central extension of the abelian normal subgroup to be precisely the one dimensional extension that is the Weyl-Heisenberg group.

The group product and inverse are given by (69-70) with $\delta=1$.

2.3.1. *Symplectic group factorization.* The defining condition for the real symplectic group $\mathcal{S}p(2n)$ is

$$\Sigma^{\mathrm{t}}\zeta\Sigma=\zeta \tag{73}$$

where ζ is the symplectic matrix defined in (41). Matrix realizations of elements of the real symplectic group may be written as

$$\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{pmatrix} \tag{74}$$

where Σ_a , $a = 1, .., 4$ are $n \times n$ submatrices. The symplectic condition (73) immediately results in the relations

$$\begin{array}{l} \Sigma_1^{\rm t}\Sigma_4 - \Sigma_3^{\rm t}\Sigma_2 = 1_n, \\ \Sigma_1^{\rm t}\Sigma_3 = (\Sigma_1^{\rm t}\Sigma_3)^{\rm t}, \\ \Sigma_2^{\rm t}\Sigma_4 = (\Sigma_2^{\rm t}\Sigma_4)^{\rm t}. \end{array} \tag{75}$$

A matrix realization of a Lie group is a coordinate system. As $\mathrm{Det}(\Sigma) = 1$, it follows that the determinate of at least one of the Σ_a, $a = 1, ..., 4,$ must be nonzero. These correspond to different coordinate patches for the manifold underlying the symplectic group. Assume $\mathrm{Det}(\Sigma_1) \neq 0$. Then [9],

$$\Sigma(\alpha, \beta, \gamma) = \begin{pmatrix} 1_n & 0 \\ \gamma & 1_n \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^{\rm t} \end{pmatrix} \begin{pmatrix} 1_n & \beta \\ 0 & 1_n \end{pmatrix}, \tag{76}$$

where we define

$$\alpha = (\Sigma_1)^{-1}, \beta = (\Sigma_1)^{-1}\Sigma_2, \gamma = \Sigma_3(\Sigma_1)^{-1}. \tag{77}$$

It follows from (75) that $\beta = \beta^{\rm t}$ and $\gamma = \gamma^{\rm t}$. The matrix realizations of elements of the symplectic group factor as

$$\Sigma(\alpha, \beta, \gamma) = \Sigma^-(\gamma)\Sigma^\circ(\alpha)\Sigma^+(\beta) \tag{78}$$

where

$$\begin{array}{l} \Sigma^\circ(\alpha) \equiv \Sigma(\alpha, 1_n, 1_n) \in \mathcal{U}(n), \\ \Sigma^+(\beta) \equiv \Sigma(1_n, \beta, 1_n) \in \mathcal{A}(m), \\ \Sigma^-(\gamma) \equiv \Sigma(1_n, 1_n, \gamma) \in \mathcal{A}(m). \end{array} \tag{79}$$

and $m = \frac{n(n-1)}{2}$. Furthermore, note that

$$\zeta\Sigma^-(\gamma)\zeta^{-1} = \Sigma^+(-\gamma). \tag{80}$$

A similar argument applies if we instead assume $\mathrm{Det}(\Sigma_4) \neq 0$. Both of these coordinate patches contain the identity, 1_{2n} but neither contains the element ζ. These require us to consider the case with either Σ_2 or Σ_3 to be assumed to be nonsingular. In this case, define

$$\widetilde{\Sigma} = \Sigma\zeta^{-1} = \begin{pmatrix} \Sigma_2 & -\Sigma_1 \\ \Sigma_4 & -\Sigma_3 \end{pmatrix}. \tag{81}$$

The $\widetilde{\Sigma}$ also satisfy the symplectic condition as

$$\zeta = \Sigma^{\rm t}\zeta\Sigma = \zeta^{\rm t}\widetilde{\Sigma}^{\rm t}\zeta\widetilde{\Sigma}\zeta \Rightarrow \widetilde{\Sigma}^{\rm t}\zeta\widetilde{\Sigma} = \zeta. \tag{82}$$

This symplectic condition results in the identities

$$\begin{array}{l} \Sigma_4^{\rm t}\Sigma_1 - \Sigma_2^{\rm t}\Sigma_3 = 1_n, \\ \Sigma_2^{\rm t}\widetilde{\Sigma}_4 = (\Sigma_2^{\rm t}\Sigma_4)^{\rm t}, \\ \Sigma_1^{\rm t}\Sigma_3 = (\Sigma_1^{\rm t}\Sigma_3)^{\rm t}. \end{array} \tag{83}$$

We can now assume $\mathrm{Det}(\Sigma_2) \neq 0$ and the analysis proceeds as before with

$$\alpha = (\Sigma_2)^{-1}, \beta = -(\Sigma_2)^{-1}\Sigma_1, \gamma = \Sigma_4(\Sigma_2)^{-1}, \tag{84}$$

In this case the factorization must include the symplectic matrix from (82)

$$\Sigma(\alpha, \beta, \gamma) = \Sigma^-(\gamma)\Sigma^\circ(\alpha)\Sigma^+(\beta)\zeta. \tag{85}$$

Finally a similar argument applies for the coordinate patch $\mathrm{Det}(\Sigma_3) \neq 0$. Both of these coordinate patches contain the element ζ but do not contain the identity

The expressions (78) and (85) can be combined into a single expression

$$\Sigma^{\epsilon}(\alpha,\beta,\gamma) = \Sigma^{-}(\gamma)\Sigma^{\circ}(\alpha)\Sigma^{+}(\beta)\zeta^{\epsilon}. \tag{86}$$

where $\epsilon \in \{0,1\}$.

2.3.2. *Lie Algebra.* The Lie algebra of the symmetry group $\mathcal{H}\overline{\mathcal{S}p}(2n)$ is the same as the Lie algebra of $\mathcal{H}\mathcal{S}p(2n)$. It may be directly computed from its matrix realization. It is convenient to use a basis for the algebra of the symplectic group corresponding to the factorized form (78). Let the $A_{i,j}$ be the generators of the unitary subgroup with elements $\Sigma(\alpha) \in \mathcal{U}(n)$, and $B_{i,j}$ the generators of the abelian subgroup with elements $\Sigma(\beta) \in \mathcal{A}(m)$ and $C_{i,j}$ the generators of the abelian subgroup with elements $\Sigma(\gamma) \in \mathcal{A}(m)$. The abelian generators are symmetric, $B_{i,j} = B_{j,i}$ and $C_{i,j} = C_{j,i}$. A general element is written as

$$Z = \alpha^{i,j}A_{i,j} + \beta^{i,j}B_{i,j} + \gamma^{i,j}C_{i,j} + p^iQ_i + q^iP_i + \iota I. \tag{87}$$

Straightforward computation shows that these generators of $\mathcal{S}p(2n)$ satisfy the Lie algebra

$$\begin{aligned}
&[A_{i,j}, A_{k,l}] = \delta_{i,l}A_{j,k} - \delta_{j,k}A_{i,l},\\
&[A_{i,j}, B_{k,l}] = \delta_{j,k}B_{i,l} + \delta_{j,l}B_{i,k},\\
&[A_{i,j}, C_{k,l}] = -\delta_{i,k}C_{j,l} - \delta_{i,l}C_{k,j},\\
&[B_{i,j}, C_k] = \delta_{i,k}A_{j,l} + \delta_{i,l}A_{j,k} + \delta_{j,k}A_{i,l} + \delta_{j,l}A_{i,k}.
\end{aligned} \tag{88}$$

The nonzero commutators of the algebra of $\mathcal{H}\mathcal{S}p(2n)$ are the above relations for the symplectic generators together with the Weyl-Heisenberg generators are

$$\begin{aligned}
&[A_{i,j}, Q_k] = \delta_{j,k}Q_i, \quad [C_{i,j}, Q_k] = \delta_{j,k}P_i + \delta_{i,k}P_j,\\
&[A_{i,j}, P_k] = -\delta_{i,k}P_j, \quad [B_{i,j}, P_k] = \delta_{j,k}Q_i + \delta_{i,k}Q_j,\\
&[P_i, Q_j] = \delta_{i,j}I.
\end{aligned} \tag{89}$$

The symplectic generators may be realized in the enveloping algebra up to a central element [10]. This will be important when we discuss the representations in Section 3.2.

$$\widetilde{A}_{i,j} = Q_iP_j, \quad \widetilde{B}_{i,j} = Q_iQ_j, \widetilde{C}_{i,j} = P_iP_j. \tag{90}$$

Clearly $B_{i,j} = B_{j,i}$ and $C_{i,j} = C_{j,i}$. Then, using the Weyl-Heisenberg commutation relations (5), this defines the commutation relations, up to the central element, I,

$$\begin{aligned}
&\left[\widetilde{A}_{i,j}, \widetilde{A}_{k,l}\right] = I(\delta_{i,l}\widetilde{A}_{j,k} - \delta_{j,k}\widetilde{A}_{i,l}),\\
&\left[A_{i,j}, \widetilde{B}_{k,l}\right] = I(\delta_{j,k}\widetilde{B}_{i,l} + \delta_{j,l}\widetilde{B}_{i,k}),\\
&\left[A_{i,j}, \widetilde{C}_{k,l}\right] = -I(\delta_{i,k}\widetilde{C}_{j,l} + \delta_{i,l}\widetilde{C}_{k,j}),\\
&\left[\widetilde{B}_{i,j}, \widetilde{C}_k\right] = I(\delta_{i,k}\widetilde{A}_{j,l} + \delta_{i,l}\widetilde{A}_{j,k} + \delta_{j,k}\widetilde{A}_{i,l} + \delta_{j,l}\widetilde{A}_{i,k}).
\end{aligned} \tag{91}$$

3. Quantum symmetry: Projective representations

The projective representations of the maximal symmetry group $\mathcal{I}\mathcal{S}p(2n)$ are equivalent to the ordinary unitary representations of its central extension $\mathcal{H}\overline{\mathcal{S}p}(2n)$. These unitary irreducible representations may be determined using the Mackey theorems for semidirect product groups.

The first step in applying the Mackey theorem for semidirect products is to determine the unitary irreducible representations of the Weyl-Heisenberg normal

subgroup. While these are well known, the method of constructing them using the Mackey theorems applied to the semidirect product of two abelian groups (2) does not appear to be as well known [11]. We review this briefly in order to introduce the Mackey theorems and also to show how the unitary irreducible representations of the symmetry group can be constructed completely from first principles.

3.1. **Unitary irreducible representations of the Weyl-Heisenberg group.** The Mackey theorem for semidirect products with an abelian normal subgroup are given in Theorem 10 in Appendix A [12]. We choose the normal subgroup (27) with elements $\Upsilon(p,0,\iota) \in \mathcal{A}(n+1)$. The unitary irreducible representations ξ of the abelian normal subgroup are the phases acting on the Hilbert space $\mathbf{H}^{\xi} = \mathbb{C}$

$$\xi(\Upsilon(p,0,\iota))|\phi\rangle = e^{i(\iota\widehat{I}+p^i\widehat{Q}_i)}\,|\phi\rangle = e^{i(\iota\lambda+p\cdot\alpha)}\,|\phi\rangle, \quad |\phi\rangle \in \mathbb{C}. \tag{92}$$

The hermitian representation of the algebra has the eigenvalues that are given by

$$\widehat{Q}_i\,|\phi\rangle = \xi'(Q_i)\,|\phi\rangle = \alpha_i\,|\phi\rangle, \quad \widehat{I}\,|\phi\rangle = \xi'(I)\,|\phi\rangle = \lambda\,|\phi\rangle, \tag{93}$$

where $\alpha \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The characters $\xi_{\alpha,\lambda}$ are parameterized by the eigenvalues α, λ and the equivalence classes that are elements of the unitary dual, $[\xi_{\alpha,\lambda}] \in \boldsymbol{U}_{\mathcal{A}(n+1)} \simeq \mathbb{R}^{n+1}$. Each equivalence class has the single element $[\xi_{\alpha,\lambda}] = \xi_{\alpha,\lambda}$.

The action of the elements $\Upsilon(0,q,0) \in \mathcal{A}(n)$ of the homogeneous group on these representations is given by the dual automorphisms

$$\begin{aligned}&\left(\widehat{\varsigma}_{\Upsilon(0,q,0)}\xi_{\alpha,\lambda}\right)(\Upsilon(p,0,\iota))\,|\phi\rangle = \xi_{\alpha,\lambda}(\varsigma_{\Upsilon(0,q,0)}\Upsilon(p,0,\iota))|\phi\rangle \\ &= \xi_{\alpha-\lambda q,\lambda}(\Upsilon(p,0,\iota))\,|\phi\rangle.\end{aligned} \tag{94}$$

In simplifying this expression, we have used (30) and (92). The little group is the set of $\Upsilon(0,q,0) \in \mathcal{K}^{\circ}$ that satisfy the fixed point equation (134),

$$\widehat{\varsigma}_{\Upsilon(0,q,0)}\xi_{\alpha,\lambda} = \xi_{\alpha-\lambda q,\lambda} = \xi_{\alpha,\lambda}. \tag{95}$$

The solution of the fixed point condition requires that $\alpha - \lambda q \equiv \alpha$. The $\lambda = 0$ solution for which the little group is $\mathcal{A}(n)$ is the degenerate case corresponding to the homomorphism $\mathcal{H}(n) \to \mathcal{A}(2n)$ with kernel $\mathcal{A}(1)$. This is just the abelian group that is not considered further here. The faithful representation with $\lambda \neq 0$ requires $p = 0$, and therefore has the trivial little group $\mathcal{K}^{\circ} \simeq \boldsymbol{e} \simeq \{\Upsilon(0,0,0)\}$. The stabilizer is $\mathcal{G}^{\circ} \simeq \mathcal{A}(n+1)$. The orbits are

$$\mathbb{O}_{\lambda} = \left\{\widehat{\varsigma}_{\Upsilon(0,q,0)}[\xi_{\alpha,\lambda}]|q \in \mathbb{R}^n\right\} = \{\xi_{q,\lambda}|q \in \mathbb{R}^n\}, \quad \lambda \in \mathbb{R}\backslash\{0\}. \tag{96}$$

All representations in the orbit are equivalent for the determination of the semidirect product unitary irreducible representations. A convenient representative of the equivalence class is $\xi_{0,\lambda}$. The unitary representations σ of the trivial little group are trivial and therefore the representations of the stabilizer are just $\varrho^{\circ} = \xi_{0,\lambda}$. The Hilbert space $\mathbf{H}^{\sigma}$ is also trivial and therefore the Hilbert space of the stabilizer is $\mathbf{H}^{\varrho^{\circ}} = \mathbf{H}^{\sigma} \otimes \mathbf{H}^{\xi} \simeq \mathbb{C}$.

3.1.1. *Mackey induction.* The final step is to apply the Mackey induction theorem to determine the faithful unitary irreducible representations of the full $\mathcal{H}(n)$ group. The induction requires the definition of the symmetric space

$$\mathbb{K} = \mathcal{G}/\mathcal{G}^{\circ} = \mathcal{H}(n)/\mathcal{A}(n+1) \simeq \mathcal{A}(n) \simeq \mathbb{R}^n, \tag{97}$$

with the natural projection π and a section Θ

$$\begin{aligned}&\pi:\mathcal{H}(n)\to\mathbb{K}:\Upsilon(p,q,\iota)\mapsto \mathrm{k}_q,\\&\Theta:\mathbb{K}\to\mathcal{H}(n):\mathrm{k}_q\mapsto\Theta(\mathrm{k}_q)=\Upsilon(0,q,0).\end{aligned}\tag{98}$$

These satisfy $\pi(\Theta(\mathrm{a}_q))=\mathrm{a}_q$ and so $\pi\circ\Theta=\mathrm{Id}_{\mathbb{K}}$ as required. Using (2), an element of the Weyl-Heisenberg group $\mathcal{H}(n)$ can be written as,

$$\Upsilon(p,q,\iota)=\Upsilon(0,q,0)\Upsilon(p,0,\iota+\frac{1}{2}p\cdot q).\tag{99}$$

The cosets are therefore defined by

$$\begin{aligned}\mathrm{k}_q&=\left\{\Upsilon(0,q,0)\Upsilon(p,0,\iota+\tfrac{1}{2}p\cdot q)|p\in\mathbb{R}^n,\iota\in\mathbb{R}\right\}\\&=\{\Upsilon(0,q,0)\mathcal{A}(n+1)\}\end{aligned}\tag{100}$$

Note that

$$\Upsilon(p,q,\iota)\mathrm{k}_x=\mathrm{k}_{x+q},\qquad x\in\mathbb{R}^n.\tag{101}$$

The Mackey induced representation theorem can now be applied straightforwardly. First, the Hilbert space is

$$\mathbf{H}^\varrho=\boldsymbol{L}^2(\mathbb{K},\mathbf{H}^{\varrho^\circ})\simeq\boldsymbol{L}^2(\mathbb{R}^n,\mathbb{C}).\tag{102}$$

Next the Mackey induction Theorem 8 yields

$$\psi'(\mathrm{k}_x)=(\varrho(\Upsilon(p,q,\iota))\psi)\left(\Upsilon(p,q,\iota)^{-1}\mathrm{k}_x\right)=\varrho^\circ(\Upsilon(a^\circ,0,\iota^\circ))\psi(\mathrm{k}_{x-q})\tag{103}$$

Using the Weyl-Heisenberg group product (2),

$$\begin{aligned}\Upsilon(p^\circ,q^\circ,\iota^\circ)&=\Theta(\mathrm{k}_x)^{-1}\Upsilon(p,q,\iota)\Theta(\Upsilon(p,q,\iota)^{-1}\mathrm{k}_x)\\&=\Upsilon(0,-x,0)\Upsilon(p,q,\iota)\Upsilon(0,x-q,0)\\&=\Upsilon(p,0,\iota+p\cdot\left(x-\tfrac{1}{2}q\right)).\end{aligned}\tag{104}$$

We lighten notation using the isomorphism $\mathrm{k}_x\mapsto x$. The induced representation theorem then yields

$$\begin{aligned}\psi'(x)&=\xi_{0,\lambda}(\Upsilon(p,0,\iota+x\cdot p-\tfrac{1}{2}p\cdot q)\psi(x-q)\\&=e^{i\lambda(\iota+x\cdot p-\frac{1}{2}p\cdot q)}\psi(x-q).\end{aligned}\tag{105}$$

Using Taylor expansion, we can write

$$\psi(x-q)=e^{-q^i\frac{\partial}{\partial x^i}}\psi(x).\tag{106}$$

The Baker Campbell-Hausdorff formula [20] enables us to combine the exponentials

$$\psi'(x)=e^{i\left(\lambda\iota+\lambda p^ix_i+q^ii\frac{\partial}{\partial x^i}\right)}\psi(x)=e^{i\left(\iota\widehat{I}+p^i\widehat{Q}_i+q^i\widehat{P}_i\right)}\psi(x).\tag{107}$$

The representation of the algebra is therefore

$$\widehat{I}\psi(x)=\lambda\psi(x),\quad\widehat{Q}_i\psi(x)=\lambda x_i\psi(x),\quad\widehat{P}_i\psi(x)=i\frac{\partial}{\partial x^i}\psi(x),\tag{108}$$

that satisfies the Heisenberg commutation relations (1).

This analysis can also be carried out choosing $\Upsilon(0,q,\iota)\in\mathcal{A}(n+1)$ to be the elements of the normal subgroup and this yields the representation with $\widehat{P}_i$ diagonal.

3.2. **Unitary irreducible representations of $\mathcal{H}\overline{\mathcal{S}p}(2n)$.** We consider next the unitary irreducible representations of the $\mathcal{H}\overline{\mathcal{S}p}(2n)$ group

$$\mathcal{H}\overline{\mathcal{S}p}(2n) \simeq \mathcal{S}p(2n) \otimes_s \mathcal{H}(n). \tag{109}$$

As $\mathcal{H}\overline{\mathcal{S}p}(2n)$ is the central extension of $\mathcal{IS}p(2n)$, the projective representations of $\mathcal{IS}p(2n)$ are equivalent to the ordinary unitary representations of $\mathcal{H}\overline{\mathcal{S}p}(2n)$.

The unitary irreducible representations of $\mathcal{H}\overline{\mathcal{S}p}(2n)$ may be determined using Mackey Theorem 9 for the nonabelian normal subgroup case. The faithful unitary representations of the Weyl-Heisenberg group are given in the previous section (105). The next step in applying the Mackey's theorem is to determine the ρ representation of the stabilizer $\mathcal{G}^\circ \subset \mathcal{H}\overline{\mathcal{S}p}(2n)$.

3.2.1. *Stabilizer and ρ representation.* The representation ρ of the stabilizer $\mathcal{G}^\circ$ acts on the Hilbert space $\mathbf{H}^\xi$ and therefore the hermitian representations ρ' of the algebra of the stabilizer must be realized in the enveloping algebra of the Weyl-Heisenberg group. The ρ representation restricted to the Weyl-Heisenberg group are given by $\rho|_{\mathcal{H}(n)} = \xi$ where ξ are the unitary irreducible representations of the Weyl Heisenberg group. The faithful representations ξ are given in (105).

The unitary representation ρ acts on $\mathbf{H}^\xi \simeq \boldsymbol{L}^2(\mathbb{R}^n, \mathbb{C})$ such that

$$\rho(\Omega^\circ)\xi(\Upsilon(z,\iota))\rho(\Omega^\circ)^{-1} = \xi(\varsigma_{\Omega^\circ}\Upsilon(z,\iota)), \quad \Omega^\circ \in \mathcal{G}^\circ. \tag{110}$$

The representation ρ factors into

$$\rho(\Omega^\circ(\delta,\Sigma,w,r)) = \xi(\Upsilon(w,r))\rho(\Sigma), \tag{111}$$

where again for notational brevity $\Sigma \equiv \Omega(1,\Sigma,0,0)$.

We already have characterized the inner automorphisms. The automorphisms corresponding factor as

$$\begin{aligned} &\xi(\Upsilon(w,r))\xi(\Upsilon(z,\iota))\xi(\Upsilon(w,r))^{-1} = \xi(\varsigma_{\Upsilon(w,r)}\Upsilon(z,\iota)), \\ &\rho(\Sigma)\xi(\Upsilon(z,\iota))\rho(\Sigma)^{-1} = \xi(\varsigma_{\Omega(\Sigma)}\Upsilon(z,\iota)) = \xi(\Upsilon(\pi(\Sigma)z,\iota)). \end{aligned} \tag{112}$$

where $\Sigma \in \overline{\mathcal{S}p}(2n)$ and $\pi : \overline{\mathcal{S}p}(2n) \to \mathcal{S}p(2n)$.

The inner automorphisms are already characterized as we know the unitary irreducible representations ξ. Consider next the representation $\rho(\Sigma)$ of the symplectic group $\overline{\mathcal{S}p}(2n)$. The hermitian representation of the symplectic generators is

$$\begin{aligned} \widehat{A}_{i,j} &= \rho'(A_{i,j}) = \lambda\widehat{Q}_i\widehat{P}_j, \\ \widehat{B}_{i,j} &= \rho'(B_{i,j}) = \lambda\widehat{Q}_i\widehat{Q}_j, \\ \widehat{C}_{i,j} &= \rho'(C_{i,j}) = \lambda\widehat{P}_i\widehat{P}_j. \end{aligned} \tag{113}$$

Clearly $\widehat{B}_{i,j} = \widehat{B}_{j,i}$ and $\widehat{C}_{i,j} = \widehat{C}_{j,i}$. Then, using the Heisenberg commutation relations (1), this defines a hermitian realization of the Lie algebra of the automorphism group acting on the Hilbert space $\mathbf{H}^{\xi} \simeq \boldsymbol{L}^2(\mathbb{R}^n, \mathbb{C})$.

$$\begin{aligned}
\left[\widehat{A}_{i,j}, \widehat{A}_{k,l}\right] &= i(\delta_{i,l}\widehat{A}_{j,k} - \delta_{j,k}\widehat{A}_{i,l}),\\
\left[\widehat{A}_{i,j}, \widehat{B}_{k,l}\right] &= i(\delta_{j,k}\widehat{B}_{i,l} + \delta_{j,l}\widehat{B}_{i,k}),\\
\left[\widehat{A}_{i,j}, \widehat{C}_{k,l}\right] &= -i(\delta_{i,k}\widehat{C}_{j,l} + \delta_{i,l}\widehat{C}_{k,j}),\\
\left[\widehat{B}_{i,j}, \widehat{C}_{k}\right] &= i(\delta_{i,k}\widehat{A}_{j,l} + \delta_{i,l}\widehat{A}_{j,k} + \delta_{j,k}\widehat{A}_{i,l} + \delta_{j,l}\widehat{A}_{i,k}),
\end{aligned} \tag{114}$$

$$\begin{aligned}
&\left[\widehat{A}_{i,j}, \widehat{Q}_{k}\right] = i\delta_{j,k}\widehat{Q}_{i}, \quad \left[\widehat{C}_{i,j}, \widehat{Q}_{k}\right] = i(\delta_{j,k}\widehat{P}_{i} + \delta_{i,k}\widehat{P}_{j}),\\
&\left[\widehat{A}_{i,j}, \widehat{P}_{k}\right] = -i\delta_{i,k}\widehat{P}_{j}, \quad \left[\widehat{B}_{i,j}, \widehat{P}_{k}\right] = i(\delta_{j,k}\widehat{Q}_{i} + \delta_{i,k}\widehat{Q}_{j}),\\
&\left[\widehat{P}_{i}, \widehat{Q}_{j}\right] = i\delta_{i,j}\widehat{I}.
\end{aligned} \tag{115}$$

Therefore, there exists a ρ' representation for the entire algebra of $\mathcal{H}\overline{\mathcal{S}p}(2n)$ and therefore the stabilizer is the group itself, $\mathcal{G}^{\circ} \simeq \mathcal{H}\overline{\mathcal{S}p}(2n)$. This explicate construction of the algebra shows that the representation $\rho(\Sigma)$ exists. Consequently, the Mackey induction theorem is not required.

The $\rho(\Sigma)$ representation is precisely (up to an overall phase) the metaplectic representation originally studied by Weil [13], [2]. We can construct this explicitly using the factorization of the symplectic group (86). We can consider each of the factors separately as

$$\rho(\Sigma(\epsilon, \alpha, \beta, \gamma) = \rho(\Sigma^{-}(\gamma))\rho(\Sigma^{\circ}(\alpha))\rho(\Sigma^{+}(\beta))\rho(\zeta^{\epsilon}), \tag{116}$$

and each of these factors can be applied separately to determine the ρ representation. The unitary representations of $\Sigma(\beta) \in \mathcal{A}(m)$, $m = \frac{n(n+1)}{2}$ in a basis with $\widehat{Q}_i$ diagonal are

$$\rho(\Sigma^{+}(\beta))|\psi_{\lambda}(x)\rangle = e^{i\alpha^{i,j}\widehat{B}_{i,j}}\,|\psi_{\lambda}(x)\rangle = e^{\frac{i}{\lambda}\beta^{i,j}x_i x_j}\,|\psi_{\lambda}(x)\rangle\ . \tag{117}$$

The representations of the elements of the unitary group $\Sigma(\alpha) \in \mathcal{U}(n)$ are

$$\rho(\Sigma^{\circ}(\alpha))|\psi_{\lambda}(x)\rangle = |\det A|^{-\frac{1}{2}}\,|\psi_{\lambda}(A^{-1}x)\rangle\ . \tag{118}$$

The symplectic matrix exchanges the p and q degrees of freedom, $\varsigma_{\zeta}\Upsilon(p, q, \iota) = \Upsilon(q, -p, \iota)$. As is well known, the unitary representation of this is the Fourier transform, $\rho(\zeta) = \boldsymbol{f}$ where

$$\rho(\Upsilon(p, q, \iota))\boldsymbol{f}|\psi_{\lambda}(x)\rangle = \boldsymbol{f}\rho(\Upsilon(q, -p, \iota))|\psi_{\lambda}(x)\rangle, \tag{119}$$

where the Fourier transform is defined as usual by

$$\widetilde{\psi}(y) = \boldsymbol{f}\psi(x) = (2\pi i)^{-\frac{n}{2}}\int e^{-ix\cdot y}\psi(x)d^n x, \tag{120}$$

and where

$$\widehat{Q}_i\,|\psi_{\lambda}(x)\rangle = \lambda x_i\,|\psi_{\lambda}(x)\rangle, \quad \widehat{P}_i\,|\widetilde{\psi}_{\lambda}(y)\Big\rangle = y_i\,|\widetilde{\psi_{\lambda}}(y)\Big\rangle. \tag{121}$$

Finally, the $\rho(\Sigma^{+}(\beta))$ representation can be computed using (80) in a basis with $\widehat{Q}_i$ diagonal giving

$$\rho(\Sigma^{-}(\gamma))\,|\psi_{\lambda}(x)\rangle = \boldsymbol{f}\rho(\Sigma^{+}(-\gamma))\boldsymbol{f}^{-1}|\psi_{\lambda}(x), \tag{122}$$

and the $\rho(\Sigma^+(-\gamma))$ is given by (116). Putting all of these together gives the representation $\rho(\Sigma)$ up to a phase. While one would expect the phase to be $m \in \mathbb{Z}$ dependent, it actually only is two valued $\pm 1 \in \mathbb{Z}_2$. The unitary representations of the double cover metaplectic group $\mathcal{M}p(2n)$ are also a representation of $\overline{\mathcal{S}p}(2n)$ due to the homomorphism (141).

Of course, all of these calculations could also be done in a basis with $\widehat{P}_i$ diagonal.

As the stabilizer is the full group, Mackey induction is not required and the unitary irreducible representations υ of $\mathcal{H}\overline{\mathcal{S}p}(2n)$ are given by

$$\upsilon(\Omega(1,\Sigma,z,\iota))|\psi(x)\rangle = \sigma(\Sigma)\otimes\xi(\Upsilon(z,\iota))\rho(\Sigma)|\psi(x)\rangle \tag{123}$$

where σ are ordinary unitary irreducible representations of $\overline{\mathcal{S}p}(2n)$, ρ are the metaplectic representation of $\overline{\mathcal{S}p}(2n)$ given above and ξ are the unitary irreducible representations of $\mathcal{H}(n)$ given in Section 3.1.

The ordinary unitary representations of the symplectic group have been partially characterized [8-9]. A complete set of unitary irreducible representations of the covering group $\overline{\mathcal{S}p}(2n)$ appears to be an open problem.

4. Summary

We have determined the projective representations of the inhomogeneous symplectic group. This is the maximal symmetry whose projective representations transform physical states such that the Heisenberg commutation relations are valid in all of the transformed states.

The inhomogeneous symplectic symmetry is well known from classical mechanics. It acts on classical phase space with position and momentum degrees of freedom. The projective representations that define the quantum symmetry require its central extension which introduces the non-abelian structure of the Weyl-Heisenberg group, $\mathcal{I}\breve{\mathcal{S}p}(2n) \simeq \overline{\mathcal{S}p}(2n)\otimes_s \mathcal{H}(n)$. The non-abelian structure is a direct result of the fact that transition probabilities are the square of the norm of physical states. Consequently, the physical states are defined up to a phase and the action of a symmetry group is given by the projective representations. This is the underlying reason for the non-abelian structure, or *quantization*. Any symmetry of quantum mechanics that preserves the position, momentum Heisenberg commutation relations must be a subgroup of this maximal symmetry.

On the other hand, we now understand special relativistic quantum mechanics as the projective representations of the inhomogeneous Lorentz group [5],[6]. The central extension of this group does not admit an algebraic extension. For the connected component, the central extension is therefore the cover that we call the Poincaré group which for $n = 3$ is $\mathcal{P} = \mathcal{SL}(2,\mathbb{C})\otimes_s \mathcal{A}(4)$.[7] Special relativistic quantum mechanics is formulated in terms of the unitary representations of the Poincaré group. There is however, no mention of the Weyl-Heisenberg group which plays a fundamental role in the original formulation of quantum mechanics.

Symmetry is one of the most fundamental concepts of physics. We have the case where we have a quantum symmetry for the Weyl-Heisenberg of quantum

[7]The full inhomogeneous group is given in terms of the orthogonal group $\mathcal{O}(1,n)$ that has 4 disconnected components. The discrete $\mathbb{Z}_{2,2}$ symmetry is P, T and PT symmetry. Its central extension is not unique and it gives rise to the $\mathcal{P}in$ group ambiguity. On the other hand the $\mathcal{SO}(1,n)$ group has 2 components but does have a unique central extension that is the $\mathcal{S}pin$ group. The discrete $\mathbb{Z}_2$ symmetry is the PT symmetry.

mechanics that is the projective representations of a classical symmetry on phase space. On the other hand, the quantum symmetry for the Minkowski metric of special relativity is given in terms of a classical symmetry on position-time space, that is, spacetime.

Quantum mechanics and special relativity have at best, an uneasy marriage. Perhaps it is due to this underlying disparity in the most basic symmetries of these theories. The standard approach is to ignore the quantum symmetry described in this paper and formulate special relativistic quantum mechanics as the projective representations of the inhomogeneous group.

If we truly are to bring together quantum mechanics and special relativity, we must first reconcile these basic symmetries and find a symmetry that encompasses both. This can be done in a remarkably straightforward manner and results in a theory that, in a physical limit, results in the usual formulation of special relativistic quantum mechanics. But, before the limit is taken, it points to a theory incorporating both symmetries that may give further understanding of the unification of quantum mechanics and relativity [14],[15],[16]. In this theory, physics takes place in extended phase space and there is no invariant global projection that gives physics in position-time space (i.e. space-time). Generally, local observers with general non-inertial trajectories construct different space-times as subspaces of extended phase space. The usual Lorentz symmetry continues to hold exactly for inertial trajectories but is generalized in a remarkable manner for non-inertial trajectories.

5. Appendix A: Key Theorems

In this appendix we review a set of definitions and theorems that are fundamental for the application of symmetry groups in quantum mechanics. We state the theorems only and refer the reader to the cited literature for full proofs.

Definition 1. *A group $\mathcal{G}$ is a semidirect product if it has a subgroup $\mathcal{K}$ (referred to as the homogeneous subgroup) and a normal subgroup $\mathcal{N}$ such that $\mathcal{K} \cap \mathcal{N} = \boldsymbol{e}$ and $\mathcal{G} \simeq \mathcal{N}\mathcal{K}$. Our notation for a semidirect product is $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$*[8][17].

It follows directly that a semidirect product is right associative in the sense that $\mathcal{D} \simeq (\mathcal{A} \otimes_s \mathcal{B}) \otimes_s \mathcal{C}$ implies that $\mathcal{D} \simeq \mathcal{A} \otimes_s (\mathcal{B} \otimes_s \mathcal{C})$ and so brackets can be removed. However $\mathcal{D} \simeq \mathcal{A} \otimes_s (\mathcal{B} \otimes_s \mathcal{C})$ does not necessarily imply $\mathcal{D} \simeq (\mathcal{A} \otimes_s \mathcal{B}) \otimes_s \mathcal{C}$ as $\mathcal{B}$ is not necessarily a normal subgroup of $\mathcal{A}$.

Definition 2. *An algebraic central extension of a Lie algebra g is the Lie algebra $\breve{g}$ that satisfies the following short exact sequence where z is the maximal abelian algebra that is central in $\breve{g}$,*

$$\mathbf{0} \to z \to \breve{g} \to g \to \mathbf{0}. \tag{124}$$

where $\mathbf{0}$ is the trivial algebra. Suppose $\{X_a\}$ is a basis of the Lie algebra g with commutation relations $[X_a, X_b] = c^c_{a,b} X_c$, $a, b = 1, ...r$. Then an algebraic central

[8]Our notation follows [17]. Another notation commonly used is $\mathcal{N} \rtimes \mathcal{K}$. It is just notation; the definition remains the same for both notations.

extension is a maximal set of central abelian generators $\{A_\alpha\}$, where $\alpha, \beta, ... = 1, ..m$, such that

$$[A_\alpha, A_\beta] = 0, \quad [X_a, A_\alpha] = 0, \quad [X_a, X_b] = c^c_{a,b} X_c + c^\alpha_{a,b} A_\alpha. \tag{125}$$

The basis $\{X_a, A_\alpha\}$ of the centrally extended Lie algebra must also satisfy the Jacobi identities. The Jacobi identities constrain the admissible central extensions of the algebra. The choice $X_a \mapsto X_a + A_a$ will always satisfy these relations and this trivial case is excluded. The algebra $\breve{g}$ constructed in this manner is equivalent to the central extension of g given in Definition 2.

Definition 3. *The central extension of a connected Lie group $\mathcal{G}$ is the Lie group $\breve{\mathcal{G}}$ that satisfies the following short exact sequence where $\mathcal{Z}$ is a maximal abelian group that is central in $\breve{\mathcal{G}}$*

$$\boldsymbol{e} \to \mathcal{Z} \to \breve{\mathcal{G}} \xrightarrow{\pi} \mathcal{G} \to \boldsymbol{e}. \tag{126}$$

The abelian group $\mathcal{Z}$ may always be written as the direct product $\mathcal{Z} \simeq \mathcal{A}(m) \otimes \mathbb{A}$ of a connected continuous abelian Lie group $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$ and a discrete abelian group $\mathbb{A}$ that may have a finite or countable dimension [10].

The exact sequence may be decomposed into an exact sequence for the *topological* central extension and the *algebraic* central extension,

$$\boldsymbol{e} \to \mathbb{A} \to \overline{\mathcal{G}} \xrightarrow{\pi^\circ} \mathcal{G} \to \boldsymbol{e}, \;\; \boldsymbol{e} \to \mathcal{A}(m) \to \breve{\mathcal{G}} \xrightarrow{\widetilde{\pi}} \overline{\mathcal{G}} \to \boldsymbol{e}. \tag{127}$$

where $\pi = \pi^\circ \circ \widetilde{\pi}$. The first exact sequence defines the universal cover where $\mathbb{A} \simeq \ker \pi^\circ$ is the fundamental homotopy group. All of the groups is in the second sequence are simply connected and therefore may be defined by the exponential map of the central extension of the algebra given by Definition 2. In other words, the full central extension may be computed by determining the universal covering group of the algebraic central extension.

Definition 4. *A ray Ψ is the equivalence class of states $|\psi_\gamma\rangle$ that are elements of a Hilbert space* **H** *up to a phase,*

$$\Psi = \left\{ e^{i\omega} \left| \psi \right\rangle | \omega \in \mathbb{R} \right\}, \quad |\psi\rangle \in H. \tag{128}$$

Note that the physical probabilities that are the square of the modulus depend only on the ray

$$|\left(\Psi_\beta, \Psi_\alpha\right)|^2 = |\langle \psi_\beta | \psi_\alpha \rangle|^2$$

for all $|\psi_\gamma\rangle \in \Psi$. For this reason, physical states in quantum mechanics are defined to be rays rather than states in the Hilbert space

Definition 5. *A projective representation ϱ of a symmetry group $\mathcal{G}$ is the maximal representation such that for $|\widetilde{\psi}_\gamma\rangle = \varrho(g)|\psi_\gamma\rangle$, the modulus is invariant $|\langle \widetilde{\psi}_\beta | \widetilde{\psi}_\alpha \rangle|^2 = |\langle \psi_\beta | \psi_\alpha \rangle|^2$ for all $|\psi_\gamma\rangle, |\widetilde{\psi}_\gamma\rangle \in \Psi$.*

Theorem 2. ***(Wigner, Weinberg):*** *Any projective representation of a Lie symmetry group $\mathcal{G}$ on a separable Hilbert space is equivalent to a representation that is either linear and unitary or anti-linear and anti-unitary. Furthermore, if $\mathcal{G}$ is connected, the projective representations are equivalent to a representation that is linear and unitary [1],[11].*

This is the generalization of the well known theorem that the ordinary representation of any compact group is equivalent to a representation that is unitary. For a projective representation, the phase degrees of freedom of the central extension enables the equivalent linear unitary or anti-linear anti-unitary representation to be constructed for this much more general class of Lie groups that admit representations on separable Hilbert spaces. (A proof of the theorem is given in Appendix A of Chapter 2 of [6]). The set of groups that this theorem applies to include all the groups that are studied in this paper.

Theorem 3. ***(Bargmann, Mackey)*** *The projective representations of a connected Lie group $\mathcal{G}$ are equivalent to the ordinary unitary representations of its central extension $\breve{\mathcal{G}}$* [7], [8].

Theorem 2 states that are all projective representations of a connected Lie group are equivalent to a projective representation that is unitary. A phase is the unitary representation of a central abelian subgroup. Therefore, the maximal representation is given in terms of the central extension of the group.

Theorem 4. *Let $\mathcal{G}$,$\mathcal{H}$ be Lie groups and $\pi : \mathcal{G} \to \mathcal{H}$ be a homomorphism. Then, for every unitary representation $\widetilde{\varrho}$ of $\mathcal{H}$ there exists a degenerate unitary representation ϱ of $\mathcal{G}$ defined by $\varrho = \widetilde{\varrho} \circ \pi$. Conversely, for every degenerate unitary representation of a Lie group $\mathcal{G}$ there exists a Lie subgroup $\mathcal{H}$ and a homomorphism $\pi : \mathcal{G} \to \mathcal{H}$ where* $\ker(\pi) \neq \boldsymbol{e}$ *such that $\varrho = \widetilde{\varrho} \circ \pi$ where $\widetilde{\varrho}$ is a unitary representation of $\mathcal{H}$.*

Noting that a representation is a homomorphism, This theorem follows straightforwardly from the properties of homomorphisms. As a consequence, the set of degenerate representations of a group is characterized by its set of normal subgroups. A *faithful* representation is the case that the representation is an isomorphism.

Theorem 5. ***(Levi)*** *Any simply connected Lie group is equivalent to the semidirect product of a semisimple group and a maximal solvable normal subgroup*[18]

As the central extension of any connected group is simply connected, the problem of computing the projective representations of a group always can be reduced to computing the unitary irreducible representations of a semidirect product group with a semisimple homogeneous group and a solvable normal subgroup. The unitary irreducible representations of the semisimple groups are known and the solvable groups that we are interested in turn out to be the semidirect product of abelian groups.

Theorem 6. *Any semidirect product group $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ is a subgroup of a group homomorphic to the group of automorphisms of $\mathcal{N}$ [13].*

The proof follows directly from the definition of the semidirect product and an automorphism group.

Theorem 7. *The automorphism group of a simply connected group is isomorphic to the automorphism group of its Lie algebra.*[18]

5.1. **Mackey theorems for the representations of semidirect product groups.** The Mackey theorems are valid for a general class of topological groups but we will only require the more restricted case $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ where the group $\mathcal{G}$ and subgroups $\mathcal{K}, \mathcal{N}$ are smooth Lie groups. The central extension of any connected Lie group is simply connected and therefore generally has the form of a semidirect product

due to Theorem 5 (Levi). Theorem 6 further constrains the possible homogeneous groups $\mathcal{K}$ of the semidirect product given the normal subgroup $\mathcal{N}$.

The first Mackey theorem is the induced representation theorem that gives a method of constructing a unitary representation of a group (that is not necessarily a semidirect product group) from a unitary representation of a closed subgroup. The second theorem gives a construction of certain representations of a certain subgroup of a semidirect product group from which the complete set of unitary irreducible representations of the group can be induced. This theorem is valid for the general case where the normal subgroup $\mathcal{N}$ is a nonabelian group. In the special case where the normal subgroup $\mathcal{N}$ is abelian, the theorem may be stated in a simpler form.

Theorem 8. (Mackey). Induced representation theorem. *Suppose that $\mathcal{G}$ is a Lie group and $\mathcal{H}$ is a Lie subgroup, $\mathcal{H} \subset \mathcal{G}$ such that $\mathbb{K} \simeq \mathcal{G}/\mathcal{H}$ is a homogeneous space with a natural projection $\pi : \mathcal{G} \to \mathbb{K}$, an invariant measure and a canonical section $\Theta : \mathbb{K} \to \mathcal{G} : \mathrm{k} \mapsto g$ such that $\pi \circ \Theta = \mathrm{Id}_{\mathbb{K}}$ where $\mathrm{Id}_{\mathbb{K}}$ is the identity map on $\mathbb{K}$. Let ρ be a unitary representation of $\mathcal{H}$ on the Hilbert space $\mathbf{H}^{\rho}$:*

$$\rho(h) : \mathbf{H}^{\rho} \to \mathbf{H}^{\rho} : \ |\varphi\rangle \mapsto |\widetilde{\varphi}\rangle = \rho(h)\,|\varphi\rangle, \quad h \in \mathcal{H}.$$

Then a unitary representation ϱ of a Lie group $\mathcal{G}$ on the Hilbert space $\mathbf{H}^{\varrho}$,

$$\varrho(g) : \mathbf{H}^{\varrho} \to \mathbf{H}^{\varrho} : \ |\psi\rangle \mapsto \left|\widetilde{\psi}\right\rangle = \varrho(g)\,|\psi\rangle, \quad g \in \mathcal{G},$$

may be induced from the representation ρ of $\mathcal{H}$ by defining

$$\widetilde{\psi}(\mathrm{k}) = (\varrho(g)\psi)(\mathrm{k}) = \rho(g^{\circ})\psi(g^{-1}\mathrm{k}), \quad g^{\circ} = \Theta(\mathrm{k})^{-1} g \Theta(g^{-1}\mathrm{k}), \tag{129}$$

where the Hilbert space on which the induced representation ϱ acts is given by $\mathbf{H}^{\varrho} \simeq \mathbf{L}^2(\mathbb{K}, H^{\rho})$ [14], [13].

The proof is straightforward given that the section Θ exists by showing first that $g^{\circ} \in \ker(\pi) \simeq \mathcal{H}$ and therefore $\rho(g^{\circ})$ is well defined.

Definition 6. *(Little groups): Let $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{N}$ be a semidirect product. Let $[\xi] \in \boldsymbol{U}_{\mathcal{N}}$ where $\boldsymbol{U}_{\mathcal{N}}$ denotes the unitary dual whose elements are equivalence classes of unitary representations of $\mathcal{N}$ on a Hilbert space $\mathbf{H}^{\xi}$. Let ρ be a unitary representation of a subgroup $\mathcal{G}^{\circ} = \mathcal{K}^{\circ} \otimes_s \mathcal{N}$ on the Hilbert space $\mathbf{H}^{\xi}$ such that $\rho_{|\mathcal{N}} = \xi$. The little groups are the set of maximal subgroups $\mathcal{K}^{\circ}$ such that ρ exists on the corresponding stabilizer $\mathcal{G}^{\circ} \simeq \mathcal{K}^{\circ} \otimes_s \mathcal{N}$ and satisfies the fixed point equation*

$$\widehat{\varsigma}_{\rho(k)}[\xi] = [\xi], k \in \mathcal{K}^{\circ}. \tag{130}$$

In this definition the dual automorphism is defined by

$$\left(\widehat{\varsigma}_{\rho(g)}\xi\right)(h) \quad = \rho(g)\rho(h)\rho(g)^{-1} = \rho(ghg^{-1}) = \xi(\varsigma_g h) \tag{131}$$

for all $g \in \mathcal{G}^{\circ}$ and $h \in \mathcal{N}$. The equivalence classes of the unitary representations of $\mathcal{N}$ are defined by

$$[\xi] = \left\{\widehat{\varsigma}_{\xi(h)}\xi | h \in \mathcal{N}\right\}. \tag{132}$$

A group $\mathcal{G}$ may have multiple little groups ${\mathcal{K}^{\circ}}_{\alpha}$ whose intersection is the identity element only. We will generally leave the label α implicit.

Theorem 9. (Mackey). Unitary irreducible representations of semidirect products. *Suppose that we have a semidirect product Lie group $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$, where $\mathcal{K}, \mathcal{N}$ are Lie subgroups. Let ξ be the unitary irreducible representation of $\mathcal{N}$ on the Hilbert space $\mathbf{H}^\xi$. Let $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{N}$ be a maximal stabilizer on which there exists a representation ρ on $\mathbf{H}^\xi$ such that $\rho|_{\mathcal{N}} = \xi$. Let σ be a unitary irreducible representation of $\mathcal{K}^\circ$ on the Hilbert space $\mathbf{H}^\sigma$. Define the representation $\varrho^\circ = \sigma \otimes \rho$ that acts on the Hilbert space $\mathbf{H}^{\varrho^\circ} \simeq \mathbf{H}^\sigma \otimes \mathbf{H}^\xi$. Determine the complete set of stabilizers and representations ρ and little groups that satisfy these properties, that we label by α,$\{(\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})_\alpha\}$. If for some member of this set $\mathcal{G}^\circ \simeq \mathcal{G}$ then for this case the representations are $(\mathcal{G}, \varrho, \mathbf{H}^\varrho) \simeq (\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})$. For the cases where the stabilizer $\mathcal{G}^\circ$ is a proper subgroup of $\mathcal{G}$ then the unitary irreducible representations $(\mathcal{G}, \varrho, \mathbf{H}^\varrho)$ are the representations induced (using Theorem 8) by the representations $(\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})$ of the stabilizer subgroup. The complete set of unitary irreducible representations is the union of the representations $\cup_\alpha \{(\mathcal{G}, \varrho, \mathbf{H}^\varrho)_\alpha\}$ over the set of all the stabilizers and corresponding little groups.*

This major result and its proof are due to Mackey[14]. Our focus in this paper is on applying this theorem.

5.1.1. *Abelian normal subgroup.* The theorem simplifies for special cases where the normal subgroup $\mathcal{N}$ is an abelian group, $\mathcal{N} \simeq \mathcal{A}(n)$. An abelian group has the property that its unitary irreducible representations ξ are the characters acting on the Hilbert space $\mathbf{H}^\xi \simeq \mathbb{C}$,

$$\xi(a) \left|\phi\right\rangle = e^{ia\cdot\nu} \left|\phi\right\rangle, \nu \in \mathbb{R}^n \tag{133}$$

The unitary irreducible representations are labeled by the ν_i that are the eigenvalues of the hermitian representation of the basis $\{A_i\}$ of the abelian Lie algebra,

$$\widehat{A}_i \left|\phi\right\rangle = \xi'(A_i) \left|\phi\right\rangle = \nu_i \left|\phi\right\rangle. \tag{134}$$

The equivalence classes $[\xi] \in \boldsymbol{U}_{\mathcal{A}(n)}$ each have a single element $[\xi] \simeq \xi$ as, for the abelian group, the expression (131) is trivial. The representations ρ act on $\mathbf{H}^\xi \simeq \mathbb{C}$ and are one dimensional and therefore must commute with the ξ. Therefore, in equation (130), $\rho(g)\xi(h)\rho(g)^{-1} = \xi(h)$ and (129) simplifies to

$$\xi(a) = \xi(\varsigma_k a) = \xi(kak^{-1}), a \in \mathcal{A}(m), \quad k \in \mathcal{K}^\circ. \tag{135}$$

Theorem 10. (Mackey). Unitary irreducible representations of a semidirect product with an abelian normal subgroup. *Suppose that we have a semidirect product group $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{A}$ where $\mathcal{A}$ is abelian. Let ξ be the unitary irreducible representation (that are the characters) of $\mathcal{A}$ on $\mathbf{H}^\xi \simeq \mathbb{C}$. Let $\mathcal{K}^\circ \subseteq \mathcal{K}$ be a Little group defined by (134) with the corresponding stabilizers $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{A}$. Let σ be the unitary irreducible representations of $\mathcal{K}^\circ$ on the Hilbert space $\mathbf{H}^\sigma$. Define the representation $\varrho^\circ = \sigma \otimes \xi$ of the stabilizer that acts on the Hilbert space $\mathbf{H}^{\varrho^\circ} \simeq \mathbf{H}^\sigma \otimes \mathbb{C}$. The theorem then proceeds as in the case of the general Theorem 9.*

6. Appendix B: Polarized Realization of the Weyl-Heisenberg group

The maps $\varphi^\pm$ defined in (35) is an isomorphism. Therefore the $\Upsilon^\pm(p, q, \iota^\pm)$ are elements of the Weyl-Heisenberg group realized in another coordinate system of

matrices. These realizations are referred to as the polarized realizations [2]. The group products in these coordinates are computed directly from (3-4) to be

$$\begin{array}{l}\Upsilon^+(p',q',\iota')\Upsilon^+(p,q,\iota)=\Upsilon^+(p'+p,q'+q,\iota+\iota+p'\cdot q),\\ \Upsilon^-(p',q',\iota')\Upsilon^-(p,q,\iota)=\Upsilon^-(p'+p,q'+q,\iota+\iota-q'\cdot p),\\ \Upsilon^\pm(p,q,\iota)^{-1}=\Upsilon^\pm(-p,-q,-\iota\pm p\cdot q).\end{array} \tag{136}$$

Note that the polarized realizations factor directly

$$\Upsilon^+(0,q,\iota)\Upsilon^+(p,0,0)=\Upsilon^+(p,q,\iota), \tag{137}$$

$$\Upsilon^-(p,0,\iota)\Upsilon^-(0,q,0)=\Upsilon^-(p,q,\iota). \tag{138}$$

The existence of the isomorphisms $\varphi^\pm$ and these two different normal subgroups $\mathcal{A}(n+1)$ with elements $\Upsilon(p,0,\iota)$ and $\Upsilon(0,q,\iota)$ whose intersection is the center $\mathcal{Z}\simeq\mathcal{A}(1)$ is responsible for many of the remarkable properties of the Weyl-Heisenberg group. In fact, we shall see shortly that the choice of the normal subgroup in determining the unitary representations when applying the Mackey theorems results in unitary representations with either p or q diagonal.
The matrix realization corresponds to a coordinate system of the Lie group and is therefore not unique. The polarized matrix realizations are given by the $n+2$ dimensional square matrices

$$\Upsilon^+(p,q,\iota)=\begin{pmatrix}1 & q^{\mathrm{t}} & \iota\\ 0 & 1_n & p\\ 0 & 0 & 1\end{pmatrix},\Upsilon^-(p,q,\iota)=\begin{pmatrix}1 & p^{\mathrm{t}} & \iota\\ 0 & 1_n & q\\ 0 & 0 & 1\end{pmatrix}. \tag{139}$$

7. Appendix C: Extended Central Extension

The central extension for a group that is not connected group is not necessarily unique. The central extension for a group that is not connected may be defined by requiring exact sequences both for the cover of the group and the homomorphisms onto the discrete group for the components. For the $\mathbb{Z}_2\otimes_s\mathcal{H}\overline{\mathcal{S}p}(2n)$, these sequences are [19]

$$\begin{array}{ccccccccc}
 & & e & & e & & e & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \to & \mathbb{Z}\otimes\mathcal{A}(1) & \to & \mathcal{H}\overline{\mathcal{S}p}(2n) & \to & \mathcal{IS}p(2n) & \to & e\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \to & \mathbb{D} & \to & \mathbb{Z}_2\otimes_s\mathcal{H}\overline{\mathcal{S}p}(2n) & \to & \mathbb{Z}_2\otimes_s\mathcal{IS}p(2n) & \to & e\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & e & & e & & e & &
\end{array} \tag{140}$$

The solution is $\mathbb{D}\simeq\mathbb{Z}_2\otimes\mathbb{Z}\otimes\mathcal{A}(1)$. Therefore the central extension of $\mathbb{Z}_2\otimes_s\mathcal{IS}p(2n)$ is unique and is given by $\mathbb{Z}_2\otimes_s\mathcal{H}\overline{\mathcal{S}p}(2n)$.

8. Appendix D: Homomorphisms

Representations are homomorphisms of a group $\mathcal{G}$. If the homomorphism is an isomorphism, then the representation is said to be faithful and otherwise it is degenerate. Theorem 4 establishes that degenerate representations are faithful representations of groups homomorphic to $\mathcal{G}$. The homomorphisms can be characterized by the normal subgroups that are the kernel of the homomorphism.

First we consider the subgroup $\mathcal{H}\overline{\mathcal{S}p}(2n)$ that we have noted in (22) is the central extension of $\mathcal{IS}p(2n)$ with center

$$\mathcal{Z} = \mathbb{Z} \otimes \mathcal{A}(1) \tag{141}$$

where $\mathbb{Z}$ is the center of $\overline{\mathcal{S}p}(2n)$ and $\mathcal{A}(1)$ is the center of $\mathcal{H}(n)$ (31). The double cover of $\mathcal{S}p(2n)$ is the metaplectic group $\mathcal{M}p(2n)$. As $\mathbb{Z}_2$ is a normal subgroup of $\mathbb{Z}$, that there is also a homomorphism from the cover of the symplectic group to the metaplectic group

$$\pi : \overline{\mathcal{S}p}(2n) \to \mathcal{M}p(2n), \qquad \ker(\pi) \simeq \mathbb{Z}/\mathbb{Z}_2. \tag{142}$$

This gives the sequence of homomorphic groups where the homomorphisms have kernels that are subgroups of the center $\mathcal{Z}$.

$$\begin{array}{ccccccc} \mathcal{H}\overline{\mathcal{S}p}(2n) & \to & \mathcal{H}\mathcal{M}p(2n) & \to & \mathcal{H}\mathcal{S}p(2n) & & \\ & \searrow & & \searrow & & \searrow & \\ & & \mathcal{I}\overline{\mathcal{S}p}(2n) & \to & \mathcal{I}\mathcal{M}p(2n) & \to & \mathcal{I}\mathcal{S}p(2n). \end{array} \tag{143}$$

The group $\mathcal{IS}p(2n)$ that has a trivial center terminates the sequence. It is the maximal *classical* symmetry group. The projective representations of any of the groups in this sequence is equivalent to the unitary representations of the $\mathcal{H}\overline{\mathcal{S}p}(2n)$. The above expressions also apply to the full group $\mathbb{Z}_2 \otimes_s \mathcal{H}\overline{\mathcal{S}p}(2n)$ by prefixing "$\mathbb{Z}_2 \otimes_s$" onto each of the groups that appear in (142).

In addition to the above homomorphisms that have abelian kernels, we have the additional homomorphisms

$$\pi : \mathbb{Z}_2 \otimes_s \mathcal{H}\overline{\mathcal{S}p}(2n) \to \mathcal{K}, \ker(\pi) = \mathcal{N}, \tag{144}$$

with

$$\begin{array}{ll} \mathcal{N} & \mathcal{K} \\ \mathcal{H}(n) & \mathbb{Z}_2 \otimes \overline{\mathcal{S}p}(2n) \\ \mathbb{Z}/\mathbb{Z}_2 \otimes \mathcal{H}(n) & \mathbb{Z}_2 \otimes \mathcal{M}p(2n) \\ \mathbb{Z} \otimes \mathcal{H}(n) & \mathbb{Z}_2 \otimes \mathcal{S}p(2n) \\ \mathcal{H}\mathcal{S}p(2n) & \mathbb{Z} \otimes \mathbb{Z}_2 \\ \mathcal{H}\mathcal{M}p(2n) & \mathbb{Z}_2 \otimes \mathbb{Z}_2 \\ \mathcal{H}\overline{\mathcal{S}p}(2n) & \mathbb{Z}_2 \end{array} \tag{145}$$

References

[1] Weyl, H. (1927). *Quantenmechanik und Gruppentheorie.* Zeitscrift fur Physik, **46**, 1–46.

[2] Folland, G. B. (1989). *Harmonic Analysis on Phase Space.* Princeton: Princeton University Press.

[3] Stone, M. H. (1932). *On one-parameter unitary groups in Hilbert Space.* Annals Math. , **33**, 643–648.

[4] von Neumann, J. (1932). *Ueber Einen Satz Von Herrn M. H. Stone.* Annals. Math., **33**, 567–573.

[5] Wigner, E. P. (1939). *On the unitary representations of the inhomogeneous Lorentz group.* Annals of Math., **40**, 149–204 .
[6] Weinberg, S. (1995). *The Quantum Theory of Fields, Volume 1.* Cambridge: Cambridge.
[7] Bargmann, V. (1954). *On Unitary Ray Representations of Continuous Groups.* Annals Math., **59**, 1–46.
[8] Mackey, G. W. (1958). *Unitary Representations of Group Extensions. I.* Acta Math., **99**, 265–311.
[9] de Gosson, M. (2006). *Symplectic Geometry and Quantum Mechanics.* Berlin: Birkhauser.
[10] Campoamor-Stursberg, R., & Low, S. G. (2009). *Virtual copies of semisimple Lie algebras in enveloping algebras of semidirect products and Casimir operators.* J. Phys. A, **42**, 065205. http://arxiv.org/abs/0810.4596
[11] Major, M. E. (1977). *The quantum mechanical representations of the anisotropic harmonic oscillator group.* J. Math. Phys., **18**, 1938–1943.
[12] Mackey, G. W. (1976). *The theory of unitary group representations.* Chicago: University of Chicago Press.
[13] Weil, A. (1964). *Sur certains groupes d'opérateurs unitaires.* Acta *Math.*, **111**, 143–211.
[14] Low, S. G. (2006). *Reciprocal relativity of noninertial frames and the quaplectic group.* Found. Phys., **36**(6), 1036–1069. http://arxiv.org/abs/math-ph/0506031
[15] Low, S. G. (2007). *Reciprocal relativity of noninertial frames: quantum mechanics.* J. Phys A, **40**, 3999–4016. http://arxiv.org/abs/math-ph/0606015
[16] Low, S. G. (2012). *Relativity Implications of the Quantum Phase.* J. Phys.: Conf. Ser., **343** , 012069.
[17] Sternberg, S. (1994). *Group theory and physics.* Cambridge: Cambridge Press.
[18] Barut, A. O., & Raczka, R. (1986). *Theory of Group Representations and Applications.* Singapore: World Scientific.
[19] Azcarraga, J. A., & Izquierdo, J. M. (1998). *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics.* Cambridge: Cambridge University Press.