On Orbit Equivalence of Quasiconformal Anosov Flows

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Abstract – *We classify up to C^∞ orbit equivalence the volume-preserving quasiconformal Anosov flows whose strong stable and strong unstable distributions are at least three-dimensional. If one of the strong distributions is two-dimensional, then we get a partial classification. Using these classification results, we obtain the following rigidity result :*

Let Φ the the orbit foliation of the geodesic flow of a closed hyperbolic manifold of dimension at least three. Let Ψ be another C^∞ one-dimensional foliation. If Φ is C^1 conjuguate to Ψ, then Φ is C^∞ conjuguate to Ψ.

Résumé – *Nous classifions, à équivalence orbitale C^∞ près, les flots d'Anosov quasiconformes topologiquement transitifs dont les distributions stable forte et instable forte sont de dimension au moins 3. Si ces deux distributions sont de dimension au moins 2, alors nous obtenons une classification partielle. Nous déduisons de ces résultats de classification le résultat de rigidité suivant :*

Soit Φ le feuilletage orbitale du flot géodésique d'une variété hyperbolique fermée de dimension au moins 3. Soit Ψ un autre feuilletage C^∞ de dimension 1. Si Φ et Ψ sont C^1 conjugué, alors Φ et Ψ sont C^∞ conjugué.

1. Introduction

1.1. Motivation

Let M be a C^∞-closed manifold. A C^∞-flow ϕ_t generated by the non-singular vector field X is said to be an *Anosov flow* if there exists a ϕ_t-invariant splitting of the tangent bundle

$$TM = \mathbb{R}X \oplus E^+ \oplus E^-,$$

a Riemannian metric on M and two positive numbers a and b such that for any $u^{\pm} \in E^{\pm}$ and for any $t \geq 0$,

$$\| D\phi_{\mp t}(u^{\pm}) \| \leq ae^{-bt} \| u^{\pm} \|,$$

where E^{-} and E^{+} are said to be the *strong stable* and *strong unstable* distributions of the flow. For any $x \in M$ the leaves containing x of E^{+} and E^{-} are denoted respectively by W_x^{+} and W_x^{-}.

Define two functions on $M \times \mathbb{R}$ as following:

$$K^{+}(x,t) = \frac{max\{\| D\phi_t(u) \| \mid u \in E_x^{+},\ \| u \|= 1\}}{min\{\| D\phi_t(u) \| \mid u \in E_x^{+},\ \| u \|= 1\}}$$

and

$$K^{-}(x,t) = \frac{max\{\| D\phi_t(u) \| \mid u \in E_x^{-},\ \| u \|= 1\}}{min\{\| D\phi_t(u) \| \mid u \in E_x^{-},\ \| u \|= 1\}}.$$

If K^{-} (K^{+}) is bounded, then the Anosov flow ϕ_t is said to be *quasiconformal on the stable (unstable) distribution.* If K^{+} and K^{-} are both bounded, then ϕ_t is said to be *quasiconformal.* If it is the case, then the superior bound of K^{+} and K^{-} is said to be the *distortion* of ϕ_t. The corresponding notions for Anosov diffeomorphisms are defined similarly (see **[Sa]**).

Recall that two C^{∞} Anosov flows $\phi_t : M \to M$ and $\psi_t : N \to N$ are said to be C^k *flow equivalent* ($k \geq 0$) if there exists a C^k diffeomorphism $h : M \to N$ such that $\phi_t = h^{-1} \circ \psi_t \circ h$ for all $t \in \mathbb{R}$. They are said to be C^k *orbit equivalent* ($k \geq 0$) if there exists a C^k diffeomorphism $h : M \to N$ sending the orbits of ϕ_t onto the orbits of ψ_t such that the orientations of the orbits are preserved. Similarly, two C^{∞} foliations Φ and Ψ are said to be C^k *conjuguate* if there exists a C^k diffeomorphism $h : M \to N$ sending the leaves of Φ onto those of Ψ. By convention, a C^0 diffeomorphism means a homeomorphism. We can prove easily the following

Lemma 1.1. *Let ϕ_t and ψ_t be two C^{∞} Anosov flows. If they are C^1 orbit equivalent and ψ_t is quasiconformal, then ϕ_t is also quasiconformal.*

Proof. Denote by $\phi : M \to N$ the C^1 orbit conjugacy between ϕ_t and ψ_t and by $\bar{\mathcal{F}}^{\pm}$ and $\bar{\mathcal{F}}^{\pm,0}$ the Anosov foliations of ψ_t. Then we have the following

Sublemma. *Under the notations above, $\phi(\mathcal{F}^{\pm,0}) = \bar{\mathcal{F}}^{\pm,0}$.*

Proof. Define $\hat{\phi}_t = \phi \circ \phi_t \circ \phi^{-1}$. Then $\hat{\phi}_t$ is a C^1 flow on N with the same orbits as ψ_t. So there exists a C^1 map $\alpha : \mathbb{R} \times N \to \mathbb{R}$ such that

$\hat{\phi}_t(x) = \psi_{\alpha(t,x)}(x)$. Define $\hat{E}^- = (D\phi)(E^-)$. Then it is the C^0 tangent bundle of the C^1 foliation $\hat{\mathcal{F}}^- = \phi(\mathcal{F}^-)$.

Let us prove first that $\hat{E}^- \subseteq \bar{E}^{-,0}$. Fix a C^0 Riemannian metric g on N such that $\bar{E}^+$ and $\bar{E}^-$ and $\bar{X}$ are orthogonal to each other. Since ϕ is C^1, then it is bi-Lipschitz. We deduce that for all $x \in N$ and $\hat{u} \in \hat{E}^-_x$, $\| D\hat{\phi}_t(\hat{u}) \| \to 0$ if $t \to +\infty$.

If $\hat{u} = \bar{u}^+ + a\bar{X}_x + \bar{u}^-$ and $\bar{u}^+ \neq 0$, then by a simple calculation we get for a certain function $b_x : \mathbb{R} \to \mathbb{R}$,

$$D\hat{\phi}_t(\hat{u}) = D(\psi_{\alpha(t,x)})(\bar{u}^+) + b_x(t) \cdot \bar{X}_{\hat{\phi}_t(x)} + D(\psi_{\alpha(t,x)})(\bar{u}^-).$$

So we get $\| D\hat{\phi}_t(\hat{u}) \| \geq \| D(\psi_{\alpha(t,x)})(\bar{u}^+) \| \to +\infty$ if $t \to +\infty$, which is a contradiction. We deduce that $\hat{E}^- \subseteq \bar{E}^{-,0}$. So $\phi(\mathcal{F}^0) \subseteq \bar{\mathcal{F}}^{-,0}$. Then it is easy to see that $\phi(\mathcal{F}^{-,0}) = \bar{\mathcal{F}}^{-,0}$, i.e. ϕ sends C^1 diffeomorphically each leaf of $\mathcal{F}^{-,0}$ onto a leaf of $\bar{\mathcal{F}}^{-,0}$. Similarly we have $\phi(\mathcal{F}^{+,0}) = \bar{\mathcal{F}}^{+,0}$. □

In particular we deduce from the sublemma above that $\hat{E}^\pm \oplus \mathbb{R}\bar{X} = \bar{E}^\pm \oplus \mathbb{R}\bar{X}$. By projecting parallel to the direction of $\bar{X}$, we get a C^0 section P of $End(\hat{E}^+, \bar{E}^+)$ and two positive constants A_1 and A_2 such that

$$A_1 \| u \| \leq \| P(u) \| \leq A_2 \| u \|, \ \forall \ u \in \hat{E}^+.$$

Then it is easy to verify that for all $x \in N$, $P \circ D_x\hat{\phi}_t = D_x\psi_{\alpha(t,x)} \circ P$.

If ψ_t is quasiconformal with distortion K, then we have the following estimation

$$\hat{K}^+(x,t) = \frac{sup\{\| D\hat{\phi}_t(\hat{u}^+) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}{inf\{\| D\hat{\phi}_t(\hat{u}^+) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}$$

$$\leq \frac{A_2}{A_1} \cdot \frac{sup\{\| P(D\hat{\phi}_t(\hat{u}^+)) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}{inf\{\| P(D\hat{\phi}_t(\hat{u}^+)) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}$$

$$\leq (\frac{A_2}{A_1})^2 \cdot \frac{sup\{\| D_x\psi_{\alpha(t,x)}(\frac{P(\hat{u}^+)}{\|P(\hat{u}^+)\|}) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}{inf\{\| D_x\psi_{\alpha(t,x)}(\frac{P(\hat{u}^+)}{\|P(\hat{u}^+)\|}) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}$$

$$\leq (\frac{A_2}{A_1})^2 \cdot K.$$

Similarly $\hat{K}^-$ is also bounded. So $\hat{\phi}_t$ is quasiconformal. Since ϕ is a C^1 diffeomorphism, then it is bi-Lipschitz. We deduce that ϕ_t is also quasiconformal. □

It is easy to see that the geodesic flow of a closed hyperbolic manifold is quasiconformal (even conformal). Then by the previous lemma, each C^∞ time change of such a flow is also quasiconformal. It is easily seen that an Anosov diffeomorphism is quasiconformal iff its suspension is quasiconformal. So if ϕ denotes a semisimple hyperbolic automorphism of a torus with two enginvalues, then its suspension is a quasiconformal Anosov flow.

It seems to be a common phenomena in mathematics that things can only be effectively studied and understood when placed in a suitable and flexible environment. Conformal structures (Anosov flows) are pretty rigid while quasiconformal structures (Anosov flows) seem to be much more flexible. We wish to better understand the classical conformal Anosov flows, notably the geodesic flows of closed hyperbolic manifolds, by using quasiconformal techniques, which is our motivation to study general quasiconformal Anosov systems.

1.2. Main theorems

In our previous paper **[Fa1]**, we have studied the rigidity of volume-preserving Anosov flows with smooth $E^+ \oplus E^-$. In particular we have obtained the following

Theorem 1.1. ([**Fa1**], Corollary 1) *Let ϕ be a C^∞ volume-preserving quasiconformal Anosov diffeomorphism on a closed manifold Σ. If the dimensions of E^+ and E^- are at least two, then up to finite covers ϕ is C^∞ conjugate to a hyperbolic automorphism of a torus.*

In **[K-Sa]**, B. Kalinin and V. Sadovskaya have classified the topologically transitive quasiconformal Anosov diffeomorphisms whose strong stable and unstable distributions are of dimension at least 3. Their argument, though quite elegant, meets an essential difficulty in the case that one of the strong stable and unstable distributions is two dimensional.

Based on the classification result in **[K-Sa]**, we classify in this paper completely the quasiconformal Anosov flows whose strong stable and unstable distributions have relatively high dimensions. More precisely, we prove

Theorem 1.2. *Let ϕ_t be a C^∞ topologically transitive quasiconformal Anosov flow such that E^+ and E^- are at least three dimensional. Then up to finite covers, ϕ_t is C^∞ orbit equivalent either to the geodesic flow of a hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.*

Under the conditions above, if $E^+ \oplus E^-$ is in addition C^1, then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to a canonical time change of the geodesic flow of a hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.

Recall that if α is a closed C^∞ 1-form on M such that $1 + \alpha(X) > 0$, then the flow of $\frac{X}{1+\alpha(X)}$ is said to be a *canonical time change* of ϕ_t, where X denotes the generator of ϕ_t.

If one of the strong distributions is two-dimensional, then by using our Theorem 1.1, we get the following partial result.

Theorem 1.3. *Let ϕ_t be a C^∞ volume-preserving quasiconformal Anosov flow such that E^+ is of dimension two and E^- is of dimension at least two. If ϕ_t has the sphere-extension property, then up to finite covers, ϕ_t is C^∞ orbit equivalent either to the geodesic flow of a three-dimensional hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.*

Under the conditions above, if $E^+ \oplus E^-$ is in addition C^1, then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to a canonical time change of the geodesic flow of a three-dimensional hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.

The sphere-extension property will be defined in Subsection 2.3. Let us just mention that this property is invariant under C^1 orbit equivalence.

Now we can get some concrete applicaions of our results above. Recall at first that flow conjugacies between Anosov flows have been and being extensively studied. The philosophical conclusion is that they exist rarely, even C^0 ones. Let us just mention two of the most beautiful supporting results (see also **[L1]** and **[L2]**) :

Theorem 1.4. (U. Hamenstädt, **[Ham2]**) *Let M be a closed negatively curved manifold. If the geodesic flow of M is C^0 flow equivalent to that of a locally symmetric space of rank one N, then M is isometric to N.*

Theorem 1.5. (R. de la Llave and R. Moriyon, **[LM]**) *Let ϕ_t and ψ_t be two C^∞ three-dimensional volume-preserving Anosov flows. If they are C^1 flow equivalent, then they are C^∞ flow equivalent.*

However there exist plenty of C^0 orbit conjugacies between Anosov flows. For example, if two C^∞ Anosov flows are sufficiently C^1-near, then they are

Hölder-continuous orbit equivalent by the celebrated structural stability (see [**An**]). A natural question to ask is whether C^1 orbit conjugacies between Anosov flows are rare.

We can deduce from Theorems 1.1 and 1.2 the following result showing that C^1 orbit conjugacies are surely rare in some cases, while Hölder-continuous orbit conjugacies are abundant.

Theorem 1.6. *Let ϕ_t be a C^∞ Anosov flow and ψ_t be the geodesic flow of a closed hyperbolic manifold of dimension at least three. If ϕ_t and ψ_t are C^1 orbit equivalent, then they are C^∞ orbit equivalent.*

In order to state the next result, let us recall firstly some notions.

Definition 1.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. Then they are said to be *quasi-isometric* if there a map $f : X \to Y$ and two positive numbers C and D such that the following two conditions are satisfied:
(1) $\frac{1}{C} d_X(x,y) - D \leq d_Y(f(x), f(y)) \leq C d_X(x,y) + D, \ \forall \ x, y \in X$.
(2) For any $y \in Y$, there exists $x \in X$ such that $d_Y(y, f(x)) \leq D$.

Roughly speaking, two metric spaces are quasi-isometric if and only if they are bi-Lipschitz equivalent in the large scale. Recall that for any $n \geq 2$, a n-dimensional Riemannian manifold M is said to be *hyperbolic* if it has constant sectional curvature -1. We denote by $\mathbb{H}^n$ the unique simply connected hyperbolic manifold. By combining some classical results with our previous theorem, we get the following

Corollary 1.1. *Let M be a n-dimensional closed Riemannian manifold of negative curvature such that $n \geq 3$. Then we have the following relations between dynamics and geometry:*
(1) *The geodesic flow of M is Hölder-continuously orbit equivalent to that of a hyperbolic manifold if and only if the universal covering space $\widetilde{M}$ with its lifted metric is quasi-isometric to $\mathbb{H}^n$.*
(2) *The geodesic flow of M is C^1 orbit equivalent to that of a hyperbolic manifold if and only if M has constant negative curvature.*

Let g be the Riemannian metric of a closed hyperbolic manifold of dimension at least three. Let g' be a perturbed Riemannian metric of g with non-constant negative curvature. Thus by the proof of the previous corollary, the orbit foliation of the geodesic flow of g is C^0 conjuguate to that of g'. However, by the previous corollary, these two C^∞ one-dimensional foliations are not C^1 conjuguate.

By combining a result of R. Manẽ with our Theorem 1.6, we get finally the following corollary, which is the key objective of this article.

Corollary 1.2. *Let Φ be the orbit foliation of the geodesic flow of a closed hyperbolic manifold of dimension at least three. Let Ψ be another C^∞ one-dimensional foliation. If Φ and Ψ are C^1 conjuguate, then they are C^∞ conjuguate.*

1.3. The organization of the paper

In Section two we recall some fundamental facts concerning quasiconformal Anosov flows and some properties of transverse (G,T)-structures of foliations. Then in Section three we prove Theorems 1.2 and 1.3. Finally in Section four we apply these two results to geodesic flows to deduce Theorem 1.6 and Corollaries 1.1 and 1.2.

2. Preliminaries

2.1. Linearizations and smooth conformal structures

Let us recall firstly the following two results established in [**Sa**]:

Theorem 2.1. ([**Sa**], Theorem 1.3) *Let f be a topologically transitive C^∞ Anosov diffeomorphism (ϕ_t be a topologically mixing C^∞ Anosov flow) on a closed manifold M which is quasiconformal on the unstable distribution. Then it is conformal with respect to a Riemannian metric on this distribution which is continuous on M and C^∞ along the leaves of the unstable foliation.*

Theorem 2.2. ([**Sa**], Theorem 1.4) *Let f (ϕ_t) be a C^∞ Anosov diffeomorphism (flow) on a closed manifold M with $dimE^+ \geq 2$. Suppose that it is conformal with respect to a Riemannian metric on the unstable distribution which is continuous on M and C^∞ along the leaves of the unstable foliation. Then the (weak) stable holonomy maps are conformal and the (weak) stable distribution is C^∞.*

Let us recall briefly the steps to prove these two theorems in the case of flow. Denote by ϕ_t a topologically mixing quasiconformal Anosov flow. Then by some classical arguments (see [**Su**] and [**Tu**]), V. Sadovskaya found two measurable ϕ_t-invariant conformal structures $\bar{\tau}^+$ and $\bar{\tau}^-$ along respectively the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$. Then as is usual for Anosov flows, these two conformal structures were pertubated to continuous ϕ_t-invariant ones denoted by τ^+ and τ^-. Using the linearizations, she proved that along each leaf of $\mathcal{F}^+$, τ^+ is isometric to a vector space with its canonical conformal struture, which has permitted her to blow up the smoothness of weak holonomy maps. Then by using a result of J. L. Journé, she proved the smoothness of the weak stable and unstable distributions.

In **[Fa1]**, we have proved the following lemma based on **[Pl1]**. For the sake of completeness, let us recall the arguments.

Lemma 2.1. *Let ϕ_t be a C^∞ topologically transitive Anosov flow. Then we have the following alternative:*
(1) *ϕ_t is topologically mixing,*
(2) *ϕ_t admits a C^∞ closed global section with constant return time.*

Proof. If Case (2) is true, then up to a constant change of time scale, ϕ_t is C^∞ flow equivalent to the suspension of a C^∞ Anosov diffeomorphism. Thus it is not topologically mixing. So the alternative is exclusive.

If there exists $x \in M$ such that W_x^+ is not dense in M, then by Theorem 1.8 of **[Pl1]** $E^+ \oplus E^-$ is the tangent bundle of a C^1 foliation $\mathcal{F}$. In addition the leaves of $\mathcal{F}$ are all comact. So ϕ_t admits a C^1 closed global section with constant return time. Then to realize Case (2) we need only prove that $E^+ \oplus E^-$ is C^∞.

Denote by λ the canonical 1-form of ϕ_t. Then λ is, a priori, a continuous 1-form on M. For each point $y \in M$ we take a small neighborhood F_y of y in the leaf containing y of $\mathcal{F}$. Then we can construct a local C^1 chart $\theta_y : (-\epsilon, \epsilon) \times F_y \to M$ such that $\theta_y(t, z) = \phi_t(z)$. In this chart we have $\lambda = dt$. We deduce that $\int_\gamma \lambda = 0$ for each piecewise C^1 closed curve γ contained in the image of θ_y. So λ is locally closed (see Section two of **[Pl1]** for the definition). Then by Proposition 2.1 of **[Pl1]**, λ is seen to be closed in a weak sense, i.e. for every C^1 immersed two-disk σ such that $\partial\sigma$ is piecewise C^1,

$$\int_{\partial\sigma} \lambda = 0.$$

So by integrating along the closed curves, λ gives an element in $Hom(\pi_1(M), \mathbb{R})$, i.e. the space of group homomorphisms of $\pi_1(M)$ into $\mathbb{R}$, where $\pi_1(M)$ denotes the fundamental group of M. However we have naturally

$$Hom(\pi_1(M), \mathbb{R}) \cong (H_1(M, \mathbb{R}))^* \cong H^1(M, \mathbb{R}),$$

where $H^1(M, \mathbb{R})$ denotes the first de Rham cohomology group of M. So there exists a C^∞ 1-form β such that for each C^∞ closed curve γ,

$$\int_\gamma \lambda = \int_\gamma \beta.$$

So by integrating $(\lambda - \beta)$ along curves, we get on M a well-defined continuous function f. Then for any $y \in M$ and any $t \in \mathbb{R}$ we have

$$(f \circ \phi_t)(y) - f(y) = \int_0^t (1 - \beta(X))(\phi_s(y))ds.$$

Since the right-hand side of this identity is a C^∞ $\mathbb{R}$-cocycle, then by **[LMM]** f is smooth. However by the definition of f we have

$$\lambda - \beta = df.$$

So λ is also smooth. We deduce that $E^+ \oplus E^- (= Ker\lambda)$ is C^∞. So Case (2) is realized if W_x^+ is not dense for a certain point $x \in M$.

Now suppose that for all $x \in M$, W_x^+ is dense in M. Fix a Riemannian metric on M. For any $x \in M$ and any $r > 0$ we denote by $M_{x,r}$ and $W_{x,r}^+$ the balls of center x and radius r in M and W_x^+. Take arbitrarily two open subsets U and V in M and a small ball $M_{y,\epsilon}$ in V. Since M is closed and each strong unstable leaf is supposed to be dense in M, then we can find $R < +\infty$ such that

$$W_{x,R}^+ \cap M_{y,\epsilon} \neq \emptyset, \ \forall\, x \in M.$$

Take a small disk $W_{x,\delta}^+$ in U. Then by the Anosov property there exists $T > 0$ such that

$$\phi_t(W_{x,\delta}^+) \supseteq W_{\phi_t(x),R}^+, \ \forall\, t \geq T.$$

So $\phi_t U \cap V \neq \emptyset$, $\forall\, t \geq T$, i.e. ϕ_t is topologically mixing. $\Box$

If ϕ_t is a topologically transitive quasiconformal Anosov flow such that the dimensions of E^+ and E^- are at least two, then by the previous lemma and Theorem 2.1, it preserves two continuous conformal strutures τ^+ and τ^- which are C^∞ along the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$. Then by Theorem 2.2, these two conformal structures are invariant under the weak holonomy maps. In addition we can deduce from the previous lemma and Theorem 2.2 the following

Lemma 2.2. *Let ϕ_t be a C^∞ topologically transitive quasiconformal Anosov flow such that the dimensions of E^+ and E^- are at least two. Then $E^{+,0}$ and $E^{-,0}$ are both C^∞.*

Based on **[Sa]**, we have found in **[Fa1]** for each quasiconformal Anosov flow ϕ_t an *unstable linearization* $\{h_x^+\}_{x\in M}$ such that for any $x \in M$, $h_x^+ : W_x^+ \to E_x^+$ is a C^∞ diffeomorphism and the following conditions are satisfied:
(1) $h_{\phi_s(x)}^+ \circ \phi_s = D_x\phi_s \circ h_x^+,\ \forall\ s \in \mathbb{R}$,
(2) $h_x^+(x) = 0$ and $(Dh_x^+)_x$ is the identity map,
(3) h_x^+ depends continuously on x in the C^∞ topology.

Recall that the family of diffeomorphisms satisfying these three conditions is unique. Similarly we have the *stable linearization* $\{h_x^-\}_{x\in M}$ of ϕ_t.

Suppose that ϕ_t satisfies the conditions of Lemma 2.2. For any $x \in M$ we can extend the conformal structure τ_x^+ at $0 \in E_x^+$ to all other points of E_x^+ via linear translations. If the resulting translation-invariant conformal structure on E_x^+ is denoted by σ_x^+, then (E_x^+, σ_x^+) is isometric to the canonical conformal structure of $\mathbb{R}^n$ if E^+ is n-dimensional. Similarly we have (E_x^-, σ_x^-) for any $x \in M$.

Then by Lemma 3.1 of **[Sa]** we get for any $x \in M$, $(h_x^+)_*\tau^+ \mid_{W_x^+} = \sigma_x^+$ and $(h_x^-)_*\tau^- \mid_{W_x^-} = \sigma_x^-$.

2.2. Transverse (G, T)-structures

In this subsection, we consider the transverse (G,T)-structures of foliations. Let $\mathcal{F}$ be a C^∞ foliation on a connected manifold M. Denote by $Q_\mathcal{F}$ the leaf space of $\mathcal{F}$ and by $\mathcal{F}(A)$ the saturation of $\mathcal{F}$ on A for any $A \subseteq M$. We assume that the holonomy maps of $\mathcal{F}$ are defined on connected transverse sections.

Let G be a real Lie group acting effectively and transitively on a connected manifold T. If Σ is a C^∞ transverse section of $\mathcal{F}$ and ϕ is a C^∞ diffeomorphism of Σ onto its open image in T, then (Σ, ϕ) is said to be a *transverse T-chart*. Two transverse T-charts (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are said to be *compatible* if for each holonomy map h of a germ of Σ_1 to a germ of Σ_2, the map $\phi_1 \circ h \circ \phi_2^{-1}$ is locally the restriction of elements of G.

A family of transverse sections is said to be *covering* if each leaf of $\mathcal{F}$ intersects at least one of the sections in this family. By definition, a *transverse (G,T)-structure* on $\mathcal{F}$ is a maximal family of compatible transverse T-charts of which the underlying family of transverse sections is covering.

In order to define a transverse (G,T)-structure, we need just seperate out a family of covering compatible T-charts. Then by considering all the T-charts compatible with this family, we get automatically a transverse (G,T)-structure.

Denote by $\widetilde{\mathcal{F}}$ the lifted foliation on the universal covering space $\widetilde{M}$ of M and denote by π the projection of $\widetilde{M}$ onto M. For each transverse (G,T)-structure on $\mathcal{F}$, we get naturally a lifted transverse (G,T)-structure on $\widetilde{\mathcal{F}}$ by considering the composition of π with the T-charts of the given transverse (G,T)-structure on $\mathcal{F}$. Then by **[Go]** there exists a C^∞ submersion $\mathcal{D} : \widetilde{M} \to T$ and a group homomorphism $\mathcal{H} : \pi_1(M) \to G$ satisfying the following two conditions:
(1) $\mathcal{D}(\gamma x) = \mathcal{H}(\gamma)\mathcal{D}(x)$, $\forall\, x \in \widetilde{M}$, $\forall\, \gamma \in \pi_1(M)$.
(2) The lifted foliation $\widetilde{\mathcal{F}}$ is defined by $\mathcal{D}$.
This submersion $\mathcal{D}$ is said to be the *developing map* of the transverse (G,T)-structure of $\mathcal{F}$ and $\mathcal{H}$ is said to be the *holonomy representation* of $\mathcal{D}$. The transverse (G,T)-structure of $\mathcal{F}$ is said to be *complete* if $\mathcal{D}$ is a C^∞ fibre bundle over $\mathcal{D}(\widetilde{M})$.

If $\mathcal{D}'$ denotes another developing map with holonomy representation $\mathcal{H}'$, then by **[Go]** there exists a unique element $g \in G$ such that $\mathcal{D}' = g \circ \mathcal{D}$ and $\mathcal{H}' = g \cdot \mathcal{H} \cdot g^{-1}$.

Since $\mathcal{D}$ is obtained by analytic continuation along curves (see **[Go]**), then for each transverse section Σ of $\widetilde{\mathcal{F}}$ such that $\mathcal{D}\mid_\Sigma$ is a C^∞ diffeomorphism onto its image, $\mathcal{D}\mid_\Sigma$ is a transverse T-chart of the lifted transverse (G,T)-structure of $\widetilde{\mathcal{F}}$.

Since $\widetilde{\mathcal{F}}$ is defined by the submersion $\mathcal{D}$, then $\mathcal{D}$ sends each leaf of $\widetilde{F}$ to a point of T. Thus for any $x \in \widetilde{M}$ there exists a small C^∞ transverse section Σ containing x of $\widetilde{\mathcal{F}}$ such that each leaf of $\widetilde{\mathcal{F}}$ intersects Σ at most once. Then it is easily seen that each leaf of $\widetilde{\mathcal{F}}$ is closed in $\widetilde{M}$.

Denote by $Q_{\widetilde{\mathcal{F}}}$ the leaf space of $\widetilde{\mathcal{F}}$. Then we have the quotient map $\bar{\mathcal{D}} : Q_{\widetilde{\mathcal{F}}} \to \mathcal{D}(\widetilde{M})$. Since each leaf of $\widetilde{\mathcal{F}}$ is closed, then $\bar{\mathcal{D}}$ is bijective iff the $\mathcal{D}$-inverse image of each point of T is connected. If this is the case, then by considering the projections of small transverse T-charts, $Q_{\widetilde{\mathcal{F}}}$ becomes naturally a C^∞ (seperable) manifold such that $\bar{\mathcal{D}}$ is a C^∞ diffeomorphism of $Q_{\widetilde{\mathcal{F}}}$ onto $\mathcal{D}(\widetilde{M})$. In addition, the fundamental group $\pi_1(M)$ of M acts naturally on $Q_{\widetilde{\mathcal{F}}}$.

By **[Hae]** we have the following

Proposition 2.1. *Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be two C^∞ foliations with complete transverse (G,T)-structures. Suppose that their developing maps have both connected fibres and the holonomy covers of their leaves are all contractible. If the $\pi_1(M_1)$-action on $Q_{\widetilde{\mathcal{F}_1}}$ is C^∞ conjugate to the $\pi_1(M_2)$-action on $Q_{\widetilde{\mathcal{F}_2}}$, then there exists a C^∞ map $f : M_1 \to M_2$ such that the following conditions are satisfied:*

(1) *f is a surjective homotopy equivalence.*
(2) *f sends each leaf of $\mathcal{F}_1$ onto a leaf of $\mathcal{F}_2$ and f sends different leaves to different leaves.*
(3) *f is transversally a local C^∞ diffeomorphism conjugating the two transverse (G,T)-structures.*

The lemma below is self-evident and will be used several times in the following.

Lemma 2.3. *Let $(M,\mathcal{F})$ be a C^∞ foliation with a transverse (G,T)-structure. If T_1 is an open subset of T and G_1 is a closed Lie subgroup of G acting transitively on T_1 such that $\mathcal{D}(\widetilde{M}) \subseteq T_1$ and $\mathcal{H}(\pi_1(M)) \subseteq G_1$, then $\mathcal{F}$ admits a transverse (G_1,T_1)-structure with the same developing map $\mathcal{D}$ and the same holonomy representation $\mathcal{H}$, which is compatible with the initial transverse (G,T)-structure.*

By adapting the arguments in **[Gh2]** we can prove the following

Lemma 2.4. *Let $(M,\mathcal{F})$ be a C^∞ foliation with a transverse (G,T)-structure. If the closed leaves of $\mathcal{F}$ are dense in M and the $\mathcal{D}$-inverse image of each point of T is connected, then $\mathcal{H}(\pi_1(M))$ is a discrete subgroup of G.*

Proof. Denote by π_1 the projection of $\widetilde{M}$ onto $Q_{\widetilde{\mathcal{F}}}$ and by π_2 the projection of $Q_{\widetilde{\mathcal{F}}}$ onto $Q_{\mathcal{F}}$ the leaf space of $\mathcal{F}$. Take a closed leaf F_x of $\mathcal{F}$ and $\widetilde{x} \in \widetilde{M}$ such that $\pi(\widetilde{x}) = x$. Since F_x is closed, then we can find a fine transverse section Σ passing through $\widetilde{x}$ such that π sends Σ diffeomorphically onto its image and $F_x \cap \pi(\Sigma) = \{x\}$. So for each $y \in \Sigma$ and $y \neq \widetilde{x}$, $\pi(y)$ is not in F_x. Thus $\pi_2^{-1}(F_x)$ is discrete in $Q_{\widetilde{\mathcal{F}}}$.

Since F_x is closed and $\pi^{-1}(F_x) = \pi_1^{-1}(\pi_2^{-1}(F_x))$, then $\pi_2^{-1}(F_x)$ is closed in $\widetilde{Q}_{\mathcal{F}}$. So the $\pi_1(M)$-orbit of $\widetilde{F}_{\widetilde{x}}$, i.e. $\pi_2^{-1}(F_x)$ is closed and discrete in $Q_{\widetilde{\mathcal{F}}}$.

Since the closed leaves of $\mathcal{F}$ are dense in M i.e. the union of all the closed leaves is dense, then π_2-inverse images of these closed leaves form a dense subset P of $Q_{\widetilde{\mathcal{F}}}$ such that the $\pi_1(M)$-orbit of each point of P is closed and discrete.

Suppose on the contrary that $\mathcal{H}(\pi_1(M))$ is not discrete in G. Since the $\mathcal{D}$-inverse image of each point of T is connected, then $\mathcal{D}$ induces a C^∞ diffeomorphism $\bar{\mathcal{D}} : Q_{\widetilde{\mathcal{F}}} \to \mathcal{D}(\widetilde{M})$. So $\bar{\mathcal{D}}(P)$ is dense in $\mathcal{D}(\widetilde{M})$ and the $\mathcal{H}(\pi_1(M))$-orbit of each point of $\bar{\mathcal{D}}(P)$ is discrete and closed in $\mathcal{D}(\widetilde{M})$.

Take a non-trivial one-parameter subgroup g_t of the closure of $\mathcal{H}(\pi_1(M))$

in G. For each $t \in \mathbb{R}$, g_t preserves the closed complement of $\mathcal{D}(\widetilde{M})$. So we have

$$g_t(\mathcal{D}(\widetilde{M})) = \mathcal{D}(\widetilde{M}).$$

Thus g_t fixes each point in $\bar{\mathcal{D}}(P)$. We deduce that g_t is a trivial one-parameter subgroup, which is a contradiction. $\square$

2.3. Sphere-extension property

Let M be a C^∞ manifold. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous foliations with C^1 leaves on M such that

$$T\mathcal{F}_1 \oplus T\mathcal{F}_2 = TM.$$

If $\mathcal{F}_1$ is a foliation by planes, i.e. each leaf of $\mathcal{F}_1$ is C^1 diffeomorphic to a certain $\mathbb{R}^n$, then $(\mathcal{F}_1, \mathcal{F}_2)$ is said to be a *plane foliation couple.* The local leaves of $\mathcal{F}_1$ are natural transverse sections of $\mathcal{F}_2$ and we consider only the holonomy maps of $\mathcal{F}_2$ with respect to these special transverse sections.

For each leaf $F_{1,x}$ of $\mathcal{F}_1$ we denote by $S_{1,x}$ its one-point compactification which is homeomorphic to a standard sphere. The point at infinity of $S_{1,x}$ is denoted by ∞.

Definition 2.1. Under the notations above, a plane foliation couple $(\mathcal{F}_1, \mathcal{F}_2)$ is said to have the *sphere-extension* property if for each holonomy map θ of $\mathcal{F}_2$ sending x to y there exists a homeomorphism $\Theta : S_{1,x} \to S_{1,y}$ which coincides locally with $\mathcal{F}_2$-holonomy maps on $S_{1,x} \setminus \{\infty, \Theta^{-1}(\infty)\}$ and extends the germ of θ at x.

If $\bar{\phi}_t$ is a lifted flow of an C^∞ Anosov flow ϕ_t, then $\bar{\phi}_t$ is said to have the *sphere-extension* property if $(\bar{\mathcal{F}}^+, \bar{\mathcal{F}}^{-,0})$ and $(\bar{\mathcal{F}}^-, \bar{\mathcal{F}}^{+,0})$ have both the sphere-extension property.

Recall that $\bar{\mathcal{F}}^+$ and $\bar{\mathcal{F}}^-$ are both foliations by planes. The corresponding notion for Anosov diffeomorphisms is defined similarly.

Denote by $(\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ the lifted couple on $\widetilde{M}$ of $(\mathcal{F}_1, \mathcal{F}_2)$. Then it is easily seen that if $(\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ has the sphere-extension property, then $(\mathcal{F}_1, \mathcal{F}_2)$ has also this property. So by considering the lifted flows and drawing pictures, we can easily see that the geodesic flows of closed negatively curved manifolds have the sphere-extension property.

It is easily verified that hyperbolic infra-nilautomorphisms have the sphere-extension property. However by [**Man**] each Anosov diffeomorphism defined on a infra-nilmanifold is topologically conjugate to a hyperbolic infra-nilautomorphism. We deduce that the suspensions of Anosov diffeomorphisms on infra-nilmanifolds have the sphere-extension property.

Lemma 2.5. *Let ϕ_t and ψ_t be two C^1 orbit equivalent C^∞ Anosov flows. Suppose that the strong stable and strong unstable distributions of ψ_t are of dimension at least two. If ψ_t has the sphere-extension property, then ϕ_t has also this property.*

Proof. Denote by ϕ the C^1 orbit equivalence. Define $\hat{\phi}_t = \phi \circ \phi_t \circ \phi^{-1}$ and denote by $\bar{E}^\pm$ the strong distributions of ψ_t. Define $\hat{\mathcal{F}}^\pm = \phi(\mathcal{F}^\pm)$ and $\hat{\mathcal{F}}^{\pm,0} = \phi(\mathcal{F}^{\pm,0})$. It is clear that ϕ_t has the sphere-extension property iff $\hat{\phi}_t$ has this property in the natural sense.

Since ϕ_t and ψ_t are C^1 orbit equivalent, then $\hat{\mathcal{F}}^{\pm,0} = \bar{\mathcal{F}}^{\pm,0}$. Take a leaf of $\mathcal{F}^+$ and take a non-periodic point x in this leaf. We can identify $\hat{W}^+_x$ and $\bar{W}^+_x$ naturally as following.

For all $y \in \hat{W}^+_x$, there exists $t \in \mathbb{R}$ such that $\psi_t(y) \in \bar{W}^+_x$. If $\hat{W}^{+,0}_x$ contains no periodic orbit, this number t is unique for each y in $\hat{W}^+_x$. If $\hat{W}^{+,0}_x$ contains a periodic orbit, then it contains exactly one periodic orbit. Denote by T its mininal positive period with respect to ψ_t. So if $\psi_t(y) \in W^+_x$, then for all $k \in \mathbb{Z}$, $\psi_{t+k\cdot T}(y) \in \bar{W}^+_x$.

Conversely if $\psi_{t_1}(y) \in \bar{W}^+_x$ and $\psi_{t_2}(y) \in \bar{W}^+_x$, then $\psi_{t_2-t_1}\bar{W}^+_x = \bar{W}^+_x$. Thus $t_2 - t_1 \in T \cdot \mathbb{Z}$. So by associating $t + T \cdot \mathbb{Z}$ to y, we get a well-defined C^∞ map from $\hat{W}^+_x$ to $\mathbb{R}/T\mathbb{Z}$.

Thus by taking a lift if necessary, there exists a unique C^1 map $\theta_x : \hat{W}^+ \to \mathbb{R}$ such that

$$\theta_x(x) = 0,\ \psi_{\theta_x(y)}(y) \in \bar{W}^+_x,\ \forall\ y \in \hat{W}^+_x.$$

Define a C^1 map $\hat{\eta}_x : \hat{W}^+ \to \bar{W}^+_x$ such that

$$\hat{\eta}_x(y) = \psi_{\theta_x(y)}(y).$$

Then $\hat{\eta}_x$ is easily seen to be a local C^1 diffeomorphism such that $\hat{\eta}(x) = x$. Similar we get $\bar{\eta} : \bar{W}^+_x \to \hat{W}^+_x$ such that $\bar{\eta}(x) = x$. If $\hat{W}^{+,0}_x$ contains no ψ_t-periodic orbit, then $\hat{\eta}$ and $\bar{\eta}$ are both C^1 diffeomorphisms.

Suppose that $\bar{W}^{+,0}_x$ contains a unique ψ_t-periodic orbit of period T. Denote by z the unique intersection point of this periodic orbit with $\bar{W}^+_x$. For each $k \in \mathbb{Z}$, we define

$$\Lambda_k = \{y \in \bar{W}^+_x \setminus \{z\} \mid \hat{\eta} \circ \bar{\eta}(y) = \psi_{kT}(y)\}.$$

Then $\bar{W}^+_x \setminus \{z\}$ is the disjoint union of $\{\Lambda_k\}_{k\in\mathbb{Z}}$. Each Λ_k is closed in $\bar{W}^+_x \setminus \{z\}$ and $x \in \Lambda_0$. Take $y \in \Lambda_0$ and since ψ_{-T} is a contracting diffeomorphism of $\bar{W}^+_x$, then a small ball containing y intersects with at most finitely many Λ_l non-trivially. We deduce that Λ_0 is open.

Since $\bar{E}^+$ is at least two-dimensional, then $\bar{W}_x^+ \setminus \{z\}$ is connected . We deduce that $\Lambda_0 = \bar{W}_x^+ \setminus \{z\}$, i.e. $\hat{\eta} \circ \bar{\eta} = Id$. Similarly we have $\bar{\eta} \circ \hat{\eta} = Id$. We identify $\hat{W}_x^+$ and $\bar{W}_x^+$ under these two sliding C^1 diffeomorphisms $\bar{\eta}$ and $\hat{\eta}$. We can identify $\hat{W}_x^-$ and $\bar{W}_x^-$ similarly.

Since these identifications conjugate the holonomy maps and ψ_t has the sphere extension property, then $\hat{\phi}_t$ has also this property. We deduce that ϕ_t has the sphere-extension property. □

3. Proofs of Theorems 1.2 and 1.3

3.1. Construction of a transverse geometric structure

Denote by ϕ_t a C^∞ topologically transitive quasiconformal Anosov flow such that E^+ and E^- are of dimensions at least three. Then by Theorem 2.1 and Lemma 2.2, $E^{+,0}$ and $E^{-,0}$ are both C^∞ and there exist τ^+ and τ^- two continuous ϕ_t-invariant conformal structures on E^+ and E^- which are C^∞ along the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$.

Denote by Φ the orbit foliation of ϕ_t. For each transverse section Σ of Φ we get two C^∞ foliations $\mathcal{F}_\Sigma^+$ and $\mathcal{F}_\Sigma^-$ on Σ by intersecting $\mathcal{F}^{\pm,0}$ with Σ. Denote their tangent distributions by E_Σ^+ and E_Σ^- respectively.

We can identify $E_\Sigma^\pm$ and $E^\pm$ by projecting $E^\pm$ onto $E_\Sigma^\pm$ parallel to the direction of the flow. Under this identification, we get two conformal structures τ_Σ^+ and τ_Σ^- on E_Σ^+ and E_Σ^-. Since τ_Σ^- is easily seen to be invariant under the Φ-holonomy maps and the $\mathcal{F}_\Sigma^+$-holonomy maps, then τ_Σ^- is C^∞ on Σ. Similarly we can see that τ_Σ^+ is also C^∞ on Σ. So we get on each transverse section Σ a C^∞ geometric structure $(\mathcal{F}_\Sigma^\pm, \tau_\Sigma^\pm)$ which is invariant under the Φ-holonomy maps.

Denote by c_n the canonical conformal structure on the n-dimensional sphere S^n and by M_n the isometry group of c_n. Then M_n acts transitively on S^n and is called the Möbius group. Suppose that E^+ is of dimension n and E^- is of dimension m. Then we can construct as following a transverse $(M_n \times M_m, S^n \times S^m)$-structure on Φ.

For any $x \in M$ we denote by $\bar{S}_x^+$ and $\bar{S}_x^-$ the one-point compactifications of E_x^+ and E_x^-. Then they admit naturally C^∞ conformal structures extending σ_x^+ and σ_x^-. Since

$$(h_x^+)_*(\tau^+) = \sigma_x^+ \quad \text{and} \quad (h_x^-)_*(\tau^-) = \sigma_x^-,$$

then S_x^+ and S_x^-, i.e. the one-point compactifications of W_x^+ and W_x^- admit also natural conformal structures isometric to those of $\bar{S}_x^+$ and $\bar{S}_x^-$ under the natural extensions of h_x^+ and h_x^-, which are denoted by $\bar{h}_x^+$ and $\bar{h}_x^-$.

By fixing two conformal frames of E_x^+ and E_x^- we get two C^∞ conformal isometries $\phi_x^+ : \bar{S}_x^+ \to S^n$ and $\phi_x^- : \bar{S}_x^- \to S^m$.

Take a C^∞ small transverse section Σ_x containing x and pieces of W_x^+ and W_x^-. Thus for $\delta \ll 1$ we get the local diffeomorphism

$$\theta_x : W_{x,\delta}^+ \times W_{x,\delta}^- \to \Sigma_x$$

$$(y,z) \to W_{\Sigma_x,y,2\delta}^- \cap W_{\Sigma_x,z,2\delta}^+.$$

Then we define $\phi_x : \Sigma_x \to S^n \times S^m$ such that $\phi_x = (\phi_x^+ \times \phi_x^-) \circ (\bar{h}_x^+ \times \bar{h}_x^-) \circ \theta_x^{-1}$.

Since τ_Σ^+ and τ_Σ^- are invariant under respectively the $\mathcal{F}_\Sigma^-$-holonomy maps and the $\mathcal{F}_\Sigma^+$-holonomy maps, then by its definition, ϕ_x is easily seen to be a local isometry of $(\mathcal{F}_{\Sigma_x}^\pm, \tau_{\Sigma_x}^\pm)$ to $(\{S^n \times *\}, \{* \times S^m\}, c_n \times c_m)$.

Let h be any Φ-holonomy map from a germ of Σ_x to a germ of Σ_y. Then it is easy to see that $\theta_y^{-1} \circ h \circ \theta_x$ is given by weak holonomy maps. We deduce that $\phi_y \circ h \circ \phi_x^{-1} = \phi \times \psi$, where ϕ and ψ are respectively local conformal isomertries of S^n and S^m. Since $n, m \geq 3$, then by the following classical theorem of Liouville, ϕ and ψ can be both extended to global conformal isometries of S^n and S^m. So $\{(\Sigma_x, \phi_x)\}_{x \in M}$ gives a transverse $(M_n \times M_m, S^n \times S^m)$-structure of Φ.

Theorem 3.1. (Liouville) *For $n \geq 3$, each local conformal isometry of S^n defined on a connected open subset can be extended uniquely to a global conformal isometry.*

3.2. Completeness

Fix a developing map $\mathcal{D}$ of the transverse $(M_n \times M_m, S^n \times S^m)$-structure of Φ defined in the previous subsection. Denote by $\mathcal{H}$ the associated holonomy representation.

Lemma 3.1. *Under the notations above, each leaf of $\widetilde{\mathcal{F}}^+$ intersects each leaf of $\widetilde{\mathcal{F}}^{-,0}$ at most once.*

Proof. By the definition of $\mathcal{D}$, for any $x \in \widetilde{M}$, $\mathcal{D}(\widetilde{W}_x^+)$ must be contained in a certain subset $S^n \times b$. Then for any $y \in \widetilde{W}_x^+$, there exists $\gamma_1 \times \gamma_2 \in M_n \times M_m$ such that

$$(\gamma_1 \times \gamma_2) \circ \mathcal{D} = \widetilde{\phi}_y,$$

where $\widetilde{\phi}_y$ is defined similarly as above. Denote by $\mathcal{D}_1$ the composition $pr_1 \circ \mathcal{D}$. Then there exists an open neighborhood V_y of y in $\widetilde{W}_x^+$ such that

$$\gamma_1 \circ \mathcal{D}_1 \mid_{V_y} = \widetilde{\phi}_y^+ \circ \widetilde{h}_y^+.$$

Since $\widetilde{h}_x^+ \circ \widetilde{h_y^+}^{-1}$ sends $\widetilde{\sigma}_y^+$ to $\widetilde{\sigma}_x^+$ and the dimension of E^+ is at least two, then $\widetilde{h}_x^+ \circ \widetilde{h_y^+}^{-1}$ is an affine map. Thus there exists $\gamma \in M_n$ such that

$$\gamma \circ \mathcal{D}_1 \mid_{\widetilde{W}_x^+} = \widetilde{\phi}_x^+ \circ \widetilde{h}_x^+.$$

So $\mathcal{D}$ sends $\widetilde{W}_x^+$ diffeomorphically onto a set of the form $(S^n \setminus a) \times b$.

For any $y \in S^n \setminus a$ such that $\mathcal{D}(z) = y$ and $z \in \widetilde{W}_x^+$, $\mathcal{D}$ sends $\widetilde{W}_z^-$ diffeomorphically onto a set of the form $y \times (S^m \setminus \omega(y))$. So we get a well-defined map $\omega : S^n \setminus a \to S^m$.

Now suppose that $\widetilde{W}_x^+$ intersects $\widetilde{W}_x^{-,0}$ at a point x' other than x. Then there exist $y, y' \in S^n$ such that $y \neq y'$ and

$$\mathcal{D}(W_x^-) \subseteq y \times S^m, \quad \mathcal{D}(W_{x'}^-) \subseteq y' \times S^m.$$

Denote by x'' the intersection of the $\widetilde{\phi}_t$-orbit of x' with $\widetilde{W}_x^-$. Then we have $\mathcal{D}(x'') \neq \mathcal{D}(x')$. However by the definition of $\mathcal{D}$, $\mathcal{D}(x'') = \mathcal{D}(x')$, which is a contradiction. We deduce that each leaf of $\widetilde{\mathcal{F}}^+$ intersects each leaf of $\widetilde{\mathcal{F}}^{-,0}$ at most once. □

The following lemma is a direct consequence of the previous lemma, which is firstly observed by T. Barbot in [**Ba**].

Lemma 3.2. *Under the notations above, the lifted orbit space $Q_{\widetilde{\Phi}}$ is Hausdoff.*

Proof. Suppose on the contrary that there exist two different orbits $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ such that each $\widetilde{\Phi}$-saturated open neighborhood of $\widetilde{\Phi}_1$ intersects that of $\widetilde{\Phi}_2$. We want to see that these two orbits are contained in the same leaf of $\widetilde{\mathcal{F}}^{-,0}$.

Suppose that it is not the case. Denote by F_1 and F_2 the leaves of $\widetilde{\mathcal{F}}^{-,0}$ containing respectively these two orbits. Then by assumption the $\widetilde{\mathcal{F}}^+$-saturated sets of F_1 and F_2 intersect non-trivially. We deduce that there exists a leaf $\widetilde{W}_x^+$ intersecting F_1 and F_2. Denote by V_1 and V_2 two disjoint open subsets of $\widetilde{W}_x^+$ containing respectively the intersection of $\widetilde{W}_x^+$ with F_1 and that of $\widetilde{W}_x^+$ with F_2. Then by assumption the $\widetilde{\mathcal{F}}^{-,0}$-saturated set of V_1 intersects that of V_2 non-trivially, which contradicts Lemma 3.1.

Thus $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ are contained in the same leaf of $\widetilde{\mathcal{F}}^{-,0}$. Similar we can prove that they are contained in the same leaf of $\widetilde{\mathcal{F}}^{+,0}$. Then by Lemma 3.1, we have $\widetilde{\Phi}_1 = \widetilde{\Phi}_2$, which is a contradiction. □

For each $x \in \widetilde{M}$ we construct an open subset U_x of $\widetilde{M}$ such that U_x is the union of the leaves of $\widetilde{\mathcal{F}}^{-,0}$ intersecting $\widetilde{W}_x^{+,0}$. Then we can find a sequence $\{x_i\}_{i=1}^{\infty} \subseteq \widetilde{M}$ satisfying the following conditions:
(1) $\cup_{i\geq 1} U_{x_i} = \widetilde{M}$.
(2) For each $k \geq 1$, $\Omega_k = \cup_{i=1}^{k} U_{x_i}$ is connected.
In the following we denote U_{x_i} by U_i. Largely inspired by the arguments in **[Gh1]**, we prove the following lemma.

Lemma 3.3. *For each $k \geq 1$, $\mathcal{D}\mid_{\Omega_k}: \Omega \to \mathcal{D}(\Omega_k)$ is a C^∞ fiber bundle with fiber $\mathbb{R}$ over $\mathcal{D}(\Omega_k)$. In addition $\mathcal{D}(\Omega_k)$ is either the complement in $S^n \times S^m$ of the graph of a continuous map from S^n to S^m or the complement of the union of $\{*\} \times S^m$ and of the graph of a continuous map from $(S^n \setminus \{*\})$ to S^m.*

Proof. We prove this lemma by induction. For $k = 1$ we have $\Omega_1 = U_1$. In the proof of Lemma 3.1 we have seen that $\mathcal{D}$ sends $\widetilde{W}_x^{+}$ diffeomorphically onto a set of the form $(S^n \setminus a) \times b$. For any $y \in S^n \setminus a$ such that $\mathcal{D}(z) = y$ and $z \in \widetilde{W}_{x_1}^{+}$, $\mathcal{D}$ sends $\widetilde{W}_z^{-}$ diffeomorphically onto a set of the form $y \times (S^m \setminus \omega(y))$. So we get a well-defined map $\omega : S^n \setminus a \to S^m$.

Denote by $Gr(\omega)$ the graph of ω. Then the complement of $Gr(\omega)$ in $(S^n \setminus a) \times S^m$ is the open set $\mathcal{D}(U_1)$. So ω is continuous. By the definition of U_1 the inverse images of $\mathcal{D}\mid_{U_1}$ are all connected. Then by the existence of fine transverse sections $\mathcal{D}\mid_{U_1}$ is seen to be a fiber bundle of fiber $\mathbb{R}$. So the lemma is true for $k = 1$.

Suppose that the lemma is true for Ω_k. Then $\mathcal{D}\mid_{\Omega_k}$ is a fiber bundle with fiber $\mathbb{R}$ and $\mathcal{D}(\Omega_k)$ is the complement in $S^n \times S^m$ of the graph of a C^0 map $u_k : S^n \to S^m$ or of the union of a vertical $a_k \times S^m$ and the graph of a C^0 map $u_k : S^n \setminus a_k \to S^m$.

In addition by the argument above, we know that $\mathcal{D}(U_{k+1})$ is the complement of the union of $b_{k+1} \times S^m$ and of the graph of a C^0 map $v_{k+1} : S^n \setminus b_{k+1} \to S^m$ and $\mathcal{D}\mid_{U_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$.

Since $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ is the complement in $S^n \times S^m$ of a finite union of topological submanifolds of codimension at least two, then $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ is connected and open.

Firstly we want to see that $\mathcal{D}\mid_{\Omega_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$. Take $x \in \Omega_k$ and $y \in U_{k+1}$ such that $\mathcal{D}(x) = \mathcal{D}(y)$. Since Ω_{k+1} is connected, then $\Omega_k \cap U_{k+1} \neq \emptyset$. So we can take $z \in \Omega_k \cap U_{k+1}$ and a C^0 curve γ in $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ connecting $\mathcal{D}(z)$ and $\mathcal{D}(x)$.

Since $\mathcal{D}\mid_{\Omega_k}$ and $\mathcal{D}\mid_{U_{k+1}}$ are fiber bundles with fiber $\mathbb{R}$, then we can lift γ to two C^0 curves $\gamma_1 \subseteq \Omega_k$ and $\gamma_2 \subseteq U_{k+1}$ such that $\gamma_1(0) = \gamma_2(0) = z$. Thus $\gamma_1(1)$ and x are contained in the same $\widetilde{\phi}_t$-orbit and so are $\gamma_2(1)$ and y.

Denote by Λ the subset of $t \in [0,1]$ such that $\gamma_1(t)$ and $\gamma_2(t)$ are in the same orbit of $\widetilde{\phi}_t$. By the section property, Λ is easily seen to be open in $[0,1]$. Suppose that $\{t_n\}_{n=1}^{\infty} \subseteq \Lambda$ and $t_n \to t$. If $\gamma_1(t)$ and $\gamma_2(t)$ are not in the same $\widetilde{\phi}_t$-orbit, then by Lemma 3.2 there exist disjoint $\widetilde{\phi}_t$-saturated open neighborhoods of $\gamma_1(t)$ and $\gamma_2(t)$. Thus for $n \gg 1$, $\gamma_1(t_n)$ and $\gamma_2(t_n)$ are not in the same orbit of $\widetilde{\phi}_t$, which is a contradiction. We deduce that Λ is closed. Thus $\Lambda = [0,1]$. So x and y are contained in the same $\widetilde{\phi}_t$-orbit. We deduce that $\mathcal{D}\mid_{\Omega_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$.

Now we want to see the form of $\mathcal{D}(\Omega_{k+1})$. Suppose at first that $\mathcal{D}(\Omega_k) = (Gr(u_k))^c$. Take $p \in S^n \setminus b_{k+1}$. If $u_k(p) \neq v_{k+1}(p)$, then $\mathcal{D}(\Omega_{k+1})$ contains the vertical $p \times S^m$. Since $p \neq b_{k+1}$, then there exists $x \in U_{k+1}$ such that $\mathcal{D}(\widetilde{W_x^-}) = p \times (S^m \setminus v_{k+1}(p))$. So there exists $y \in \Omega_k$ such that $p \times v_{k+1}(p) \in \mathcal{D}(\widetilde{W_y^-})$. In particular, $\mathcal{D}(\widetilde{W_x^-}) \cap \mathcal{D}(\widetilde{W_y^-}) \neq \emptyset$. So there exists $t \in \mathbb{R}$ such that $\widetilde{\phi}_t(\widetilde{W_x^-}) = \widetilde{W_y^-}$. We deduce that $\mathcal{D}(\widetilde{W_x^-}) = p \times S^m$, which is absurd. So in this case, $\mathcal{D}(\Omega_{k+1}) = (Gr(u_k))^c$.

Suppose that $\mathcal{D}(\Omega_k) = (Gr(u_k) \cup (a_k \times S^m))^c$. For each $p \in S^n \setminus \{a_k, b_{k+1}\}$ we get as above that $u_k(p) = v_{k+1}(p)$.

If $a_k \neq b_{k+1}$, then u_k and v_{k+1} can be extended to the same continuous map $\bar{u}_k$ on S^n. In this case $\mathcal{D}(\Omega_{k+1}) = (Gr(\bar{u}_k))^c$.

If $a_k = b_{k+1}$, then we certainly have $\mathcal{D}(\Omega_{k+1}) = (Gr(u_k) \cup (a_k \times S^m))^c$.

□

We deduce from the previous lemma that $\mathcal{D} : \widetilde{M} \to \mathcal{D}(\widetilde{M})$ is a C^∞ fiber bundle with fiber $\mathbb{R}$. So the transverse $(M_n \times M_m, S^n \times S^m)$-structure of Φ is complete. In addition by the proof of the previous lemma we see that if b_k is equal to b_1 for each $k \geq 1$ then $\mathcal{D}(\widetilde{M}) = (Gr(u_1) \cup (a_1 \times S^m))^c$. If there exists $k > 1$ such that $b_k \neq b_1$ then $\mathcal{D}(\widetilde{M}) = (Gr(\bar{u}_1))^c$.

By exchanging the roles of E^+ and E^- in the previous lemma, we get the following two cases:

(1) $\mathcal{D}(\widetilde{M}) = (S^n \setminus a) \times (S^m \setminus b)$.

(2) $\mathcal{D}(\widetilde{M}) = (Gr(f))^c$ where f is a homeomorphism of S^n onto S^m. In particular $n = m$ in this case.

Let us consider firstly Case (1). By changing the developing map we can suppose that $a = b = \infty$. Denote by CO_n the isometry group of the canonical conformal structure of $\mathbb{R}^n$. Then by Lemma 2.3 we get a compatible transverse $(CO_n \times CO_m, \mathbb{R}^n \times \mathbb{R}^m)$-structure of Φ. In particular, the weak stable and weak unstable foliations admit transverse affine structures. So by **[Pl2]** the flow ϕ_t admits a C^∞ global section Σ. Since the Poincaré map ϕ of Σ is also topologically transitive and quasiconformal, then by **[K-Sa]**, ϕ_t is C^∞ conjugate to a finite factor of a hyperbolic automorphism of a torus. We deduce that up to finite covers, ϕ_t is C^∞ orbit equivalent to the suspension of a hyperbolic automorphism of a torus.

Now we consider Case (2). Denote by Γ the fundamental group of M. Then by Lemma 2.4 the group $\mathcal{H}(\Gamma)$ is discrete in $M_n \times M_n$. Define $\mathcal{H}_1 = pr_1 \circ \mathcal{H}$ and $\mathcal{H}_2 = pr_2 \circ \mathcal{H}$. Then we have

$$f \circ \mathcal{H}_1(\gamma) \circ f^{-1} = \mathcal{H}_2(\gamma), \ \forall\, \gamma \in \Gamma.$$

We deduce that $\mathcal{H}_1(\Gamma)$ and $\mathcal{H}_2(\Gamma)$ are both discrete in M_n. Denote $\mathcal{H}_1(\Gamma)$ by Γ_1 and $\mathcal{H}_2(\Gamma)$ by Γ_2. Since ϕ_t is topologically transitive, then Φ admits at least a simply connected leaf. We deduce that $\mathcal{H}$ is injective. So Γ and Γ_1 and Γ_2 are all isomorphic.

We can prove that Γ_1 is uniform in M_n as following. Suppose on the contrary that Γ_1 is not uniform. Then Γ_1 admits a finite index torsion free subgroup Γ_1' such that $cd(\Gamma_1') \le n$, where $cd(\Gamma_1')$ denotes the cohomological dimension of Γ_1'. So by passing to a finite index subgroup if necessary, we can suppose that $cd(\Gamma) \le n$.

Denote by $B\Gamma$ the classifying space of Γ and by $E\Gamma$ the universal covering space of $B\Gamma$. Then we have

$$B\Gamma \cong \Gamma_1 \diagdown \mathbb{H}^{n+1}, \ E\Gamma \cong \mathbb{H}^{n+1},$$

where $\mathbb{H}^{n+1}$ denotes the simply connected hyperbolic space of dimension $n+1$. Denote by $E\Gamma \times_\Gamma \widetilde{M}$ the quotient manifold of $E\Gamma \times \widetilde{M}$ under the diagonal action of Γ. Then we have the following fibre bundle with fiber $\widetilde{M}$

$$\pi_1 : E\Gamma \times_\Gamma \widetilde{M} \to B\Gamma,$$

$$\Gamma((a,x)) \to \Gamma(a).$$

By using the cohomology Leray-Serre spectral sequence to this fibre bundle (see **[Mc]**), we get that

$$E_2^{p,q} = H^p(\Gamma, H^q(\widetilde{M}))$$

converges to $H^{p+q}(E\Gamma \times_\Gamma \widetilde{M})$. Since $\widetilde{M}$ is a fibre bundle with fiber $\mathbb{R}$ and base $(S^n \times S^n) \diagdown (Gr(f))$, then $\widetilde{M}$ is homotopically equivalent to the sphere

S^n. Since we have in addition $cd(\Gamma) \le n$, then we deduce from the spectral sequence above that $H^{2n+1}(E\Gamma \times_\Gamma \widetilde{M})$ is trivial.

However by projecting onto the second factor $E\Gamma \times_\Gamma \widetilde{M}$ is easily seen to be also a fibre bundle over M and with contractible fiber $E\Gamma$. So $E\Gamma \times_\Gamma \widetilde{M}$ is homotopically equivalent to M. We deduce that $H^{2n+1}(M)$ is trivial, which is absurd. So Γ_1 is uniform in M_n. Similarly Γ_2 is also uniform in M_n.

Since f conjugates Γ_1 to Γ_2, then by Mostow's rigidity theorem (see [**Mo**]) f is contained in M_n. So by replacing $\mathcal{D}$ by $(Id \times f^{-1}) \circ \mathcal{D}$, we can suppose that $f = Id$ and

$$\mathcal{D}(\widetilde{M}) = (S^n \times S^n) \diagdown \Delta,$$

where Δ denotes the diagonal of $S^n \times S^n$. In addition, we have $\mathcal{H}_1 = \mathcal{H}_2$. So by Lemma 6.3.1, Φ admits a compatible transverse $(M_n, (S^n \times S^n)\diagdown\Delta)$-structure with respect to the diagonal action of M_n on $(S^n \times S^n)\diagdown\Delta$.

Lift ϕ_t to a finite cover to eliminate the torsion of Γ and define $V = \mathcal{H}(\Gamma)\diagdown\mathbb{H}^{n+1}$. Then V is a closed hyperbolic manifold. In addition, the Γ-action on $Q_{\widetilde{\Phi}}$ is C^∞ conjugate to the $\mathcal{H}(\Gamma)$-action on the leaf space of the lifted geodesic flow of V under $\mathcal{D}$ and $\mathcal{H}$. Since the holonomy of each periodic orbit of ϕ_t is non-trivial, then the holonomy covering of each leaf of Φ is contractible. Denote by ψ_t the geodesic flow of V. So by Proposition 2.1 there exists a C^∞ homotopy equivalence h conjugating the leaf space of ϕ_t with that of ψ_t. However h is not in general a C^∞ diffeomorphism. In order to get a C^∞ orbit conjugacy between ϕ_t and ψ_t, we use a classical diffusion argument discovered by É. Ghys. Let us recall briefly this argument (see [**Gh2**] and [**Ba**] for details):

There exists a C^∞ function $u : \mathbb{R} \times M \to \mathbb{R}$ such that

$$h(\phi_t(x)) = \psi_{u(t,x)}(h(x)),\ \forall\ t \in \mathbb{R},\ \forall\ x \in M.$$

Define for $T \gg 1$, $u_T(x) = \frac{1}{T}\int_0^T u(s,x)ds$ and $h_T : M \to T^1V$ such that $h_T(x) = \psi_{u_T(x)}(h(x))$. If $T \gg 1$, then we can see that h_T satisfies the same conditions as h and is a C^∞ diffeomorphism.

So up to finite covers, ϕ_t is C^∞ orbit equivalent to the geodesic flow of a closed hyperbolic manifold, which finishes the proof of the first part of Theorem 1.2.

3.3. Smoothness blowing up

In this subsection we prove the second part of Theorem 1.2. Suppose that ϕ_t satisifes the conditions of Theorem 1.2 such that $E^+ \oplus E^-$ is in addition C^1. Then because of the first part of Theorem 1.2, ϕ_t is seen to be volume-preserving. So in order to prove the second part of Theorem 1.2, we need only prove the C^∞ smoothness of $E^+ \oplus E^-$ and then use the following classification result established in [**Fa1**] :

Theorem 3.2. ([**Fa1**], Theorem 1) *Let ϕ_t be a C^∞ volume-preserving uniformly quasiconformal Anosov flow on a closed manifold M. If $E^+ \oplus E^-$ is C^∞ and the dimensions of E^+ and E^- are at least 2, then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical perturbation of the geodesic flow of a hyperbolic manifold.*

Lemma 3.4. *Under the notations above, $E^+ \oplus E^-$ is C^∞.*

Proof. Suppose at first that ϕ_t is C^∞ orbit equivalent to the geodesic flow ψ_t of a hyperbolic manifold (up to finite covers). Denote by λ the canonical 1-form of ϕ_t and by X the generator of ψ_t. Up to C^∞ flow conjugacy we suppose that ϕ_t is generated by fX.

Since $E^+ \oplus E^-$ is supposed to be C^1, then λ is C^1 and $\lambda(X) = \frac{1}{f}$ is C^∞. It is easily seen that $d\lambda$ is ϕ_t-invariant. Then by the Anosov property of ϕ_t, we get $i_{fX} d\lambda = 0$. Thus $i_X d\lambda = 0$. We deduce that $d\lambda$ is ψ_t-invariant. Denote by λ' the canonical 1-form of ψ_t. Then by [**Ham1**] there exists $a \in \mathbb{R}$ such that $d\lambda = a \cdot d\lambda'$.

Define $\beta = \lambda - a \cdot \lambda'$. Then β is a C^1 1-form such that $d\beta = 0$. In addition $\beta(X) = \lambda(X) - a$ is C^∞.

Sublemma. *Let ϕ_t be a C^∞ volume-preserving Anosov flow on M. If α is a C^1 1-form on M such that $d\alpha = 0$ and $\alpha(X)$ is C^∞, then α is C^∞.*

Proof. Since $d\alpha = 0$ and the Stokes formula is valid for C^1 forms (even for Lipchitz forms), then there exists a C^∞ 1-form β giving the same element of $(H_1(M, \mathbb{R}))^*$ as that of α. So by integrating $(\alpha - \beta)$ along curves, we get a well-defined C^2 function f on M. Thus for any $x \in M$ and any $t \in \mathbb{R}$ we have

$$f(\phi_t(x)) - f(x) = \int_0^t (\alpha(X) - \beta(X)) \circ \phi_s(x) ds.$$

Since $\alpha(X)$ is supposed to be C^∞, then by [**LMM**], f is seen to be C^∞. However by the definition of f, we have $\alpha - \beta = df$. Thus α is C^∞. $\square$

We deduce from this sublemma that β is C^∞. Thus λ is C^∞. So $E^+ \oplus E^-$ is also C^∞.

If ϕ_t is C^∞ orbit equivalent to the suspension ψ_t of a hyperbolic automorphism of a torus (up to finite covers), then by similar arguments as above, we can see that $d\lambda$ is ψ_t-invariant.

Take a leaf Σ of the foliation of the sum of the strong stable and the strong unstable distributions of ψ_t and denote by ψ its Poincaré map. Then $\lambda \mid_\Sigma$ is C^1 and $d(\lambda \mid_\Sigma)$ is ψ-invariant. Thus by the same arguments as in **[Fa2]** we get $d(\lambda \mid_\Sigma) = 0$. We deduce that $d\lambda = 0$. Since in addition $\lambda(X) = \frac{1}{f}$ is C^∞, then by the previous sublemma λ is C^∞. Thus $E^+ \oplus E^-$ is also C^∞. □

Proof of Theorem 1.3. Suppose that ϕ_t satisfies the conditions of Theorem 1.3. Similar to the previous section, we can construct a C^∞ geometric structure $(\mathcal{F}_\Sigma^\pm, \tau_\Sigma^\pm)$ on each transverse section Σ of Φ. Similarly we can construct a family of transverse charts $\{(\Sigma_x, \phi_x)\}_{x\in M}$. Then because of the sphere-extension property, the chart changes of these charts are easily seen to be given by the restrictions of the elements of $M_2 \times M_m$ with respect to the natural action of $M_2 \times M_m$ on $S^2 \times S^m$. So in this way we get a transverse $(M_2 \times M_m, S^2 \times S^m)$-structure on Φ. Then as in the previous subsection the proof splitts into Case (1) and Case (2). Each of them is understood in the same manner as in the previous subsection.

4. Applications to the geodesic flows of hyperbolic manifolds

Now let us begin to prove Theorem 1.6.

Lemma 4.1. *Let ϕ_t and ψ_t be two C^∞ Anosov flows which are C^1 orbit equivalent. If ψ_t is volume-preserving, then so is ϕ_t.*

Proof. By conjugating ϕ_t by the C^1 orbit conjugacy, we can suppose that ϕ_t is a C^1 flow and a time change of ψ_t. Denote by ν the ψ_t-invariant volume form and by X the generator of ψ_t. Then by taking $i_X\nu$ we get a family of Ψ-holonomy invariant volume forms on the transverse sections of Ψ. This family of transversal volume forms is also Φ-holonomy invariant. Denote by dt_ϕ the normalized foliated measure along the leaves of Φ such that $dt_\phi(Y) \equiv 1$, where Y denotes the generator of ϕ_t. In each flow box of ϕ_t we take the product measure $\nu_\Sigma \otimes dt_\phi$. Then it is easily seen that in the intersection of two flow boxes the two measures coincide. Then we can extend this family of local measures to a measure μ on M which is in the Lebesgue class and easily seen to be ϕ_t-invariant. □

Proof of Theorem 1.6. Since ψ_t is conformal, then by Lemma 1.1, ϕ_t is quasiconformal. In addition by lemmas 2.5 and 4.1, ϕ_t verifies the conditions of Theorem 1.2 or Theorem 1.3. So up to finite covers, ϕ_t is C^∞ orbit equivalent either to a suspension or to the geodesic flow of a hyperbolic manifold $\hat{\psi}_t$. Since ψ_t is contact, then it admits no C^1 global section. So up to finite covers, ϕ_t is C^∞ orbit equivalent to $\hat{\psi}_t$.

However in the proofs of Theorems 1.2 and 1.3, we passed to a finite cover only in order to eliminate the torsion in the fundamental group of M. But in the current case, the fundamental group has no torsion by the classical Cartan theorem. So ϕ_t is C^∞ orbit equivalent to $\hat{\psi}_t$. Then by Mostow's rigidity theorem (see [**Mo**] and [**M**]), $\hat{\psi}_t$ is C^∞ flow equivalent to ψ_t. We deduce that ϕ_t is C^∞ orbit equivalent to ψ_t. $\square$

Proof of Corollary 1.1. Let us prove firstly (1). Suppose that the geodesic flow of M is Hölder-continuously orbit equivalent to that of a hyperbolic manifold N. Since $n \geq 3$, then the fundamental group of M is isomorphic to that of N. Since $\pi_1(M)$ with its word metric is quasi-isometric to $\widetilde{M}$ and $\pi_1(N)$ is quasi-isometric to $\mathbb{H}^n$, then we deduce that $\widetilde{M}$ is quasi-isometric to $\mathbb{H}^n$.

Conversely, if $\widetilde{M}$ is quasi-isometric to $\mathbb{H}^n$, then $\pi_1(M)$ is also quasi-isometric to $\mathbb{H}^n$. Thus by [**Su**] and [**Tu**], there exists a uniform lattice Γ in the isometric group of $\mathbb{H}^n$ and a surjective group homomorphism $\rho : \pi_1(M) \to \Gamma$ such that the kernal of ρ is finite. However by a classical result of É. Cartan, $\pi_1(M)$ is without torsion. We deduce that $\pi_1(M)$ is isomorphic to Γ. In particular, Γ is also without torsion. So $N = \Gamma \diagdown \mathbb{H}^n$ is a closed hyperbolic manifold.

Denote respectively by ϕ_t and ψ_t the geodesic flows of M and N. Since $\pi_1(M) \cong \pi_1(N)$, then by [**Gr**], ϕ_t is C^0 orbit equivalent to ψ_t. Since each continuous orbit conjugacy between Anosov flows can be C^0 approximated by Hölder-continuous orbit conjugacies (see [**HK**]), then (1) is true.

Now let us prove (2). We need only prove the necessarity. Suppose that the geodesic flow ϕ_t of M is C^1 orbit equivalent to the geodesic flow ψ_t of a closed hyperbolic manifold N. Since ψ_t is conformal, then by Lemma 1.1, ϕ_t is quasiconformal. Thus by Corollary 1.1, it is C^∞ orbit equivalent to the geodesic flow of N. Since C^∞ orbit conjugacy preserves weak stable and weak unstable distributions, then ϕ_t is Anosov-smooth. So by [**BFL**], it is C^∞ flow equivalent to the geodesic flow of a Riemannian manifold of constant negative curvature. Then by [**BCG**], M has constant negative curvature. □

Before the proof of Corollary 1.2, let us recall firstly some notions. A C^∞ flow ϕ_t defined on a closed n-dimensional manifold is said to be *quasi-Anosov* if there exists a continuous $(n-1)$-dimensional distribution ν transversal to the flow, such that for any non-zero vector v in ν, the set $\{\| D\phi_t(v) \|,\ t \in \mathbb{R}\}$ is unbounded with respect to a certain (then all) Riemannian metric. In **[Ma]**, R. Mañẽ proved the following important result.

Theorem 4.1. (R. Mañẽ) *If ϕ_t is quasi-Anosov and volume-preserving, then ϕ_t is Anosov.*

Proof of Corollary 1.2. Denote $h : M \to N$ a C^1 conjuguacy sending the leaves of Φ onto those of Ψ. Since Φ is the orbit foliation of the geodesic flow of a hyperbolic manifold, then Ψ is orientable. Thus we can find a C^∞ non-where vanishing vector field Y tangent to Ψ, whose flow is C^1 orbit equivalent to ϕ_t under h. We denote by ψ_t the flow of Y.

There exists a C^1 map $\alpha : M \times \mathbb{R} \to \mathbb{R}$ such that $\psi_t \circ h = h \circ \phi_{\alpha(t,\cdot)}$. Since h is C^1, then there exists $A > 0$ such that for any $u \in TM$, we have

$$\frac{1}{A} \| u \| \leq \| Dh(u) \| \leq A \| u \| .$$

Denote by E^+ and E^- the strong unstable and stable distributions of ϕ_t. Then for any $x \in M$ and any $(u^+ + u^-) \in (E^+ \oplus E^-)_x$,

$$\| D\psi_t(Dh(u^+ + u^-)) \| = \| Dh(D(\phi_{\alpha(t,\cdot)})(u^+ + u^-)) \|$$

$$\geq \frac{1}{A} \| D(\phi_{\alpha(t,x)})(u^+) + D(\phi_{\alpha(t,x)})(u^-) + b(t) \cdot X_{\phi_{\alpha(t,x)}}(x) \|,$$

where X denotes the generator of ϕ. Since ϕ_t is Anosov, then it is easy to see that for any $v \in Dh(E^+ \oplus E^-)$, $\{\| D\psi_t(v) \|,\ t \in \mathbb{R}\}$ is unbounded. Thus ψ_t is quasi-Anosov. In addition, we know by Lemma 4.1 that ψ_t is volume-preserving. Thus we deduce from Theorem 4.1 that ψ_t is a C^∞ Anosov flow, which is C^1 orbit equivalent to ϕ_t. Then we deduce from Theorem 1.6 that ψ_t is C^∞ orbit equivalent to ϕ_t. Thus their orbit foliations Ψ and Φ are C^∞ conjuguate.

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