

On generalizations of Gowers norms and their geometry

Hamed Hatami

Department of Computer Science

University of Toronto

e-mail: hamed@cs.toronto.edu

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Abstract

Motivated by the definition of the Gowers uniformity norms, we introduce and study a wide class of norms. Our aim is to establish them as a natural generalization of the L_p norms. We shall prove that these normed spaces share many of the nice properties of the L_p spaces. Some examples of these norms are L_p norms, trace norms S_p when p is an even integer, and Gowers uniformity norms.

Every such norm is defined through a pair of weighted hypergraphs. In regard to a question of László Lovász, we prove several results in the direction of characterizing all hypergraph pairs that correspond to norms.

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1 Introduction

Consider a measurable function $f : [0, 1] \rightarrow \mathbb{C}$. For $1 \leq p < \infty$, the L_p norm of f is defined as

$$\|f\|_p = \left(\int |f(x)|^p dx \right)^{1/p} = \left(\int f(x)^{p/2} \overline{f(x)}^{p/2} dx \right)^{1/p}. \quad (1)$$

Next consider a measurable function $f : [0, 1]^2 \rightarrow \mathbb{C}$. The Gowers 2-uniformity norm of f is defined as

$$\|f\|_{U_2} = \left(\int f(x_0, y_0) f(x_1, y_1) \overline{f(x_0, y_1)} \overline{f(x_1, y_0)} dx_0 dx_1 dy_0 dy_1 \right)^{1/4}. \quad (2)$$

Note that there are similarities between (1) and (2): Their underlying vector space is a function space, and the norm of a function f is defined by a formula of the form $(\int \Pi)^{1/p}$, where $p > 0$ and Π is a product which involves different copies of powers of f and \bar{f} . The purpose of this article is to use a common framework to study the norms that are defined in a similar fashion. Our aim is to establish this class of norms as a natural generalization of the L_p norms. We shall prove that they share many of the nice properties of the L_p norms.

An important class of norms that fall into our setting are Gowers norms. They are introduced by Gowers [8, 9] as a measurement of pseudo-randomness in his proof for Szémeredi's theorem on arithmetic progressions. The discovery of these norms resulted in a better understanding of the concept of pseudo-randomness, and this led to an enormous amount of progress in the area, and establishment of remarkable results such as Green and Tao's theorem [10] that the primes contain arbitrarily long arithmetic progressions. Although Gowers norms are very special case of our framework, surprisingly some of their key properties, and ideas from pseudo-randomness theory will be needed in our proofs.

For now let us focus on two-variable functions $f : [0, 1]^2 \rightarrow \mathbb{C}$. For finite sets V_1, V_2 and functions $\alpha, \beta : V_1 \times V_2 \rightarrow \mathbb{R}^+$, consider

$$\|f\|_{(\alpha, \beta)} := \left(\int \prod_{(i, j) \in V_1 \times V_2} f(x_i, y_j)^{\alpha(i, j)} \prod_{(i, j) \in V_1 \times V_2} \overline{f(x_i, y_j)}^{\beta(i, j)} \right)^{1/t},$$

where $t := \sum_{(i, j) \in V} \alpha(i, j) + \beta(i, j)$. A natural question is that for which α, β , the function $\|\cdot\|_{(\alpha, \beta)}$ defines a norm. For example both formulas

$$\|f\|_{2U_2} := \|f^2\|_{U_2}^{1/2} = \left(\int f(x_0, y_0)^2 f(x_1, y_1)^2 \overline{f(x_0, y_1)}^2 \overline{f(x_1, y_0)}^2 dx_0 dx_1 dy_0 dy_1 \right)^{1/8}, \quad (3)$$

and

$$\left(\int |f(x_0, y_0)|^{\sqrt{2}} |f(x_1, y_1)|^{\sqrt{2}} |f(x_0, y_1)| |f(x_1, y_0)| dx_0 dx_1 dy_0 dy_1 \right)^{1/(2\sqrt{2}+2)}, \quad (4)$$

can be defined as $\|\cdot\|_{(\alpha, \beta)}$ for proper choices of functions α and β . They are both always nonnegative, and homogenous with respect to scaling. But do they satisfy the triangle inequality? One of our main results, Theorem 2.1, says that if $\|\cdot\|_{(\alpha, \beta)}$ satisfies the triangle inequality, then one of the following two conditions hold:

- **Type I:** There exists a constant $s \geq 1$ such that $\alpha(i, j) = \beta(i, j) \in \{0, s/2\}$, for every $(i, j) \in V_1 \times V_2$;
- **Type II:** For every $(i, j) \in V_1 \times V_2$, $\alpha(i, j) = \beta(i, j) = 0$, $\alpha(i, j) = 1 - \beta(i, j) = 0$, or $1 - \alpha(i, j) = \beta(i, j) = 0$.

It follows from the above theorem that neither of (3) and (4) satisfies the triangle inequality. The L_p norm $\|f\|_p = (\int |f(x, y)|^p)^{1/p}$ is an example of a norm of Type I, and $\|\cdot\|_{U_2}$ defined in (2) is an example of a norm of Type II.

Among the key ingredients in the proof of Theorem 2.1 is a Hölder type inequality that we prove in Lemma 2.10. This inequality is extremely useful in this article and shall be applied frequently. One can think of it as a common generalization of the classical Hölder inequality and the Gowers-Cauchy-Schwarz inequality.

We also study the norms $\|\cdot\|_{(\alpha, \beta)}$ from a geometric point of view, and determine their moduli of smoothness and convexity. These two parameters are among the most important invariants in Banach

space theory. Our results in particular determine the moduli of smoothness and convexity of Gowers norms. They also provide a unified proof for some previously known facts about L_p and Schatten spaces, and generalize them to a wider class of norms. When the norm is of Type II we can show that the corresponding normed space satisfies the so called Hanner inequality. This inequality has been proven to hold only for a few spaces, namely the L_p spaces by Hanner [11], and the Schatten spaces S_p for $p \geq 4$ and $1 \leq p \leq 4/3$ by Ball, Carlen and Lieb [1]. We also prove a complex interpolation theorem for normed spaces of Type I, and use it together with the Hanner inequality to obtain various optimum results in terms of the constants involved in the definition of moduli of smoothness and convexity.

The norms studied here are generalizations of the graph norms studied in [12]. For an integer $k > 0$, it is well-known that the $2k$ -trace norm of a matrix can be defined through the graph C_{2k} , the cycle of length $2k$. This gives a combinatorial interpretation of the $2k$ -trace norm with many applications in graph theory. A remarkable recent example is the work of Bourgain and Gamburd [3] on expanders. Inspired by the fact that the cycles of even length correspond to norms, and the numerous applications of these norms in graph theory, László Lovász posed the problem of characterizing all graphs that correspond to norms. The study of this problem is initiated by the author in [12], where among other things, a rather surprising application to Erdős-Simonovits-Sidorenko conjecture has been proven.

Although the framework of the present article is a generalization of [12], almost all of the results proven here are new even in the context of the graph norms. In particular we settle an open question posed in [12].

1.1 Notations and Definitions

In this section we give the formal definition of a hypergraph pair, and introduce the notations and conventions used throughout the article. A measure in this article is always a positive measure.

Let $k > 0$ be an integer, V_1, \dots, V_k be finite nonempty sets and $V := V_1 \times \dots \times V_k$. For $\alpha, \beta : V \rightarrow \mathbb{R}$, the pair $H = (\alpha, \beta)$ is called a *k-hypergraph pair*. The size of H is defined as

$$|H| := \sum_{\omega \in V} |\alpha(\omega)| + |\beta(\omega)|.$$

When we say $H = (\alpha, \beta)$ takes only integer values, we mean that $\text{ran}(\alpha), \text{ran}(\beta) \subseteq \mathbb{Z}$.

Consider two k -hypergraph pairs: $H = (\alpha, \beta)$ over $V = V_1 \times \dots \times V_k$, and $H' = (\alpha', \beta')$ over $W = W_1 \times \dots \times W_k$. An *isomorphism* from H to H' is a k -tuple $h = (h_1, \dots, h_k)$ such that $h_i : V_i \rightarrow W_i$ are bijections satisfying

$$\alpha(\omega) = \alpha'(h(\omega)), \quad \beta(\omega) = \beta'(h(\omega)),$$

for every $\omega = (\omega_1, \dots, \omega_k) \in V$, where $h(\omega) := (h_1(\omega_1), \dots, h_k(\omega_k))$. We say H is *isomorphic* to H' , and denote it by $H \cong H'$, if there exists an isomorphism from H to H' .

Let $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space. Every $\omega \in V$ defines a projection from $\Omega^{V_1} \times \dots \times \Omega^{V_k}$ to Ω^k in a natural way. For a measurable function $f : \Omega^k \rightarrow \mathbb{C}$, let $f^H : \Omega^{V_1} \times \dots \times \Omega^{V_k} \rightarrow \mathbb{C}$ be defined as

$$f^H(x) := \left(\prod_{\omega \in V} f(\omega(x))^{\alpha(\omega)} \right) \left(\prod_{\omega \in V} \overline{f(\omega(x))^{\beta(\omega)}} \right),$$

where here, and in the sequel we always assume $0^0 = 1$. As we discussed above we want to use hypergraph pairs to construct normed spaces.

Definition 1.1 Consider a k -hypergraph pair $H = (\alpha, \beta)$ with $\alpha, \beta \geq 0$, and a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. Let $L_H(\mathcal{M})$ be the set of functions $f : \Omega^k \rightarrow \mathbb{C}$ with $\| |f| \|_H < \infty$, where for a measurable function $f : \Omega^k \rightarrow \mathbb{C}$,

$$\|f\|_H := \left(\int f^H \right)^{1/|H|}. \quad (5)$$

A hypergraph pair is called *norming* (semi-norming), if $\|\cdot\|_H$ defines a norm (semi-norm) on $L_H(\mathcal{M})$ for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$.

Remark 1.2 As the reader might have noticed, the variables and the infinitesimals are missing from the integral in (5). To keep the notation simple, here and in the sequel when there is no ambiguity we will omit the variables and infinitesimals from the integrals. ■

Remark 1.3 Note that if $H \cong H'$, then for every function f we have $\int f^H = \int f^{H'}$. ■

Remark 1.4 Note that a hypergraph pair is norming (semi-norming), if $\|\cdot\|_H$ defines a norm (semi-norm) on $L_H(\mathcal{M})$, for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ with $|\Omega| < \infty$. ■

As one would suspect from Definition 1.1, the function $\|\cdot\|_H$ is not a priori a norm. We will pursue the question: “Which hypergraph pairs are norming (semi-norming), and what are the properties of the normed spaces induced by them?”

Remark 1.5 Let V_1, \dots, V_k be arbitrary finite sets. For $\psi \in V_1 \times \dots \times V_k$, we denote by 1_ψ the k -hypergraph pair $(\delta_\psi, 0)$, where δ_ψ is the Dirac’s delta function: $\delta_\psi(\omega) = 1$ if $\omega = \psi$, and $\delta_\psi(\omega) = 0$ otherwise.

We will apply arithmetic operations to hypergraph pairs: For example for two hypergraph pairs $H_1 = (\alpha_1, \beta_1)$ and $H_2 = (\alpha_2, \beta_2)$, their sum $H_1 + H_2$ and their difference $H_1 - H_2$ are defined respectively as the pairs $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$ and $(\alpha_1 - \alpha_2, \beta_1 - \beta_2)$. For a hypergraph pair $H = (\alpha, \beta)$ define $\overline{H} := (\beta, \alpha)$, and $rH := (r\alpha, r\beta)$ for every $r \in \mathbb{R}$. Now let $H_1 = (\alpha_1, \beta_1)$ be a hypergraph pair over $V_1 \times \dots \times V_k$ and $H_2 = (\alpha_2, \beta_2)$ be a hypergraph pair over $W_1 \times \dots \times W_k$. By considering proper isomorphisms we can assume that W_i and V_i are all disjoint. Then the disjoint union $H_1 \dot{\cup} H_2$ is defined as a hypergraph pair over $(V_1 \dot{\cup} W_1) \times \dots \times (V_k \dot{\cup} W_k)$ whose restrictions to $V_1 \times \dots \times V_k$ and $W_1 \times \dots \times W_k$ are respectively H_1 and H_2 , and is defined to be zero everywhere else. With these definitions, it is easy to verify that for a measurable function $f : \Omega^k \rightarrow \mathbb{C}$, we have

$$\begin{aligned} f^{H_1+H_2} &= f^{H_1} f^{H_2} \\ f^{H_1-H_2} &= f^{H_1} / f^{H_2} \\ f^{\overline{H}} &= \overline{f^H} \\ f^{rH} &= (f^H)^r = (f^r)^H \\ \int f^{H_1 \dot{\cup} H_2} &= \int f^{H_1} \int f^{H_2}. \end{aligned}$$

■

Consider a hypergraph pair H , and note that $\|\cdot\|_H = \|\cdot\|_{H \dot{\cup} H}$. Thus in order to characterize all norming (semi-norming) hypergraph pairs it suffices to consider hypergraph pairs that are minimal according to the following definition:

Definition 1.6 A hypergraph pair H over $V_1 \times \dots \times V_k$ is called *minimal* if

- For every $i \in [k]$ and $v_i \in V_i$, there exists at least one $\omega \in \text{supp}(\alpha) \cup \text{supp}(\beta)$ such that $\omega_i = v_i$.
- There is no k -hypergraph pair H' such that $H \cong H' \dot{\cup} H'$.

The next couple of examples show that some well-known families of normed spaces fall in the framework defined above.

Example 1.7 Let $L_p = (\alpha, \beta)$ be the 1-hypergraph pair defined as $\alpha = \beta = p/2$ over V_1 which contains only one element. Then for a measurable function $f : \Omega \rightarrow \mathbb{C}$, we have

$$\|f\|_{L_p} = \left(\int f^{p/2} \overline{f^{p/2}} \right)^{1/p} = \left(\int |f|^p \right)^{1/p} = \|f\|_p.$$

Hence in this case the $\|\cdot\|_{L_p}$ norm is the usual L_p norm. ■

Example 1.8 Let $k = 2$, $V_1 = V_2 = \{0, 1, \dots, m-1\}$, for some positive integer m . Define the 2-hypergraph pair $S_{2m} = (\alpha, \beta)$ as

$$\alpha(i, j) := \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(i, j) := \begin{cases} 1 & i = j + 1 \pmod{m} \\ 0 & \text{otherwise} \end{cases}$$

Let μ be the counting measure on a finite set Ω . Then for $A : \Omega^2 \rightarrow \mathbb{C}$ we have

$$\begin{aligned} \|A\|_{S_{2m}} &= \left(\sum A(x_0, y_0) \overline{A(x_1, y_0)} A(x_1, y_1) \overline{A(x_2, y_1)} \dots A(x_{m-1}, y_{m-1}) \overline{A(x_0, y_{m-1})} \right)^{1/2m} \\ &= (\text{Tr}(AA^*)^m)^{1/2m}, \end{aligned}$$

which shows that in this case the $L_{S_{2m}}$ norm coincides with the usual $2m$ -trace norm of matrices. \blacksquare

Example 1.9 Let k be a positive integer and $V_1 = \dots = V_k = \{0, 1\}$, and for $\omega \in V_1 \times \dots \times V_k$,

$$\alpha(\omega) := \sum_{i=1}^k \omega_i \pmod{2}$$

and

$$\beta(\omega) := 1 - \alpha(\omega).$$

Then for the k -hypergraph pair $U_k = (\alpha, \beta)$, $\|\cdot\|_{U_k}$ is called the Gowers k -uniformity norm. \blacksquare

1.2 Graph norms and subgraph densities

Hypergraph norms are important in the study of subgraph and sub-hypergraph densities. In fact this was one of the main motivations for studying the graph norms in [12]. We refer the reader to [12] for the details, but for now let us define the graph norms in our notation. Recall that a bipartite graph is a triple $H = (V_1, V_2, E)$ where $E \subseteq V_1 \times V_2$. Note that every such graph can be identified with a 2-hypergraph pair $H = (\alpha, 0)$ over $V_1 \times V_2$ where α is the indicator function of E . In [12] two candidates for being norms are corresponded to H . In our notation, they are defined by the formulas

$$\left| \int f^H \right|^{1/|H|}, \tag{6}$$

and

$$\left(\int |f|^H \right)^{1/|H|}, \tag{7}$$

where in (6) f is assumed to be a real-valued function. In our notation (6) = $\|f\|_{H \cup \overline{H}}$ and (7) = $\|f\|_{\frac{H+\overline{H}}{2}}$ which shows that our framework in this article is sufficiently general to include the graph norms.

An important conjecture due to Erdős and Simonovits [5] (See also Sidorenko [18, 19]) can be formulated in the language of the graph norms. Consider an arbitrary bipartite graph $H = (V_1, V_2, E)$, a probability space $\mathcal{P} = (\Omega, \mathcal{F}, \mu)$ and a measurable function $f : \Omega^2 \rightarrow \mathbb{R}^+$. It is conjectured in [5] that ¹

$$\|f\|_1 \leq \|f\|_H.$$

It has been shown in [12] that if the formula in (7) corresponds to a norm, then the statement of the conjecture is true for H . The same arguments hold in the setting of hypergraph pairs as well, and similar inequalities can be obtained for norming hypergraph pairs. This follows from Corollary 2.12

¹This form of the conjecture is due to Sidorenko, but it is equivalent to what is conjectured in [5].

below. However it should be noted that the analogue of Erdős-Simonovits-Sidorenko conjecture for k -variable functions where $k > 2$ is false (See [19]).

The moduli of smoothness and convexity are two dual parameters assigned to a normed space that play a fundamental role in Banach space theory. We will discuss them extensively in Section 3. In [12] the moduli of smoothness and convexity of the normed spaces defined by (6) are determined, but for the normed spaces defined by (7) it was left open. This question will be answered in Theorem 3.11.

1.3 Constructing norming hypergraph pairs

The following definition introduces the tensor product of two hypergraph pairs.

Definition 1.10 *Let $H_1 = (\alpha_1, \beta_1)$ be a k -hypergraph pair over $V_1 \times \dots \times V_k$ and $H_2 = (\alpha_2, \beta_2)$ be a k -hypergraph pair over $W_1 \times \dots \times W_k$. Then the tensor product of H_1 and H_2 , is a k -hypergraph pair over $U_1 \times \dots \times U_k$ where $U_i := V_i \times W_i$, defined as*

$$H_1 \otimes H_2 := (\alpha_1 \otimes \alpha_2 + \beta_1 \otimes \beta_2, \alpha_1 \otimes \beta_2 + \beta_1 \otimes \alpha_2).$$

We have already seen in Examples 1.7, 1.8, 1.9 that norming hypergraph pairs do exist. Theorem 1.11 below shows that it is possible to combine two norming hypergraph pairs to construct a new one.

Theorem 1.11 *Let H_1 and H_2 be two hypergraph pairs. If H_1 and H_2 are norming (semi-norming), then $H_1 \otimes H_2$ is also norming (semi-norming).*

The proof of Theorem 1.11 is parallel to the proof of Theorem 2.9 in [12], and thus we omit it.

The following Lemma which we state without a proof is a generalization of Theorem 2.8 (ii) in [12]. It can be easily derived using a similar argument to the proof of Theorem 2.8 (ii) in [12].

Lemma 1.12 *Consider finite sets V_1, \dots, V_k . For $\frac{1}{2} \leq p < \infty$, the hypergraph pair $K = (p, p)$ over $V_1 \times \dots \times V_k$ is norming.*

2 Structure of Norming hypergraph pairs

In this section we study the structure of semi-norming hypergraph pairs. The main result that we prove in this direction is the following.

Theorem 2.1 *Let $H = (\alpha, \beta)$ be a semi-norming hypergraph pair. Then $H \cong \overline{H}$, and one of the following two cases hold*

- **Type I:** *There exists a real $s \geq 1$, such that for every $\psi \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, $\alpha(\psi) = \beta(\psi) = s/2$. In this case, s is called the parameter of H .*
- **Type II:** *For every $\psi \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, we have $\{\alpha(\psi), \beta(\psi)\} = \{0, 1\}$.*

Note that the condition $H \cong \overline{H}$ is trivially satisfied for every hypergraph pair that satisfies the requirements of Type I hypergraph pairs. This is not true for Type II hypergraph pairs, and in this case $H \cong \overline{H}$ implies a further restriction on the structure of the hypergraph pair.

Remark 2.2 Note that if H is of Type I, then for every measure space \mathcal{M} and every $f \in L_H(\mathcal{M})$, we have $\|f\|_H = \| |f| \|_H$. This fact will be used frequently in the sequel. ■

Suppose that $H = (\alpha, \beta)$ is a k -hypergraph pair over $V_1 \times \dots \times V_k$. For a subset $S \subseteq [k]$, we use the notation π_S to denote the natural projection from $V_1 \times \dots \times V_k$ to $\prod_{i \in S} V_i$. We can construct a hypergraph pair $H_S := (\alpha_S, \beta_S)$ where $\alpha_S, \beta_S : \prod_{i \in S} V_i \rightarrow \mathbb{C}$ are defined as

$$\alpha_S : \omega \mapsto \sum \{ \alpha(\omega') : \pi_S(\omega') = \omega \},$$

and

$$\beta_S : \omega \mapsto \sum \{\beta(\omega') : \pi_S(\omega') = \omega\}.$$

By Remark 1.4, we have The following trivial observation:

Observation 2.3 *If $H = (\alpha, \beta)$ is a norming (semi-norming) k -hypergraph pair, then for every $S \subseteq [k]$, H_S is norming (semi-norming).*

Remark 2.4 The importance of Observation 2.3 is in that one can apply Theorem 2.1 to H_S to deduce more conditions on the structure of the original semi-norming hypergraph pair H . For example applying Theorem 2.1 to H_S when S has only one element implies that for every $1 \leq i \leq k$, there exists a number d_i such that for every $v_i \in V_i$, we have $\sum \{\alpha(\omega) : \omega_i = v_i\} = \sum \{\beta(\omega) : \omega_i = v_i\} = d_i$. ■

The next theorem gives another necessary condition on the structure of a semi-norming hypergraph pair.

Theorem 2.5 *Suppose that $H = (\alpha, \beta)$ is a semi-norming k -hypergraph pair over $V_1 \times \dots \times V_k$. Let $W_i \subseteq V_i$ for $i = 1, \dots, k$, and H' be the restriction of H to $W_1 \times \dots \times W_k$. Then*

$$\frac{|H'|}{|W_1| + \dots + |W_k| - 1} \leq \frac{|H|}{|V_1| + \dots + |V_k| - 1}.$$

We present the proofs of Theorems 2.1 and 2.5 in Section 2.5, but first we need to develop some tools.

2.1 Two Hölder type inequalities

One of our main tools in the study of hypergraph norms is the trick of amplification by taking tensor powers. This trick has been used successfully in many places (see for example [17]).

Definition 2.6 *For $f, g : \Omega^k \rightarrow \mathbb{C}$, the tensor product of f and g is defined as $f \otimes g : (\Omega^2)^k \rightarrow \mathbb{C}$ where $f \otimes g[(x_1, y_1), \dots, (x_k, y_k)] = f(x_1, \dots, x_k)g(y_1, \dots, y_k)$.*

We have the following trivial observation.

Observation 2.7 *Let H_1, H_2 be two k -hypergraph pairs, and $f_1, f_2, g_1, g_2 : \Omega^k \rightarrow \mathbb{C}$. Then*

$$\int (f_1 \otimes f_2)^{H_1} (g_1 \otimes g_2)^{H_2} = \left(\int f_1^{H_1} g_1^{H_2} \right) \left(\int f_2^{H_1} g_2^{H_2} \right).$$

Now with Observation 2.7 in hand, we can prove our first result about semi-norming hypergraph pairs.

Lemma 2.8 *Let $H = (\alpha, \beta)$ be a semi-norming hypergraph pair. Then for every measurable space \mathcal{M} , and every $f, g \in L_H(\mathcal{M})$ the following holds. For every $\psi \in \text{supp}(\alpha)$,*

$$\left| \int f^{H-1\psi} g^{1\psi} \right| \leq \|f\|_H^{|H|-1} \|g\|_H, \quad (8)$$

and for every $\psi \in \text{supp}(\beta)$

$$\left| \int f^{H-1\bar{\psi}} g^{1\bar{\psi}} \right| \leq \|f\|_H^{|H|-1} \|g\|_H. \quad (9)$$

Conversely, if for a measure space \mathcal{M} , and every $f, g \in L_H(\mathcal{M})$, $\int f^H \in \mathbb{R}^+$, and at least one of (8) or (9) holds for some $\psi \in V_1 \times \dots \times V_k$, then $\|\cdot\|_H$ is a semi-norm on $L_H(\mathcal{M})$.

Proof. First we prove the converse direction which is easier. Consider two measurable functions $f, g : \Omega^k \rightarrow \mathbb{C}$ and suppose that (8) holds for some $\psi \in V_1 \times \dots \times V_k$. Then

$$\begin{aligned} \|f + g\|_H^{|H|} &= \int (f + g)^H = \int (f + g)^{H-1\psi} (f + g)^{1\psi} \\ &= \int (f + g)^{H-1\psi} f^{1\psi} + \int (f + g)^{H-1\psi} g^{1\psi} \leq \|f + g\|_H^{|H|-1} \|f\|_H + \|f + g\|_H^{|H|-1} \|g\|_H, \end{aligned}$$

which simplifies to the triangle inequality. The proof of the case where (9) holds is similar.

Now let us turn to the other direction. Suppose that H is a semi-norming hypergraph pair. Consider $f, g \in L_H(\mathcal{M})$. We might assume that $\|f\|_H \neq 0$, as otherwise one can instead consider a small perturbation of f . Since $\|\cdot\|_H$ is a semi-norm, for every $t \in \mathbb{R}^+$ and every $f, g : \Omega \rightarrow \mathbb{C}$, we have $\|f + tg\|_H \leq \|f\|_H + t\|g\|_H$ which implies that

$$\left. \frac{d\|f + tg\|_H}{dt} \right|_0 \leq \|g\|_H. \quad (10)$$

Computing the derivative

$$\frac{d(f + tg)^H}{dt} = \sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) (f + tg)^{H-1\psi} g^{1\psi} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) (f + tg)^{H-\overline{1}\psi} g^{\overline{1}\psi},$$

shows that

$$\frac{d\|f + tg\|_H}{dt} = \frac{1}{|H|} \|f + tg\|_H^{1-|H|} \left(\int \sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) (f + tg)^{H-1\psi} g^{1\psi} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) (f + tg)^{H-\overline{1}\psi} g^{\overline{1}\psi} \right).$$

Thus by (10),

$$\frac{1}{|H|} \|f\|_H^{1-|H|} \left(\int \sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) f^{H-1\psi} g^{1\psi} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) f^{H-\overline{1}\psi} g^{\overline{1}\psi} \right) \leq \|g\|_H,$$

or equivalently

$$\frac{1}{|H|} \left(\int \sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) f^{H-1\psi} g^{1\psi} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) f^{H-\overline{1}\psi} g^{\overline{1}\psi} \right) \leq \|f\|_H^{|H|-1} \|g\|_H. \quad (11)$$

Since (11) holds for every measure space and every pair of measurable functions, for every integer $m > 0$, we can replace f and g in (11), respectively with $f^{\otimes m} \otimes \overline{f}^{\otimes m}$ and $g^{\otimes m} \otimes \overline{g}^{\otimes m}$, and apply Observation 2.7 to obtain

$$\frac{1}{|H|} \left(\sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) \left| \int f^{H-1\psi} g^{1\psi} \right|^{2m} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) \left| \int f^{H-\overline{1}\psi} g^{\overline{1}\psi} \right|^{2m} \right) \leq \left(\|f\|_H^{|H|-1} \|g\|_H \right)^{2m}. \quad (12)$$

But since (12) holds for every m , it establishes (8) and (9) as

$$\frac{1}{|H|} \left(\sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) + \sum_{\psi \in \text{supp}(\beta)} \alpha(\psi) \right) = 1.$$

We have the following corollary to Lemma 2.8. ■

Corollary 2.9 *If H is a semi-norming hypergraph pair, then $\alpha(\omega) + \beta(\omega) \geq 1$, for every $\omega \in \text{supp}(\alpha) \cup \text{supp}(\beta)$.*

Proof. Let the underlying measure space be the set $\{0, 1\}$ with the counting measure. Consider $\omega \in \text{supp}(\alpha)$, and note that by (8), for every pair of functions $f, g : \{0, 1\}^k \rightarrow \mathbb{C}$, we have

$$\left| \int f^{H-1_\psi} g^{1_\psi} \right| \leq \|f\|_H^{|H|-1} \|g\|_H. \quad (13)$$

For every $x = (x_1, \dots, x_k) \in \{0, 1\}^k$, define $g(x) := 1$ and

$$f(x) := \begin{cases} \epsilon & x_1 = \dots = x_k = 1 \\ 1 & \text{otherwise} \end{cases}$$

Then $|\int f^{H-1_\psi} g^{1_\psi}| = |\int f^{H-1_\psi}| \geq \epsilon^{\alpha(\omega) + \beta(\omega) - 1}$, while $\|f\|_H \leq \|g\|_H = \|1\|_H$, which contradicts (13) for sufficiently small $\epsilon > 0$, if $\alpha(\omega) + \beta(\omega) < 1$. \blacksquare

Under some extra conditions it is possible to extend (8) and (9) to a much more powerful inequality.

Lemma 2.10 *Let H be a semi-norming hypergraph pair, and H_1, \dots, H_n be nonzero² hypergraph pairs satisfying $H_1 + H_2 + \dots + H_n = H$. Then for every measure space \mathcal{M} and functions $f_1, f_2, \dots, f_n \in L_H(\mathcal{M})$, we have*

$$\left| \int f_1^{H_1} f_2^{H_2} \dots f_n^{H_n} \right| \leq \|f_1\|_H^{|H_1|} \dots \|f_n\|_H^{|H_n|},$$

provided that at least one of the following two conditions hold:

- (a) We have $f_1, \dots, f_n \geq 0$.
- (b) For every $H_i = (\alpha_i, \beta_i)$, the functions α_i, β_i take only integer values.

Proof. Let us first assume that $f_1, \dots, f_n \geq 0$. Suppose to the contrary that

$$\int f_1^{H_1} f_2^{H_2} \dots f_n^{H_n} > \|f_1\|_H^{|H_1|} \dots \|f_n\|_H^{|H_n|}. \quad (14)$$

After normalization we can assume that $\|f_1\|_H, \|f_2\|_H, \dots, \|f_n\|_H \leq 1$ while the right-hand side of (14) is strictly greater than 1. Since (14) remains valid after small perturbations of f_i 's, without loss of generality we might also assume that for every $1 \leq i \leq n$, f_i does not take the zero value on any point. Consider a positive integer m , and note that by Observation 2.7

$$\begin{aligned} \int \left(\sum_{i=1}^n f_i^{\otimes m} \right)^H &= \int \prod_{i=1}^n \left(\sum_{i=1}^n f_i^{\otimes m} \right)^{H_i} = \int (f_1^{\otimes m})^{H_1} \dots (f_n^{\otimes m})^{H_n} \left(\prod_{i=1}^n \left(\frac{f_1^{\otimes m} + \dots + f_n^{\otimes m}}{f_i^{\otimes m}} \right)^{H_i} \right) \\ &\geq \int (f_1^{\otimes m})^{H_1} \dots (f_n^{\otimes m})^{H_n} = \left(\int f_1^{H_1} \dots f_n^{H_n} \right)^m. \end{aligned}$$

On the other hand, Observation 2.7 shows that $\|f_i^{\otimes m}\|_H = \|f_i\|_H^m \leq 1$ for every $i \in [n]$. Then for sufficiently large m we get a contradiction:

$$\left\| \sum f_i^{\otimes m} \right\|_H \geq \left(\int f_1^{H_1} \dots f_n^{H_n} \right)^{m/|H|} > n.$$

Next consider the case where f_i are not necessarily positive, but we know that α_i, β_i all take only integer values. Again to get a contradiction assume that

$$\left| \int f_1^{H_1} \dots f_n^{H_n} \right| > 1 \geq \|f_1\|_H^{|H_1|} \dots \|f_n\|_H^{|H_n|}.$$

²i.e. $H_i \neq (0, 0)$ for every $1 \leq i \leq n$.

where $\|f_1\|_H, \dots, \|f_n\|_H \leq 1$. In this case for every $i \in [n]$, we will consider $f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m}$. Let \mathcal{H} denote the set of all n -tuples of nonzero hypergraph pairs $(H'_1, H'_2, \dots, H'_n)$ where H'_i 's take only nonnegative integer values and $H'_1 + H'_2 + \dots + H'_n = H$. By Observation 2.7

$$\int \prod_{i=1}^n (f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m})^{H'_i} = \left| \int \prod_{i=1}^n f_i^{H'_i} \right|^{\otimes 2m} \geq 0.$$

Now by expanding the product defined by H , we have

$$\begin{aligned} \int \left(\sum_{i=1}^n f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m} \right)^H &= \sum_{(H'_1, \dots, H'_n) \in \mathcal{H}} \int \prod_{i=1}^n (f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m})^{H'_i} \\ &\geq \int \prod_{i=1}^n (f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m})^{H_i} = \left| \int \prod_{i=1}^n f_i^{H_i} \right|^{2m}, \end{aligned}$$

which leads to a contradiction similar to the previous case. \blacksquare

Remark 2.11 It is possible to show that Lemma 2.10 does not necessarily hold in the general case where none of the two conditions are satisfied. To see this consider S_4 from Example 1.8. By Lemma 2.8, if Lemma 2.10 holds for the decomposition $S_4 = \frac{1}{3}S_4 + \frac{1}{3}S_4 + \frac{1}{3}S_4$, then $3S_4$ would be a semi-norming hypergraph pair. But Theorem 2.1 implies that $3S_4$ is not a semi-norming hypergraph pair. \blacksquare

Consider a *probability space* $\mathcal{P} = (\Omega, \mathcal{F}, \mu)$. It is well-known that for every $1 \leq p \leq q$, and for every $f \in L_q(\mathcal{P})$, we have $\|f\|_p \leq \|f\|_q$. The next corollary generalizes this to hypergraph pairs.

Corollary 2.12 *Let $H = (\alpha, \beta)$ be a semi-norming k -hypergraph pair. Consider a probability space $\mathcal{P} = (\Omega, \mathcal{F}, \mu)$ and $f \in L_H(\mathcal{P})$. Let $K = (\alpha', \beta')$ be a nonzero k -hypergraph pair over the same domain as H such that $\alpha' \leq \alpha$ and $\beta' \leq \beta$. Then*

$$\|f\|_K \leq \|f\|_H,$$

provided that at least one of the following three conditions holds:

- (a) $f \geq 0$.
- (b) H is of type I.
- (c) The functions $\alpha, \beta, \alpha', \beta'$ take only integer values.

Proof. Parts (a) and (b) follow from applying Lemma 2.10 (a), with parameters $n := 2$, $H_1 := K$, $H_2 := H - K$, $f_1 := |f|$ and $f_2 := 1$.

Part (c) follows from applying Lemma 2.10 (b), with parameters $n := 2$, $H_1 := K$, $H_2 := H - K$, $f_1 := f$ and $f_2 := 1$. \blacksquare

2.2 Factorizable hypergraph pairs

In this section we characterize all norming and semi-norming 1-hypergraph pairs. As it is mentioned before, it suffices to consider the hypergraph pairs that are minimal according to Definition 1.6. We have already seen one class of examples of norming 1-hypergraph pairs, namely the 1-hypergraph pairs L_p of Example 1.7. There exists also a semi-norming 1-hypergraph pair that is not norming. Let $G = (1, 0)$ be the 1-hypergraph pair over a set V_1 of size 1. Then for a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and a measurable $f : \Omega \rightarrow \mathbb{C}$ we have $\|f\|_{G \cup \overline{G}} = |f|$ which defines a semi-norm. The next proposition shows that these are the only examples.

Proposition 2.13 *If H is a minimal norming 1-hypergraph pair, then there exists $1 \leq p < \infty$ such that $H \cong L_p$. If H is a minimal semi-norming 1-hypergraph pair that is not norming, then $H \cong G \dot{\cup} \overline{G}$, where $G = (1, 0)$ is a 1-hypergraph pair over a set V_1 of size 1.*

To prove Proposition 2.13 we need to study the hypergraph pairs which are decomposable into disjoint union of other hypergraph pairs.

Definition 2.14 *A hypergraph pair $H = (\alpha, \beta)$ is called factorizable, if it is the disjoint union of two hypergraph pairs.*

The next proposition shows that two non-factorizable hypergraph pairs define identical norms, if and only if they are isomorphic. For the proof, we need an easy fact stated in the following Remark.

Remark 2.15 Let x_1, \dots, x_n be n complex variables. Define a term as a product $\prod_{i=1}^n x_i^{p_i} \overline{x_i}^{q_i}$, where p_i, q_i are nonnegative reals. Now let P and Q be two formal finite sums of terms. It is easy to see that P and Q are equal as functions on \mathbb{C}^n , if and only if they are equal as formal sums. ■

Proposition 2.16 *Let H_1 and H_2 be two minimal k -hypergraph pairs. Suppose that either H_1 and H_2 are both non-factorizable, or we have $|H_1| = |H_2|$. Then*

- *If for every measure space $(\Omega, \mathcal{F}, \mu)$, and every $f : \Omega^k \rightarrow \mathbb{C}$, $\|f\|_{H_1} = \|f\|_{H_2}$, then $H_1 \cong H_2$.*
- *If for every measure space $(\Omega, \mathcal{F}, \mu)$, and every $f : \Omega^k \rightarrow \mathbb{C}$, $\|f\|_{H_1} = \overline{\|f\|_{H_2}}$, then $H_1 \cong \overline{H_2}$.*

Proof. Suppose that H_1 and H_2 are respectively defined over $V_1 \times \dots \times V_k$ and $W_1 \times \dots \times W_k$. First assume that H_1 and H_2 are both non-factorizable. Let μ be the counting measure on $\Omega = [m]$, where $m > \sum_{i=1}^k |V_i| + |W_i|$ is a positive integer. Suppose that for every $f : \Omega^k \rightarrow \mathbb{C}$ with $\|f\|_{H_1} = \|f\|_{H_2}$. Then define $f(x_1, \dots, x_k)$ to be equal to 1, if $x_1 = \dots = x_k$, and equal to 0 otherwise. Since H_1 and H_2 are non-factorizable it is easy to see that $\int f^{H_1} = \int f^{H_2} = |\Omega|$ and we deduce that $|H_1| = |H_2|$. So it is sufficient to prove the proposition for the case where $|H_1| = |H_2|$.

Now for every $f : \Omega^k \rightarrow \mathbb{C}$, we have $\int f^{H_1} = \int f^{H_2}$. For $1 \leq i \leq k$, consider $f_i : \Omega^k \rightarrow \{0, 1\}$ defined as $f_i(x_1, \dots, x_k) = 1$ if and only if $x_1 = \dots = x_{i-1} = x_{i+1} = \dots = x_k = 1$. Then it is easy to see that $\int f_i^{H_1} = |\Omega|^{|V_i|}$ and $\int f_i^{H_2} = |\Omega|^{|W_i|}$ which implies $|V_i| = |W_i|$. Thus without loss of generality we may assume that $V_i = W_i = \{1, \dots, |V_i|\}$, for every $1 \leq i \leq k$. Now for every $f : \Omega^k \rightarrow \mathbb{C}$ we have

$$\sum_{x \in \Omega^{V_1} \times \dots \times \Omega^{V_k}} \prod_{\omega \in V} f(\omega(x))^{\alpha(\omega)} \overline{f(\omega(x))}^{\beta(\omega)} = \sum_{x \in \Omega^{V_1} \times \dots \times \Omega^{V_k}} \prod_{\omega \in V} f(\omega(x))^{\alpha'(\omega)} \overline{f(\omega(x))}^{\beta'(\omega)} \quad (15)$$

Consider $x = [(1, \dots, |V_1|), (1, \dots, |V_2|), \dots, (1, \dots, |V_k|)] \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$. Then $\omega(x) = \omega$ for every $\omega \in V$, and hence

$$\prod_{\omega \in V} f(\omega(x))^{\alpha(\omega)} \overline{f(\omega(x))}^{\beta(\omega)} = \prod_{\omega \in V} f(\omega)^{\alpha(\omega)} \overline{f(\omega)}^{\beta(\omega)}. \quad (16)$$

Since (16) appears in the sum in the left-hand side of (15), by Remark 2.15 it must also appear as a term in the right-hand side of (15). Hence there exists $y = [(y_{1,1}, \dots, y_{1,|V_1|}), \dots, (y_{k,1}, \dots, y_{k,|V_k|})] \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$ such that

$$\prod_{\omega \in V} f(\omega(y))^{\alpha'(\omega)} \overline{f(\omega(y))}^{\beta'(\omega)} = \prod_{\omega \in V} f(\omega)^{\alpha(\omega)} \overline{f(\omega)}^{\beta(\omega)}. \quad (17)$$

By minimality (see Definition 1.6), for every $v \in V_i$, there exists $\omega = (\omega_1, \dots, \omega_k) \in \text{supp}(\alpha) \cup \text{supp}(\beta)$ such that $\omega_i = v$. This implies $\{y_{i,1}, \dots, y_{i,|V_i|}\} = V_i$, for every $1 \leq i \leq k$. Now $h = (h_1, \dots, h_k)$ defined as $h_i : j \mapsto y_{i,j}$ (for every $1 \leq i \leq k$ and $1 \leq j \leq |V_i|$) is an isomorphism between H_1 and H_2 .

In the second part of the proposition where it is assumed $\|f\|_{H_1} = \overline{\|f\|_{H_2}}$, instead of (15) one obtains that the left-hand side of (15) is equal to the conjugate of the right-hand side. The proof then proceeds similar to the previous case. ■

Theorem 2.17 Let $H = H_1 \dot{\cup} H_2 \dot{\cup} \dots \dot{\cup} H_m$ be a semi-norming hypergraph pair such that H_i are all non-factorizable. Then for every measure space \mathcal{M} and every $f \in L_H(\mathcal{M})$ we have

$$\|f\|_{H_1 \dot{\cup} \overline{H_1}} = \|f\|_{H_2 \dot{\cup} \overline{H_2}} = \dots = \|f\|_{H_m \dot{\cup} \overline{H_m}} = \|f\|_H.$$

Proof. Let $H = G_1 \dot{\cup} G_2$ be semi-norming, where G_1 and G_2 are not necessarily non-factorizable, $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space, and $f \in L_H(\mathcal{M})$. Note that

$$\|f\|_H = \|f\|_{\frac{|G_1|}{|H|}} \|f\|_{\frac{|G_2|}{|H|}} = \|f\|_{\frac{|G_1|}{|G_1|+|G_2|}} \|f\|_{\frac{|G_2|}{|G_1|+|G_2|}}.$$

It follows from Theorem 2.1 that either H is of Type I, or H and G_1 both take only integer values. Hence by Corollary 2.12

$$\left| \|f\|_{G_1}^{|G_1|} \right| \leq \|f\|_H^{|G_1|} = \|f\|_{\frac{|G_1|^2}{|G_1|+|G_2|}} \|f\|_{\frac{|G_2|^2}{|G_1|+|G_2|}},$$

which simplifies to

$$\| \|f\|_{G_1} \| \leq \| \|f\|_{G_2} \|.$$

Similarly one can show that $\| \|f\|_{G_2} \| \leq \| \|f\|_{G_1} \|$ and thus $\| \|f\|_{G_1} \| = \| \|f\|_{G_2} \|$.

By induction we conclude that $\| \|f\|_{H_1} \| = \dots = \| \|f\|_{H_2} \|$, for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and every $f \in L_H(\mathcal{M})$, and this completes the proof as $\| \|f\|_{H_i} \| = \|f\|_{H_i \dot{\cup} \overline{H_i}}$. ■

Now we can state the proof of Proposition 2.13.

Proof.[Proposition 2.13] Consider a semi-norming 1-hypergraph pair H over a set $V_1 = \{v_1, \dots, v_m\}$. Consider the factorization $H = H_1 \dot{\cup} H_2 \dot{\cup} \dots \dot{\cup} H_m$, where H_i is a 1-hypergraph pair over $\{v_i\}$. By Theorem 2.17, always $\|f\|_{H_1 \dot{\cup} \overline{H_1}} = \|f\|_{H_2 \dot{\cup} \overline{H_2}} = \dots = \|f\|_{H_m \dot{\cup} \overline{H_m}} = \|f\|_H$. By Theorem 2.1, for every $1 \leq i \leq m$, either $H_i \dot{\cup} \overline{H_i} \cong L_p \dot{\cup} \overline{L_p}$ for some $1 \leq p < \infty$, or $H_i \dot{\cup} \overline{H_i} \cong G \dot{\cup} \overline{G}$ which completes the proof. ■

2.3 Semi-norming hypergraph pairs that are not norming

In this section we study the structure of the semi-norming hypergraph pairs which are *not* norming. Consider a semi-norming k -hypergraph pair $H = (\alpha, \beta)$ over $V := V_1 \times \dots \times V_k$ of Type I with parameter $s = 2m$, where m is a positive integer. Since H is of Type I, it is trivially norming. Consider an arbitrary positive integer k' . We want to use H to construct a semi-norming $(k+k')$ -hypergraph pair that is not norming. For $k+1 \leq i \leq k+k'$, let $V_i := \text{supp}(\alpha) \times \{1, \dots, s\}$. Now $G = (\alpha', \beta')$ is defined by

$$\alpha(v_1, \dots, v_{k+k'}) := \begin{cases} 1 & v_{k+1} = \dots = v_{k+k'} = ([v_1, \dots, v_k], i) \text{ where } 1 \leq i \leq m \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta(v_1, \dots, v_{k+k'}) := \begin{cases} 1 & v_{k+1} = \dots = v_{k+k'} = ([v_1, \dots, v_k], i) \text{ where } m+1 \leq i \leq 2m \\ 0 & \text{otherwise} \end{cases}$$

Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, and an integrable function $f : \Omega^{k+k'} \rightarrow \mathbb{C}$. Let $F : \Omega^k \rightarrow \mathbb{C}$ be defined as $F(x_1, \dots, x_k) = \int f(x_1, \dots, x_{k+k'}) dx_{k+1} \dots dx_{k+k'}$. It is not difficult to see that $\|f\|_G = \|F\|_H$, which shows that G is semi-norming. On the other-hand if $\int f dx_{k+1} \dots dx_{k+k'} = 0$, then $\|f\|_G = \|F\|_H = \|0\|_H = 0$ which implies that G is not norming. The next proposition shows that in fact every semi-norming hypergraph pair which is not norming is of this form.

Proposition 2.18 Let $H = (\alpha, \beta)$ be a semi-norming k -hypergraph pair of Type II over $V := V_1 \times \dots \times V_k$. Define S to be the set of all $1 \leq i \leq k$ such that for every $v_i \in V_i$,

$$\sum \{ \alpha(\omega) + \beta(\omega) : \omega \in V, \omega_i = v_i \} = 1.$$

Then $H_{[k] \setminus S}$ is a norming hypergraph pair of Type I.

Proof. Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. Note that if $S \neq \emptyset$, then for every $i \in S$, every $f \in L_H(\mathcal{M})$ with $\int f(x_1, \dots, x_k) dx_i = 0$ satisfies $\|f\|_H = 0$. So if H is norming, then $H_{[k] \setminus S} = H$, and the proposition holds. Consider a k -hypergraph pair $H = (\alpha, \beta)$ over $V := V_1 \times \dots \times V_k$ which is not norming. Then there exists a function $f \in L_H(\mathcal{M})$, for some measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, such that $\int f^H = 0$ and $f \neq 0$. Lemma 2.8, then shows that for every $g \in L_H(\mathcal{M})$, and every $\psi \in \text{supp}(\alpha)$,

$$\int g^{H-1\psi} f^{1\psi} = 0. \quad (18)$$

Since $f \neq 0$, there exists measurable sets $\Gamma_1, \dots, \Gamma_k \subseteq \Omega$ such that $\int_{\Gamma_1 \times \dots \times \Gamma_k} f \neq 0$. Define $g : \Omega^k \rightarrow \{0, 1\}$, as

$$g(x_1, \dots, x_k) = \begin{cases} 1 & (x_1, \dots, x_k) \in \Gamma_1 \times \dots \times \Gamma_k \\ 0 & \text{otherwise} \end{cases}$$

Suppose that for every $i \in [k]$, there exists $\omega \in \text{supp}(\alpha) \cup \text{supp}(\beta)$ such that $\omega \neq \psi$ but $\omega_i = \psi_i$. Then it is easy to see that for every $x \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$,

$$g^{H-1\psi}(x) = \begin{cases} 1 & \psi(x) \in \Gamma_1 \times \dots \times \Gamma_k \\ 0 & \text{otherwise} \end{cases}$$

But then $\int g^{H-1\psi} f^{1\psi} = \int_{\Gamma_1 \times \dots \times \Gamma_k} f \neq 0$ contradicting (18).

It follows from (18) and its analogue for $\psi \in \text{supp}(\beta)$ that the following holds: For every $\psi = (\psi_1, \dots, \psi_k) \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, there exists $i \in [k]$ such that

$$\{\omega \in \text{supp}(\alpha) \cup \text{supp}(\beta) : \omega_i = \psi_i \text{ and } \omega \neq \psi\} = \emptyset,$$

or in other words: $\sum \{\alpha(\omega) + \beta(\omega) : \omega \in V, \omega_i = \psi_i\} = 1$. Now Remark 2.4 shows that $i \in S$. By Observation 2.3, $H_{[k] \setminus S}$ is semi-norming, but then maximality of S shows that it is also norming. ■

2.4 Some facts about Gowers norms

In this section we prove some facts about Gowers norms that are needed in the subsequent sections. These facts are only proved as auxiliary results, and thus our aim is not to obtain the best possible bounds or to prove them in the most general possible setting.

Let $V_1 = \dots = V_k = \{0, 1\}$, and U_k be the Gowers k -hypergraph pair defined in Example 1.9. Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and measurable functions $f_\omega : \Omega^k \rightarrow \mathbb{C}$ for $\omega \in V := V_1 \times \dots \times V_k$. The following inequality due to Gowers [8] (see also [20]) can be proven by iterated applications of the Cauchy-Schwarz inequality:

$$\left| \int \prod_{\omega \in V} f_\omega^{1\omega} \right| \leq \prod_{\omega \in V} \|f_\omega\|_{U_k}. \quad (19)$$

Since always $\|f\|_{U_k} \leq \|f\|_\infty$, we have the following easy corollary.

Corollary 2.19 *Let $H = (\alpha, \beta)$ be a k -hypergraph pair over $W := W_1 \times W_2 \times \dots \times W_k$, and $\psi \in W$ be such that $\alpha(\psi) = \beta(\psi) = 0$. Then for the measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and every pair of measurable functions $f, g : \Omega^k \rightarrow \mathbb{C}$, we have*

$$\left| \int f^H g^{1\psi} \right| \leq \|g\|_{U_k} \|f\|_\infty^{|H|}.$$

The next Lemma shows that there exists a function g such that its range is $\{-1, 1\}$ but its Gowers norm is arbitrarily small.

Lemma 2.20 *For every $\epsilon > 0$, there exists a probability space $(\Omega, \mathcal{F}, \mu)$ and a function $g : \Omega^k \rightarrow \{-1, 1\}$ such that $\|g\|_{U_k} \leq \epsilon$ and $\int g = 0$.*

Proof. Consider a sufficiently large even integer m , set $\Omega = [m]$, and let μ be the uniform probability measure on Ω . Define g randomly so that $\{g(\omega)\}_{\omega \in \Omega^k}$ are independent Bernoulli random variables taking values uniformly in $\{-1, 1\}$. Then it is easy to see that

$$\mathbb{E}(\int g)^2 = o_{m \rightarrow \infty}(1) \quad \text{and} \quad \mathbb{E}(\int g^{U_k})^2 = o_{m \rightarrow \infty}(1).$$

Hence for sufficiently large m , there exists $g_0 : \Omega^k \rightarrow \{-1, 1\}$ such that $|\int g_0| \leq (\epsilon/4)^{2^k}$ and $\|g_0\|_{U_k} \leq \epsilon/2$. Trivially there exists $g_1 : \Omega^k \rightarrow \{-1, 1\}$ such that $\int g_1 = 0$ and $\int |g_1 - g_0| \leq (\epsilon/4)^{2^k}$. Then by Hölder's inequality

$$\|g_0 - g_1\|_{U_k} = \left(\int (g_0 - g_1)^{U_k} \right)^{-2^k} \leq \|g_0 - g_1\|_{2^k} \leq 2(\epsilon/4) = \epsilon/2,$$

where in the last inequality we used the fact that the range of $g_0 - g_1$ is $\{-2, 0, 2\}$. Now

$$\|g_1\|_{U_k} \leq \|g_0\|_{U_k} + \|g_0 - g_1\|_{U_k} \leq \epsilon,$$

which shows that g_1 is the desired function. ■

Lemma 2.21 *For a k -hypergraph pair H over $V := V_1 \times \dots \times V_k$, a probability space \mathcal{P} , and a zero-one function $f \in L_H(\mathcal{P})$ we have*

$$\int f^H \geq \|f\|_1^{-|V_1| \dots |V_k|}.$$

Proof. Consider the k -hypergraph pair $K = (\frac{1}{2}, \frac{1}{2})$ over V . Lemma 1.12 shows that K is a norming hypergraph pair. Since f is a zero-one function, we have $f^H \geq f^K$, and thus by Corollary 2.12

$$\int f^H \geq \int f^K \geq \|f\|_1^{|K|} \geq \|f\|_1^{-|V_1| \dots |V_k|}.$$

■

Lemma 2.22 *Let $f, g : \Omega^k \rightarrow \mathbb{C}$ be two measurable functions with respect to the probability space $(\Omega, \mathcal{F}, \mu)$. Let $H = (\alpha, 0)$ be a hypergraph pair such that $\text{ran}(\alpha) \subseteq \{0, 1\}$. Then*

$$\left| \int f^H - g^H \right| \leq |H| \|f - g\|_{U_k} \max(\|f\|_\infty, \|g\|_\infty)^{|H|-1}.$$

Proof. Let us label the elements of $\text{supp}(\alpha)$ as $\omega_1, \dots, \omega_{|H|}$. Then for $0 \leq i \leq |H|$ define $H_i := \sum_{j=1}^i 1_{\omega_j}$, so that $H_0 = (0, 0)$ and $H_{|H|} = H$. Now by telescoping and applying Corollary 2.19 we have

$$\begin{aligned} \left| \int f^H - g^H \right| &\leq \sum_{i=1}^{|H|} \left| \int f^{H-H_{i-1}} g^{H_{i-1}} - f^{H-H_i} g^{H_i} \right| = \sum_{i=1}^{|H|} \left| \int f^{H-H_i} g^{H_{i-1}} (f^{1_{\omega_i}} - g^{1_{\omega_i}}) \right| = \\ &= \sum_{i=1}^{|H|} \left| \int f^{H-H_i} g^{H_{i-1}} (f - g)^{1_{\omega_i}} \right| \leq \sum_{i=1}^{|H|} \|f - g\|_{U_k} \|f\|_\infty^{|H|-i} \|g\|_\infty^{i-1} \leq \\ &\leq |H| \|f - g\|_{U_k} \max(\|f\|_\infty, \|g\|_\infty)^{|H|-1}. \end{aligned}$$

■

2.5 Proofs of Theorems 2.1 and 2.5

Proof.[Theorem 2.1] Suppose that H is a semi-norming k -hypergraph pair over $V = V_1 \times \dots \times V_k$. The fact that $H \cong \overline{H}$ follows from Proposition 2.16 because trivially $|H| = |\overline{H}|$ and $\|f\|_H = \|f\|_{\overline{H}}$.

Now let $\epsilon > 0$ be sufficiently small, and $h : \Omega^k \rightarrow \{-1, 1\}$ be such that $\|h\|_{U_k} \leq \epsilon$ and $\int h = 0$, where here $(\Omega, \mathcal{F}, \mu)$ is a *probability* space. The existence of h is guaranteed by Lemma 2.20.

First we show that it is either the case that for every $\psi \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, $\alpha(\psi) = \beta(\psi)$ or for every $\psi \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, $\{\alpha(\psi), \beta(\psi)\} = \{0, 1\}$, and we will handle the existence of a universal s later. Suppose that this statement fails for some ψ . Note that at least one of $\alpha(\psi)$ or $\beta(\psi)$ is not equal to 0. We will assume that $\alpha(\psi) > \beta(\psi)$, and the proof of the case $\alpha(\psi) < \beta(\psi)$ will be similar. Since it is not the case that $\beta(\psi) = 1 - \alpha(\psi) = 0$, denoting $H - 1_\psi = (\alpha', \beta')$ we have

$$\psi \in \text{supp}(\alpha') \cup \text{supp}(\beta'). \quad (20)$$

For $p := \alpha(\psi) - \beta(\psi) \geq 0$, define $g := h^{1/p}$, and

$$f := \begin{cases} 1 & h = 1 \\ 0 & h = -1 \end{cases}.$$

Since $\int h = 0$, we have $\int f = 1/2$ and

$$\int f^{H-1_\psi} g^{1_\psi} = \int f^H \geq 2^{-|V_1| \dots |V_k|}, \quad (21)$$

where the equality follows from (20) and the definition of f , and the inequality follows from Lemma 2.21. Denote by K the hypergraph pair obtained from H by setting $\alpha(\psi) = \beta(\psi) = 0$, i.e. $K := H - \alpha(\psi)1_\psi - \beta(\psi)\overline{1}_\psi$. Now since $|g| = 1$, applying Corollary 2.19, we have

$$\left| \int g^H \right| = \left| \int g^K g^{\alpha(\psi)1_\psi + \beta(\psi)\overline{1}_\psi} \right| = \left| \int g^K |g|^{\beta(\psi)1_\psi} g^{p1_\psi} \right| = \left| \int g^K h^{1_\psi} \right| \leq \|h\|_{U_k} \leq \epsilon,$$

which shows that

$$\|f\|_H^{H-1} \|g\|_H \leq \|f\|_H^{H-1} \epsilon^{1/|H|}. \quad (22)$$

For sufficiently small ϵ , (21) and (22) contradict Lemma 2.8.

Next we will prove the existence of a universal s . So suppose that $H = (\alpha, \beta)$ is semi-norming and $\alpha = \beta$. Let $s = \max\{\alpha(\omega) + \beta(\omega) : \omega \in V\}$. We will show that $\frac{1}{s}H$ is semi-norming, and then Corollary 2.9 implies that $\alpha(\omega) + \beta(\omega) \in \{0, s\}$. Let ψ be such that $\alpha(\psi) + \beta(\psi) = s$, and let $\tilde{H}_\psi = \frac{1_\psi + \overline{1}_\psi}{2}$. Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and measurable functions $f, g : \Omega^k \rightarrow \mathbb{C}$, and note that

$$\begin{aligned} \left| \int f^{(\frac{1}{s}H)-1_\psi} g^{1_\psi} \right| &\leq \int |f|^{(\frac{1}{s}H)-1_\psi} |g|^{1_\psi} = \int \left(|f|^{1/s} \right)^{H-s\tilde{H}_\psi} \left(|g|^{1/s} \right)^{s\tilde{H}_\psi} \leq \\ &\leq \| |f|^{1/s} \|_H^{H-s} \| |g|^{1/s} \|_H^s = \|f\|_{\frac{1}{s}H}^{|\frac{1}{s}H|-1} \|g\|_{\frac{1}{s}H}, \end{aligned}$$

where in the second inequality we used Lemma 2.10. Now Lemma 2.8 shows that $\frac{1}{s}H$ is a semi-norming hypergraph pair, and this finishes the proof. \blacksquare

Next we give the proof of Theorem 2.5.

Proof.[Theorem 2.5] Suppose that $H = \dot{\cup}_{i=1}^m H_i$ where H_i are non-factorizable. Define $f : [0, 1]^k \rightarrow \mathbb{R}$ as in the following: $f(x_1, \dots, x_k) = 1$ if $[kx_1] = \dots = [kx_k]$, and $f(x) = 0$ otherwise. Then by Corollary 2.12 we have

$$\|f\|_{H'} \leq \|f\|_H. \quad (23)$$

It is easy to see that

$$\int f^{H'} \geq k \left(\frac{1}{k} \right)^{|W_1| + \dots + |W_k|} \times m,$$

while

$$\int f^H = k \left(\frac{1}{k} \right)^{|V_1| + \dots + |V_k|} \times m.$$

Plugging these into (23), and simplifying it, we obtain the assertion of the theorem. \blacksquare

3 Geometry of the Hypergraph Norms

3.1 Moduli of Smoothness and Convexity

Let us start by recalling the definition of moduli of smoothness and convexity of a normed space. For a normed space X , define the modulus of smoothness as the function

$$\rho_X(\tau) = \sup \left\{ \frac{\|x - \tau y\| + \|x + \tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\}, \quad (24)$$

and the modulus of convexity as

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1, \|x - y\| \geq 2\epsilon \right\}, \quad (25)$$

where $0 \leq \epsilon \leq 1$. It should be noticed that the function δ_X is frequently defined with ϵ in place of 2ϵ . The following observation of Lindenstrauss [13] shows that these two functions behave in a dual form via Legendre transform:

$$\rho_{X^*}(\tau) = \sup \{ \tau\epsilon - \delta_X(\epsilon) : 0 \leq \epsilon \leq 1 \}, \quad (26)$$

where X^* is the dual of X .

A normed space X is called *uniformly smooth*, if $\lim_{\tau \rightarrow 0} \rho_X(\tau)/\tau = 0$, and it is called *uniformly convex*, if for every $\epsilon > 0$, $\delta_X(\epsilon) > 0$. For $t \in (1, 2]$ a normed space X is said to be *t-uniformly smooth*, if there exists a constant $C > 0$ such that $\rho_X(\tau) \leq (C\tau)^t$, and for $r \in [2, \infty)$, a normed space is said to be *r-uniformly convex*, if there exists a constant $C > 0$ such that $\delta_X(\epsilon) \geq (\epsilon/C)^r$. It is known that $\rho_{\ell_2}(\tau) = (1 + \tau^2)^{1/2} - 1 = \tau^2/2 + O(\tau^4)$, $\tau > 0$ and $\delta_{\ell_2}(\epsilon) = 1 - (1 - \epsilon^2)^{1/2} = \epsilon^2/2 + O(\epsilon^4)$ for $0 < \epsilon < 1$. Dvoretzky's theorem (see for example [16]) implies that for every infinite dimensional normed space X , we have $\rho_X(\tau) \geq \rho_{\ell_2}(\tau)$ and $\delta_X(\epsilon) \leq \delta_{\ell_2}(\epsilon)$, and this was the reason for requiring $t \in (1, 2]$ and $r \in [2, \infty)$ in the definition of *t-uniform smoothness* and *r-uniform convexity*. We will give another equivalent definition for the notions of *t-uniform smoothness* and *r-uniform convexity* due to Ball *et al* [1]. First we need two simple lemmas.

Lemma 3.1 *Let $1 < p \leq q < \infty$ and $\rho = \sqrt{\frac{p-1}{q-1}}$. Then for every two vectors x and y in an arbitrary normed space X , we have*

$$\left(\frac{\|x + \rho y\|^q + \|x - \rho y\|^q}{2} \right)^{1/q} \leq \left(\frac{\|x + y\|^p + \|x - y\|^p}{2} \right)^{1/p}.$$

For the proof of Lemma 3.1 see Corollary 1.e.14 in [14].

Lemma 3.2 *Let $t \in (1, 2]$, $r \in [2, \infty)$, and $1 < p, q < \infty$. Then there exists constants $C = C(t, p)$ and $C^* = C^*(r, q)$ such that for every $x, y \in \mathbb{C}$,*

$$\left(\frac{|x + y|^p + |x - y|^p}{2} \right)^{1/p} \leq (|x|^t + |Cy|^t)^{1/t}, \quad (27)$$

and

$$\left(\frac{|x + y|^q + |x - y|^q}{2} \right)^{1/q} \geq \left(|x|^r + \left| \frac{1}{C^*} y \right|^r \right)^{1/r}. \quad (28)$$

Furthermore for the best constants one can assume $C(t, p) = C^*(r, q)$, if $\frac{1}{r} + \frac{1}{t} = 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We only prove (27), and (28) as well as the last assertion of the lemma will follow from duality by Proposition 3.5 below. It suffices to prove the theorem for $t = 2$ as the right-hand side of (27) is a decreasing function in t . By Lemma 3.1, we have

$$\left(\frac{|x+y|^p + |x-y|^p}{2}\right)^{1/p} \leq \left(\frac{|x+\rho y|^2 + |x-\rho y|^2}{2}\right)^{1/2} \leq (|x|^2 + |\rho y|^2)^{1/2},$$

where $\rho = \max(1, \sqrt{p-1})$. ■

Now for a normed space X , inspired by Lemma 3.2, for $1 < t \leq 2 \leq r < \infty$, and $1 < p, q < \infty$, one can investigate the validity of the following two inequalities:

$$\left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} \leq (\|x\|^t + \|Ky\|^t)^{1/t}, \quad (29)$$

and

$$\left(\frac{\|x+y\|^q + \|x-y\|^q}{2}\right)^{1/q} \geq (\|x\|^r + \|K^{-1}y\|^r)^{1/r} \quad (30)$$

where K is a constant. We denote the smallest constant K such that (29) is satisfied for all $x, y \in X$ by $K_{t,p}(X)$ and similarly the smallest constant such that (30) is satisfied by $K_{r,q}^*(X)$. Trivially $K_{t,p}(X) \geq C(t,p)$ and $K_{r,q}^*(X) \geq C^*(r,q)$ where $C(t,p)$ and $C^*(r,q)$ are the constants defined in Lemma 3.2.

Remark 3.3 In the sequel $C(t,p)$ and $C^*(r,q)$ always refer to the constants from Lemma 3.2. Note that $C(t,p)$ and $K_{t,p}(X)$ are both increasing in t and p , and $C^*(r,q)$ and $K_{r,q}^*(X)$ are both decreasing in r and q . Since Lemma 3.1 is valid for every normed space X , for $1 < p_2 \leq p_1 < \infty$,

$$\begin{aligned} \left(\frac{\|x+y\|^{p_1} + \|x-y\|^{p_1}}{2}\right)^{1/p_1} &\leq \left(\frac{\|x + \sqrt{\frac{p_1-1}{p_2-1}}y\|^{p_2} + \|x - \sqrt{\frac{p_1-1}{p_2-1}}y\|^{p_2}}{2}\right)^{1/p_2} \\ &\leq \left(\|x\|^t + \left\|K_{t,p_2}(X) \sqrt{\frac{p_1-1}{p_2-1}}y\right\|^t\right)^{1/t}, \end{aligned}$$

which implies $K_{t,p_1}(X) \leq \sqrt{\frac{p_1-1}{p_2-1}}K_{t,p_2}(X)$. Similarly for $1 < q_2 \leq q_1 < \infty$,

$$\begin{aligned} \left(\frac{\|x+y\|^{q_2} + \|x-y\|^{q_2}}{2}\right)^{1/q_2} &\geq \left(\frac{\|x + \sqrt{\frac{q_2-1}{q_1-1}}y\|^{q_1} + \|x - \sqrt{\frac{q_2-1}{q_1-1}}y\|^{q_1}}{2}\right)^{1/q_1} \\ &\geq \left(\|x\|^r + \left\|\frac{1}{K_{r,q_1}^*(X)} \sqrt{\frac{q_2-1}{q_1-1}}y\right\|^r\right)^{1/r}, \end{aligned}$$

which shows that $K_{r,q_2}^*(X) \leq \sqrt{\frac{q_1-1}{q_2-1}}K_{r,q_1}^*(X)$. ■

The following proposition which follows from Remark 3.3, and Proposition 7 in [1] shows that one can use (29) and (30) to give an alternative definition of t -uniform smoothness and r -uniform convexity.

Proposition 3.4 *Let X be a t -uniformly smooth normed space. Then for every $1 < p < \infty$, we have $K_{t,p}(X) < \infty$. Conversely if $K_{t,p}(X) < \infty$ for some $1 < p < \infty$, then X is t -uniformly smooth.*

Similarly let Y be an r -uniformly convex normed space. Then for every $1 < q < \infty$, we have $K_{r,q}^(Y) < \infty$. Conversely if $K_{r,q}^*(Y) < \infty$ for some $1 < q < \infty$, then Y is r -uniformly convex.*

The constants $K_{t,p}$ and $K_{r,q}^*$ behave nicely with respect to the duality. The proof of the following proposition is identical to the proof of Lemma 5 from [1], and thus we omit it.

Proposition 3.5 *Consider a normed space X and its dual X^* . Suppose that $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r} + \frac{1}{t} = 1$. Then $K_{r,p}(X) = K_{t,q}^*(X^*)$.*

The notion of uniform convexity is first defined by Clarkson in [4], where he studied the smoothness and convexity of L_p spaces. To this end he established four inequalities known as Clarkson inequalities. Let $1 < p \leq 2 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. In our notation the Clarkson inequalities are the following: $K_{p,p}(L_p) = 1$, $K_{q,q}^*(L_q) = 1$, $K_{q,p}^*(L_p) = 1$, and $K_{p,q}(L_q) = 1$. The first two are easier to prove and known as “easy” Clarkson inequalities, and the latter two are known as “strong” Clarkson inequalities. The following observation shows that the strong Clarkson inequalities imply the easy Clarkson inequalities.

Observation 3.6 *Let $1 < t \leq 2 \leq r < \infty$ be such that $\frac{1}{t} + \frac{1}{r} = 1$. Then $K_{t,r}(X) = 1$ if and only if $K_{r,t}^*(X) = 1$.*

Proof. Suppose that $K_{t,r}(X) = 1$. Then for every $x, y \in X$, we have

$$\left(\frac{\|x+y\|^r + \|x-y\|^r}{2} \right)^{1/r} \leq (\|x\|^t + \|y\|^t)^{1/t}.$$

Now consider $x', y' \in X$. Replacing x and y in the above inequality, respectively with $\frac{x'+y'}{2}$ and $\frac{x'-y'}{2}$ we get

$$\left(\frac{\|x'\|^r + \|y'\|^r}{2} \right)^{1/r} \leq \left(\left\| \frac{x'+y'}{2} \right\|^t + \left\| \frac{x'-y'}{2} \right\|^t \right)^{1/t}.$$

which simplifies to

$$(\|x'\|^r + \|y'\|^r)^{1/r} \leq \left(\frac{\|x'+y'\|^t + \|x'-y'\|^t}{2} \right)^{1/t},$$

showing that $K_{r,t}^*(X) = 1$. The proof of the converse direction is similar. ■

Consider $1 < p \leq 2 \leq q < \infty$. As we have already seen in Proposition 3.4, Clarkson’s inequalities imply that L_p and L_q spaces are both p -uniformly smooth and q -uniformly convex. However this is not in general the best possible. The actual situation is the following. The L_p spaces are p -uniformly smooth and 2-uniformly convex, and the L_q spaces are 2-uniformly smooth and q -uniformly convex. These facts are proved by Hanner [11] through the so called Hanner inequality. For $1 < p \leq 2$, we say that a normed space satisfies the p -Hanner inequality, if

$$\|x+y\|^p + \|x-y\|^p \geq (\|x\| + \|y\|)^p + \left| \|x\| - \|y\| \right|^p,$$

and for $2 \leq q < \infty$, it satisfies the q -Hanner inequality if

$$\|x+y\|^q + \|x-y\|^q \leq (\|x\| + \|y\|)^q + \left| \|x\| - \|y\| \right|^q.$$

It is shown in [1] that if X satisfies the p -Hanner inequality, then X^* satisfies the q -Hanner inequality where $\frac{1}{p} + \frac{1}{q} = 1$. The following proposition reveals the relation between the Hanner inequality and the notions of uniform smoothness and uniform convexity.

Proposition 3.7 *If a normed space X satisfies the t -Hanner inequality for $1 < t \leq 2$, then for every $2 \leq q < \infty$, we have $K_{q,t}^*(X) = C^*(q, t)$, and for every $1 < p \leq t'$, we have $K_{t,p}(X) = 1$ where $\frac{1}{t} + \frac{1}{t'} = 1$.*

Similarly if a normed space X satisfies the r -Hanner inequality for $2 \leq r < \infty$, then for every $1 < p \leq 2$, we have $K_{p,r}(X) = C(p, r)$, and for every $r' \leq q < \infty$, we have $K_{r,q}^(X) = 1$, where $\frac{1}{r} + \frac{1}{r'} = 1$.*

Proof. Suppose that X satisfies the t -Hanner inequality for $1 < t \leq 2$. Consider $2 \leq q < \infty$, and $x, y \in X$. By the t -Hanner inequality

$$\left(\frac{\|x + y\|^t + \|x - y\|^t}{2} \right)^{1/t} \geq \left(\frac{(\|x\| + \|y\|)^t + \left| \|x\| - \|y\| \right|^t}{2} \right)^{1/t} \geq \left(\|x\|^q + \left\| \frac{1}{C^*(q,t)} y \right\|^q \right)^{1/q},$$

which shows that $K_{q,t}^*(X) \leq C^*(q,t)$. But from this, and Observation 3.6 we also get $K_{t,t'}(X) = 1$ as $K_{t',t}^*(X) \leq C^*(t',t) = 1$. Hence for $1 < p \leq t'$ we have $K_{t,p}(X) = 1$. The second assertion follows from the first one by duality. \blacksquare

Inequalities (29) and (30) are first appeared in [1], where for $q \geq 2$, the equalities $K_{2,q}(\ell_q) = K_{2,q}(S_q) = K_{2,2}(\ell_q) = K_{2,2}(S_q) = \sqrt{q-1}$ are proved, where S_q corresponds to the q -trace norm.

Proposition 3.8 *For $1 < t \leq 2 \leq r < \infty$, $1 < t_1 \leq 2 \leq r_1 < \infty$, and $1 < p < \infty$, we have*

$$K_{t_1,p}(\ell_r) = \begin{cases} C(t_1, r) & p \leq r \\ C(t_1, p) \leq \cdot \leq C(t_1, r) \sqrt{\frac{p-1}{r-1}} & p \geq r \end{cases} \quad (31)$$

and

$$K_{r_1,p}^*(\ell_r) = \begin{cases} C^*(r_1, t) & p \geq t \\ C^*(r_1, p) \leq \cdot \leq C^*(r_1, t) \sqrt{\frac{t-1}{p-1}} & p \leq t \end{cases} \quad (32)$$

In particular $K_{2,p}(\ell_r) = \max(\sqrt{p-1}, \sqrt{r-1})$, and $K_{2,p}^*(\ell_t) = \max\left(\sqrt{\frac{1}{p-1}}, \sqrt{\frac{1}{r-1}}\right)$.

Proof. It suffices to prove (31), and then (32) will follow from duality. Since ℓ_r satisfies the r -Hanner inequality, by Proposition 3.7 we have $K_{t_1,r}(\ell_r) = C(t_1, r)$. Then it follows from Lemma 3.1 that for $p \geq r$, $K_{t_1,p}(\ell_r) \leq C(t_1, r) \sqrt{\frac{p-1}{r-1}}$. Furthermore since $K_{t_1,p}(\ell_r)$ is increasing in p , we have $K_{t_1,p}(\ell_r) \leq C(t_1, r)$, for $p \leq r$. It remains to show that $K_{t_1,p}(\ell_r) \geq C(t_1, r)$ for $p \leq r$. Consider two complex numbers a and b , and let $x, y \in \ell_r$ be as $x = (a, a)$ and $y = (b, -b)$. Then since $\|x + y\|_r = \|x - y\|_r = (|a + b|^r + |a - b|^r)^{1/r}$, plugging these two vectors in

$$\left(\frac{\|x + y\|_r^p + \|x - y\|_r^p}{2} \right)^{1/p} \leq (\|x\|_r^{t_1} + \|K_{t_1,p}(\ell_r)y\|_r^{t_1})^{1/t_1},$$

we get

$$\left(\frac{|a + b|^r + |a - b|^r}{2} \right)^{1/r} \leq (|a|^{t_1} + |K_{t_1,p}(\ell_r)b|^{t_1})^{1/t_1},$$

which shows that $K_{t_1,p}(\ell_r) \geq C(t_1, r)$. \blacksquare

Let $1 < t \leq 2 \leq r < \infty$ with $\frac{1}{t} + \frac{1}{r} = 1$. The spaces ℓ_t and ℓ_r are respectively 2-uniformly convex and 2-uniformly smooth. Proposition 3.8 determines the optimum value of all corresponding constants. In terms of the constants corresponding to t -uniformly smoothness of ℓ_t and r -uniformly convexity of ℓ_r , by Remark 3.3 and Clarkson's inequalities we have

$$K_{t,p}(\ell_t) = \begin{cases} 1 & p \leq r \\ C(t, p) \leq \cdot \leq \sqrt{\frac{p-1}{r-1}} & p \geq r \end{cases}$$

and

$$K_{r,p}^*(\ell_t) = \begin{cases} 1 & p \geq t \\ C^*(r, p) \leq \cdot \leq \sqrt{\frac{t-1}{p-1}} & p \leq t \end{cases}$$

The moduli of smoothness and convexity of a Banach space are only isometric invariant, and they may change considerably under an equivalent renorming. This leads to the definition of type and cotype.

	$1 < p \leq t$	$t \leq p \leq 2$	$2 \leq p \leq r$	$r \leq p < \infty$
$C(2, p)$	1	1	$\sqrt{p-1}$	$\sqrt{p-1}$
$C^*(2, p)$	$\sqrt{\frac{1}{p-1}}$	$\sqrt{\frac{1}{p-1}}$	1	1
$K_{2,p}(\ell_r)$	$\sqrt{r-1}$	$\sqrt{r-1}$	$\sqrt{r-1}$	$\sqrt{p-1}$
$K_{2,p}^*(\ell_t)$	$\sqrt{\frac{1}{p-1}}$	$\sqrt{\frac{1}{t-1}}$	$\sqrt{\frac{1}{t-1}}$	$\sqrt{\frac{1}{t-1}}$
$C(t, p)$	1	1	1	$\leq \sqrt{\frac{p-1}{r-1}}$
$K_{t,p}(\ell_t)$	1	1	1	$\leq \sqrt{\frac{p-1}{r-1}}$
$C^*(r, p)$	$\leq \sqrt{\frac{t-1}{p-1}}$	1	1	1
$K_{r,p}^*(\ell_r)$	$\leq \sqrt{\frac{t-1}{p-1}}$	1	1	1
$K_{t_1,p}(\ell_r)$	$C(t_1, r)$	$C(t_1, r)$	$C(t_1, r)$	$\leq C(t_1, r) \sqrt{\frac{p-1}{r-1}}$
$K_{r_1,p}^*(\ell_t)$	$\leq C^*(r_1, t) \sqrt{\frac{t-1}{p-1}}$	$C^*(r_1, t)$	$C^*(r_1, t)$	$C^*(r_1, t)$

Figure 1: Here $1 < t \leq 2 \leq r < \infty$ are such that $\frac{1}{t} + \frac{1}{r} = 1$, and $1 < t_1 \leq 2 \leq r_1 < \infty$ are arbitrary.

A normed space is of *type* $1 \leq t \leq 2$ if there exists a constant T_t such that for every integer $n \geq 0$, and every set of vectors x_1, \dots, x_n ,

$$\mathbb{E} \left\| \sum_{i=1}^n \epsilon_i x_i \right\|^t \leq T_t \left(\sum_{i=1}^n \|x_i\|^t \right)^{1/t},$$

where ϵ_i are independent Bernoulli random variables taking values uniformly in $\{-1, 1\}$. Similarly a normed space is said to be of *cotype* $2 \leq r \leq \infty$ if there exists a constant C_r such that for every integer $n \geq 0$, and every set of vectors x_1, \dots, x_n ,

$$\left(\sum_{i=1}^n \|x_i\|^r \right)^{1/r} \leq C_r \mathbb{E} \left\| \sum_{i=1}^n \epsilon_i x_i \right\|,$$

where in the case $r = \infty$ the left hand-side must be replaced by $\max_{i=1}^n \|x_i\|$.

Trivially every normed space is of type 1 and of cotype ∞ . If a normed space is of type t_0 and cotype r_0 , then it is also of type t and cotype r provided that $t \leq t_0 \leq 2 \leq r_0 \leq r$. Note that type and cotype do not change under an equivalent norm. Figiel and Pisier [6, 7] proved that t -uniform smoothness implies type t , and r -uniform convexity implies cotype r . The reverse is of course not true as for example every finite dimensional space is of type and cotype 2.

For $\lambda \geq 1$, a normed space X is said to be λ -finitely representable in a normed space Y , if for every finite dimensional subspace $E \subseteq X$, there exists a linear map $T : E \rightarrow Y$ such that $\|T\| \|T^{-1}\| \leq \lambda$. If for every $\lambda > 1$, X is λ -finitely representable in Y , then we simply say X is finitely representable in Y .

It is well-known that infinite dimensional L_p spaces are of type $\min(p, 2)$ and cotype $\max(2, p)$, and nothing better. Thus if ℓ_p is λ -finitely representable in an space X of type t and cotype r , then $t \leq \min(2, p)$ and $r \geq \max(2, p)$. A beautiful theorem due to Maurey and Pisier [15] says that the converse is also true, i.e. ℓ_p and ℓ_q are finitely representable in X where $p = \sup\{t : X \text{ is of type } t\}$ and $q = \inf\{r : X \text{ is of cotype } r\}$.

Thus in order to study the type, cotype, modulus of smoothness, and modulus of convexity of a normed space X , it is natural therefore to first try to find the smallest $p \geq 1$ and largest q that ℓ_p and ℓ_q are finitely representable in X .

For a hypergraph pair H , define $\ell_H := L_H(\mathbb{N})$ where \mathbb{N} is endowed with the counting measure.

Theorem 3.9 *If $H = (\alpha, \beta)$ is a non-factorizable semi-norming hypergraph pair, then $\ell_{|H|}$ is a subspace of ℓ_H . Furthermore if H is of Type I with parameter $s \leq 2$, then ℓ_s is finitely representable in ℓ_H .*

The first part of the theorem which is trivial, shows that any infinite dimensional L_H space is not of any cotype $q < \min(2, |H|)$. The second part which is more interesting and was unknown to the author in [12] shows that if H is of Type I with parameter $s < 2$, then every infinite dimensional L_H space is not of any type $p > s$. In particular in the case $s = 1$, an infinite dimensional L_H space has no nontrivial type, and is not uniformly smooth and convex. The next theorem shows that every such space is of cotype $\min(2, |H|)$ which is the best possible by Theorem 3.9.

Theorem 3.10 *Let H be a non-factorizable semi-norming hypergraph pair of Type I, then ℓ_H is of cotype $\min(2, |H|)$.*

In Theorem 3.10, only the case $s = 1$ is interesting to us, as for $s > 1$ we will prove something stronger in Theorem 3.11. The key to prove Theorem 3.10 is the following observation. Consider a non-factorizable semi-norming k -hypergraph pair $H = (\alpha, \alpha)$ of Type I over $V := V_1 \times \dots \times V_k$, and functions $f_1, f_2, \dots, f_n \in \ell_H$. Then

$$\sum_{i=1}^n f_i^H = \sum_{i=1}^n \prod_{\omega \in V} |f_i \circ \omega|^{2\alpha(\omega)} \leq \prod_{\omega \in V} \left(\sum_{i=1}^n |f_i \circ \omega|^{|H|} \right)^{1/|H|} = \left(\sum_{i=1}^n |f_i|^{|H|} \right)^{\frac{H}{|H|}},$$

where in the inequality above we used the classical Hölder inequality. Hence ³

$$\sum_{i=1}^n \|f_i\|_H^{|H|} \leq \left\| \left(\sum_{i=1}^n |f_i|^{|H|} \right)^{1/|H|} \right\|_H. \quad (33)$$

We will also need the following inequality ⁴ in the sequel:

$$\left\| \left(\sum_{i=1}^n |f_i|^s \right)^{\frac{1}{s}} \right\|_H = \left(\int \left(\sum_{i=1}^n |f_i|^s \right)^{\frac{H}{s}} \right)^{1/|H|} = \left\| \sum_{i=1}^n |f_i|^s \right\|_{H/s}^{1/s} \leq \left(\sum_{i=1}^n \| |f_i|^s \|_{H/s} \right)^{1/s} = \left(\sum_{i=1}^n \|f_i\|_H^s \right)^{1/s}, \quad (34)$$

where we used the fact that H/s is also norming. Now we can state the proof of Theorem 3.10.

Proof.[Theorem 3.10] Consider functions $f_1, \dots, f_n \in \ell_H$, and let $m := \max(|H|, 2)$. By applying Minkowski's inequality, Khintchine's inequality, and then (33), there exists a constant C such that

$$\begin{aligned} \mathbb{E} \left\| \sum_{i=1}^n \epsilon_i f_i \right\|_H &= \mathbb{E} \left\| \sum_{i=1}^n \epsilon_i f_i \right\|_H \geq \left\| \sum_{i=1}^n \epsilon_i f_i \right\|_H \geq C \left\| \left(\sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_H \\ &\geq C \left\| \left(\sum_{i=1}^n |f_i|^m \right)^{1/m} \right\|_H \geq C \left(\sum_{i=1}^n \|f_i\|_H^m \right)^{1/m}. \end{aligned}$$

■

Now let us turn to the other hypergraph pairs, i.e. the ones which are not of Type I with parameter 1. From Theorem 3.9, in terms of the four parameters type, cotype, modulus of smoothness, and of convexity, the following theorem is the strongest statement one can hope to prove about them, and in particular implies Theorem 3.10 for H of Type I with parameter $s > 1$.

³Inequality (33) says that ℓ_H is $|H|$ -concave as a Banach lattice when H is of Type I. For the definition of Banach lattice convexity and concavity we refer the reader to [14].

⁴Inequality (34) says that ℓ_H is s -convex as a Banach lattice (see [14]).

Theorem 3.11 *Let H be a non-factorizable semi-norming hypergraph pair such that $|H| \geq 2$.*

- *If H is of Type II or Type I with parameter $s \geq 2$, then ℓ_H is 2-uniformly smooth and $|H|$ -uniformly convex;*
- *If H is of Type I with parameter $1 < s \leq 2$, then ℓ_H is s -uniformly smooth and $|H|$ -uniformly convex.*

Remark 3.12 If $1 < |H| < 2$, then it is easy to see by the previous results that $\|\cdot\|_H$ corresponds to the L_p norm where $p = |H|$, and thus the Banach space properties of the norm are well-understood. The case $|H| = 1$ is also trivial. \blacksquare

As it is discussed above, the notions of t -uniform smoothness and r -uniform convexity can be further refined by looking at the constants $K_{t,p}$ and $K_{r,q}^*$. In proving Theorem 3.11 we will try to obtain the best possible constants. This is treated and discussed in more details in Section 3.4. Next we prove Theorems 3.9.

3.2 Proof of Theorem 3.9

Define $T : \ell_{|H|} \rightarrow \ell_H$ as $T : a \mapsto f_a$, where for $a = \{a_i\}_{i \in \mathbb{N}}$, $f_a : \mathbb{N}^k \rightarrow \mathbb{C}$ is defined as

$$f_a(i_1, \dots, i_k) = \begin{cases} a_i & i_1 = i_2 = \dots = i_k = i \\ 0 & \text{otherwise} \end{cases}$$

Since H is non-factorizable, it is easy to see that T is an isometry.

Next we show that ℓ_s is finitely representable in ℓ_H . Since $L_H([0, 1])$ is finitely representable in ℓ_H , it suffices to find a map $T : \ell_s([n]) \rightarrow L_H([0, 1])$ with $\|T\| \|T^{-1}\| \leq 1 + \epsilon$, for every $n \in \mathbb{N}$ and every $\epsilon > 0$. To this end we find $f_1, \dots, f_n : [0, 1]^k \rightarrow \mathbb{C}$, such that for every $x = (x_1, \dots, x_n) \in \ell_s([n])$ with $\|x\|_s = n^{1/s}$,

$$1 - \epsilon/4 \leq \left\| \sum_{i=1}^n x_i f_i \right\|_H \leq 1 + \epsilon/4,$$

and then the map $T : \ell_s([n]) \rightarrow L_H([0, 1])$ defined by $T : e_i \mapsto f_i$, for $i \in [n]$, satisfies $\|T\| \|T^{-1}\| \leq \frac{1+\epsilon/4}{1-\epsilon/4} \leq 1 + \epsilon$, for $\epsilon < 1$. An argument similar to the proof of Lemma 2.20, shows that there exists $f_1, \dots, f_n : [0, 1]^k \rightarrow \{0, 1\}$ such that $\sum f_i = 1$, and for every $i \in [n]$, $\int f_i = \frac{1}{n}$ and $\|f_i - \frac{1}{n}\|_{U_k} \leq \delta$. Note that since f_i are zero-one valued functions, $\sum_{i=1}^n f_i = 1$ implies that the supports of f_i are pairwise disjoint. Then we have

$$\int \left(\sum_{i=1}^n x_i f_i \right)^H = \int \left(\sum_{i=1}^n |x_i|^s f_i \right)^{\tilde{H}},$$

where $\tilde{H} = (\frac{\alpha+\beta}{s}, 0)$. Furthermore if $\|x\|_s = n^{1/s}$, then

$$\left\| \left(\sum_{i=1}^n |x_i|^s f_i \right) - 1 \right\|_{U_k} = \left\| \sum_{i=1}^n \left(|x_i|^s f_i - \frac{|x_i|^s}{n} \right) \right\|_{U_k} \leq \sum_{i=1}^n |x_i|^s \left\| f_i - \frac{1}{n} \right\|_{U_k} \leq \delta \|x\|_s = \delta n^{1/s}.$$

Now by Lemma 2.22

$$\left| \int \left(\sum_{i=1}^n |x_i|^s f_i \right)^{\tilde{H}} - 1 \right| = \left| \int \left(\sum_{i=1}^n |x_i|^s f_i \right)^{\tilde{H}} - 1^{\tilde{H}} \right| \leq \delta n^{1/s} |\tilde{H}| \max \left(\left\| \sum_{i=1}^n |x_i|^s f_i \right\|_{\infty}, 1 \right)^{|\tilde{H}|-1} \leq \delta n^{|\tilde{H}|} |\tilde{H}|.$$

Now taking δ sufficiently small finishes the proof.

3.3 Complex Interpolation

Let us recall the definition of the complex interpolation spaces. Two topological vector spaces are called compatible, if there exists a Hausdorff topological vector space containing both of these spaces as subspaces. Consider two compatible normed space X_0 and X_1 and endow the space $X_0 + X_1$ with the norm $\|f\|_{X_0+X_1} = \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + \|f_1\|_{X_1})$. For every $0 \leq \theta \leq 1$, one constructs the corresponding complex interpolation space $[X_0, X_1]_\theta$, as in the following.

Let $\mathcal{F}(X_0, X_1)$ be the set of all analytic function $v : \{z : 0 \leq \operatorname{Re} z \leq 1\} \rightarrow X_0 + X_1$ which are continuous and bounded on the boundary, and moreover the function $t \rightarrow v(j + it)$ ($j = 0, 1$) are continuous functions from the real line into X_j which tend to zero as $|t| \rightarrow \infty$. We provide the vector space \mathcal{F} with a norm

$$\|v\|_{\mathcal{F}} := \max \left\{ \sup_{x \in \mathbb{R}} \|v(ix)\|_{X_0}, \sup_{x \in \mathbb{R}} \|v(1 + ix)\|_{X_1} \right\}.$$

Then for every $0 \leq \theta \leq 1$, the complex interpolation space of X_0 and X_1 is a normed space $X_0 \cap X_1 \subseteq [X_0, X_1]_\theta \subseteq X_0 + X_1$ defined as

$$[X_0, X_1]_\theta := \{f \in X_0 + X_1 : v(\theta) = f \exists v \in \mathcal{F}(X_0, X_1)\},$$

with the following norm:

$$\|f\|_\theta = \inf \{ \|v\|_{\mathcal{F}} : f = v(\theta), v \in \mathcal{F}(X_0, X_1) \}.$$

The space $[X_0, X_1]_\theta$ has an interesting property. Consider compatible pairs X_0, X_1 and Y_0, Y_1 . Let $T : X_0 + X_1 \rightarrow Y_0 + Y_1$ be a bounded linear map. Then (see [2]),

$$\|T\|_{[X_0, X_1]_\theta \rightarrow [Y_0, Y_1]_\theta} \leq \|T\|_{X_0 \rightarrow Y_0}^{1-\theta} \|T\|_{X_1 \rightarrow Y_1}^\theta. \quad (35)$$

Theorem 3.13 *Let $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space and H be a norming hypergraph pair of Type I with parameter 1. Then for every $0 \leq \theta \leq 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, where $p_0, p_1 \geq 1$,*

$$[L_{p_0 H}(\mathcal{M}), L_{p_1 H}(\mathcal{M})]_\theta = L_{p H}(\mathcal{M}).$$

Proof. Let $f : \Omega^k \rightarrow \mathbb{C}$ be a measurable function with $\|f\|_{pH} = 1$. Define

$$v : \{z : 0 \leq \operatorname{Re} z \leq 1\} \rightarrow L_{p_0 H}(\mathcal{M}) + L_{p_1 H}(\mathcal{M})$$

by

$$v(z) = |f|^{p(\frac{1-z}{p_0} + \frac{z}{p_1})}.$$

Then $v(\theta) = |f|$ which shows that

$$\|f\|_\theta \leq \max \left\{ \sup_{x \in \mathbb{R}} \|v(ix)\|_{p_0 H}, \sup_{x \in \mathbb{R}} \|v(1 + ix)\|_{p_1 H} \right\}.$$

But note that

$$\|v(ix)\|_{p_0 H} = \left(\int |v(ix)|^{p_0 H} \right)^{1/p_0 H} = \left(\int (|f|^{p/p_0})^{p_0 H} \right)^{1/p_0 H} = \left(\int |f|^{pH} \right)^{1/p_0 H} = 1,$$

and similarly $\|v(1 + ix)\|_{p_1 H} \leq 1$ which shows that $\|f\|_\theta \leq \|f\|_{pH}$.

Now for the other direction assume that $\|f\|_\theta = 1$. Then for every $\epsilon > 0$, there exists v_ϵ such that $f = v_\epsilon(\theta)$ and $\|v_\epsilon\|_{\mathcal{F}} \leq 1 + \epsilon$. By Hölder's inequality,

$$\|f\|_{pH}^{|H|} = \sup \left\{ \int f^H g^H : \|g\|_{qH} \leq 1 \right\},$$

where $1 = \frac{1}{p} + \frac{1}{q}$. Fix $g : \Omega^k \rightarrow \mathbb{C}$ with $\|g\|_{qH} \leq 1$, and define

$$u : \{z : 0 \leq \operatorname{Re} z \leq 1\} \rightarrow L_{q_0 H}(\mathcal{M}) + L_{q_1 H}(\mathcal{M})$$

by

$$u(z) = |g|^{q(\frac{1-z}{q_0} + \frac{z}{q_1})},$$

where $\frac{1}{q_0} + \frac{1}{p_0} = 1$ and $\frac{1}{q_1} + \frac{1}{p_1} = 1$. Let

$$F_\epsilon(z) = \int v_\epsilon(z)^H u(z)^H,$$

and notice that

$$|F_\epsilon(ix)| = \int v_\epsilon(ix)^H u(ix)^H \leq \|v_\epsilon(ix)\|_{p_0 H}^{|H|} \|u(ix)\|_{q_0 H}^{|H|} \leq \|v_\epsilon\|_{\mathcal{F}}^{|H|} \times \|g^{q/q_0}\|_{q_0 H}^{|H|} \leq (1 + \epsilon)^{|H|}.$$

Similarly

$$|F_\epsilon(1+ix)| = \int v_\epsilon(1+ix)^H u(1+ix)^H \leq \|v_\epsilon(1+ix)\|_{p_1 H}^{|H|} \|u(1+ix)\|_{q_1 H}^{|H|} \leq \|v_\epsilon\|_{\mathcal{F}}^{|H|} \times \|g^{q/q_1}\|_{q_1 H}^{|H|} \leq (1 + \epsilon)^{|H|}.$$

Then

$$\left| \int f^H g^H \right| = |F_\epsilon(\theta)| \leq 1 + \epsilon,$$

which by tending ϵ to zero leads to $\|f\|_{pH} \leq 1$. We conclude that $\|f\|_{pH} = \|f\|_\theta$. \blacksquare

3.4 Proof of Theorem 3.11

In this section we give sharp bounds on the moduli of smoothness and convexity of the norms defined by semi-norming hypergraph pairs. This of course will prove Theorem 3.11.

Consider a non-factorizable semi-norming hypergraph pair H , and an infinite dimensional space L_H . Theorem 3.9 shows that L_H contains $\ell_{|H|}$ as a subspace, and thus $K_{t,p}(\ell_{|H|}) \leq K_{t,p}(L_H)$ and $K_{r,q}^*(\ell_{|H|}) \leq K_{r,q}^*(L_H)$, for $1 < t \leq 2 \leq r < \infty$ and $1 < p, q < \infty$. Comparing Proposition 3.7 with Figure 1 shows that proving the $|H|$ -Hanner inequality for L_H spaces, gives the optimal values of $K_{2,p}(L_H)$ and $K_{|H|,|H|}^*(L_H)$, for every $p > 1$.

Theorem 3.14 (Hanner Inequality) *Let H be a non-factorizable semi-norming hypergraph pair which is either of Type II, or of Type I with an even integer parameter. Then for every $f, g \in \ell_H$, we have*

$$\|f + g\|_H^{|H|} + \|f - g\|_H^{|H|} \leq (\|f\|_H + \|g\|_H)^{|H|} + \left| \|f\|_H - \|g\|_H \right|^{|H|}.$$

Proof. Without loss of generality assume that $\|f\|_H \geq \|g\|_H$. Let \mathcal{H} be the set of all pairs (H_1, H_2) such that H_1 and H_2 are hypergraph pairs taking only nonnegative integer values, and furthermore $H_1 + H_2 = H$ and $|H_2|$ is an even integer. Then

$$\begin{aligned} \|f + g\|_H^{|H|} + \|f - g\|_H^{|H|} &= \int (f + g)^H + (f - g)^H = \sum_{(H_1, H_2) \in \mathcal{H}} \int f^{H_1} g^{H_2} \\ &\leq \sum_{(H_1, H_2) \in \mathcal{H}} \|f\|_H^{|H_1|} \|g\|_H^{|H_2|} = (\|f\|_H + \|g\|_H)^{|H|} + (\|f\|_H - \|g\|_H)^{|H|}, \end{aligned}$$

where in the inequality we used Lemma 2.10. \blacksquare

Consider a norming hypergraph pair H of Type I with parameter $s < 2$ and $|H| \geq 2$. Note that for every $2 \leq q < \infty$, ℓ_s does not satisfy the q -Hanner inequality, as otherwise it would be 2-uniformly convex. Hence it follows from Theorem 3.9 that ℓ_H does not satisfy the q -Hanner inequality for any $2 \leq q < \infty$. However we conjecture the following.

Conjecture 3.15 *Let $H = (\alpha, \beta)$ be a non-factorizable semi-norming hypergraph pair of Type I with parameter $s \geq 2$. Then every L_H space satisfies the $|H|$ -Hanner inequality.*

Since we could not establish the $|H|$ -Hanner inequality for all norming hypergraph pairs of Type I we have to treat some of them separately. The next two lemmas which give the optimum bounds for uniform smoothness and convexity constants of ℓ_H when H is a non-factorizable hypergraph pair of Type I with parameter $s \geq 2$ would have been followed from a positive answer to Conjecture 3.15.

Lemma 3.16 (2-Smoothness) *Let $H = (\alpha, \beta)$ be a non-factorizable semi-norming k -hypergraph pair with $|H| \geq 2$. If H is of Type II, or of Type I with parameter $s \geq 2$, then*

$$K_{2,p}(\ell_H) = K_{2,p}(\ell_{|H|}) = \begin{cases} \sqrt{|H|-1} & p \leq |H| \\ \sqrt{p-1} & p \geq |H| \end{cases}$$

Proof. It suffices to prove $K_{2,|H|}(\ell_H) \leq \sqrt{|H|-1}$, and the rest will follow from Remark 3.3. Suppose that H is defined over $V := V_1 \times \dots \times V_k$. For $f, g \in \ell_H$, we have to prove

$$\left(\frac{\|f+g\|_H^{|H|} + \|f-g\|_H^{|H|}}{2} \right)^{2/|H|} \leq \|f\|_H^2 + (|H|-1)\|g\|_H^2. \quad (36)$$

Consider the counting measure on $\{-1, 1\}$, and define the two functions $\epsilon_1, \epsilon_2 : \{-1, 1\}^k \rightarrow \{-1, 0, 1\}$ as

$$\epsilon_1(x_1, \dots, x_k) = \begin{cases} 1 & x_1 = \dots = x_k \\ 0 & \text{otherwise} \end{cases},$$

and

$$\epsilon_2(x_1, \dots, x_k) = \begin{cases} x_1 & x_1 = \dots = x_k \\ 0 & \text{otherwise} \end{cases}.$$

Note that since H is non-factorizable, for $x \in \{-1, 1\}^{V_1} \times \dots \times \{-1, 1\}^{V_1}$, we have

$$\epsilon_1^H(x) = \begin{cases} 1 & x = (1, \dots, 1) \\ 0 & \text{otherwise} \end{cases}, \quad (37)$$

and

$$\epsilon_2^H(x) = \begin{cases} \eta & x = (\eta, \dots, \eta) \\ 0 & \text{otherwise} \end{cases}, \quad (38)$$

Let $\tilde{f} = f \otimes \epsilon_1$ and $\tilde{g} = g \otimes \epsilon_2$. From (37) and (38) it is easy to see that

$$\int (\tilde{f} + \tilde{g})^H = \int (\tilde{f} - \tilde{g})^H = \int (f + g)^H + (f - g)^H,$$

and $\int \tilde{f}^H = 2 \int f^H$ and $\int \tilde{g}^H = 2 \int g^H$. Hence it suffices to prove

$$\left(\frac{\int (\tilde{f} + \tilde{g})^H}{2} \right)^{2/|H|} \geq \left(\frac{\int \tilde{f}^H}{2} \right)^{2/|H|} + (|H|-1) \left(\frac{\int \tilde{g}^H}{2} \right)^{2/|H|}.$$

which simplifies to

$$\left(\int (\tilde{f} + \tilde{g})^H \right)^{2/|H|} \geq \left(\int \tilde{f}^H \right)^{2/|H|} + (|H|-1) \left(\int \tilde{g}^H \right)^{2/|H|}. \quad (39)$$

We will show that for $0 \leq t \leq 1$

$$\left(\int (\tilde{f} + t\tilde{g})^H \right)^{2/|H|} \geq \left(\int \tilde{f}^H \right)^{2/|H|} + t^2(|H|-1) \left(\int \tilde{g}^H \right)^{2/|H|}. \quad (40)$$

Note that (40) reduces to (39) for $t = 1$. Consider the functions $L, R : [0, 1] \rightarrow \mathbb{R}$, defined as

$$L(t) = \left(\int (\tilde{f} + t\tilde{g})^H \right),$$

and

$$R(t) = \left(\int \tilde{f}^H \right)^{2/|H|} + t^2(|H| - 1) \left(\int \tilde{g}^H \right)^{2/|H|}.$$

We have

$$\frac{d}{dt}L(t) = \int \sum_{\psi \in V} \alpha(\psi)(\tilde{f} + t\tilde{g})^{H-1\psi} \tilde{g}^{1\psi} + \beta(\psi)(\tilde{f} + t\tilde{g})^{H-\overline{1\psi}} \tilde{g}^{\overline{1\psi}}.$$

Then

$$\frac{d}{dt}L(t)^{2/|H|} = \frac{2}{|H|} \left(\int \sum_{\psi \in V} \alpha(\psi)(\tilde{f} + t\tilde{g})^{H-1\psi} \tilde{g}^{1\psi} + \beta(\psi)(\tilde{f} + t\tilde{g})^{H-\overline{1\psi}} \tilde{g}^{\overline{1\psi}} \right) L(t)^{\frac{2-|H|}{|H|}}.$$

We want to compute the second derivative. Denote $\mathcal{H} = \{1_\psi : \psi \in V\} \cup \{\overline{1_\psi} : \psi \in V\}$, and define $\gamma : \mathcal{H} \rightarrow \mathbb{R}$ by $\gamma : 1_\psi \mapsto \alpha(\psi)$ and $\gamma : \overline{1_\psi} \mapsto \beta(\psi)$. We have

$$\begin{aligned} \frac{d^2}{dt^2}L(t)^{2/|H|} &= \frac{2}{|H|} \left(\int \sum_{H_1 \neq H_2 \in \mathcal{H}} \gamma(H_1)\gamma(H_2)(\tilde{f} + t\tilde{g})^{H-H_1-H_2} \tilde{g}^{H_1+H_2} \right. \\ &\quad \left. + \sum_{H_1 \in \mathcal{H}} \gamma(H_1)(\gamma(H_1) - 1)(\tilde{f} + t\tilde{g})^{H-2H_1} \tilde{g}^{2H_1} \right) L(t)^{\frac{2-|H|}{|H|}} + \\ &\quad + \left(\frac{d}{dt}L(t) \right)^2 \frac{2(2-|H|)}{|H|^2} L(t)^{\frac{2-2|H|}{|H|}}. \end{aligned}$$

Recalling the definition of \tilde{f} and \tilde{g} , it is easy to see that

$$L(0)^{2/|H|} = R(0),$$

and since $\int \tilde{f}^{H-1\psi} \tilde{g}^{1\psi} = \int f^{H-1\psi} g^{1\psi} - \int f^{H-1\psi} g^{1\psi} = 0$ and $\int \tilde{f}^{H-\overline{1\psi}} \tilde{g}^{\overline{1\psi}} = \int f^{H-\overline{1\psi}} g^{\overline{1\psi}} - \int f^{H-\overline{1\psi}} g^{\overline{1\psi}} = 0$, we have

$$\left. \frac{d}{dt}L(t)^{2/|H|} \right|_{t=0} = \left. \frac{d}{dt}R(t) \right|_{t=0} = 0.$$

Furthermore since H is of Type II or of Type I with parameter $s \geq 2$, by Lemma 2.10, we have

$$\begin{aligned} \frac{d^2}{dt^2}L(t)^{2/|H|}|_{t=0} &= \frac{2}{|H|} \left(\int \sum_{H_1 \neq H_2 \in \mathcal{H}} \gamma(H_1)\gamma(H_2) \tilde{f}^{H-H_1-H_2} \tilde{g}^{H_1+H_2} + \right. \\ &\quad \left. \sum_{H_1 \in \mathcal{H}} \gamma(H_1)(\gamma(H_1) - 1) \tilde{f}^{H-2H_1} \tilde{g}^{2H_1} \right) L(0)^{\frac{2-|H|}{|H|}} \\ &\leq \frac{2}{|H|} \left(\sum_{H_1 \neq H_2 \in \mathcal{H}} \gamma(H_1)\gamma(H_2) + \sum_{H_1 \in \mathcal{H}} \gamma(H_1)(\gamma(H_1) - 1) \right) (\|\tilde{f}\|_H^{|H|-2} \|\tilde{g}\|_H^2) \|\tilde{f}\|_H^{2-|H|} \\ &= 2(|H| - 1) \|\tilde{g}\|_H^2 = \frac{d^2}{dt^2}R(t)|_{t=0}. \end{aligned} \tag{41}$$

Now for every $0 \leq t_0 \leq 1$, one can replace \tilde{f} with $\tilde{f} + t_0\tilde{g}$ in (41) and obtain that for every $0 \leq t_0 \leq 1$

$$\left. \frac{d^2}{dt^2}L(t)^{2/|H|} \right|_{t=t_0} \leq \left. \frac{d^2}{dt^2}R(t) \right|_{t=t_0}. \tag{42}$$

We conclude (40). ■

Next we prove Clarkson's inequalities for ℓ_H when H is a semi-norming hypergraph pair of Type II or of Type I with parameter $s \geq 2$. As it is mentioned above this would follow from Conjecture 3.15.

Lemma 3.17 (Clarkson's Inequalities) *Let H be a non-factorizable semi-norming hypergraph pair of Type II or Type I with parameter $s \geq 2$ such that $q := |H| \geq 2$. Then*

$$K_{p,q}(\ell_H) = K_{q,p}^*(\ell_H) = K_{q,q}^*(\ell_H) = 1,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Recall that always $K_{q,q}^* \leq K_{q,p}^*$. Hence it suffices to prove $K_{p,q}(\ell_H) = 1$, as by Observation 3.6 this would imply $K_{q,p}^*(\ell_H) = 1$. To this end, we need to show that for $f, g \in \ell_H$, we have

$$\left(\frac{\|f+g\|_H^q + \|f-g\|_H^q}{2} \right)^{1/q} \leq (\|f\|_H^p + \|g\|_H^p)^{1/p}, \quad (43)$$

which is equivalent to

$$\left(\left\| \frac{f+g}{2} \right\|_H^q + \left\| \frac{f-g}{2} \right\|_H^q \right)^{1/q} \leq \left(\frac{\|f\|_H^p + \|g\|_H^p}{2} \right)^{1/p}. \quad (44)$$

Proposition 3.7 shows that (44) follows from the $|H|$ -Hanner inequality. Hence Theorem 3.14 implies (44) when H is of Type II or it is of Type I with parameter s where s is an even integer. Next assume that H is of Type I with parameter $s \geq 2$.

For a real $1 \leq t < \infty$, and a norming hypergraph pair G , define the norm $L_t(\ell_G)$ on the set of pairs (f, g) where $f, g \in \ell_G$ as

$$\|(f, g)\|_{L_t(\ell_G)} := (\|f\|_G^t + \|g\|_G^t)^{1/t}.$$

Consider the linear map $T : (f, g) \mapsto (\frac{f+g}{2}, \frac{f-g}{2})$. Then (44) says that

$$\|T\|_{L_p(\ell_H) \rightarrow L_q(\ell_H)} \leq 2^{-\frac{1}{p}}. \quad (45)$$

We will prove this by interpolation. Let $\tilde{H} = \frac{1}{s}H$, and s_0 and s_1 be two even integers satisfying $2 \leq s_0 \leq s \leq s_1$, and θ be such that $\frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}$. Then $\frac{1}{p} = \frac{1-\theta}{t_0} + \frac{\theta}{t_1}$, where $\frac{1}{t_0} + \frac{1}{s_0|\tilde{H}|} = 1$ and $\frac{1}{t_1} + \frac{1}{s_1|\tilde{H}|} = 1$. Theorem 3.13 above, together with Theorem 5.1.2 from [2] imply that

$$\left[L_{s_0|\tilde{H}|}(\ell_{s_0\tilde{H}}), L_{s_1|\tilde{H}|}(\ell_{s_1\tilde{H}}) \right]_{\theta} = L_{s|\tilde{H}|}([\ell_{s_0\tilde{H}}, \ell_{s_1\tilde{H}}]_{\theta}) = L_q(\ell_H),$$

and

$$\left[L_{t_0}(\ell_{s_0\tilde{H}}), L_{t_1}(\ell_{s_1\tilde{H}}) \right]_{\theta} = L_p([\ell_{s_0\tilde{H}}, \ell_{s_1\tilde{H}}]_{\theta}) = L_p(\ell_H).$$

Furthermore

$$\left(2^{-\frac{1}{t_0}} \right)^{1-\theta} \left(2^{-\frac{1}{t_1}} \right)^{\theta} = 2^{-\frac{1}{p}}.$$

Now since we know that (45) holds for even values of $s \geq 2$, we have

$$\|T\|_{L_{t_0}(\ell_{s_0\tilde{H}}) \rightarrow L_{s_0|\tilde{H}|}(\ell_{s_0\tilde{H}})} \leq 2^{-\frac{1}{t_0}},$$

and

$$\|T\|_{L_{t_1}(\ell_{s_1\tilde{H}}) \rightarrow L_{s_1|\tilde{H}|}(\ell_{s_1\tilde{H}})} \leq 2^{-\frac{1}{t_1}}.$$

Then interpolation (35), implies (45). ■

Next Lemma determines the moduli of smoothness and convexity of non-factorizable semi-norming hypergraph pairs of Type I with parameter $1 < s \leq 2$.

Lemma 3.18 *Let H be a non-factorizable semi-norming hypergraph pair of Type I with parameter $s > 1$ with $|H| \geq 1$. Then $K_{s,|H|}(\ell_H) = C(s, |H|)$ and $K_{|H|,s}^*(X) = C^*(|H|, s)$.*

Proof. Let $C := C(s, |H|)$ and $C^* := C^*(|H|, s)$. Consider $f, g \in \ell_H$. By (33) and (34) we have

$$\begin{aligned} \left(\frac{\|f + g\|_H^{|H|} + \|f - g\|_H^{|H|}}{2} \right)^{1/|H|} &\leq \left\| \left(\frac{|f + g|^{|H|} + |f - g|^{|H|}}{2} \right)^{1/|H|} \right\|_H \\ &\leq \left\| (|f|^s + |Cg|^s)^{1/s} \right\|_H \\ &\leq (\|f\|_H^s + \|Cg\|_H^s)^{1/s}, \end{aligned}$$

which shows that $K_{s,|H|}(\ell_H) \leq C$. To prove $K_{|H|,s}^* = C^*$, note that by (34) and (33) we have

$$\begin{aligned} \left(\frac{\|f + g\|_H^s + \|f - g\|_H^s}{2} \right)^{1/s} &\geq \left\| \left(\frac{|f + g|^s + |f - g|^s}{2} \right)^{1/s} \right\|_H \\ &\geq \left\| \left(|f|^{|H|} + \left| \frac{1}{C^*} g \right|^{|H|} \right)^{1/|H|} \right\|_H \\ &\geq \left(\|f\|_H^{|H|} + \left\| \frac{1}{C^*} g \right\|_H^{|H|} \right)^{1/|H|}. \end{aligned}$$

■

Remark 3.19 Note that all results in Section 3.4 are stated for *non-factorizable* semi-norming hypergraph pairs. Consider a semi-norming hypergraph pair $H = H_1 \dot{\cup} \dots \dot{\cup} H_m$, where H_i 's are non-factorizable. If H is of Type I, then by Theorem 2.17, $\|\cdot\|_H = \|\cdot\|_{H_1}$, and thus one can apply the results of Section 3.4 to H_1 instead. However some of our results do not cover the case where H is factorizable and of Type II. ■

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References

- [1] Keith Ball, Eric A. Carlen, and Elliott H. Lieb. Sharp uniform convexity and smoothness inequalities for trace norms. *Invent. Math.*, 115(3):463–482, 1994.
- [2] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.
- [3] Jean Bourgain and Alex Gamburd. Uniform expansion bounds for Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$. *Ann. of Math. (2)*, 167(2):625–642, 2008.
- [4] James A. Clarkson. Uniformly convex spaces. *Trans. Amer. Math. Soc.*, 40(3):396–414, 1936.
- [5] P. Erdős and M. Simonovits. Cube-supersaturated graphs and related problems. In *Progress in graph theory (Waterloo, Ont., 1982)*, pages 203–218. Academic Press, Toronto, ON, 1984.
- [6] Tadeusz Figiel. On the moduli of convexity and smoothness. *Studia Math.*, 56(2):121–155, 1976.

- [7] Tadeusz Figiel and Gilles Pisier. Séries aléatoires dans les espaces uniformément convexes ou uniformément lisses. *C. R. Acad. Sci. Paris Sér. A*, 279:611–614, 1974.
- [8] Timothy Gowers. A new proof of Szemerédi’s theorem for arithmetic progressions of length four. *Geom. Funct. Anal.*, 8(3):529–551, 1998.
- [9] Timothy Gowers. Hypergraph regularity and the multidimensional Szemerédi theorem. *Ann. of Math. (2)*, 166(3):897–946, 2007.
- [10] Ben Green and Terence Tao. The primes contain arbitrarily long arithmetic progressions. *Ann. of Math. (2)*, 167(2):481–547, 2008.
- [11] Olof Hanner. On the uniform convexity of L^p and ℓ^p . *Ark. Mat.*, 3:239–244, 1956.
- [12] Hamed Hatami. Graph norms and Sidorenko’s conjecture. *Israel J. Math.*, to appear.
- [13] Joram Lindenstrauss. On the modulus of smoothness and divergent series in Banach spaces. *Michigan Math. J.*, 10:241–252, 1963.
- [14] Joram Lindenstrauss and Lior Tzafriri. *Classical Banach spaces. II*, volume 97 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin, 1979. Function spaces.
- [15] Bernard Maurey and Gilles Pisier. Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. *Studia Math.*, 58(1):45–90, 1976.
- [16] Vitali D. Milman and Gideon Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.
- [17] Imre Z. Ruzsa. Sums of finite sets. In *Number theory (New York, 1991–1995)*, pages 281–293. Springer, New York, 1996.
- [18] Alexander Sidorenko. Inequalities for functionals generated by bipartite graphs. *Diskret. Mat.*, 3(3):50–65, 1991.
- [19] Alexander Sidorenko. A correlation inequality for bipartite graphs. *Graphs Combin.*, 9(2):201–204, 1993.
- [20] Terence Tao. The ergodic and combinatorial approaches to Szemerédi’s theorem. In *Proceedings of the Montreal workshop on additive combinatorics and number theory*, to appear.