

Complex order fractional derivatives in viscoelasticity

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Abstract

We introduce complex order fractional derivatives in models that describe viscoelastic materials. This can not be carried out unrestrictedly, and therefore we derive, for the first time, real valued compatibility constraints, as well as physical constraints that lead to acceptable models. As a result, we introduce a new form of complex order fractional derivative. Also, we consider a fractional differential equation with complex derivatives, and study its solvability. Results obtained for stress relaxation and creep are illustrated by several numerical examples.

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1 Introduction

Fractional calculus is a powerful tool for modeling various phenomena in mechanics, physics, biology, chemistry, medicine, economy, etc. Last few decades have brought a rapid expansion of the non-integer order differential and integral calculus, from which both the theory and its applications benefit significantly. However, most of the work done in this field so far has been based on the use of real order fractional derivatives and integrals. It is worth to mention that there are several authors who also applied complex order fractional derivatives to model various phenomena, see the work of Machado or Makris, [18, 20, 21]. In all of these papers, restrictions on constitutive parameters that follow from the Second Law of Thermodynamics were not examined. In the analysis that follows this issue will be addressed.

The main goal of this paper is to motivate and explain basic concepts of fractional calculus with complex order fractional derivatives. Throughout the paper we will investigate constitutive equations in the dimensionless form, for all independent (t and x) and dependent (σ

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and ε) variables. Thus, consider a constitutive equation, given by (1), connecting the stress $\sigma(t, x)$ at the point $x \in \mathbb{R}$ and time $t \in \mathbb{R}_+$, with the strain $\varepsilon(t, x)$:

$$\sum_{n=0}^N a_n {}_0D_t^{\alpha_n} \sigma(t, x) = \sum_{m=0}^M b_m {}_0D_t^{\beta_m} \varepsilon(t, x), \quad (1)$$

that contains fractional derivatives of complex order $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_M$. The precise definition of the operator ${}_0D_t^\eta$ of fractional differentiation with respect to t is given below. In order to make a useful framework for the study of (1) we involve two types of conditions: 1. real valued compatibility constraints, and 2. thermodynamical constraints. Since this paper deals only with the well-posedness of constitutive equations of type (1) and their solvability for strain if stress is prescribed, we may, without loss of generality, assume that both σ and ε are functions only of t . Also, equation (1) can be seen as a generalization of different models considered in the literature so far (see e.g. [5, 7, 8, 12, 14, 15, 19, 22]), since by taking all α_n and β_m to be real numbers, the problem is reduced to the real case studied in the mentioned papers.

Our results will show that constitutive equations of the form (1) may lead to creep and stress relaxation curves that are not monotonic. Non-monotonic creep curves were observed experimentally, however such a behavior is attributed to inertia either of the rod itself or of the rheometer.

The paper is organized as follows. In Section 2 we investigate conditions leading to constitutive equations containing complex derivatives of stress and strain that can be used in viscoelastic models of the wave equation. More precisely, we derive restrictions on parameters in constitutive equation of the form (1) under which the Laplace and Fourier transforms, as well as their inverses, of real-valued functions will remain real-valued. Also, in order to impose the validity of the Second Law of Thermodynamics we follow the procedure presented in [6], and obtain additional restrictions on model parameters in constitutive equations. As a result, we shall introduce a new form of the fractional derivative of complex order. Then, in Section 3, we treat the fractional Kelvin-Voigt complex order constitutive equation for viscoelastic body, find thermodynamical restrictions and prove sufficient conditions for its invertibility. Several numerical examples are presented in Section 4, as an illustration of creep and stress relaxation in viscoelastic materials.

To the end of this section we recall basic definitions and results that will be used in our work. Fractional operators of complex order are introduced as follows (see [17, 23]): For $\eta \in \mathbb{C}$ with $0 < \operatorname{Re} \eta < 1$, definition of the left Riemann-Liouville fractional integral of an absolutely continuous function on $[0, T]$, $T > 0$ ($y \in AC([0, T])$) coincides with the case of real η , i.e., ${}_0I_t^\eta y(t) := \frac{1}{\Gamma(\eta)} \int_0^t \frac{y(\tau)}{(t-\tau)^{1-\eta}} d\tau$, $t \in [0, T]$, where Γ is the Euler gamma function. If $\eta = i\theta$, $\theta \in \mathbb{R}$, then the latter integral diverges, and hence one introduces the fractional integration of imaginary order as

$${}_0I_t^{i\theta} y(t) := \frac{d}{dt} {}_0I_t^{1+i\theta} y(t) = \frac{1}{\Gamma(1+i\theta)} \frac{d}{dt} \int_0^t (t-\tau)^{i\theta} y(\tau) d\tau, \quad t \in [0, T].$$

However, in both cases the left Riemann-Liouville fractional derivative of order $\eta \in \mathbb{C}$ with $0 \leq \operatorname{Re} \eta < 1$ is given by

$${}_0D_t^\eta y(t) := \frac{d}{dt} {}_0I_t^{1-\eta} y(t) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dt} \int_0^t \frac{y(\tau)}{(t-\tau)^\eta} d\tau, \quad t \in [0, T].$$

The basic tool for our study will be the Laplace and Fourier transforms. In order to have a good framework we will perform these transforms in $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions. It is the dual space for the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$. In particular, we are interested in the space $\mathcal{S}'_+(\mathbb{R})$ whose elements are of the form $y = P(D)Y_0$, where Y_0 is a locally integrable polynomial bounded function on \mathbb{R} that vanishes on $(-\infty, 0)$, and $P(D)$ denotes a partial differential operator.

The Fourier transform of $y \in L^1(\mathbb{R})$ (or $y \in L^2(\mathbb{R})$) is defined as

$$\mathcal{F}y(\omega) = \hat{y}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} y(x) dx, \quad \omega \in \mathbb{R}. \quad (2)$$

In the distributional setting, one has $\langle \mathcal{F}y, \varphi \rangle = \langle y, \mathcal{F}\varphi \rangle$, $y \in \mathcal{S}'(\mathbb{R})$, $\varphi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{F}\varphi$ is defined by (2). For $y \in L^1(\mathbb{R})$ with $y(t) = 0$, $t < 0$, and $|y(t)| \leq Ae^{at}$, $a, A > 0$, the Laplace transform is given by

$$\mathcal{L}y(s) = \tilde{y}(s) = \int_0^{\infty} e^{-st} y(t) dt, \quad \operatorname{Re} s > a.$$

If $y \in \mathcal{S}'_+(\mathbb{R})$ then $a = 0$ (since y is bounded by a polynomial). Then $\mathcal{L}y$ is a holomorphic function in the half plane $\operatorname{Re} s > 0$ (see e.g. [24]).

Let $Y(s)$, $\operatorname{Re} s > 0$, be a holomorphic function bounded by a polynomial in that domain. Then, for a suitable polynomial P , $Y(s)/P(s)$ is integrable along the line $\Gamma = (a - i\infty, a + i\infty)$, and the inverse Laplace transform of Y is a tempered distribution $y(t) = P(\frac{d}{dt})Y_0(t)$, where $Y_0(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Y(s)}{P(s)} e^{st} ds$.

Let $y \in \mathcal{S}'_+$. Recall:

$$\begin{aligned} \mathcal{F} \left[\frac{d^n}{dx^n} y \right] (\omega) &= (i\omega)^n \mathcal{F}y(\omega) \quad (\omega \in \mathbb{R}), & \mathcal{L} \left[\frac{d^n}{dt^n} y \right] (s) &= s^n \mathcal{L}y(s) \quad (\operatorname{Re} s > 0), \quad n \in \mathbb{N}, \\ \mathcal{F}[{}_0D_x^\alpha y] (\omega) &= (i\omega)^\alpha \mathcal{F}y(\omega) \quad (\omega \in \mathbb{R}), & \mathcal{L}[{}_0D_t^\alpha y] (s) &= s^\alpha \mathcal{L}y(s) \quad (\operatorname{Re} s > 0), \quad \alpha \in \mathbb{C}. \end{aligned}$$

2 Linear fractional constitutive equations with complex derivatives

In what follows, we shall denote by α and β the orders of fractional derivatives. α will be assumed to be a real number, while β will be an element of \mathbb{C} which is not real, i.e., $\beta = A + iB$, with $B \neq 0$. Also, we shall assume that $0 < \alpha, A < 1$.

2.1 Real valued compatibility constraints

Similarly as in the real case, see [2], it is quite difficult to begin with a study of the most general case of (1). Therefore, in order to try to recognize the essence of the problem and find possibilities for overcoming it, we shall first concentrate to simpler forms of constitutive equations that contain complex derivatives. Consider the following generalization of the Hooke law in the complex setting:

$$\sigma(t) = b_0 D_t^\beta \varepsilon(t), \quad (3)$$

where $\beta \in \mathbb{C}$ and $b \in \mathbb{R}$. In order to find restrictions on parameters b and β in (3) which yield a physically acceptable constitutive equation, we shall verify the next two conditions: For real strain ε , the stress σ has to be real valued function of t . We call this a real valued compatibility requirement. Thermodynamical restrictions will result from the Second Law of Thermodynamics, and will be studied in the next section. Note that in the case of constitutive equations with only real-valued fractional derivatives, the real valued compatibility requirement always holds true, while the thermodynamical restrictions had to be investigated (cf. [2]).

Theorem 2.1 *Let $\varepsilon \in AC([0, T])$ be real-valued, for all $T > 0$, $0 < A < 1$ and $b \neq 0$. Then function σ defined by (3) is real valued if and only if $\beta \in \mathbb{R}$.*

Proof. It follows from (3), with $\beta = A + iB$, and $1/\Gamma(1 - \beta) = h + ir$, that

$$\text{Im } \sigma(t) = b \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A} \left(r \cos(B \ln \tau) - h \sin(B \ln \tau) \right) d\tau, \quad t \geq 0.$$

Denoting by $\frac{r}{h} := \text{tg } \phi$, for $h \neq 0$, we obtain

$$\text{Im } \sigma(t) = \frac{bh}{\cos \phi} \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A} \sin(\phi - B \ln \tau) d\tau, \quad t \geq 0.$$

In the case $h = 0$, the imaginary part of σ reduces to $br \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A} \cos(B \ln \tau) d\tau$.

If $B = 0$ then $r = 0$ and $\phi = 0$, hence $\text{Im } \sigma = 0$ and σ is a real-valued function.

Next, suppose that $B \neq 0$. But then one can find a subinterval (t_1, t_2) of $(0, T)$ where $\sin(\phi - B \ln \tau)$, respectively $\cos(B \ln \tau)$, is positive (or negative), and choose $\varepsilon \in AC([0, T])$ which is compactly supported in (t_1, t_2) and strictly positive. This leads to a contradiction with the assumption $\text{Im } \sigma = 0$. \square

The previous theorem implies that equations of form (3) with $\beta \in \mathbb{C} \setminus \mathbb{R}$ can not be a constitutive equation for a viscoelastic body.

Next, consider the equation

$$\sigma(t) = b_1 {}_0D_t^{\beta_1} \varepsilon(t) + b_2 {}_0D_t^{\beta_2} \varepsilon(t), \quad t \geq 0, \quad (4)$$

where $b_1, b_2 \in \mathbb{R}$, and $\beta_1, \beta_2 \in \mathbb{C} \setminus \mathbb{R}$, i.e., $\beta_k = A_k + iB_k$ and $B_k \neq 0$ ($k = 1, 2$). Suppose again that $\varepsilon \in AC([0, T])$ is a real-valued function, for every $T > 0$.

Remark 2.2 Note that dimension $[{}_0D_t^{\beta_1} \varepsilon]$ is $T^{-\beta_1}$, where T is time unit. Therefore, (4) makes sense if $[b_1] = T^{\beta_1}$ and $[b_2] = T^{\beta_2}$.

Theorem 2.3 *Function σ given by (4) is real-valued for all real-valued positive $\varepsilon \in AC([0, T])$ if and only if $b_1 = b_2$ and $\beta_2 = \bar{\beta}_1$.*

Proof. We continue with the notation of Theorem 2.1. Let $t \geq 0$. Then

$$\sigma(t) = \frac{b_1}{\Gamma(1 - \beta_1)} \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-\beta_1} d\tau + \frac{b_2}{\Gamma(1 - \beta_2)} \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-\beta_2} d\tau$$

Denote by $h_k + ir_k := 1/\Gamma(1 - \beta_k)$, $k = 1, 2$. Then the imaginary part of the right hand side reads:

$$\begin{aligned} \operatorname{Im} \sigma(t) &= \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A_1} \left(b_1 r_1 \cos(B_1 \ln \tau) - b_1 h_1 \sin(B_1 \ln \tau) \right) d\tau \\ &\quad + \frac{d}{dt} \int_0^t \varepsilon(t - \tau) \tau^{-A_2} \left(b_2 r_2 \cos(B_2 \ln \tau) - b_2 h_2 \sin(B_2 \ln \tau) \right) d\tau \end{aligned} \quad (5)$$

Using the identity $\Gamma(\bar{z}) = \overline{\Gamma(z)}$, it is straight forward to check that $b_1 = b_2$ and $\beta_2 = \bar{\beta}_1$ imply that $\operatorname{Im} \sigma = 0$, and hence σ is a real-valued function on $[0, T]$, for every $T > 0$.

Conversely, we want to find conditions on parameters which yield a real-valued function σ . Thus, we look at (5), with the change of variables $p = \ln \tau$, $\tau \in (0, t)$, $t \leq T$, and solutions of the equation

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\ln t} e^p \varepsilon(t - e^p) \left(b_1 r_1 e^{-A_1 p} \cos(B_1 p) - b_1 h_1 e^{-A_1 p} \sin(B_1 p) \right. \\ \left. + b_2 r_2 e^{-A_2 p} \cos(B_2 p) - b_2 h_2 e^{-A_2 p} \sin(B_2 p) \right) dp = 0, \quad t \in [0, T]. \end{aligned} \quad (6)$$

Set $\frac{r_k}{h_k} := \operatorname{tg} \phi_k$, $k = 1, 2$, for $h_1, h_2 \neq 0$. (For $h_k = 0$ set $\phi_k := \frac{\pi}{2}$, $k = 1, 2$.) Then (6) gives

$$\frac{d}{dt} \int_{-\infty}^{\ln t} e^p \varepsilon(t - e^p) \left(\frac{b_1 h_1}{\cos \phi_1} e^{-A_1 p} \sin(\phi_1 - B_1 p) + \frac{b_2 h_2}{\cos \phi_2} e^{-A_2 p} \sin(\phi_2 - B_2 p) \right) dp = 0.$$

Assume first that $|B_1| \neq |B_2|$, say $|B_1| > |B_2|$. Since the basic period of $\sin(\phi_1 - B_1 p)$ ($T_0 = 2\pi/|B_1|$) is smaller than for $\sin(\phi_2 - B_2 p)$, it follows that for every $k \in \mathbb{N}$, $k > k_0$, where k_0 depends on $\ln t$, the function $\sin(\phi_1 - B_1 p)$ changes its sign at least three times in the interval $(k\pi, k\pi + 2\pi/|B_2|)$. Thus, on that interval, there exist at least two intervals where $\sin(\phi_1 - B_1 p)$ and $\sin(\phi_2 - B_2 p)$ have the same sign, and two intervals where they have opposite signs. We conclude that there exists an interval $[a, b] \subseteq (k\pi, k\pi + 2\pi/|B_2|) \subset (-\infty, \ln t)$, so that

$$\frac{b_1 h_1}{\cos \phi_1} e^{-A_1 p} \sin(\phi_1 - B_1 p) + \frac{b_2 h_2}{\cos \phi_2} e^{-A_2 p} \sin(\phi_2 - B_2 p) > 0, \quad p \in [a, b]. \quad (7)$$

Choose $\delta > 0$ and $k \in \mathbb{N}$ so that

$$\{t - e^p; t \in (T/2 - \delta, T/2 + \delta), p \in [a, b]\} = (T/2 - \delta - e^b, T/2 + \delta - e^a) = I \subset (0, T).$$

Now, we choose a non-negative function $\varepsilon \in AC([0, T])$ with the following properties: $\operatorname{supp} \varepsilon \subseteq I$, so that the function $p \mapsto \varepsilon(t - e^p)$, $p \in [a, b]$, is strictly positive on some $[a_1, b_1] \subseteq (a, b)$. This implies that for $t \in (T/2 - \delta, T/2 + \delta)$,

$$\int_a^b e^p \varepsilon(t - e^p) \left(\frac{b_1 h_1}{\cos \phi_1} e^{-A_1 p} \sin(\phi_1 - B_1 p) + \frac{b_2 h_2}{\cos \phi_2} e^{-A_2 p} \sin(\phi_2 - B_2 p) \right) dp$$

is not a constant function. This is in contradiction with (6).

Therefore, in order to have (6), one must have $|B_1| = |B_2|$. Moreover, arguing as above, one concludes that $|\phi_1 - B_1 p| = |\phi_2 - B_2 p|$ must hold, for all $p \in (-\infty, \ln t)$, $t \geq 0$. Then we examine the equation

$$b_1 h_1 e^{-A_1 p} - b_2 h_2 e^{-A_2 p} = 0, \quad \text{or} \quad b_1 h_1 e^{-A_1 p} + b_2 h_2 e^{-A_2 p} = 0, \quad p \in [k\pi, k\pi + 2\pi/|B_1|].$$

Now in both cases, $b_1 b_2 > 0$ or $b_1 b_2 < 0$, it is easy to conclude that $A_1 = A_2$, $b_1 = b_2$ and $B_1 = -B_2$ have to be satisfied. This proves the theorem. \square

Remark 2.4 (i) Theorem 2.3 states that a real valued compatibility constraint for constitutive equations of form (4) holds if they contain complex fractional derivatives of strain, whose orders have to be complex conjugated numbers. Therefore, we may assume in the sequel, without loss of generality, that $B > 0$.

(ii) According to the above analysis, one can take arbitrary linear combination of pairs of complex conjugated fractional derivatives of strain. Moreover, one can also allow the same type of fractional derivatives of stress. Thus, one can consider the most general stress-strain constitutive equation with fractional derivatives of complex order:

$$\sigma(t) + \sum_{i=1}^N c_i \left({}_0D_t^{\gamma_i} + {}_0D_t^{\bar{\gamma}_i} \right) \sigma(t) = \varepsilon(t) + \sum_{j=1}^M b_j \left({}_0D_t^{\beta_j} + {}_0D_t^{\bar{\beta}_j} \right) \varepsilon(t),$$

where $c_i, b_j \in \mathbb{R}$ and $\gamma_i, \beta_j \in \mathbb{C}$, $i = 1, \dots, N$, $j = 1, \dots, M$.

(iii) As a consequence one has that stress-strain relations can also contain arbitrary real order fractional derivatives, without any additional restrictions. This fact has already been known from previous work.

(iv) The same conclusions can also be obtained using a different approach. One can apply the result from [11, p. 293, Satz 2], which tells that a function F is real-valued (almost everywhere), if its Laplace transform is real-valued, for all real s in the half-plane of convergence right from some real x_0 , in order to show that an admissible fractional constitutive equation (1) may be of complex order only if it contains pairs of complex conjugated fractional derivatives of stress and strain.

2.2 Thermodynamical restrictions

In the analysis that follows we shall consider the isothermal processes only. For such processes the Second Law of Thermodynamics, i.e., the entropy of the system increases, is equivalent to the dissipativity condition

$$\int_0^T \sigma(t) \dot{\varepsilon}(t) dt \geq 0. \quad (8)$$

Equation (8) is one-dimensional version of the Second Principle of Thermodynamics for simple materials under isothermal conditions (see [1, p. 83]). Also, in writing (8) we observe the fact that for $t \in (-\infty, 0)$ the system is in virginal state. Note that $T = \infty$ in [10, p. 113]. Since we consider arbitrary T , condition (8) is stronger than the condition in [10].

In [1] stress is assumed as

$$\sigma(t) = G_0 \varepsilon(t) + \int_0^\infty G'(u) \varepsilon(t-u) du, \quad (9)$$

where G_0 is instantaneous elastic modulus, and G is relaxation function, while strain is taken in the form

$$\varepsilon(s) = \varepsilon_0 \cos(\omega s) + \varepsilon_0 \sin(\omega s), \quad s \leq t, \quad \omega > 0,$$

where ε_0 is the amplitude of strain. These assumptions along with (8) and $T = \frac{2n\pi}{\omega}$ lead to

$$\int_0^\infty G'(u) \sin(\omega u) du < 0. \quad (10)$$

Note that (10) is equivalent to

$$\mathcal{F}_s[G'(t)](\omega) = \text{Im} \left(\mathcal{F}[G'(t)](\omega) \right) > 0,$$

where $\mathcal{F}_s[f(t)](\omega) = \int_0^\infty f(t) \sin(\omega t) dt$.

Since constitutive equations containing fractional derivatives that we shall treat in this work do not obey (9), we follow the approach of [6]. Namely, in [6] it was assumed that periodic strain ε results in periodic stress σ after the end of transient regime. Positivity of dissipation work (8) for a cycle and each time instant within the cycle leads to

$$\text{Re } \hat{E}(\omega) \geq 0 \quad \text{and} \quad \text{Im } \hat{E}(\omega) \geq 0, \quad \forall \omega > 0,$$

where $\hat{E} = \frac{\hat{\sigma}}{\hat{\varepsilon}}$ is the complex modulus obtained from the constitutive equation by applying the Fourier transform (cf. [6, Eq. (20), (21)]). Note that $\text{Re } \hat{E}$ and $\text{Im } \hat{E}$ are referred as storage and loss modulus respectively.

We start with the constitutive equation

$$\sigma(t) = 2b {}_0\bar{D}_t^\beta \varepsilon(t), \quad {}_0\bar{D}_t^\beta := \frac{1}{2} \left({}_0D_t^\beta + {}_0D_t^{\bar{\beta}} \right), \quad t \geq 0, \quad (11)$$

where we assume that $b > 0$ and $\beta = A + iB$, $0 < A < 1$, $B > 0$. Note that (11) generalizes the Hooke law in the complex fractional framework. In the case $\beta \in \mathbb{R}$, this new complex fractional operator ${}_0\bar{D}_t^\beta$ coincides with the usual left Riemann-Liouville fractional derivative.

We apply the Fourier transform to (11): $\hat{\sigma}(\omega) = b((i\omega)^\beta + (i\omega)^{\bar{\beta}})\hat{\varepsilon}(\omega)$, $\omega \in \mathbb{R}$. Then, define the complex modulus \hat{E} such that $\hat{\sigma}(\omega) = \hat{E}(\omega) \cdot \hat{\varepsilon}(\omega)$, $\omega \in \mathbb{R}$, i.e.,

$$\hat{E}(\omega) := b \left((i\omega)^\beta + (i\omega)^{\bar{\beta}} \right) = b\omega^A \left(e^{-\frac{B\pi}{2}} e^{i(\frac{A\pi}{2} + B \ln \omega)} + e^{\frac{B\pi}{2}} e^{i(\frac{A\pi}{2} - B \ln \omega)} \right), \quad \omega \in \mathbb{R}.$$

Thermodynamical restrictions involve, for $\omega \in \mathbb{R}_+$,

$$\text{Re } \hat{E}(\omega) = b\omega^A \left(e^{-\frac{B\pi}{2}} \cos \left(\frac{A\pi}{2} + B \ln \omega \right) + e^{\frac{B\pi}{2}} \cos \left(\frac{A\pi}{2} - B \ln \omega \right) \right) \geq 0, \quad (12)$$

$$\text{Im } \hat{E}(\omega) = b\omega^A \left(e^{-\frac{B\pi}{2}} \sin \left(\frac{A\pi}{2} + B \ln \omega \right) + e^{\frac{B\pi}{2}} \sin \left(\frac{A\pi}{2} - B \ln \omega \right) \right) \geq 0. \quad (13)$$

But this is in contradiction with $B > 0$, since for $\omega > 0$, (12) and (13) imply $B = 0$.

In order not to confront the real valued compatibility requirement and the Second Law of Thermodynamics for (11), one may require that (12) and (13) hold for ω in some bounded interval instead of in all of \mathbb{R} . Alternatively, as we shall do in the sequel, one can modify (11) by adding additional terms, in order to preserve the Second Law of Thermodynamics.

Thus, we proceed by proposing the following constitutive equation

$$\sigma(t) = a {}_0D_t^\alpha \varepsilon(t) + 2b {}_0\bar{D}_t^\beta \varepsilon(t), \quad t \geq 0, \quad (14)$$

where we assume that $a, b > 0$, $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, and $\beta = A + iB$, $0 < A < 1$, $B > 0$. Again we follow the procedure described above for deriving thermodynamical restrictions: $\hat{\sigma}(\omega) = [a(i\omega)^\alpha + b((i\omega)^\beta + (i\omega)^{\bar{\beta}})]\hat{\varepsilon}(\omega)$, $\omega \in \mathbb{R}$. Consider the complex module ($\omega \in \mathbb{R}$)

$$\hat{E}(\omega) = a\omega^\alpha e^{i\frac{\alpha\pi}{2}} + b\omega^A \left(e^{-\frac{B\pi}{2}} e^{i(\frac{A\pi}{2} + B \ln \omega)} + e^{\frac{B\pi}{2}} e^{i(\frac{A\pi}{2} - B \ln \omega)} \right); \quad (15)$$

$$\operatorname{Re} \hat{E}(\omega) = a \omega^\alpha \cos \frac{\alpha\pi}{2} + b \omega^A \left(e^{-\frac{B\pi}{2}} \cos \left(\frac{A\pi}{2} + B \ln \omega \right) + e^{\frac{B\pi}{2}} \cos \left(\frac{A\pi}{2} - B \ln \omega \right) \right) \quad (16)$$

$$\operatorname{Im} \hat{E}(\omega) = a \omega^\alpha \sin \frac{\alpha\pi}{2} + b \omega^A \left(e^{-\frac{B\pi}{2}} \sin \left(\frac{A\pi}{2} + B \ln \omega \right) + e^{\frac{B\pi}{2}} \sin \left(\frac{A\pi}{2} - B \ln \omega \right) \right). \quad (17)$$

We will investigate conditions $\operatorname{Re} \hat{E} \geq 0$ and $\operatorname{Im} \hat{E} \geq 0$ on \mathbb{R}_+ . The assumption $\alpha > A$ leads to a contradiction, since for $\omega \searrow 0$ the sign of the second term in (16) determines the sign of $\operatorname{Re} \hat{E}$, and it can be negative. Thus, we must have $\alpha \leq A$. If $\alpha < A$, then for $\omega \rightarrow \infty$, the second term in (16) could be negative. This together yields that the only possibility is $A = \alpha$. (The same conclusion is obtained if one considers $\operatorname{Im} \hat{E} \geq 0$, $\omega > 0$.)

Therefore, with $A = \alpha$, (16) and (17) become:

$$\operatorname{Re} \hat{E}(\omega) = a \omega^\alpha \cos \frac{\alpha\pi}{2} + 2b \omega^\alpha f(\omega), \quad \omega > 0, \quad (18)$$

$$\operatorname{Im} \hat{E}(\omega) = a \omega^\alpha \sin \frac{\alpha\pi}{2} + 2b \omega^\alpha g(\omega), \quad \omega > 0, \quad (19)$$

with

$$f(\omega) := \cos \frac{\alpha\pi}{2} \cos(\ln \omega^B) \cosh \frac{B\pi}{2} + \sin \frac{\alpha\pi}{2} \sin(\ln \omega^B) \sinh \frac{B\pi}{2}, \quad \omega > 0, \quad (20)$$

$$g(\omega) := \sin \frac{\alpha\pi}{2} \cos(\ln \omega^B) \cosh \frac{B\pi}{2} - \cos \frac{\alpha\pi}{2} \sin(\ln \omega^B) \sinh \frac{B\pi}{2}, \quad \omega > 0. \quad (21)$$

We further have that $\operatorname{Re} \hat{E}(\omega) \geq a \omega^\alpha \cos \frac{\alpha\pi}{2} + 2b \omega^\alpha \min_{\omega \in \mathbb{R}_+} f(\omega)$, $\omega > 0$, and the similar estimate for $\operatorname{Im} \hat{E}$, hence we now look for the minimums of functions f and g on \mathbb{R}_+ . Using the substitution $x = \ln \omega^B$ we find that $f'(x) = 0$ and $g'(x) = 0$ at x_f and x_g so that

$$\operatorname{tg} x_f = \operatorname{tg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{B\pi}{2} \quad \text{and} \quad \operatorname{ctg} x_g = -\operatorname{ctg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{B\pi}{2}.$$

Solutions x_{f1}, x_{f2}, x_{g1} and x_{g2} satisfy: $x_{f1} \in (0, \frac{\pi}{2})$, $x_{f2} \in (\pi, \frac{3\pi}{2})$, and $x_{g1} \in (\frac{\pi}{2}, \pi)$, $x_{g2} \in (\frac{3\pi}{2}, 2\pi)$, since $\operatorname{tg} \frac{\alpha\pi}{2}, \operatorname{ctg} \frac{\alpha\pi}{2}, \operatorname{tgh} \frac{B\pi}{2} > 0$, and

$$f(x_f) = \pm \cos \frac{\alpha\pi}{2} \cosh \frac{B\pi}{2} \sqrt{1 + \left(\operatorname{tg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{B\pi}{2} \right)^2},$$

$$g(x_g) = \pm \sin \frac{\alpha\pi}{2} \cosh \frac{B\pi}{2} \sqrt{1 + \left(\operatorname{ctg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{B\pi}{2} \right)^2}.$$

Therefore, we have $\min_{x \in \mathbb{R}} f(x) = f(x_{f2})$ and $\min_{x \in \mathbb{R}} g(x) = g(x_{g1})$, so that (18) and (19) can be estimated as

$$\operatorname{Re} \hat{E}(\omega) \geq \omega^\alpha \cos \frac{\alpha\pi}{2} \left(a - 2b \cosh \frac{B\pi}{2} \sqrt{1 + \left(\operatorname{tg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{B\pi}{2} \right)^2} \right), \quad \omega > 0,$$

$$\operatorname{Im} \hat{E}(\omega) \geq \omega^\alpha \sin \frac{\alpha\pi}{2} \left(a - 2b \cosh \frac{B\pi}{2} \sqrt{1 + \left(\operatorname{ctg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{B\pi}{2} \right)^2} \right), \quad \omega > 0.$$

We obtain the thermodynamical restrictions for (14) by requiring $\operatorname{Re} \hat{E}(\omega) \geq 0$ and $\operatorname{Im} \hat{E}(\omega) \geq 0$, for $\omega \in \mathbb{R}_+$:

$$a \geq 2b \cosh \frac{B\pi}{2} \sqrt{1 + \left(\operatorname{ctg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{B\pi}{2} \right)^2}, \quad \text{if } \alpha \in \left(0, \frac{1}{2} \right], \quad (22)$$

and

$$a \geq 2b \cosh \frac{B\pi}{2} \sqrt{1 + \left(\operatorname{tg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{B\pi}{2} \right)^2}, \quad \text{if } \alpha \in \left[\frac{1}{2}, 1 \right). \quad (23)$$

Notice that both restrictions further imply $a \geq 2b$.

Remark 2.5 (i) Fix a and α . Inequalities (22) and (23) imply that as B increases, the constant b has to decrease, i.e., the contribution of complex fractional derivative of strain in the constitutive equation (14) is smaller if its imaginary part is larger.

(ii) Also, inequalities (22) and (23) lead to the same restrictions on parameters a, b, α and B , since for $\alpha \in (0, \frac{1}{2}]$ one has $1 - \alpha \in [\frac{1}{2}, 1)$, and the values of (22) and (23) coincide.

(iii) Under the same conditions, constitutive equation (14) can be extended to $\sigma(t) = \varepsilon(t) + a_0 D_t^\alpha \varepsilon(t) + 2b_0 \bar{D}_t^\beta \varepsilon(t)$, $t \geq 0$, which will be investigated in the next section.

3 Complex order fractional Kelvin-Voigt model

Consider the constitutive equation involving the complex order fractional derivative

$$\sigma(t) = \varepsilon(t) + a_0 D_t^\alpha \varepsilon(t) + 2b_0 \bar{D}_t^\beta \varepsilon(t), \quad t \geq 0. \quad (24)$$

We assume $a, b, E > 0$, $0 < \alpha < 1$, $B > 0$, $\beta = \alpha + iB$, and σ and ε are real-valued functions. The Laplace transform of (24) is $\tilde{\sigma}(s) = E(1 + a s^\alpha + b(s^\beta + s^{\bar{\beta}}))\tilde{\varepsilon}(s)$, $\operatorname{Re} s > 0$, and hence

$$\tilde{\varepsilon}(s) = \frac{1}{1 + a s^\alpha + b(s^\beta + s^{\bar{\beta}})} \tilde{\sigma}(s), \quad \operatorname{Re} s > 0. \quad (25)$$

In order to determine ε from (25) we need to analyze zeros of

$$\psi(s) = 1 + a s^\alpha + b(s^\beta + s^{\bar{\beta}}), \quad s \in \mathbb{C}. \quad (26)$$

Note that if we put $s = i\omega$, $\omega \in \mathbb{R}_+$, in (26), it becomes the complex modulus:

$$\psi(i\omega) = 1 + \hat{E}(\omega) = 1 + a(i\omega)^\alpha + b((i\omega)^\beta + (i\omega)^{\bar{\beta}}), \quad \omega > 0, \quad (27)$$

where \hat{E} is given in (15).

Let $s = r e^{i\varphi}$, $r > 0$, $\varphi \in [0, 2\pi]$. Then (with $\beta = \alpha + iB$)

$$\psi(s) = 1 + ar^\alpha e^{i\alpha\varphi} + br^\alpha (e^{-B\varphi} e^{i(\ln r^B + \alpha\varphi)} + e^{B\varphi} e^{-i(\ln r^B - \alpha\varphi)}),$$

and

$$\operatorname{Re} \psi(s) = 1 + ar^\alpha \cos(\alpha\varphi) + 2br^\alpha (\cos(\ln r^B) \cos(\alpha\varphi) \cosh(B\varphi) + \sin(\ln r^B) \sin(\alpha\varphi) \sinh(B\varphi)), \quad (28)$$

$$\operatorname{Im} \psi(s) = ar^\alpha \sin(\alpha\varphi) + 2br^\alpha (\cos(\ln r^B) \sin(\alpha\varphi) \cosh(B\varphi) - \sin(\ln r^B) \cos(\alpha\varphi) \sinh(B\varphi)). \quad (29)$$

3.1 Thermodynamical restrictions

In the case of (27), using (28) and (29) we obtain:

$$\begin{aligned}\operatorname{Re} \psi(i\omega) &= 1 + \operatorname{Re} \hat{E}(\omega) \geq 1 + a\omega^\alpha \cos \frac{\alpha\pi}{2} + 2b\omega^\alpha \min_{x \in \mathbb{R}} f(x), & x = \ln \omega^B, \omega > 0, \\ \operatorname{Im} \psi(i\omega) &= \operatorname{Im} \hat{E}(\omega) \geq a\omega^\alpha \sin \frac{\alpha\pi}{2} + 2b\omega^\alpha \min_{x \in \mathbb{R}} g(x), & x = \ln \omega^B, \omega > 0,\end{aligned}$$

where f and g are as in (20) and (21). This leads to the same thermodynamical restrictions (22) and (23), as in Section 2.2.

Therefore, from now on, we shall assume (22) and (23) to hold true. Now we shall examine the zeros of ψ .

3.2 Zeros of ψ and solutions of (24)

Theorem 3.1 *Let ψ be the function given by (26). Then*

(i) ψ has no zeros in the right complex half-plane $\operatorname{Re} s \geq 0$.

(ii) ψ has no zeros in \mathbb{C} if the coefficients a, b, α and B satisfy

$$\begin{aligned}a &\geq 2b \cosh(B\pi) \sqrt{1 + (\operatorname{tg}(\alpha\pi) \operatorname{tgh}(B\pi))^2}, & \text{for } \alpha \in \left[\frac{1}{4}, \frac{3}{4}\right] \setminus \left\{\frac{1}{2}\right\}, \\ a &\geq 2b \cosh(B\pi) \sqrt{1 + (\operatorname{ctg}(\alpha\pi) \operatorname{tgh}(B\pi))^2}. & \text{for } \alpha \in \left(0, \frac{1}{4}\right) \cup \left\{\frac{1}{2}\right\} \cup \left(\frac{3}{4}, 1\right).\end{aligned}\tag{30}$$

Proof. First, we notice that if s_0 is a solution to $\psi(s) = 0$, then \bar{s}_0 (the complex conjugate of s_0) is also a solution, since $\psi(\bar{s}) = 1 + a\bar{s}^\alpha + b(\bar{s}^\beta + \bar{s}^{\bar{\beta}}) = \overline{\psi(s)}$. Thus, we restrict our attention to the upper complex half-plane $\operatorname{Im} s \geq 0$, i.e., $\varphi \in [0, \pi]$.

Using (28) and (29) we have, with $s = re^{i\varphi}$, $r > 0$, $\varphi \in [0, \pi]$, and $x = \ln r^B$,

$$\operatorname{Re} \psi(s) \geq 1 + ar^\alpha \cos(\alpha\varphi) + 2br^\alpha \min_{x \in \mathbb{R}} f(x),\tag{31}$$

$$\operatorname{Im} \psi(s) \geq ar^\alpha \sin(\alpha\varphi) + 2br^\alpha \min_{x \in \mathbb{R}} g(x),\tag{32}$$

where

$$f(x) = \cos(x) \cos(\alpha\varphi) \cosh(B\varphi) + \sin(x) \sin(\alpha\varphi) \sinh(B\varphi), \quad x \in \mathbb{R},\tag{33}$$

$$g(x) = \cos(x) \sin(\alpha\varphi) \cosh(B\varphi) - \sin(x) \cos(\alpha\varphi) \sinh(B\varphi), \quad x \in \mathbb{R}.\tag{34}$$

The critical points x_f and x_g of f and g , respectively, satisfy

$$\operatorname{tg} x_f = \operatorname{tg}(\alpha\varphi) \operatorname{tgh}(B\varphi) \geq 0 \quad \text{and} \quad \operatorname{tg} x_g = -\operatorname{ctg}(\alpha\varphi) \operatorname{tgh}(B\varphi) \leq 0,\tag{35}$$

The proof of (i) and (ii) will be given by the argument principle.

(i) Consider ψ in the case $\operatorname{Re} s, \operatorname{Im} s > 0$. Choose a contour $\Gamma = \gamma_{R1} \cup \gamma_{R2} \cup \gamma_{R3} \cup \gamma_{R4}$, as it is shown in Fig. 1.

γ_{R1} is parametrized by $s = x$, $x \in (\varepsilon, R)$ with $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$, so that (28) and (29) yield

$$\begin{aligned}\operatorname{Re} \psi(x) &= 1 + x^\alpha (a + 2b \cos(\ln x^\beta)) \geq 1 + x^\alpha (a - 2b) \geq 0, \\ \operatorname{Im} \psi(x) &= 0,\end{aligned}$$

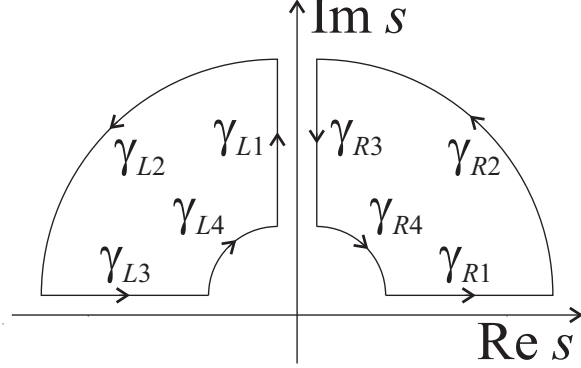


Figure 1: Contour Γ .

since both (22) and (23) imply $a \geq 2b$. Moreover, we have $\lim_{x \rightarrow 0} \psi(x) = 1$ and $\lim_{x \rightarrow \infty} \psi(x) = \infty$.

Along γ_{R2} one has $s = Re^{i\varphi}$, $\varphi \in [0, \frac{\pi}{2}]$, with $R \rightarrow \infty$. By (35) we have $\text{tg } x_f \geq 0$, so that

$$\min_{x \in \mathbb{R}} f(x) = -\cos(\alpha\varphi) \cosh(B\varphi) \sqrt{1 + (\text{tg}(\alpha\varphi) \text{tgh}(B\varphi))^2},$$

and therefore (31) becomes

$$\text{Re } \psi(s) \geq 1 + R^\alpha \cos(\alpha\varphi) (a - 2b \cosh(B\varphi) \sqrt{1 + (\text{tg}(\alpha\varphi) \text{tgh}(B\varphi))^2}) \geq 0.$$

The previous inequality holds true, since for $\varphi \in [0, \frac{\pi}{2}]$ we have that

$$p(\varphi) = \cosh(B\varphi) \sqrt{1 + (\text{tg}(\alpha\varphi) \text{tgh}(B\varphi))^2} \leq \cosh \frac{B\pi}{2} \sqrt{1 + \left(\text{tg} \frac{\alpha\pi}{2} \text{tgh} \frac{B\pi}{2} \right)^2},$$

because of the fact that the function p monotonically increases on $[0, \frac{\pi}{2}]$. Moreover, by (28) and (29), we have

$$\begin{aligned} \text{Re } \psi(s) &\rightarrow \infty \quad \text{and} \quad \text{Im } \psi(s) = 0, \quad \text{for } \varphi = 0, \quad R \rightarrow \infty, \\ \text{Re } \psi(s) &\rightarrow \infty, \quad \text{for } \varphi = \frac{\pi}{2}, \quad R \rightarrow \infty. \end{aligned}$$

The next segment is γ_{R3} , which is parametrized by $s = i\omega$, $\omega \in [R, \varepsilon]$, with $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$. Then (28) and (29) yield

$$\text{Re } \psi(i\omega) = 1 + \text{Re } \hat{E}(\omega) \geq 0 \quad \text{and} \quad \text{Im } \psi(i\omega) = \text{Im } \hat{E}(\omega) \geq 0, \quad \omega \in (\varepsilon, R),$$

due to the thermodynamical requirements. Moreover, by (28) and (29), we have

$$\begin{aligned} \text{Re } \psi(\omega) &\rightarrow 1 \quad \text{and} \quad \text{Im } \psi(\omega) \rightarrow 0, \quad \text{as } \omega \rightarrow 0, \\ \text{Re } \psi(\omega) &\rightarrow \infty \quad \text{and} \quad \text{Im } \psi(\omega) \rightarrow \infty, \quad \text{as } \omega \rightarrow \infty. \end{aligned}$$

The last part of the contour Γ is the arc γ_{R4} , with $s = \varepsilon e^{i\varphi}$, $\varphi \in [0, \frac{\pi}{2}]$, with $\varepsilon \rightarrow 0$. Using the same arguments as for the contour γ_{R2} , we have

$$\text{Re } \psi(s) \geq 1 + \varepsilon^\alpha \cos(\alpha\varphi) (a - 2b \cosh(B\varphi) \sqrt{1 + (\text{tg}(\alpha\varphi) \text{tgh}(B\varphi))^2}) \geq 1. \quad (36)$$

Also, by (29) and (36), we have

$$\operatorname{Re} \psi(s) \rightarrow 1 \quad \text{and} \quad \operatorname{Im} \psi(s) \rightarrow 0, \quad \text{as} \quad \varepsilon \rightarrow 0.$$

We conclude that $\Delta \arg \psi(s) = 0$ so that, by the argument principle, there are no zeroes of ψ in the right complex half-plane $\operatorname{Re} s \geq 0$.

(ii) In order to discuss the zeros of ψ in the left complex half-plane, we use the contour $\Gamma_L = \gamma_{L1} \cup \gamma_{L2} \cup \gamma_{L3} \cup \gamma_{L4}$, shown in Fig. 1. The contour γ_{L1} has the same parametrization as the contour γ_{R3} , so the same conclusions as for γ_{R3} hold true.

The parametrization of the contour γ_{L2} is $s = Re^{i\varphi}$, $\varphi \in [\frac{\pi}{2}, \pi]$, with $R \rightarrow \infty$. Let us distinguish two cases.

a1. $\alpha\varphi \in (0, \frac{\pi}{2})$

Then $\sin(\alpha\varphi) > 0$, $\cos(\alpha\varphi) > 0$, so the critical points of f and g , see (33) and (34), given by (35), satisfy $\operatorname{tg} x_f > 0$ and $\operatorname{tg} x_g < 0$. For the minimums of f and g in (31) and (32) we have

$$\begin{aligned} \min_{x \in \mathbb{R}} f(x) &= -\cos(\alpha\varphi) \cosh(B\varphi) \sqrt{1 + (\operatorname{tg}(\alpha\varphi) \operatorname{tgh}(B\varphi))^2}, \\ \min_{x \in \mathbb{R}} g(x) &= -\sin(\alpha\varphi) \cosh(B\varphi) \sqrt{1 + (\operatorname{ctg}(\alpha\varphi) \operatorname{tgh}(B\varphi))^2}, \end{aligned}$$

respectively, so that (31) and (32) become

$$\operatorname{Re} \psi(s) \geq 1 + R^\alpha \cos(\alpha\varphi) \left(a - 2b \cosh(B\varphi) \sqrt{1 + (\operatorname{tg}(\alpha\varphi) \operatorname{tgh}(B\varphi))^2} \right), \quad (37)$$

$$\operatorname{Im} \psi(s) \geq R^\alpha \sin(\alpha\varphi) \left(a - 2b \cosh(B\varphi) \sqrt{1 + (\operatorname{ctg}(\alpha\varphi) \operatorname{tgh}(B\varphi))^2} \right). \quad (38)$$

Function $H_f(\varphi) = \cosh(B\varphi) \sqrt{1 + (\operatorname{tg}(\alpha\varphi) \operatorname{tgh}(B\varphi))^2}$ is monotonically increasing for $\varphi \in [\frac{\pi}{2}, \pi]$, since $\alpha\varphi \in (0, \frac{\pi}{2})$, thus

$$\operatorname{Re} \psi(s) \geq 1 + R^\alpha \cos(\alpha\varphi) \left(a - 2b \cosh(B\pi) \sqrt{1 + (\operatorname{tg}(\alpha\pi) \operatorname{tgh}(B\pi))^2} \right) \geq 0,$$

if (30) is satisfied. Note that $\operatorname{Re} \psi(s) \rightarrow \infty$ and $\operatorname{Im} \psi(s) \rightarrow \infty$, for $\varphi = \pi$, $R \rightarrow \infty$.

b1. $\alpha\varphi \in [\frac{\pi}{2}, \pi)$

Then $\sin(\alpha\varphi) > 0$, $\cos(\alpha\varphi) \leq 0$, so the critical points of g , see (34), given by (35), satisfy $\operatorname{tg} x_g \geq 0$. For the minimum of g in (34) we have

$$\min_{x \in \mathbb{R}} g(x) = -\sin(\alpha\varphi) \cosh(B\varphi) \sqrt{1 + (\operatorname{ctg}(\alpha\varphi) \operatorname{tgh}(B\varphi))^2},$$

so that (32) becomes

$$\operatorname{Im} \psi(s) \geq R^\alpha \sin(\alpha\varphi) \left(a - 2b \cosh(B\varphi) \sqrt{1 + (\operatorname{ctg}(\alpha\varphi) \operatorname{tgh}(B\varphi))^2} \right).$$

Function $H_g(\varphi) = \cosh(B\varphi) \sqrt{1 + (\operatorname{ctg}(\alpha\varphi) \operatorname{tgh}(B\varphi))^2}$ is monotonically increasing for $\varphi \in [\frac{\pi}{2}, \pi]$, since $\alpha\varphi \in [\frac{\pi}{2}, \pi)$, and (30) implies

$$\operatorname{Im} \psi(s) \geq R^\alpha \sin(\alpha\varphi) \left(a - 2b \cosh(B\pi) \sqrt{1 + (\operatorname{ctg}(\alpha\pi) \operatorname{tgh}(B\pi))^2} \right) \geq 0.$$

Note that $\operatorname{Im} \psi(s) \rightarrow \infty$, for $\varphi = \pi$, $R \rightarrow \infty$.

Now we discuss possible situations for $\alpha \in (0, 1)$ and $\varphi \in [\frac{\pi}{2}, \pi]$.

If $\alpha \in (0, \frac{1}{2})$ then a1. holds so that $\operatorname{Re} \psi(s) \geq 0$.

If $\alpha \in [\frac{1}{2}, 1)$ then we distinguish two cases. For $\varphi \in [\frac{\pi}{2}, \frac{\pi}{2\alpha})$ case a1. holds, so $\operatorname{Re} \psi(s) \geq 0$. For $\varphi \in [\frac{\pi}{2\alpha}, \pi)$ case b1. holds, and $\operatorname{Im} \psi(s) \geq 0$.

Parametrization of the contour γ_{L3} is $s = xe^{i\pi}$, $x \in (\varepsilon, R)$, with $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$. Again, we have two cases.

a2. $\alpha \in (0, \frac{1}{2})$

Then, for $x \in (\varepsilon, R)$, using the same argumentation as in case a1, we have, by (37) and (38) (with $R = x$), $\operatorname{Re} \psi(s) \geq 1$ and $\operatorname{Im} \psi(s) \geq 0$, due to (30). Thus, looking at (37) and (38) (with $R = x$), we conclude

$$\begin{aligned} \operatorname{Re} \psi(s) &\rightarrow \infty & \text{and} & & \operatorname{Im} \psi(s) &\rightarrow \infty, & \text{for } & x \rightarrow \infty, \\ \operatorname{Re} \psi(s) &\rightarrow 1 & \text{and} & & \operatorname{Im} \psi(s) &\rightarrow 0, & \text{for } & x \rightarrow 0. \end{aligned}$$

b2. $\alpha \in [\frac{1}{2}, 1)$

Then, for $x \in (\varepsilon, R)$, using the same argumentation as in case b1, we have (by (38)) that $\operatorname{Im} \psi(s) \geq 0$, and

$$\operatorname{Im} \psi(s) \rightarrow \infty, \quad \text{for } x \rightarrow \infty \quad \text{and} \quad \operatorname{Im} \psi(s) \rightarrow 0, \quad \text{for } x \rightarrow 0.$$

The parametrization of the contour γ_{L4} is $s = \varepsilon e^{i\varphi}$, $\varphi \in [\frac{\pi}{2}, \pi]$, with $\varepsilon \rightarrow 0$. From (28) and (29), for sufficiently small ε , we have

$$\operatorname{Re} \psi(s) \rightarrow 1 \quad \text{and} \quad \operatorname{Im} \psi(s) \rightarrow 0, \quad \text{for } \varepsilon \rightarrow 0, \varphi \in \left[\frac{\pi}{2}, \pi\right].$$

Summing up all results from cases a1, a2, b1 and b2, we obtain the following:

- For $\alpha \in (0, \frac{1}{2})$, $\operatorname{Re} \psi(s) \geq 0$, for $s \in \Gamma_L$, which implies that $\Delta \arg \psi(s) = 0$. Therefore, using the argument principle, we conclude that in this case ψ has no zeros in the left complex half-plane.
- If $\alpha \in [\frac{1}{2}, 1)$, then for $s \in \gamma_{L1}$ and $s \in \{z \in \gamma_{L2} \mid \arg z \leq \frac{\pi}{2\alpha}\}$, we have $\operatorname{Re} \psi(s) \geq 0$, while for $s \in \{z \in \gamma_{L2} \mid \arg z > \frac{\pi}{2\alpha}\}$ and $s \in \gamma_{L3}$, we have $\operatorname{Im} \psi(s) \geq 0$. For $s \in \gamma_{L4}$ we again have $\operatorname{Re} \psi(s) \geq 0$. Hence, we conclude that $\Delta \arg \psi(s) = 0$, and therefore, using the argument principle, neither in case $\alpha \in [\frac{1}{2}, 1)$ function ψ has zeros in the left complex half-plane.

This completes the proof. □

Rewrite (25) as

$$\tilde{\varepsilon}(s) = \tilde{K}(s)\tilde{\sigma}(s), \quad \tilde{K}(s) = \frac{1}{1 + as^\alpha + b(s^\beta + s^\beta)}, \quad \operatorname{Re} s > 0. \quad (39)$$

Theorem 3.2 *Let $\tilde{\varepsilon}$ be given by (39). Then*

$$\varepsilon(t) = K(t) * \sigma(t), \quad t \geq 0. \quad (40)$$

Moreover, if (30) holds, then

$$K(t) = K_I(t) = \frac{1}{2\pi i} \int_0^\infty \left(\frac{e^{-qt}}{1 + q^\alpha e^{i\alpha\pi} [a + b(e^{i \ln q^B} e^{-B\pi} + e^{-i \ln q^B} e^{B\pi})]} - \frac{e^{-qt}}{1 + q^\alpha e^{-i\alpha\pi} [a + b(e^{i \ln q^B} e^{B\pi} + e^{-i \ln q^B} e^{-B\pi})]} \right) dq. \quad (41)$$

If condition (30) is violated, then ψ has at most finite number of zeros in the left complex half-plane, and

$$K = K_I \quad \text{or} \quad K = K_I + K_R,$$

where

$$K_R(t) = \sum_{\substack{\psi(s_i)=0 \\ i=1,2,\dots,n}} \left(\text{Res}(\tilde{K}(s)e^{st}, s_i) + \text{Res}(\tilde{K}(s)e^{st}, \bar{s}_i) \right), \quad (42)$$

with \tilde{K} given by (39).

Proof. The first part is clear. We invert now \tilde{K} , given in (39), by the use of the Cauchy residues theorem

$$\oint_{\tilde{\Gamma}} \tilde{K}(s)e^{st} ds = 2\pi i \sum_{\psi(\tilde{s})=0} \text{Res}(\tilde{K}(s)e^{st}, \tilde{s}) \quad (43)$$

and the contour $\tilde{\Gamma} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_r \cup \Gamma_3 \cup \Gamma_4 \cup \gamma_0$ shown in Fig. 2.

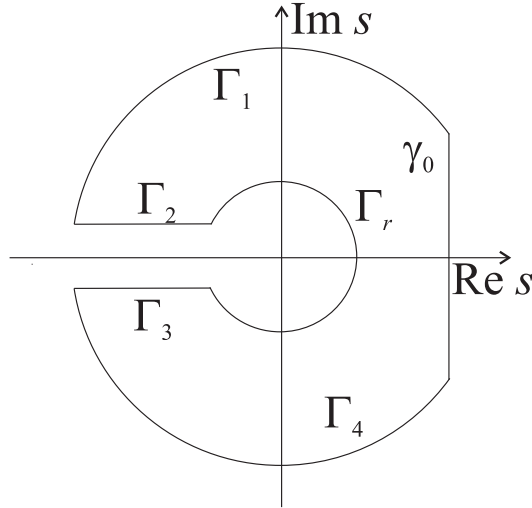


Figure 2: Contour $\tilde{\Gamma}$.

If condition (30) is satisfied, then, by Theorem 3.1, the residues equal zero. One can show that the integrals over the contours Γ_1 , Γ_r and Γ_4 tend to zero when $R \rightarrow \infty$ and $r \rightarrow 0$. The

remaining integrals give:

$$\begin{aligned} \lim_{\substack{R \rightarrow \infty \\ r \rightarrow 0}} \int_{\Gamma_2} \tilde{K}(s) e^{st} ds &= - \int_0^\infty \frac{e^{-qt}}{1 + q^\alpha e^{i\alpha\pi} [a + b(e^{i \ln q^B} e^{-B\pi} + e^{-i \ln q^B} e^{B\pi})]} dq, \\ \lim_{\substack{R \rightarrow \infty \\ r \rightarrow 0}} \int_{\Gamma_3} \tilde{K}(s) e^{st} ds &= \int_0^\infty \frac{e^{-qt}}{1 + q^\alpha e^{-i\alpha\pi} [a + b(e^{i \ln q^B} e^{B\pi} + e^{-i \ln q^B} e^{-B\pi})]} dq, \\ \lim_{\substack{R \rightarrow \infty \\ r \rightarrow 0}} \int_{\gamma_0} \tilde{K}(s) e^{st} ds &= 2\pi i K_I(t), \end{aligned}$$

which, by the Cauchy residues theorem (43), leads to (41).

If condition (30) is violated, then, by Theorem 3.1, the denominator of \tilde{K} either has no zeros in the complex plane, and so $K = K_I$, or it has zeros in the left complex half-plane, which comes in pairs with complex conjugates. We show now that ψ (which is the dominator of \tilde{K}) can have at most finite number of zeros for $\text{Im } s \leq 0$. Rewrite $\psi(s) = 0$ as $a + b(s^{iB} + s^{-iB}) = s^{-\alpha}$. If $s = r e^{i\varphi}$, $\varphi \in [\frac{\pi}{2}, \pi]$, then we have

$$a + b(r^{iB} e^{-\varphi B} + r^{-iB} e^{\varphi B}) = r^{-\alpha} e^{-\alpha\varphi i}.$$

When $r \rightarrow \infty$ the right hand side tends to zero, while the left hand side tends to a . As $r \rightarrow 0$ we see that the left hand side is bounded, while the right hand side is not bounded. Thus, the zeros in the left half-plane of function ψ , if exist, have to be bounded both from above and below. In that case we have $K = K_I + K_R$, where K_R is given by (42). \square

4 Numerical verifications

Here we present several examples of the proposed constitutive equation. We shall treat stress relaxation, creep and periodic loading cases.

4.1 Stress relaxation experiment

We take (24) with $\varepsilon(t) = H(t)$, H is the Heaviside function, and regularize it as $H_\varepsilon(t) = 1 - \exp(-t/k)$, $k \rightarrow 0$. In order to determine σ we calculate

$$\sigma(t) = H_\varepsilon(t) + a_0 D_t^\alpha H_\varepsilon(t) + b_0 \bar{D}_t^\beta H_\varepsilon(t), \quad t \geq 0, \quad (44)$$

with $\sigma(0) = 0$, using the expansion formula (see [3, 4]),

$${}_0 D_t^\gamma y(t) \approx \frac{y(t)}{t^\gamma} A(N, \gamma) - \sum_{p=1}^N C_{p-1}(\gamma) \frac{V_{p-1}(y)(t)}{t^{p+\gamma}}, \quad (45)$$

where

$$A(N, \gamma) = \frac{\Gamma(N+1+\gamma)}{\gamma \Gamma(1-\gamma) \Gamma(\gamma) N!}, \quad C_{p-1}(\gamma) = \frac{\Gamma(p+\gamma)}{\Gamma(1-\gamma) \Gamma(\gamma) (p-1)!},$$

and

$$V_{p-1}^{(1)}(y)(t) = t^{p-1} y(t), \quad V_{p-1}(y)(0) = 0, \quad p = 1, 2, 3, \dots \quad (46)$$

Inserting (45) into (44) we obtain

$$\begin{aligned} \sigma(t) \approx & \left\{ 1 + \left[\frac{a A(N, \alpha)}{t^\alpha} + b \left(\frac{A(N, \alpha + iB)}{t^{\alpha+iB}} + \frac{A(N, \alpha - iB)}{t^{\alpha-iB}} \right) \right] \right\} H_\varepsilon(t) \\ & - \sum_{p=1}^N \left\{ \frac{C_{p-1}(\alpha)}{t^{p+\alpha}} + \left[\frac{C_{p-1}(\alpha + iB)}{t^{p+\alpha+iB}} + \frac{C_{p-1}(\alpha - iB)}{t^{p+\alpha-iB}} \right] \right\} V_{p-1}(H_\varepsilon)(t), \end{aligned} \quad (47)$$

where

$$V_{p-1}^{(1)}(H_\varepsilon)(t) = t^{p-1} H_\varepsilon(t), \quad p = 1, 2, 3, \dots \quad (48)$$

We will compare (47) with the stress σ obtained by (44) and by the definition of fractional derivative (11):

$$\begin{aligned} \sigma(t) = & H_\varepsilon(t) + a \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{H_\varepsilon(\tau) d\tau}{(t-\tau)^\alpha} \\ & + b \frac{d}{dt} \left[\frac{1}{\Gamma(1-\alpha-iB)} \int_0^t \frac{H_\varepsilon(\tau) d\tau}{(t-\tau)^{\alpha+iB}} + \frac{1}{\Gamma(1-\alpha+iB)} \int_0^t \frac{H_\varepsilon(\tau) d\tau}{(t-\tau)^{\alpha-iB}} \right]. \end{aligned} \quad (49)$$

In Fig. 3 we show results obtained by determining σ from (49) for small times and different values of B . In the same figure we show, by dots, the values of σ , in several points, obtained by using (47), (48) for $k = 0.01$, $N = 100$. As could be seen from Fig. 3, the results obtained from (49) and (47), (48) agree well. The stress relaxation curves are shown in Fig. 4, for the same set of parameters and for larger times. As could be seen, regardless of the values of B , we have $\lim_{t \rightarrow \infty} \sigma(t) = 1$. Note that in all cases of B , the restriction which follows from the dissipation inequality is satisfied.

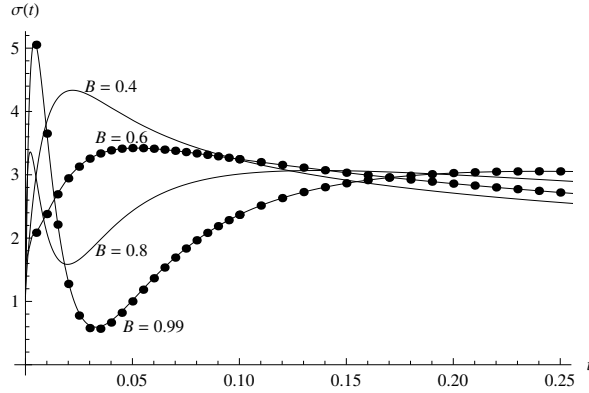


Figure 3: Stress relaxation curves for $\alpha = 0.4$ and $B \in \{0.4, 0.6, 0.8, 0.99\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 0.25]$.

4.2 Creep experiment

Suppose that $\sigma(t) = H(t)$, i.e.,

$$H(t) = (1 + a_0 D_t^\alpha + 2b_0 \bar{D}_t^\beta) \varepsilon(t), \quad t \geq 0. \quad (50)$$

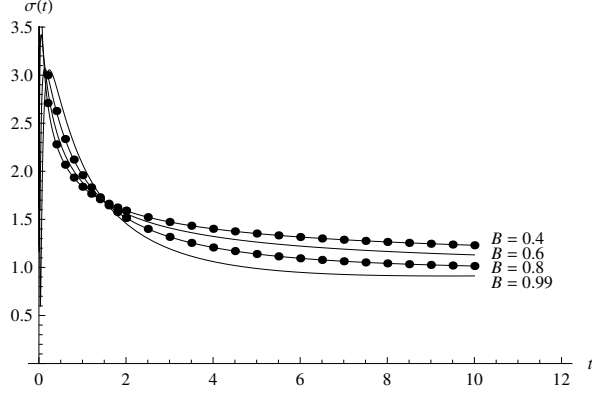


Figure 4: Stress relaxation curves for $\alpha = 0.4$ and $B \in \{0.4, 0.6, 0.8, 0.99\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 10]$.

By (45), we obtain

$$\varepsilon(t) \approx \frac{H(t) + \sum_{p=1}^N \left\{ \frac{C_{p-1}(\alpha)}{t^{p+\alpha}} + \frac{C_{p-1}(\alpha+iB)}{t^{p+\alpha+iB}} + \frac{C_{p-1}(\alpha-iB)}{t^{p+\alpha-iB}} \right\} V_{p-1}(\varepsilon)(t)}{1 + \frac{aA(N,\alpha)}{t^\alpha} + 2b \left(\frac{A(N,\alpha+iB)}{t^{\alpha+iB}} + \frac{A(N,\alpha-iB)}{t^{\alpha-iB}} \right)},$$

or

$$\varepsilon(t) \approx \frac{H(t)t^\alpha + \sum_{p=1}^N \left\{ \frac{C_{p-1}(\alpha)}{t^p} + \frac{C_{p-1}(\alpha+iB)}{t^{p+iB}} + \frac{C_{p-1}(\alpha-iB)}{t^{p-iB}} \right\} V_{p-1}(\varepsilon)(t)}{t^\alpha + aA(N,\alpha) + 2b \left(\frac{A(N,\alpha+iB)}{t^{iB}} + \frac{A(N,\alpha-iB)}{t^{-iB}} \right)}. \quad (51)$$

By using (51) in (46) we obtain

$$V_{p-1}^{(1)}(\varepsilon)(t) \approx t^{p-1} \frac{H(t)t^\alpha + \sum_{p=1}^N \left\{ \frac{C_{p-1}(\alpha)}{t^p} + \frac{C_{p-1}(\alpha+iB)}{t^{p+iB}} + \frac{C_{p-1}(\alpha-iB)}{t^{p-iB}} \right\} V_{p-1}(\varepsilon)(t)}{t^\alpha + aA(N,\alpha) + 2b \left(\frac{A(N,\alpha+iB)}{t^{iB}} + \frac{A(N,\alpha-iB)}{t^{-iB}} \right)},$$

$$V_{p-1}(\varepsilon)(0) = 0, \quad p = 1, 2, 3, \dots$$

Equation (50) may also be solved by contour integration

$$\varepsilon(t) = K(t) * H(t), \quad t \geq 0, \quad (52)$$

where K is given by (41), see (40). Finally, the values of ε , at discrete points, could be determined directly from $\tilde{\varepsilon}(s) = \frac{1}{s} \tilde{K}(s)$, $\text{Re } s > 0$, see (39), by the use of Post inversion formula, see [9]. Thus,

$$\varepsilon(t) = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} \left[s^{n+1} \frac{d^n}{ds^n} \frac{1}{s(1 + as^\alpha + b(s^\beta + s^{\bar{\beta}}))} \right]_{s=\frac{n}{t}}, \quad t \geq 0. \quad (53)$$

In Fig. 5, 6 and 7 we show ε for several values of parameters determined from (52). In Fig. 5, for specified values of t we present values of ε , determined from (51), with $N = 7$, denoted by dots, as well as the values of ε , determined by (53), with $n = 25$, denoted by squares. As could be seen the agreement of results determined by different methods is significant.

From Fig. 6 and 7 one sees that, regardless of the value of B , creep curves tend to $\varepsilon = 1$. In Fig. 6 the creep curves are monotonically increasing, while in Fig. 7 creep curves has oscillatory character, characteristic for the case when the mass of the rod is not neglected. Note that in all cases of B , the restriction determined by the dissipation inequality is satisfied.

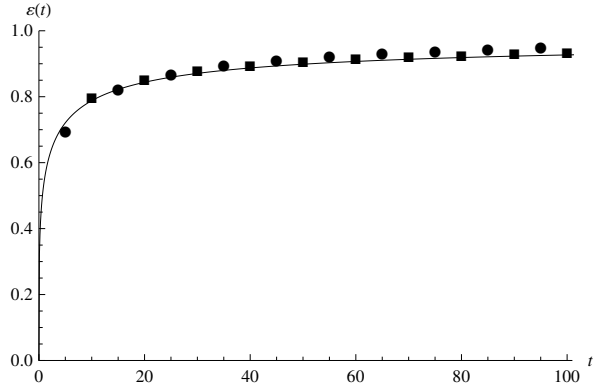


Figure 5: Creep curve for $\alpha = 0.4$ and $B = 0.4$, $a = 0.8$, $b = 0.1$, $t \in [0, 100]$.

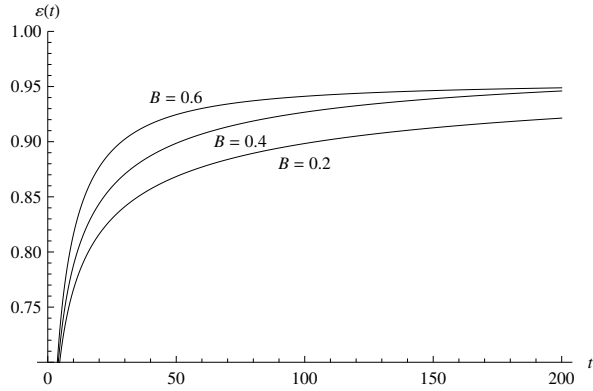


Figure 6: Creep curves for $\alpha = 0.4$ and $B \in \{0.2, 0.4, 0.6\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 200]$.

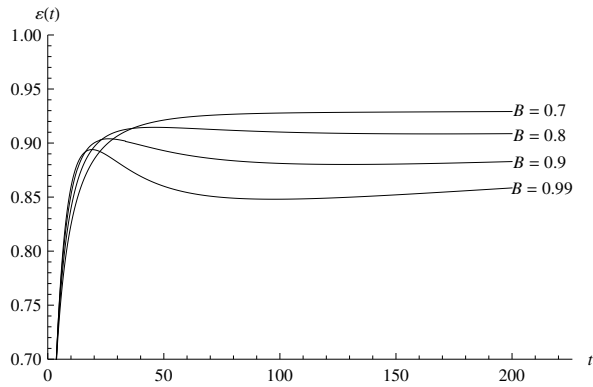


Figure 7: Creep curves for $\alpha = 0.4$ and $B \in \{0.7, 0.8, 0.9, 0.99\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 200]$.

5 Conclusion

In this work, we proposed a new constitutive equation with fractional derivatives of complex order for viscoelastic body of the Kelvin-Voigt type. The use of fractional derivatives of complex order, together with restrictions following from the Second Law of Thermodynamics, represent the main novelty of our work. Our results can be summarized as follows.

1. In order to obtain real stress for given real strain, we used two fractional derivatives of complex order that are complex conjugated numbers, see Theorem 2.3.
2. The restrictions that follow from the Second Law of Thermodynamics for isothermal process implied that the constitutive equation must additionally contain a fractional derivative of real order. Thus, the simplest constitutive equation that gives real stress for real strain and satisfies the dissipativity condition is given by (24).
3. We provided a complete analysis of solvability of the complex order fractional Kelvin-Voigt model given by (24) (see Theorems 3.1 and 3.2).
4. We studied stress relaxation and creep problems through equation (24). An increase of B implied that the stress relaxation decreases more rapidly to the limiting value of stress, i.e., $\lim_{t \rightarrow \infty} \sigma(t) = 1$.
5. We presented numerical experiments when the dissipation inequality is satisfied. The creep experiment showed that the increase in the imaginary part of the complex derivative B changes the character of creep curve from monotonic to oscillatory form, see Fig. 6 and 7. However, the creep curves never cross the value equal to 1. The creep curve resembles the form of creep curve when either the mass of the rod, see [5, p. 124], or inertia of the rheometer, see [16], is taken into account.
6. Our further study will be directed to the problems of vibration and wave propagation of a rod with finite mass and constitutive equation (24), along the lines presented in [5], see also [13].

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