Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function

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Abstract

The Tracy-Widom distribution functions involve integrals of a Painlevé II function starting from positive infinity. In this paper, we express the Tracy-Widom distribution functions in terms of integrals starting from minus infinity. There are two consequences of these new representations. The first is the evaluation of the total integral of the Hastings-McLeod solution of the Painlevé II equation. The second is the evaluation of the constant term of the asymptotic expansions of the Tracy-Widom distribution functions as the distribution parameter approaches minus infinity. For the GUE Tracy-Widom distribution function, this gives an alternative proof of the recent work of Deift, Its, and Krasovsky. The constant terms for the GOE and GSE Tracy-Widom distribution functions are new.

1 Introduction

Let $F_1(x), F_2(x)$, and $F_4(x)$ denote the GOE, GUE, and GSE Tracy-Widom distribution functions, respectively. They are defined as [24, 25]

$$F_1(x) = F(x)E(x), \qquad F_2(x) = F(x)^2, \qquad F_4(x) = \frac{1}{2}\left\{E(x) + \frac{1}{E(x)}\right\}F(x), \tag{1}$$

where

$$F(x) = \exp\left(-\frac{1}{2}\int_x^\infty R(s)ds\right), \qquad E(x) = \exp\left(-\frac{1}{2}\int_x^\infty q(s)ds\right). \tag{2}$$

Here the (real) function $q(x)$ is the solution to the Painlevé II equation

$$q'' = 2q^3 + xq, \tag{3}$$

that satisfies the boundary condition

$$q(x) \sim \mathrm{Ai}(x), \qquad x \to +\infty. \tag{4}$$

Recall [1] that the Airy function $\mathrm{Ai}(x)$ satisfies $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}}, \qquad x \to +\infty. \tag{5}$$

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There is a unique global solution $q(x)$ to the equation (3) with the condition (4) (the Hastings-McLeod solution) [19]. The function $R(s)$ is defined as

$$R(x) = \int_x^\infty (q(s))^2 ds. \tag{6}$$

By taking derivatives and using (3) and (4) (see, for example, (1.15) of [24] and (2.6) of [3]), the function $R(x)$ can also be written as

$$R(x) = (q'(x))^2 - x(q(x))^2 - (q(x))^4. \tag{7}$$

Integrating by parts, $F(x)$ can be written as

$$F(x) = \exp\left(-\frac{1}{2}\int_x^\infty (s-x)(q(s))^2 ds \right), \tag{8}$$

which is commonly used in the literature.

Notice that (2) involves integrals from x to positive infinity. The main results of this paper are the following representations of $F(x)$ and $E(x)$, which involve integrals from minus infinity to x.

Theorem 1.1. *For $x < 0$,*

$$F(x) = 2^{1/48} e^{\frac{1}{2}\zeta'(-1)} \frac{e^{-\frac{1}{24}|x|^3}}{|x|^{1/16}} \exp\left\{ \frac{1}{2}\int_{-\infty}^x \left(R(y) - \frac{1}{4}y^2 + \frac{1}{8y} \right) dy \right\}, \tag{9}$$

where $\zeta(z)$ is the Riemann-zeta function, and

$$E(x) = \frac{1}{2^{1/4}} e^{-\frac{1}{3\sqrt{2}}|x|^{3/2}} \exp\left\{ \frac{1}{2}\int_{-\infty}^x \left(q(y) - \sqrt{\frac{|y|}{2}} \right) dy \right\}. \tag{10}$$

Remark 1. *The formula* (9) *also follows from the recent work [8] of Deift, Its, and Krasovsky. See subsection 1.2 below for further discussion.*

The integrals in (9) and (10) converge. Indeed, it is known that [19, 12]

$$q(x) = \sqrt{\frac{-x}{2}} \left(1 + \frac{1}{8x^3} - \frac{73}{128x^6} + \frac{10219}{1024x^9} + O(|x|^{-12}) \right), \qquad x \to -\infty. \tag{11}$$

This asymptotic behavior of q was obtained using the integrable structure of the Painlevé II equation (see, for example, [16]). The coefficients of the higher terms in the above asymptotic expansion can also be computed recursively (see for example, Theorem 1.28 of [12]). For $R(x)$, (7) and (11) imply that

$$R(x) = \frac{x^2}{4} \left(1 - \frac{1}{2x^3} + \frac{9}{16x^6} - \frac{128}{62x^9} + O(x^{-12}) \right), \qquad x \to -\infty. \tag{12}$$

We now discuss two consequences of Theorem 1.1.

1.1 Total integrals of $q(x)$ and $R(x)$

Comparing with (2), Theorem 1.1 is equivalent to the following.

Corollary 1.2. *For $c < 0$,*

$$\int_c^\infty R(y)dy + \int_{-\infty}^c \left(R(y) - \frac{1}{4}y^2 + \frac{1}{8y} \right) dy = -\frac{1}{24}\log 2 - \zeta'(-1) + \frac{1}{12}|c|^3 + \frac{1}{8}\log|c| \tag{13}$$

and

$$\int_c^\infty q(y)dy + \int_{-\infty}^c \left(q(y) - \sqrt{\frac{|y|}{2}} \right) dy = \frac{1}{2}\log 2 + \frac{\sqrt{2}}{3}|c|^{3/2}. \tag{14}$$

These formulas should be compared with the evaluation of the total integral of the Airy function [1]:

$$\int_{-\infty}^{\infty} \mathrm{Ai}(y)dy = 1. \tag{15}$$

Recall that the Airy differential equation is the small amplitude limit of the Painlevé II equation. Unlike the Airy function, $R(x)$ and $q(x)$ do not decay as $x \to -\infty$, and hence we need to subtract out the diverging terms in order to make the integrals finite.

1.2 Asymptotics of Tracy-Widom distribution functions as $x \to -\infty$

Using formulas (2) and (12), Tracy and Widom computed that (see Section 1.D of [24]) as $x \to -\infty$,

$$F_2(x) = \tau_2 \frac{e^{-\frac{1}{12}|x|^3}}{|x|^{1/8}} \left(1 + \frac{3}{2^6|x|^3} + O(|x|^{-6})\right) \tag{16}$$

for some undetermined constant τ_2. The constant τ_2 was conjectured in the same paper [24] to be

$$\tau_2 = 2^{1/24} e^{\zeta'(-1)}. \tag{17}$$

This conjecture (17) was recently proved by Deift, Its, and Krasovsky [8]. In this paper, we present an alternative proof of (17). Moreover, we also compute the similar constants τ_1 and τ_4 for the GOE and GSE Tracy-Widom distribution functions. The asymptotics similar to (16) follow from (1) and (11): as $x \to -\infty$,

$$F_1(x) = \tau_1 \frac{e^{-\frac{1}{24}|x|^3 - \frac{1}{3\sqrt{2}}|x|^{3/2}}}{|x|^{1/16}} \left(1 - \frac{1}{24\sqrt{2}|x|^{3/2}} + O(|x|^{-3})\right), \tag{18}$$

$$F_4(x) = \tau_4 \frac{e^{-\frac{1}{24}|x|^3 + \frac{1}{3\sqrt{2}}|x|^{3/2}}}{|x|^{1/16}} \left(1 + \frac{1}{24\sqrt{2}|x|^{3/2}} + O(|x|^{-3})\right). \tag{19}$$

Using (11) and (12), Theorem 1.1 implies the following.

Corollary 1.3. *As* $x \to -\infty$,

$$F(x) = 2^{1/48} e^{\frac{1}{2}\zeta'(-1)} \frac{e^{-\frac{1}{24}|x|^3}}{|x|^{1/16}} \left(1 + \frac{3}{2^7|x|^3} + O(|x|^{-6})\right) \tag{20}$$

and

$$E(x) = \frac{1}{2^{1/4}} e^{-\frac{1}{3\sqrt{2}}|x|^{3/2}} \left(1 - \frac{1}{24\sqrt{2}|x|^{3/2}} + O(|x|^{-3})\right). \tag{21}$$

Hence

$$\tau_1 = 2^{-11/48} e^{\frac{1}{2}\zeta'(-1)}, \qquad \tau_2 = 2^{1/24} e^{\zeta'(-1)}, \qquad \tau_4 = 2^{-35/48} e^{\frac{1}{2}\zeta'(-1)}. \tag{22}$$

Conversely, using (11) and (12), this Corollary together with (2) implies Corollary 1.2, and hence Theorem 1.1.

This is one example of so-called *constant problems* in random matrix theory. One can ask the same question of evaluating the constant term in the asymptotic expansion in other distribution functions such as the limiting gap distribution in the bulk or in the hard edge. For the gap probability distribution in the bulk scaling limit which is given by the Fredholm determinant of the sine-kernel, Dyson [13] first conjectured the constant term for $\beta = 2$ in terms of $\zeta'(-1)$ using a formula in an earlier work [27] of Widom. This conjecture was proved by Ehrhardt [14] and Krasovsky [20], independently and simultaneously. A third proof was given in [9]. The constant problem for $\beta = 1$ and $\beta = 4$ ensembles in the bulk scaling limit was recently obtained by Ehrhardt [15]. For the hard edge of the β-Laguerre ensemble associated with the weight $x^m e^{-x}$, the constant was obtained by Forrester [18] (equation (2.26a)) when m is a non-negative integer and $2/\beta$ is a positive integer.

The above limiting distribution functions in random matrix theory are expressed in terms of a Fredholm determinant or an integral involving a Painlevé function. For example, the proof of [8] used the Freldhom determinant formula of the GUE Tracy-Widom distribution:

$$F_2(x) = \det(1 - \mathbb{A}_x), \tag{23}$$

where $\mathbb{A}_x$ is the operator on $L^2((x,\infty))$ whose kernel is

$$\mathbb{A}(u,v) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)}{u-v}. \tag{24}$$

In terms of the Fredholm determinant formula, the difficulty comes from the fact that even if we know all the eigenvalues $\lambda_j(x)$ of $\mathbb{A}_x$, we still need to evaluate the product $\prod_{j=1}^{\infty}(1-\lambda_j(x))$. When one uses the Painlevé function, one faces a similar difficulty of evaluating the total integral of the Painlevé function.

We remark that the asymptotics as $x \to +\infty$ of $F(x)$ and $E(x)$ (and hence $F_\beta(x)$) are, using (4),

$$F(x) = 1 - \frac{e^{-\frac{4}{3}x^{3/2}}}{32\pi x^{3/2}}\left(1 - \frac{35}{24x^{3/2}} + O(x^{-3})\right), \tag{25}$$

$$E(x) = 1 - \frac{e^{-\frac{2}{3}x^{3/2}}}{4\sqrt{\pi}x^{3/2}}\left(1 - \frac{41}{48x^{3/2}} + O(x^{-3})\right). \tag{26}$$

1.3 Outline of the proof

The Tracy-Widom distribution functions are the limits of a variety of objects such as the largest eigenvalue of certain ensembles of random matrices, the length of the longest increasing subsequence of a random permutation, the last passage time of a certain last passage percolation model, and the height of a certain random growth model (see, for example, the survey [21]). Dyson [13] exploited this notion of universality to solve the constant problem for the sine-kernel determinant. Namely, among the many different quantities whose limit is the sine-kernel determinant, he chose one for which the associated constant term is explicitly computable (specifically, a certain Toeplitz determinant on an arc for which the constant term had been obtained by Widom [27]), and then took the appropriate limit while checking the limit of the constant term. However, the rigorous proof of this idea was only obtained in the subsequent work of Ehrhardt [14] and Krasovsky [20]. In order to apply this idea for $F_2(x)$, the key step is to choose the appropriate approximate ensemble. In the work of Deift, Its, and Krasovsky [8], the authors started with the Laguerre unitary ensemble and took the appropriate limit while controlling the error terms. In this paper, we use the fact that $F_\beta(x)$ is a (double-scaling) limit of a Toeplitz/Hankel determinant.

Let $D_n(t)$ denote the $n \times n$ Toeplitz determinant with symbol $f(e^{i\theta}) = e^{2t\cos(\theta)}$ on the unit circle:

$$\begin{aligned} D_n(t) &= \det\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2t\cos\theta} e^{i(j-k)\theta}d\theta\right)_{0\le j,k\le n-1} \\ &= \frac{1}{(2\pi)^n n!}\int_{[-\pi,\pi]^n} e^{2t\sum_{j=1}^n \cos\theta_j} \prod_{1\le k<\ell\le n}\left|e^{i\theta_k} - e^{i\theta_\ell}\right|^2 \prod_{j=1}^n d\theta_j. \end{aligned} \tag{27}$$

Note that some references (e.g. [7]) define D_n as an $(n+1)\times(n+1)$ determinant, whereas others (e.g. [5]) use our convention. In studying the asymptotics of the length of the longest increasing subsequence in random permutations, in [2], the authors proved that when

$$n = [2t + xt^{1/3}], \tag{28}$$

as $t \to \infty$,

$$e^{-t^2}D_n(t) \to F_2(x). \tag{29}$$

The idea of the proof of (29) in [2] is as follows. The Toeplitz determinants are intimately related to orthogonal polynomials on the unit circle. Let $p_j(z) = \kappa_j z^j + \cdots$ be the orthonormal polynomial of degree j with respect to the weight $e^{2t\cos\theta}\frac{d\theta}{2\pi}$:

$$\int_{-\pi}^{\pi} p_j(e^{i\theta})\overline{p_k(e^{i\theta})}e^{2t\cos\theta}\frac{d\theta}{2\pi} = \delta_{jk} \qquad \text{for } j,k \geq 0. \tag{30}$$

If $\kappa_j > 0$ then p_j is unique. We denote by $\pi_j(z;t) = \pi_j(z)$ the monic orthogonal polynomial: $p_k(z) = \kappa_k \pi_j(z)$. Then (see, for example, [23]) the leading coefficient $\kappa_j = \kappa_j(t)$ is given by

$$\kappa_j(t) = \sqrt{\frac{D_j(t)}{D_{j+1}(t)}}. \tag{31}$$

As the strong Szegö limit theorem implies that $D_n(t) \to e^{t^2}$ as $n \to \infty$ for fixed t, the left-hand-side of (29) can be written as

$$e^{-t^2}D_n(t) = \prod_{q=n}^{\infty}\frac{D_q(t)}{D_{q+1}(t)} = \prod_{q=n}^{\infty}\kappa_q^2(t). \tag{32}$$

The basic result of [2] is that

$$\kappa_q^2(t) \sim 1 - \frac{R(y)}{t^{1/3}}, \qquad t \to \infty, \quad q = [2t + yt^{1/3}] \tag{33}$$

for y in a compact subset of $\mathbb{R}$. (In [2], the notations $v(x) = -R(x)$ and $u(x) = -q(x)$ are used.) Hence formally, as $t \to \infty$ with $n = 2t + xt^{1/3}$,

$$\log\left(e^{-t^2}D_n(t)\right) = \sum_{q=n}^{\infty}\log\left(\kappa_q^2(t)\right) \sim t^{1/3}\int_x^{\infty}\log\left(1 - \frac{R(y)}{t^{1/3}}\right)dy \sim -\int_x^{\infty}R(y)dy = \log F_2(x). \tag{34}$$

The first step of this paper is to write, instead of (32),

$$e^{-t^2}D_n(t) = e^{-t^2}\prod_{q=1}^{n}\frac{D_q(t)}{D_{q-1}(t)} = e^{-t^2}\prod_{q=1}^{n}\frac{1}{\kappa_{q-1}^2(t)}. \tag{35}$$

Here $D_0(t) := 1$. Then formally, we expect that as $t \to \infty$ with $n = 2t + xt^{1/3}$, (35) converges to an integral from $-\infty$ to x. For this to work, we need the asymptotics of $\kappa_q(t)$ for the whole range of q and t such that $1 \leq q \leq 2t + xt^{1/3}$ as $t \to \infty$.

It turns out it is more convenient to write, for an arbitrary fixed L,

$$e^{-t^2}D_n(t) = e^{-t^2}D_L(t)\prod_{q=L+1}^{n}\frac{D_q(t)}{D_{q-1}(t)} = e^{-t^2}D_L(t)\prod_{q=L+1}^{n}\frac{1}{\kappa_{q-1}^2(t)}. \tag{36}$$

We introduce another fixed large number $M > 0$ and write

$$\log(e^{-t^2}D_n) = -t^2 + \underbrace{\log(D_L)}_{\text{exact part}} + \underbrace{\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log(\kappa_{q-1}^{-2})}_{\text{Airy part}} + \underbrace{\sum_{q=[2t-Mt^{1/3}]}^{[2t+xt^{1/3}]}\log(\kappa_{q-1}^{-2})}_{\text{Painlevé part}}. \tag{37}$$

Since L and M are arbitrary, we can compute the desired limit by computing

$$\lim_{L,M\to\infty}\lim_{t\to\infty}\log(e^{-t^2}D_n(t)), \qquad n = [2t + xt^{1/3}]. \tag{38}$$

From (33), the Painlevé part converges to a finite integral of $R(y)$ from $y = -M$ to $y = x$ as $t \to \infty$. For the Airy part, we need the asymptotics of $\kappa_q(t)$ for $L+1 \le q \le 2t - Mt^{1/3}$ as $t \to \infty$ for fixed $L, M > 0$. The paper [2] obtains a weak one-sided bound of $\kappa_q(t)$ for $\epsilon t \le q \le 2t - Mt^{1/3}$ as $t \to \infty$, where $\epsilon > 0$ is small but fixed. The technical part of this paper is to compute the leading asymptotics of $\kappa_q(t)$ in $L+1 \le q \le 2t - Mt^{1/3}$ with proper control of the errors so that the Airy part converges. The advantage of introducing L is that we do not need small values of q, which simplifies the analysis. The calculation is carried out in Section 3. Finally, for the exact part, the asymptotics of $D_L(t)$ as $t \to \infty$ are straightforward using a steepest-descent method since the size of the determinant is fixed and only the weight varies. The limit is given in terms of the Selberg integral for the $L \times L$ Gaussian unitary ensemble, which is given by a product of Gamma functions, the Barnes G-function. The asymptotics of the Barnes G-function as $L \to \infty$ are related to the term $\zeta'(-1)$ (see (48) below). The computation is carried out in Section 2.

Now we outline the proof of the formula (10) for $E(x)$. In the study of symmetrized random permutations it was proven in [4, 5] that, in a similar double scaling limit, certain other determinants converge to $F_1(x)$ and $F_4(x)$. But it was observed in [4, 5] that these determinants can be expressed in terms of $\kappa_q(t)$ and $\pi_q(0;t)$ for the *same* orthonormal polynomials (30) above. Hence by using the same idea for $D_n(t)$, we only need to keep track of $\pi_q(0;t)$ in the asymptotic analysis of the orthogonal polynomials. See Section 5 below for more details.

This paper is organized as follows. In Section 2, the asymptotics of the exact part of (37) are computed. We compute the asymptotics of the Airy part in Section 3. The proof of (9) for $F(x)$ in Theorem 1.1 is then given in Section 4. The proof of (10) for $E(x)$ in Theorem 1.1 is given in Section 5.

While we were writing up this paper, Alexander Its told us that there is another way to compute the constant term for $E(x)$ using a formula in [5]. This idea will be explored in a later publication together with Its to compute the total integrals of other Painlevé solutions, such as the Ablowitz-Segur solution.

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2 The exact part

We compute the exact part of (37). From equation (27),

$$D_L(t) = \frac{1}{(2\pi)^L L!} \int_{[-\pi,\pi]^L} e^{2t\sum_{j=1}^L \cos\theta_j} \prod_{1\le k<\ell\le L} \left|e^{i\theta_k} - e^{i\theta_\ell}\right|^2 \prod_{j=1}^L d\theta_j. \tag{39}$$

Following the standard stationary phase method of restricting each integral to a small interval $-\epsilon \le \theta \le \epsilon$ and expanding $e^{i\theta}$ and $e^{2t\cos\theta}$ in Taylor series, $D_L(t)$ is approximately

$$\frac{1}{(2\pi)^L L!} \int_{[-\epsilon,\epsilon]^L} e^{2tL - t\sum_{j=1}^L \theta_j^2} \prod_{1\le k<\ell\le L} |\theta_k - \theta_\ell|^2 \prod_{j=1}^L d\theta_j \tag{40}$$

as $t \to \infty$. By extending the range of integration to $\mathbb{R}^L$, we obtain

$$\lim_{t\to\infty} D_L(t) \cdot \left(\frac{e^{2tL}}{(2\pi)^L} D_L^{\text{Herm}}(t) \right)^{-1} = 1, \tag{41}$$

where

$$D_L^{\text{Herm}}(t) = \frac{1}{L!} \int_{[-\infty,\infty]^L} e^{-t\sum_{j=1}^L \theta_j^2} \prod_{1\le k<\ell\le n} |\theta_k - \theta_\ell|^2 \prod_{j=1}^L d\theta_j. \tag{42}$$

This integral is known as a Selberg integral and is computed explicitly as (see for example, [22], equation (17.6.7))

$$D_L^{\mathrm{Herm}}(t) = \frac{\pi^{L/2}}{2^{L(L-1)/2} t^{L^2/2}} \prod_{q=0}^{L-1} q! = \frac{\pi^{L/2}}{2^{L(L-1)/2} t^{L^2/2}} G(L+1), \tag{43}$$

where $G(z)$ denotes the Barnes G-function, or double gamma function. Some properties of the Barnes G-functions are (see, for example, [26, 6])

$$G(z+1) = \Gamma(z)G(z) \tag{44}$$

$$G(1) = G(2) = G(3) = 1. \tag{45}$$

$$\log G\left(\frac{1}{2}\right) = \frac{1}{24}\log 2 - \frac{1}{4}\log \pi + \frac{3}{2}\zeta'(-1), \quad \log G\left(\frac{3}{2}\right) = \frac{1}{24}\log 2 + \frac{1}{4}\log \pi + \frac{3}{2}\zeta'(-1), \tag{46}$$

$$G(n) = 1!2!\cdots(n-2)!, \qquad n = 2, 3, 4, \cdots. \tag{47}$$

$$\log G(z+1) = \frac{z^2}{2}\log z - \frac{3}{4}z^2 + \frac{z}{2}\log(2\pi) - \frac{1}{12}\log z + \zeta'(-1) + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty. \tag{48}$$

Therefore,

$$\lim_{L\to\infty}\lim_{t\to\infty}\left(\log(D_L) - \left\{2Lt - \frac{L^2}{2}\log(2t) + \left(\frac{L^2}{2} - \frac{1}{12}\right)\log L - \frac{3}{4}L^2 + \zeta'(-1)\right\}\right) = 0. \tag{49}$$

3 The Airy part

The main result of this section is Lemma 3.10 which computes

$$\lim_{L,M\to\infty}\lim_{t\to\infty}\sum_{q=L+1}^{[2t-Mt^{1/3}-1]} \log(\kappa_{q-1}^{-2}), \tag{50}$$

the Airy part of (37). We use the notation

$$\gamma = \frac{2t}{q}. \tag{51}$$

It is well known that the leading coefficients κ_q^{-2} of orthonormal polynomials can be expressed in terms of the solution of a matrix Riemann-Hilbert Problem (RHP) [17]. We start with the RHP for $m^{(5)}$ defined in Section 6 (p.1156) of [2], which is obtained through a series of explicit transformations of the original RHP for orthogonal polynomials. For notational ease, we drop the tildes Let θ_c be defined such that $0 < \theta_c < \pi$ and $\sin^2 \frac{\theta_c}{2} = \frac{1}{\gamma}$. For q in the regime $L+1 \leq q \leq 2t - Mt^{1/3} - 1$, we have $\gamma > 1$. Define the contours $C_1 = \{e^{i\theta} : \theta_c < |\theta| \leq \pi\}$ and $C_2 = \{e^{i\theta} : 0 \leq |\theta| \leq \theta_c\}$ with the orientations given as in Figure 1(a). Also define the contours C_{in} and C_{out} as in Figure 1(a). Let $\Sigma^{(5)} = C_1 \cup C_2 \cup C_{\mathrm{in}} \cup C_{\mathrm{out}}$. Now let $m^{(5)}(z) = m^{(5)}(z; t, q)$ be the solution to the following RHP:

$$\begin{cases} m^{(5)} \text{ is analytic in } z \in \mathbb{C}\backslash\Sigma_5 \\ m_+^{(5)}(z) = m_-^{(5)}(z)v^{(5)}(z) \text{ for } z \in \Sigma_5 \\ m^{(5)} = I \text{ as } z \to \infty \end{cases} \tag{52}$$

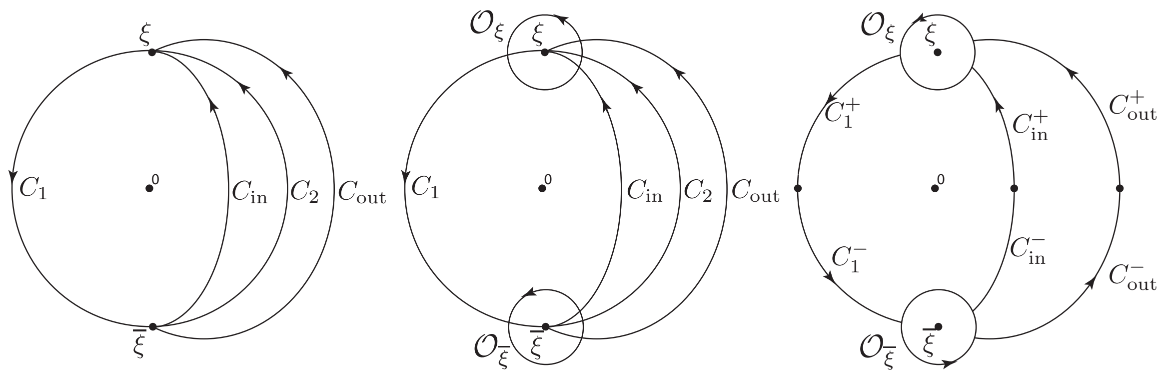
where the jump matrix $v^{(5)}(z) = v^{(5)}(z;t,q)$ is given by

$$v^{(5)}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & z \in C_2 \\ \begin{pmatrix} 1 & e^{-2q\alpha} \\ 0 & 1 \end{pmatrix} & z \in C_1 \\ \begin{pmatrix} 1 & 0 \\ e^{2q\alpha} & 1 \end{pmatrix} & z \in C_{\text{in}} \cup C_{\text{out}}. \end{cases} \tag{53}$$

Here

$$\alpha(z) = -\frac{\gamma}{4}\int_{\xi}^{z} \frac{s+1}{s^2}\sqrt{(s-\xi)(s-\overline{\xi})}ds, \tag{54}$$

where $\xi = e^{i\theta_c}$ and the branch is chosen to be analytic in $\mathbb{C}\backslash\overline{C_2}$ and $\sqrt{(s-\xi)(s-\overline{\xi})} \sim +s$ for $s \to \infty$. Then



(a) The contour $\Sigma^{(5)}$ for $m^{(5)}$. (b) Introduction of $\mathcal{O}_\xi$ and $\mathcal{O}_{\overline{\xi}}$. (c) The contour Σ for $m^{(R)}$.

Figure 1: Contours used in the definition of the RHP for $m^{(5)}$ and $m^{(R)}$.

(see (6.40) of [2])

$$\kappa_{q-1}^2 = -e^{q(-\gamma+\log\gamma+1)} m_{21}^{(5)}(0) \tag{55}$$

and

$$\pi_q(0) = (-1)^q m_{11}^{(5)}(0). \tag{56}$$

We analyze the solution $m^{(5)}$ to this RHP for the regime $L+1 \le q \le 2t - Mt^{1/3} - 1$ as $t \to \infty$. Our analysis builds on the work of [2] and makes two main technical improvements. The first is that the paper [2] only considered the regime when $\epsilon t \le q$. Hence q necessarily grows to infinity. In this work, we allow q to be finite. The second is that we compute a higher order correction explicitly to the asymptotics obtained in [2]. This higher-order correction contributes to the sum (50). In [2], only a one-sided bound of a similar sum was obtained. We merely outline the analysis for the parts that overlap with the analysis of [2].

From the construction of α in [2] we have that $\left|e^{-\alpha(z)}\right| < 1$ for $z \in C_1$ and $\left|e^{\alpha(z)}\right| < 1$ for $z \in C_{\text{in}} \cup C_{\text{out}}$. If we formally take the limit of our jump matrix $v^{(5)}$ as $q \to \infty$ the jumps on the contours C_{in} and C_{out} approach the identity matrix and the jumps on C_1 and C_2 approach constant jumps. This limiting RHP is solved explicitly by

$$m^{(5,\infty)}(z) = \begin{pmatrix} \frac{1}{2}\left(\beta+\beta^{-1}\right) & \frac{1}{2i}\left(\beta-\beta^{-1}\right) \\ -\frac{1}{2i}\left(\beta-\beta^{-1}\right) & \frac{1}{2}\left(\beta+\beta^{-1}\right) \end{pmatrix}, \tag{57}$$

where $\beta(z) = \left(\frac{z-\xi}{z-\xi^{-1}}\right)^{1/4}$, which is analytic for $z \in \mathbb{C} \setminus \overline{C_2}$ and $\beta \to 1$ as $z \to \infty$. Note that

$$m_{21}^{(5,\infty)}(0) = -\frac{1}{\sqrt{\gamma}} = -\frac{q}{2t}, \qquad m_{11}^{(5,\infty)}(0) = \sqrt{\frac{\gamma-1}{\gamma}} = \sqrt{\frac{2t-q}{2t}}. \tag{58}$$

However, the convergence of the jump matrix $v^{(5)}(z)$ is not uniform near the points ξ and $\overline{\xi}$, since $\alpha(\xi) = \alpha(\overline{\xi}) = 0$. Therefore a parametrix is introduced around these points. For fixed $\delta < \frac{1}{100}$, define

$$\mathcal{O}_\xi = \{z : |z-\xi| \le \delta|\xi - \overline{\xi}|\}, \quad \mathcal{O}_{\overline{\xi}} = \{z : |z-\overline{\xi}| \le \delta|\xi - \overline{\xi}|\}. \tag{59}$$

Note that the diameter of $\mathcal{O}_\xi$ is of order $\frac{\sqrt{\gamma-1}}{\gamma}$ and varies as t and q vary. The diameter approaches 0 as $\gamma \to 1$ or $\gamma \to \infty$, which happens when q is close to $2t - Mt^{1/3}$ or $L+1$, respectively. However, the point is that in the regime $L+1 \le q \le 2t - Mt^{1/3} - 1$, the diameter of $\mathcal{O}_\xi$ cannot shrink "too fast." Therefore, the usual Airy parametrix for a domain of fixed size still yields a good parametrix for the RHP in the regime under consideration. The case when $\gamma \to 1$ "slowly" was analyzed in [2] for the leading asymptotics of $m^{(5)}$. In this section, we also analyze the case when $\gamma \to \infty$ "slowly," and also improve the work in [2] to obtain a higher-order correction term.

Orient the boundary of both $\mathcal{O}_\xi$ and $\mathcal{O}_{\overline{\xi}}$ in the counterclockwise direction. Now as in [2] (see also [12]) for $z \in \mathcal{O}_\xi \setminus \Sigma^{(5)}$ define the matrix-valued function m_p as

$$m_p(z) = \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \sqrt{\pi} e^{i\pi/6} q^{\frac{\sigma_3}{6}} \left(\left(\frac{3}{2}\alpha(z)\right)^{\frac{2}{3}} \left(\frac{z-\overline{\xi}}{z-\xi}\right)\right)^{\frac{\sigma_3}{4}} \Psi\left(\left(\frac{3}{2} q\alpha(z)\right)^{2/3}\right) e^{q\alpha(z)\sigma_3}, \tag{60}$$

where $\omega = e^{2\pi i/3}$ and

$$\Psi(s) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(s) & \mathrm{Ai}(\omega^2 s) \\ \mathrm{Ai}'(s) & \omega^2 \mathrm{Ai}'(\omega^2 s) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & 0 < \arg(s) < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(s) & \mathrm{Ai}(\omega^2 s) \\ \mathrm{Ai}'(s) & \omega^2 \mathrm{Ai}'(\omega^2 s) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \frac{2\pi}{3} < \arg(s) < \pi, \\ \begin{pmatrix} \mathrm{Ai}(s) & -\omega^2 \mathrm{Ai}(\omega s) \\ \mathrm{Ai}'(s) & -\mathrm{Ai}'(\omega s) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \pi < \arg(s) < \frac{4\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(s) & -\omega^2 \mathrm{Ai}(\omega s) \\ \mathrm{Ai}'(s) & -\mathrm{Ai}'(\omega s) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \frac{4\pi}{3} < \arg(s) < 2\pi. \end{cases} \tag{61}$$

We can define $m_p(z) = \overline{m_p(\overline{z})}$ for $z \in \mathcal{O}_{\overline{\xi}}$ by (60). For $z \notin \Sigma^{(5)} \cup (\overline{\mathcal{O}_\xi \cup \mathcal{O}_{\overline{\xi}}})$, let $m_p(z) = m^{(5,\infty)}(z)$. It is shown in [2] that m_p then solves a RHP that has the same jump conditions as $m^{(5)}$ on the contour C_2 as well as on $\Sigma^{(5)} \cap \mathcal{O}$, where we define $\mathcal{O} = \mathcal{O}_\xi \cup \mathcal{O}_{\overline{\xi}}$.

Define $R(z) = m^{(5)}(z) m_p^{-1}(z)$. Then R solves a RHP on $\Sigma = \partial\mathcal{O} \cup ((C_1 \cup C_{\text{in}} \cup C_{\text{out}}) \cap \mathcal{O}^c)$ with jump $v_R = m_{p-} v^{(5)} v_p^{-1} m_{p-}^{-1}$. Explicitly, the jump matrix v_R is given by

$$v_R = I + \begin{cases} 0, & z \in (\Sigma^{(5)} \cap \mathcal{O}) \cup C_2, \\ m^{(5,\infty)} \begin{pmatrix} 0 & e^{-2q\alpha} \\ 0 & 0 \end{pmatrix} (m^{(5,\infty)})^{-1}, & z \in C_1 \cap \mathcal{O}^c, \\ m^{(5,\infty)} \begin{pmatrix} 0 & 0 \\ e^{2q\alpha} & 0 \end{pmatrix} (m^{(5,\infty)})^{-1}, & z \in (C_{\text{in}} \cup C_{\text{out}}) \cap \mathcal{O}^c, \\ \frac{1}{q\alpha} v_R^{q\alpha} + v_R^E, & z \in \partial\mathcal{O}, \end{cases} \tag{62}$$

where $v_R^{q\alpha}$ is given explicitly in Lemma 3.1 below, and the matrix v_R^E is defined as $v_R^E = v_R - I - \frac{1}{q\alpha}v_R^{q\alpha}$ for $z \in \partial\mathcal{O}$. Since $m^{(5)}(0) = R(0)m_p(0) = R(0)m^{(5,\infty)}(0)$, we have $m_{21}^{(5)}(0) = -\frac{1}{\sqrt{\gamma}}(R_{22}(0) - \sqrt{\gamma-1}R_{21}(0))$ and $m_{11}^{(5)}(0) = -\frac{1}{\sqrt{\gamma}}(R_{12}(0) - \sqrt{\gamma-1}R_{11}(0))$. Therefore

$$\kappa_{q-1}^2 = \frac{1}{\sqrt{\gamma}}e^{q(-\gamma+\log\gamma+1)}\big(R_{22}(0) - \sqrt{\gamma-1}R_{21}(0)\big) \tag{63}$$

and

$$\pi_q(0) = -\frac{(-1)^q}{\sqrt{\gamma}}\big(R_{12}(0) - \sqrt{\gamma-1}R_{11}(0)\big). \tag{64}$$

In [2], for $z \in \partial\mathcal{O}$, the jump matrix v_R is approximated by the identity matrix I and the terms $\frac{1}{q\alpha}v_R^{q\alpha} + v_R^E$ are treated as an error (for the case when $et \le q$). For our purpose, we need to compute the contribution from the next order term $\frac{1}{q\alpha}v_R^{q\alpha}$ explicitly.

It will be shown in the following subsections that for any $\epsilon > 0$, there are L_0 and M_0 such that for fixed $L \ge L_0$ and $M \ge M_0$, there is $t_0 = t_0(L, M)$ such that $||v_R - I||_{L^\infty(\Sigma)} < \epsilon$ for all $t \ge t_0$ and $L + 1 \le q \le 2t - Mt^{1/3} - 1$. This was shown in [2] for $\epsilon_1 t \le q \le 2t - Mt^{1/3} - 1$. Then we proceed via the standard Riemann-Hilbert analysis as, for example, in [2]. Let $\mathfrak{C}(f)(z) := \frac{1}{2\pi i}\int_\Sigma \frac{f(s)}{s-z}ds$ be the Cauchy operator defined for $z \notin \Sigma$. For $z \in \Sigma$, $\mathfrak{C}_-(f)(z)$ is defined as the nontangential limit of $\mathfrak{C}(f)(z')$ as z' approaches z from the right-hand side of Σ. Define the operator $\mathfrak{C}_R(f) = \mathfrak{C}_-(f(v_R - I))$ for $f \in L^2(\Sigma)$ and the function $\mu = I + (1 - \mathfrak{C}_R)^{-1}\mathfrak{C}_R I$. A simple scaling argument shows that $\mathfrak{C}_R$ is a uniformly bounded operator for $L + 1 \le q \le 2t - Mt^{1/3} - 1$. Since the supremum norm of $v_R - I$ can be made as small as necessary, we find that for L and M fixed but chosen large enough, $(1 - \mathfrak{C}_R)^{-1}$ is a bounded L^2 operator with norm uniformly bounded for t sufficiently large for all q such that $L + 1 \le q \le 2t - Mt^{1/3} - 1$. By the theory of Riemann-Hilbert problems,

$$R(z) - I = \frac{1}{2\pi i}\int_\Sigma \frac{\mu(s)(v_R - I)}{s - z}ds. \tag{65}$$

Define the contours $\Sigma^\pm$ as the part of Σ in the upper-half and lower-half planes, respectively. That is,

$$\Sigma^\pm = \Sigma \cap (\pm\Im(z) > 0). \tag{66}$$

Also define

$$C_1^\pm = C_1 \cap \Sigma^\pm, \quad C_{\rm in}^\pm = C_{\rm in} \cap \Sigma^\pm, \quad C_{\rm out}^\pm = C_{\rm out} \cap \Sigma^\pm \tag{67}$$

as shown in figure 1(c). Now, by the Schwartz-reflexivity of v_R (see [2], p. 1159) and μ,

$$\frac{1}{2\pi i}\int_{\Sigma^-} \frac{\mu(s)(v_R(s) - I)}{s}ds = \overline{\frac{1}{2\pi i}\int_{\Sigma^+} \frac{\mu(s)(v_R(s) - I)}{s}ds} \tag{68}$$

and therefore

$$R(0) - I = \Re\left[\frac{1}{\pi i}\int_{\Sigma^+} \frac{\mu(s)(v_R - I)}{s}ds\right]. \tag{69}$$

We write this as

$$R(0) - I = R^{(1)} + R^{(2)} + R^{(3)} + R^{(4)} + R^{(5)} \tag{70}$$

where

$$R^{(1)} = \Re\left[\frac{1}{\pi i}\int_{\partial\mathcal{O}_\xi} \frac{1}{q\alpha}v_R^{q\alpha}(s)\frac{ds}{s}\right],$$

$$R^{(2)} = \Re\left[\frac{1}{\pi i}\int_{\partial\mathcal{O}_\xi} \mu(s)\cdot v_R^E(s)\frac{ds}{s}\right], \quad R^{(3)} = \Re\left[\frac{1}{\pi i}\int_{C_1^+} \mu(s)(v_R(s) - I)\frac{ds}{s}\right], \tag{71}$$

$$R^{(4)} = \Re\left[\frac{1}{\pi i}\int_{C_{\rm in}^+\cup C_{\rm out}^+} \mu(s)(v_R(s)-I)\frac{ds}{s}\right], \quad R^{(5)} = \Re\left[\frac{1}{\pi i}\int_{\partial\mathcal{O}_\xi} (\mu(s)-I)\frac{1}{q\alpha}v_R^{q\alpha}(s)\frac{ds}{s}\right].$$

Hence using (63),

$$\begin{aligned}
&\sum_{q=L+1}^{[2t-Mt^{1/3}-1]} \log(\kappa_{q-1}^{-2}) \\
&= \sum_{q=L+1}^{[2t-Mt^{1/3}-1]} \left\{ -q(-\gamma+\log\gamma+1) + \frac{1}{2}\log\gamma - \log\left(R_{22}(0) - \sqrt{\gamma-1}R_{21}(0)\right)\right\} \\
&= \sum_{q=L+1}^{[2t-Mt^{1/3}-1]} \left\{ -q\big(-\frac{2t}{q} + \log\left(\frac{2t}{q}\right) + 1\big) + \frac{1}{2}\log\left(\frac{2t}{q}\right)\right. \\
&\quad \left. - \log\left(1+R_{22}^{(1)}(0) - \sqrt{\gamma-1}R_{21}^{(1)}(0)\right) - \log\left(1+\sum_{i=2}^{5}\frac{R_{22}^{(i)}(0)-\sqrt{\gamma-1}R_{21}^{(i)}(0)}{1+R_{22}^{(1)}(0)-\sqrt{\gamma-1}R_{21}^{(1)}(0)}\right)\right\}.
\end{aligned} \tag{72}$$

3.1 Calculation of $R^{(1)} = \Re\left[\frac{1}{\pi i}\int_{\partial\mathcal{O}_\xi}\frac{1}{q\alpha}v_R^{q\alpha}(s)\frac{ds}{s}\right]$

First, we compute $v_R^{q\alpha}$ explicitly.

Lemma 3.1. *For $z \in \partial\mathcal{O}_\xi$, the jump matrix $v_R(z)$ can be written as $v_R = I + \frac{1}{q\alpha}v_R^{q\alpha} + v_R^E$ where*

$$v_R^{q\alpha} = \frac{1}{2}\begin{pmatrix} d_1\beta^2 + c_1\beta^{-2} & i(d_1\beta^2 - c_1\beta^{-2}) \\ i(d_1\beta^2 - c_1\beta^{-2}) & -(d_1\beta^2 + c_1\beta^{-2})\end{pmatrix}, \qquad c_1 = \frac{5}{72}, \quad d_1 = -\frac{7}{72} \tag{73}$$

and

$$v_R^E = O\left(\frac{1}{|q\alpha|^2}\right). \tag{74}$$

Proof. On $\partial\mathcal{O}_\xi$, $v_R = m_{p-}m_{p+}^{-1}$. On this contour $m_{p-} = m^{(5,\infty)}$ and m_{p+} is given by (60). Thus for $z \in \partial\mathcal{O}_\xi$

$$v_R = m^{(5,\infty)}e^{-q\alpha(z)\sigma_3}\Psi^{-1}\left(\left(\frac{3}{2}q\alpha(z)\right)^{2/3}\right)\left(\left(\frac{3}{2}\alpha(z)\right)^{\frac{2}{3}}\left(\frac{z-\bar{\xi}}{z-\xi}\right)\right)^{-\frac{\sigma_3}{4}} e^{-i\pi/6}q^{-\frac{\sigma_3}{6}}\frac{1}{\sqrt{\pi}}\begin{pmatrix}1 & -1\\ -i & -i\end{pmatrix}^{-1}. \tag{75}$$

For $0 < \arg(s) < \frac{2\pi}{3}$, consider (see (61))

$$\Psi(s) = \begin{pmatrix}\mathrm{Ai}(s) & \mathrm{Ai}(\omega^2 s)\\ \mathrm{Ai}'(s) & \omega^2\mathrm{Ai}'(\omega^2 s)\end{pmatrix}e^{-(i\pi/6)\sigma_3} \tag{76}$$

with $\omega = e^{2i\pi/3}$. From Abramowitz and Stegun [1] (10.4.59) and (10.4.61), for $|\arg(s)| < \pi$,

$$\mathrm{Ai}(s) = \frac{e^{-(2/3)s^{3/2}}}{2\sqrt{\pi}s^{1/4}}\left(1 - \frac{c_1}{\frac{2}{3}s^{3/2}} + O\left(\frac{1}{|s|^3}\right)\right) \tag{77}$$

$$\mathrm{Ai}'(s) = -\frac{s^{1/4}e^{-(2/3)s^{3/2}}}{2\sqrt{\pi}}\left(1 - \frac{d_1}{\frac{2}{3}s^{3/2}} + O\left(\frac{1}{|s|^3}\right)\right) \tag{78}$$

wherein $c_1 = \frac{5}{72}$ and $d_1 = -\frac{7}{72}$. Also note the identity

$$\mathrm{Ai}(s) + \omega\mathrm{Ai}(\omega s) + \omega^2\mathrm{Ai}(\omega^2 s) = 0 \tag{79}$$

and, from Abramowitz and Stegun (10.4.11.13),

$$W[\mathrm{Ai}(s), \mathrm{Ai}(\omega s)] = \frac{1}{2\pi} e^{-i\pi/6} \tag{80}$$

$$W[\mathrm{Ai}(s), \mathrm{Ai}(\omega^2 s)] = \frac{1}{2\pi} e^{i\pi/6} \tag{81}$$

$$W[\mathrm{Ai}(\omega s), \mathrm{Ai}(\omega^2 s)] = \frac{1}{2\pi} e^{i\pi/2}. \tag{82}$$

Using $\det \Psi(s) = W[\mathrm{Ai}(s), \mathrm{Ai}(\omega^2 s)] = \frac{1}{2\pi} e^{i\pi/6}$, we have

$$\Psi^{-1}(s) = 2\pi \begin{pmatrix} \omega^2 \mathrm{Ai}'(\omega^2 s) & -\mathrm{Ai}(\omega^2 s) \\ -Ai'(s) e^{-i\pi/3} & \mathrm{Ai}(s) e^{-i\pi/3} \end{pmatrix}. \tag{83}$$

Using $\frac{2}{3}\lambda(z)^{3/2} = \alpha(z)$, equations (77) and (78) yield

$$\mathrm{Ai}(q^{2/3}\lambda(z)) = \frac{e^{-q\alpha}}{2\sqrt{\pi}(q^{2/3}\lambda)^{1/4}} \left(1 - \frac{c_1}{q\alpha} + O\left(\frac{1}{|q\alpha|^2}\right)\right) \tag{84}$$

$$\mathrm{Ai}'(q^{2/3}\lambda(z)) = -\frac{(q^{2/3}\lambda)^{1/4} e^{-q\alpha}}{2\sqrt{\pi}} \left(1 - \frac{d_1}{q\alpha} + O\left(\frac{1}{|q\alpha|^2}\right)\right) \tag{85}$$

$$\mathrm{Ai}(\omega^2 q^{2/3}\lambda(z)) = \frac{e^{i\pi/6} e^{q\alpha}}{2\sqrt{\pi}(q^{2/3}\lambda)^{1/4}} \left(1 + \frac{c_1}{q\alpha} + O\left(\frac{1}{|q\alpha|^2}\right)\right) \tag{86}$$

$$\omega^2 \mathrm{Ai}'(\omega^2 q^{2/3}\lambda(z)) = -\frac{\omega^2 (q^{2/3}\lambda)^{1/4} e^{q\alpha}}{2\sqrt{\pi} e^{i\pi/6}} \left(1 + \frac{d_1}{q\alpha} + O\left(\frac{1}{|q\alpha|^2}\right)\right). \tag{87}$$

We insert the asymptotics (84)-(87) into (83) resulting in the asymptotic formulas for $0 < \arg(q^{2/3}\lambda(z)) < \pi$ and $q\alpha$ large,

$$\Psi^{-1}(q^{2/3}\lambda(z)) = \sqrt{\pi} e^{i\pi/6} e^{q\alpha\sigma_3} \left\{ \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} + \frac{1}{q\alpha} \begin{pmatrix} d_1 & -c_1 \\ id_1 & ic_1 \end{pmatrix} + O\left(\frac{1}{|q\alpha|^2}\right) \right\} \left(\frac{3}{2} q\alpha\right)^{\sigma_3/6}. \tag{88}$$

It is straightforward to compute an analogous expansion for $\Psi^{-1}(s)$ for the other values of $\arg(q^{2/3}\lambda(z))$ in equation (76). To do this one must use the asymptotic formulas (77) and (78) as well as the additional expansions (10.4.60) and (10.4.62) from Abramowitz and Stegun [1]. Namely, for $|\arg(s)| < 2\pi/3$,

$$\mathrm{Ai}(-s) = \frac{s^{-1/4}}{\sqrt{\pi}} \left[\sin\left(\frac{2}{3} s^{3/2} + \frac{\pi}{4}\right) - \frac{3c_1}{2s^{3/2}} \cos\left(\frac{2}{3} s^{3/2} + \frac{\pi}{4}\right) + O\left(\frac{1}{|s|^3}\right) \right], \tag{89}$$

$$\mathrm{Ai}(-s) = -\frac{s^{1/4}}{\sqrt{\pi}} \left[\cos\left(\frac{2}{3} s^{3/2} + \frac{\pi}{4}\right) - \frac{3d_1}{2s^{3/2}} \sin\left(\frac{2}{3} s^{3/2} + \frac{\pi}{4}\right) + O\left(\frac{1}{|s|^3}\right) \right]. \tag{90}$$

After carrying out this computation, the first two terms in the expansion are the same in all four regions. In other words, (88) is valid not only for $0 < \arg(q^{2/3}\lambda(z)) < 2\pi/3$ but for all regions in the definition of Ψ in (76). Inserting the expansion in (88), equation (75) reduces to

$$v_R = I + \frac{1}{2q\alpha} \begin{pmatrix} d_1\beta^2 + c_1\beta^{-2} & i(d_1\beta^2 - c_1\beta^{-2}) \\ i(d_1\beta^2 - c_1\beta^{-2}) & -(d_1\beta^2 + c_1\beta^{-2}) \end{pmatrix} + O\left(\frac{1}{|q\alpha|^2}\right), \tag{91}$$

for all $z \in \partial\mathcal{O}_\xi$. □

Now we explicitly evaluate $R^{(1)}$.

Lemma 3.2. *We have*

$$R^{(1)} = \begin{pmatrix} \frac{1}{8q(\gamma-1)} - \frac{1}{24q\gamma} & \frac{1}{8q(\gamma-1)^{1/2}} - \frac{(\gamma-1)^{1/2}}{24q\gamma} \\ \frac{1}{8q(\gamma-1)^{1/2}} - \frac{(\gamma-1)^{1/2}}{24q\gamma} & -\frac{1}{8q(\gamma-1)} + \frac{1}{24q\gamma} \end{pmatrix}. \tag{92}$$

Proof. From Lemma 3.1, it is sufficient to compute the integrals

$$I_1 = \frac{1}{2\pi i}\int_{\partial\mathcal{O}_\xi} \frac{\beta(s)^2}{\alpha(s)s} ds \quad \text{and} \quad I_2 = \frac{1}{2\pi i}\int_{\partial\mathcal{O}_\xi} \frac{1}{\beta(s)^2\alpha(s)s} ds. \tag{93}$$

We will use the relations $\xi = e^{i\theta_c}$ and

$$\sin(\theta_c) = \frac{2(\gamma-1)^{1/2}}{\gamma}, \quad \cos(\theta_c) = \frac{\gamma-2}{\gamma}, \quad \sin\left(\frac{\theta_c}{2}\right) = \frac{1}{\gamma^{1/2}}, \quad \cos\left(\frac{\theta_c}{2}\right) = \left(\frac{\gamma-1}{\gamma}\right)^{1/2}. \tag{94}$$

Note that $\alpha(z) = \frac{2}{3}(z-\xi)^{3/2}G(z)$ for an analytic function $G(z)$ in $\mathcal{O}_\xi$ (see the bottom line at p.1157 of [2]). Hence by residue calculations,

$$I_1 = \frac{1}{2\pi i}\int_{\partial\mathcal{O}_\xi} \frac{3}{2(z-\xi)(z-\overline{\xi})^{1/2}G(z)z} dz = \frac{3}{2}\frac{1}{(\xi-\overline{\xi})^{1/2}G(\xi)\xi} \tag{95}$$

and

$$\begin{aligned} I_2 = \frac{1}{2\pi i}\int_{\partial\mathcal{O}_\xi} \frac{3(z-\overline{\xi})^{1/2}}{2(z-\xi)^2G(z)z} dz \quad &= \quad \frac{3}{2}\left(\frac{d}{dz}\left[\frac{(z-\overline{\xi})^{1/2}}{G(z)z}\right]\right)\Big|_{z=\xi} \\ = \frac{3}{2}\left[\frac{1}{2(\xi-\overline{\xi})^{1/2}G(\xi)\xi} - \frac{(\xi-\overline{\xi})^{1/2}G'(\xi)}{G(\xi)^2\xi} - \frac{(\xi-\overline{\xi})^{1/2}}{G(\xi)\xi^2}\right]&. \end{aligned} \tag{96}$$

But since $\alpha(z) = \frac{2}{3}(z-\xi)^{3/2}G(z)$ and $\alpha'(z) = -\frac{\gamma}{4}\frac{z+1}{z^2}\sqrt{(z-\xi)(z-\overline{\xi})}$, a straightforward computation yields that

$$G(\xi) = \lim_{z\to\xi}\frac{\alpha'(z)}{(z-\xi)^{1/2}} = -\frac{\gamma}{4}\frac{\xi+1}{\xi^2}(\xi-\overline{\xi})^{1/2}. \tag{97}$$

and

$$G'(\xi) = \frac{3\gamma}{20}\left(\frac{(\xi+2)(\xi-\overline{\xi})^{1/2}}{\xi^3} - \frac{\xi+1}{2\xi^2(\xi-\overline{\xi})^{1/2}}\right). \tag{98}$$

Using (94), we obtain

$$I_1 = \frac{-3}{4(\gamma-1)} + \frac{3}{4(\gamma-1)^{1/2}}i, \tag{99}$$

and

$$I_2 = \frac{3}{4(\gamma-1)} - \frac{3}{5\gamma} + \left(\frac{3}{4(\gamma-1)^{1/2}} - \frac{3(\gamma-1)^{1/2}}{5\gamma}\right)i. \tag{100}$$

Therefore,

$$R^{(1)}_{11} = -R^{(1)}_{22} = \frac{1}{q}\Re\left[-\frac{7}{72}I_1 + \frac{5}{72}I_2\right] = \frac{1}{8q(\gamma-1)} - \frac{1}{24q\gamma} \tag{101}$$

and

$$R^{(1)}_{12} = R^{(1)}_{21} = \frac{1}{q}\Im\left[\frac{7}{72}I_1 + \frac{5}{72}I_2\right] = \frac{1}{8q(\gamma-1)^{1/2}} - \frac{(\gamma-1)^{1/2}}{24q\gamma}. \tag{102}$$

□

3.2 Bound on $R^{(2)} = \Re\left[\frac{1}{\pi i}\int_{\partial\mathcal{O}_\xi} \mu(s)\cdot O\left(\frac{1}{|q\alpha|^2}\right)\frac{ds}{s}\right]$

We begin by establishing the leading term of $\alpha(z)$ for z near ξ.

Lemma 3.3. *For $1 \le q < 2t$ and for z such that $|z-\xi| \le \min\{\frac{1}{2}, |\xi-\overline{\xi}|\}$,*

$$\left|\alpha(z) - \frac{2}{3}(\gamma-1)^{3/4}(z-\xi)^{3/2}e^{-3i\pi/4}e^{-3i\theta_c/2}\right| \le \frac{50(\gamma-1)^{3/4}|z-\xi|^{5/2}}{|\xi-\overline{\xi}|}. \tag{103}$$

Proof. Write $z = \xi(1+\epsilon)$. Then $|\epsilon| = |z-\xi| \le \min\{\frac{1}{2}, |\xi-\overline{\xi}|\}$. Under the change of variables $s = \xi(1+\epsilon u)$, equation (54) for $\alpha(z)$ becomes

$$\alpha(z(\epsilon)) = -\frac{\gamma\epsilon^{3/2}(1+\xi)\sqrt{\xi-\overline{\xi}}}{4\sqrt{\xi}}\int_0^1 \sqrt{u}\left(1+\frac{\xi\epsilon}{\xi-\overline{\xi}}u\right)^{1/2}\frac{1+\frac{\xi\epsilon}{1+\xi}u}{(1+\epsilon u)^2}du. \tag{104}$$

Using $\xi = e^{i\theta_c}$ and (94), we have

$$-\frac{\gamma\epsilon^{3/2}(1+\xi)\sqrt{\xi-\overline{\xi}}}{4\sqrt{\xi}} = (\gamma-1)^{3/4}e^{-3i\pi/4}\epsilon^{3/2} = (\gamma-1)^{3/4}(z-\xi)^{3/2}e^{-3i\pi/4}e^{-3i\theta_c/2}. \tag{105}$$

For the integrand in (104), using the inequalities $|(1+w)^{1/2}-1| \le |w|$ for $|w| \le 1$ and $|(1+w)^{-2}-1| \le 10|w|$ for $|w| \le \frac{1}{2}$, and using the fact that $\frac{1}{|1+\xi|} \le \frac{2}{|\xi-\overline{\xi}|}$ and $1 \le \frac{2}{|\xi-\overline{\xi}|}$, we obtain

$$\left|\left(1+\frac{\xi\epsilon}{\xi-\overline{\xi}}u\right)^{1/2}\frac{1+\frac{\xi\epsilon}{1+\xi}u}{(1+\epsilon u)^2} - 1\right| \le \frac{50|\epsilon|}{|\xi-\overline{\xi}|} = \frac{50|z-\xi|}{|\xi-\overline{\xi}|}. \tag{106}$$

Therefore, we obtain (103). □

Lemma 3.4. *For $L+1 \le q \le 2t - Mt^{1/3} - 1$, there is a constant $c > 0$ such that*

$$|R^{(2)}| \le \frac{c(2t)^2}{q^{3/2}(2t-q)^{5/2}}. \tag{107}$$

Proof. On $\partial\mathcal{O}_\xi$, $|z-\xi| = \delta|\xi-\overline{\xi}| \le \frac{1}{40}|\xi-\overline{\xi}|$. Hence from Lemma 3.3, we have

$$|\alpha(z)| \ge \frac{1}{6}(\gamma-1)^{3/4}|z-\xi|^{3/2} = \frac{\delta^{3/2}}{6}(\gamma-1)^{3/4}|\xi-\overline{\xi}|^{3/2} = \frac{4}{3}\delta^{3/2}\left(\frac{\gamma-1}{\gamma}\right)^{3/2} \tag{108}$$

for $z \in \partial\mathcal{O}_\xi$. Therefore, as μ and $\frac{1}{s}$ are bounded on $\partial\mathcal{O}_\xi$,

$$|R^{(2)}| \le c'\int_{\partial\mathcal{O}_\xi} \frac{\gamma^3}{q^2(\gamma-1)^3}|ds| = \frac{2\pi c'\gamma^3\delta|\xi-\overline{\xi}|}{q^2(\gamma-1)^3} = \frac{c\gamma^2}{q^2(\gamma-1)^{5/2}} \tag{109}$$

for some constants $c', c > 0$, as $|\xi-\overline{\xi}| = \frac{4(\gamma-1)^{1/2}}{\gamma}$.

□

3.3 Bound on $R^{(3)} = \Re\left[\frac{1}{\pi i}\int_{C_1^+} \mu(s)(v_R(s)-I)\frac{ds}{s}\right]$

Since $\mu(s)$ and $1/s$ are bounded on C_1^+, $|R^{(3)}| \le c'||v_R - I||_{L^1(C_1^+)}$ for some constant $c' > 0$. But on C_1^+, $v_R(z) - I = O(e^{-2q\alpha(z)})$. Hence

$$|R^{(3)}| \le c||e^{-2q\alpha(z)}||_{L^1(C_1^+)}, \tag{110}$$

for some constant $c > 0$. For $z = e^{i\theta} \in C_+$ (hence $\theta_c < \theta \le \pi$), using

$$\sqrt{(e^{i\phi} - \xi)(e^{i\phi} - \overline{\xi})} = |(e^{i\phi} - e^{i\phi_c})(e^{i\phi} - e^{-i\phi_c})|^{1/2} e^{i(\pi+\phi)/2}, \tag{111}$$

we have

$$\begin{aligned} \alpha(e^{i\theta}) &= -\frac{\gamma}{4}\int_{\theta_c}^{\theta} \frac{1+e^{i\phi}}{e^{2i\phi}} \sqrt{(e^{i\phi} - e^{i\theta_c})(e^{i\phi} - e^{-i\theta_c})} \cdot ie^{i\phi} d\phi \\ &= \gamma \int_{\theta_c}^{\theta} \cos\left(\frac{\phi}{2}\right) \sin^{1/2}\left(\frac{\phi+\theta_c}{2}\right) \sin^{1/2}\left(\frac{\phi-\theta_c}{2}\right) d\phi. \end{aligned} \tag{112}$$

(Recall that $\gamma = \frac{2t}{q}$.) Note that $\alpha(\xi) = 0$, $\alpha(s)$ is real and positive on C_1^+, and $\alpha(e^{i\theta})$ increases as θ increases.

Lemma 3.5. *For $1 \le q \le 2t$,*

$$\alpha(e^{i\theta}) \ge \frac{1}{12\pi}\sqrt{\gamma(\gamma-1)}(\theta-\theta_c)^2 \tag{113}$$

for $\theta_c \le \theta \le \pi$.

Proof. We consider two cases separately: $\theta_c \le \frac{\pi}{3}$ and $\theta_c \ge \frac{\pi}{3}$.

Start with the case when $\theta_c \le \frac{\pi}{3}$. We consider two sub-cases: $\theta \le \frac{2\pi}{3}$ and $\theta \ge \frac{2\pi}{3}$. When $\theta \le \frac{2\pi}{3}$, $0 \le \frac{\phi}{2} \le \frac{\pi}{3}$, $0 \le \frac{\phi+\theta_c}{2} \le \frac{\pi}{2}$ and $0 \le \frac{\phi-\theta_c}{2} \le \frac{\pi}{3}$. Hence using the basic inequalities $\cos(x) \ge \frac{1}{2}$ for $0 \le x \le \frac{\pi}{3}$ and $\sin(x) \ge \frac{2}{\pi}x$ for $0 \le x \le \frac{\pi}{2}$, we find that

$$\alpha(e^{i\theta}) \ge \frac{\gamma}{2\pi}\int_{\theta_c}^{\theta} (\phi+\theta_c)^{1/2}(\phi-\theta_c)^{1/2} d\phi \ge \frac{\gamma}{2\pi}\int_{\theta_c}^{\theta}(\phi-\theta_c)d\phi = \frac{\gamma}{4\pi}(\theta-\theta_c)^2. \tag{114}$$

When $\theta \ge \frac{2\pi}{3}$, from the monotonicity of $\alpha(e^{i\theta})$ and using (114),

$$\alpha(e^{i\theta}) \ge \alpha(e^{\frac{2\pi}{3}i}) \ge \frac{\gamma}{4\pi}\left(\frac{2\pi}{3} - \theta_c\right)^2. \tag{115}$$

For $0 \le \theta_c \le \frac{\pi}{3}$ and $\frac{2\pi}{3} \le \theta \le \pi$, we have $\frac{2\pi}{3} - \theta_c \ge \frac{\pi}{3} \ge \frac{1}{3}(\theta - \theta_c)$. Therefore,

$$\alpha(e^{i\theta}) \ge \frac{\gamma}{12\pi}(\theta-\theta_c)^2. \tag{116}$$

For the second case, when $\theta_c \ge \frac{\pi}{3}$, using the change of variables $\phi \mapsto \pi - \phi$,

$$\alpha(e^{i\theta}) = \gamma\int_{\pi-\theta}^{\pi-\theta_c} \sin\left(\frac{\phi}{2}\right)\sin^{1/2}\left(\frac{\phi+(\pi-\theta_c)}{2}\right)\sin^{1/2}\left(\frac{(\pi-\theta_c)-\phi}{2}\right)d\phi. \tag{117}$$

Note that $0 \le \frac{\phi}{2} \le \frac{\pi}{3}$, $0 \le \frac{\phi+(\pi-\theta_c)}{2} \le \pi - \theta_c \le \frac{2\pi}{3}$, and $0 \le \frac{(\pi-\theta_c)-\phi}{2} \le \frac{\pi}{3}$. Using the basic inequalities $\sin(x) \ge \frac{3\sqrt{3}}{2\pi}x$ for $0 \le x \le \frac{\pi}{3}$ and $\sin(x) \ge \frac{3\sqrt{3}}{4\pi}x$ for $0 \le x \le \frac{2\pi}{3}$, we find that

$$\begin{aligned} \alpha(e^{i\theta}) &\ge \frac{27\gamma}{8\sqrt{2}\pi^2}\int_{\pi-\theta}^{\pi-\theta_c} \phi(\phi+(\pi-\theta_c))^{1/2}((\pi-\theta_c)-\phi)^{1/2}d\phi \\ &\ge \frac{27\gamma}{8\sqrt{2}\pi^2}\int_{\pi-\theta}^{\pi-\theta_c}\phi((\pi-\theta_c)-\phi)d\phi \\ &= \frac{9\gamma}{16\sqrt{2}}\big(\pi-\theta_c+2(\pi-\theta)\big)(\theta-\theta_c)^2 \ge \frac{9\gamma}{16\sqrt{2}\pi^2}(\pi-\theta_c)(\theta-\theta_c)^2. \end{aligned} \tag{118}$$

Since $\sqrt{\frac{1-\gamma}{\gamma}} = \cos\left(\frac{\theta_c}{2}\right) = \sin\left(\frac{\pi-\theta_c}{2}\right) \le \frac{\pi-\theta_c}{2}$, we have

$$\alpha(e^{i\theta}) \ge \frac{9}{8\sqrt{2}\pi^2}\sqrt{\gamma(\gamma-1)}(\theta-\theta_c)^2. \tag{119}$$

Combining (114), (116), and (119) completes the proof. ∎

Lemma 3.6. *For $L+1 \le q \le 2t - Mt^{1/3} - 1$, there is a constant $c > 0$ such that*

$$|R^{(3)}| \le \frac{c(2t)^2}{q^{3/2}(2t-q)^{5/2}}. \tag{120}$$

Proof. Let $e^{i\theta_*}$ be the endpoint of C_1^+ on $\partial\mathcal{O}_\xi$. Note that since radius of $\partial\mathcal{O}_\xi$ is $\delta|\xi - \overline{\xi}| = 4\delta\frac{\sqrt{\gamma-1}}{\gamma}$,

$$\theta_* - \theta_c \ge 4\delta\frac{\sqrt{\gamma-1}}{\gamma}. \tag{121}$$

Using Lemma 3.5 and changing variables,

$$\|e^{-2q\alpha}\|_{L^1(C_1^+)} \le \int_{\theta_*}^{\pi} e^{-\frac{q\sqrt{\gamma(\gamma-1)}}{6\pi}(\theta-\theta_c)^2} d\theta \le \left(\frac{3\pi}{q\sqrt{\gamma(\gamma-1)}}\right)^{1/2} \int_{x_*}^{\infty} e^{-\frac{1}{2}x^2} dx \tag{122}$$

where

$$x_* = \left(\frac{q\sqrt{\gamma(\gamma-1)}}{3\pi}\right)^{1/2}(\theta_* - \theta_c) \ge \frac{4\delta}{\sqrt{3\pi}}\sqrt{q}\left(\frac{\gamma-1}{\gamma}\right)^{3/4}. \tag{123}$$

Using the inequality $\int_a^\infty e^{-\frac{1}{2}x^2}dx \le \frac{1}{a^3}$ for $a > 0$,

$$\|e^{-2q\alpha}\|_{L^1(C_1^+)} \le \left(\frac{3\pi}{q\sqrt{\gamma(\gamma-1)}}\right)^{1/2} \frac{1}{x_*^3} \le \frac{9\pi^2}{128\delta^3}\frac{(2t)^2}{q^{3/2}(2t-q)^{5/2}}. \tag{124}$$

Hence from (110) we obtain (120). □

3.4 Bound on $R^{(4)} = \Re\left[\frac{1}{\pi i}\int_{C_{\text{in}}^+ \cup C_{\text{out}}^+} \mu(s)(v_R(s) - I)\frac{ds}{s}\right]$

As before, since on $C_{\text{in}}^+ \cup C_{\text{out}}^+$ the functions $\mu(s)$ and $\frac{1}{s}$ are uniformly bounded and $v_R(z) - I = O(e^{-2q\alpha(z)})$,

$$|R^{(4)}| \le c'||v_R - I||_{L^1(C_{\text{in}}^+\cup C_{\text{out}}^+)} \le c\|e^{2q\alpha}\|_{L^1(C_{\text{in}}^+\cup C_{\text{out}}^+)} \tag{125}$$

for some constants $c', c > 0$.

Lemma 3.7. *For $L+1 \le q \le 2t - Mt^{1/3} - 1$, there is a constant $c > 0$ such that*

$$|R^{(4)}| \le \frac{c(2t)^2}{q^{3/2}(2t-q)^{5/2}}. \tag{126}$$

Proof. Let $\gamma_0 = \csc^2(\frac{\pi}{24})$. Let $\delta_4 > 0$ be a small positive number defined on p. 1152 of [2]. We estimate $e^{-2q\alpha}$ in the following three cases separately: (i) $2t(1+\delta_4)^{-1} \le q \le 2t - Mt^{1/3} + 1$, (ii) $\frac{2t}{\gamma_0} \le q \le 2t(1+\delta_4)^{-1}$ and (iii) $L + 1 \le q \le \frac{2t}{\gamma_0}$.

(i) For $2t(1+\delta_4)^{-1} \le q \le 2t - Mt^{1/3} - 1$, from (6.37) of [2], there are constants $c_1, c_2, c_3, c_4, c_5 > 0$ such that

$$||e^{2q\alpha}||_{L^1(C_{\text{in}}^+\cup C_{\text{out}}^+)} \le c_1\int_{c_2\sqrt{\gamma-1}}^{\infty} e^{-c_3qx^3}dx + c_4e^{-c_5q}. \tag{127}$$

Using the change of variables $y = c_3qx^3$,

$$\begin{aligned} ||e^{2q\alpha}||_{L^1(C_{\text{in}}^+\cup C_{\text{out}}^+)} &\le \frac{c_1}{3c_3^{1/3}q^{1/3}}\int_{c_2q(\gamma-1)^{3/2}}^{\infty}\frac{e^{-y}}{y^{2/3}}dy + c_4e^{-c_5q} \\ &\le \frac{c_1}{3c_2^2c_3q(\gamma-1)}e^{-c_2^3c_3q(\gamma-1)^{3/2}} + c_4e^{-c_5q} \\ &\le \frac{c_1}{3c_2^5c_3^2q^2(\gamma-1)^{5/2}} + \frac{c_4}{c_5^2q^2}. \end{aligned} \tag{128}$$

Since $\gamma \geq 1$ and $\gamma - 1 \leq 1 + \delta_4$, we find that

$$||e^{2q\alpha}||_{L^1(C_{\rm in}^+ \cup C_{\rm out}^+)} \leq \frac{c\gamma^2}{q^2(\gamma-1)^{5/2}} = \frac{c(2t)^2}{q^{3/2}(2t-q)^{5/2}} \tag{129}$$

for a constant $c > 0$.

(ii) For $\frac{2t}{\gamma_0} \leq q \leq 2t(1+\delta_4)^{-1}$, note that the radius of $\mathcal{O}_\xi$ is of $O(1)$. Hence a standard calculation in Riemann-Hilbert steepest-descent analysis shows that for $z \in C_{\rm in}^+ \cap C_{\rm out}^+$, $\Re(\alpha(z)) \leq -c'$ for some constant $c' > 0$. Since the length of $L^1(C_{\rm in}^+ \cup C_{\rm out}^+)$ is bounded,

$$||e^{2q\alpha}||_{L^1(C_{\rm in}^+ \cup C_{\rm out}^+)} \leq c'' e^{-2c'q} \leq \frac{c''}{(2c')^2 q^2} \leq \frac{c'''\gamma^2}{q^2(\gamma-1)^{5/2}} = \frac{c'''(2t)^2}{q^{3/2}(2t-q)^{5/2}} \tag{130}$$

for some constants $c'', c''' > 0$.

(iii) Consider the case when $L + 1 \leq q \leq \frac{2t}{\gamma_0}$. Then $0 \leq \theta_c \leq \frac{\pi}{12}$. In this case, we make a specific choice of $C_{\rm in}^+$ and $C_{\rm out}^+$:

$$\begin{aligned} C_{\rm in}^+ &= \left\{ \xi + \rho \sin\theta_c e^{i(\theta_c + \frac{7}{6}\pi)} : 2\delta \leq \rho \leq \frac{1}{-\sin(\theta_c + \frac{7}{6}\pi)} \right\} \\ C_{\rm out}^+ &= \left\{ \xi + \rho \sin_c e^{i(\theta_c - \frac{1}{6}\pi)} : 2\delta \leq \rho \leq \frac{1}{-\sin(\theta_c - \frac{1}{6}\pi)} \right\}. \end{aligned} \tag{131}$$

The contours $C_{\rm in}^+$ are straight line segments from ξ to a point on the positive real axis. (Recall that $\mathcal{O}_\xi$ has the radius $\delta|\xi - \overline{\xi}| = 2\delta \sin\theta_c$.) Now we estimate $\Re(\alpha(z))$ for $z \in C_{\rm in}^+ \cup C_{\rm out}^+$. For $z \in C_{\rm in}^+$, take the contour in (54) to be the straight line from ξ to z. Then one can check from the geometry that

$$\begin{aligned} &0 \leq \arg(1+s) \leq \theta_c, \qquad 0 \leq \arg(s) \leq \theta_c, \qquad \arg(s-\xi) = \theta_c + \frac{7\pi}{6} \\ &\frac{\pi}{2} \leq \arg(s - \overline{\xi}) \leq \frac{5\pi}{6} - \theta_c, \qquad \arg(ds) = \theta_c + \frac{2\pi}{6}. \end{aligned} \tag{132}$$

Therefore the argument of the integrand in (54) is in $[2\pi, 2\pi + \frac{\pi}{6} + 2\theta_c] \subset [2\pi, 2\pi + \frac{\pi}{3}]$ since $0 \leq \theta_c \leq \frac{\pi}{12}$. Thus, the cosine of the argument is greater than or equal to $\cos(\frac{\pi}{3}) = \frac{1}{2}$. Therefore, for $z = \xi + \rho \sin\theta_c e^{i(\theta_c + \frac{7}{6}\pi)} \in C_{\rm in}^+$, by the change of variables $s = \xi + y \sin\theta_c e^{i(\theta_c + \frac{7}{6}\pi)}$,

$$\begin{aligned} \Re(\alpha(z)) &\leq -\frac{\gamma}{8} \int_\xi^z \left| \frac{1+s}{s^2} \sqrt{(s-\xi)(s-\overline{\xi})} \right| |ds| \\ &\leq -\frac{\gamma \sin^2\theta_c \cos(\frac{\theta_c}{2})}{2\sqrt{2}} \int_0^\rho \sqrt{y} \frac{|1 + \frac{y \sin\theta_c}{2\cos(\frac{\theta_2}{2})} e^{i(\frac{\theta_c}{2} + \frac{7\pi}{6})}|}{|1 + y \sin\theta_c e^{i\frac{7\pi}{6}}|^2} \left|1 - i\frac{1}{2} y e^{i(\theta_c + \frac{7\pi}{6})}\right|^{1/2} dy. \end{aligned} \tag{133}$$

Using the inequality $|1 + xe^{i\phi}| \geq |\sin\phi|$ for all $x \in \mathbb{R}$, and using $0 \leq \theta_c \leq \frac{\pi}{12}$ and $|y| \leq \frac{1}{-\sin(\theta_c + \frac{7\pi}{6})} \leq 2$, we have

$$\begin{aligned} \Re(\alpha(z)) &\leq -\frac{\gamma \sin^2\theta_c \cos(\frac{\theta_c}{2})}{36\sqrt{2}} \int_0^\rho \sqrt{y} dy \\ &= -\frac{\sqrt{2}}{27} \left(\frac{\gamma-1}{\gamma}\right)^{3/2} \rho^{3/2} \\ &\leq -\frac{\sqrt{2}}{27} \left(\frac{\gamma-1}{\gamma}\right)^{3/2} (2\delta)^{3/2} \end{aligned} \tag{134}$$

for $z \in C_{\rm in}^+$. For $z \in C_{\rm out}^+$, taking the contour in (54) to be the straight line from ξ to z, we can check that

$$\begin{aligned} &0 \le \arg(1+s) \le \theta_c, \qquad 0 \le \arg(s) \le \theta_c, \qquad \arg(s-\xi) = \theta_c - \frac{\pi}{6} \\ &\frac{\pi}{6} - \theta_c \le \arg(s-\overline{\xi}) \le \frac{\pi}{2}, \qquad \arg(ds) = \theta_c - \frac{\pi}{6}. \end{aligned} \tag{135}$$

Hence the argument of the integrand in (54) is in $[-\theta_c - \frac{\pi}{6}, 2\theta_c] \subset [-\frac{\pi}{4}, \frac{\pi}{6}]$. Therefore, for $z = \xi + \rho \sin\theta_c e^{i(\theta_c - \frac{1}{6}\pi)} \in C_{\rm out}^+$,

$$\begin{aligned} \Re(\alpha(z)) &\le -\frac{\gamma}{4\sqrt{2}} \int_\xi^z \left| \frac{1+s}{s^2} \sqrt{(s-\xi)(s-\overline{\xi})} \right| |ds| \\ &\le -\frac{\gamma \sin^2\theta_c \cos(\frac{\theta_c}{2})}{2} \int_0^\rho \sqrt{y} \frac{|1 + \frac{y \sin\theta_c}{2\cos(\frac{\theta_2}{2})} e^{i(\frac{\theta_c}{2} - \frac{\pi}{6})}|}{|1 + y \sin\theta_c e^{-i\frac{\pi}{6}}|^2} \left|1 - i\frac{1}{2} y e^{i(\theta_c - \frac{\pi}{6})}\right|^{1/2} dy \\ &\le -\frac{1}{27\sqrt{2}} \left(\frac{\gamma - 1}{\gamma} \right)^{3/2} \rho^{3/2} \\ &\le -\frac{1}{27\sqrt{2}} \left(\frac{\gamma - 1}{\gamma} \right)^{3/2} (2\delta)^{3/2}. \end{aligned} \tag{136}$$

From (134) and (136), arguing as in (130), we obtain

$$||e^{2q\alpha}||_{L^1(C_{\rm in}^+ \cup C_{\rm out}^+)} \le \frac{c''(2t)^2}{q^{3/2}(2t-q)^{5/2}} \tag{137}$$

for a constant $c'' > 0$. Hence we obtain the estimate for $|R^{(4)}|$. □

3.5 Bound on $R^{(5)} = \Re\left[\frac{1}{\pi i}\int_{\partial\mathcal{O}_\xi} (\mu(s) - I) \frac{1}{q\alpha} v_R^{q\alpha} \frac{ds}{s}\right]$

Lemma 3.8. *For $L + 1 \le q \le 2t - Mt^{1/3} - 1$, there is a constant $c > 0$ such that*

$$|R^{(5)}| \le \frac{c(2t)^2}{q^{3/2}(2t-q)^{5/2}}. \tag{138}$$

Proof. As $\mu - I = (1 - \mathfrak{C}_R)^{-1} \mathfrak{C}_R I$, and as $(1 - \mathfrak{C})^{-1}$ and $\mathfrak{C}_-$ are uniformly bounded,

$$||\mu - I||_{L^2(\Sigma)} \le c_0 ||\mathfrak{C}_R I||_{L^2(\Sigma)} = c_0 ||\mathfrak{C}_-(v_R - I)||_{L^2(\Sigma)} \le c_1 ||v_R - I||_{L^2(\Sigma)} \tag{139}$$

for some constants $c_0, c_1 > 0$ when t is large enough. Below, we assume that t is large enough so that the above estimate holds. Now

$$\begin{aligned} |R^{(5)}| &\le \int_{\partial\mathcal{O}_\xi} \left| (\mu(s) - I) \frac{v_R^{q\alpha}}{q\alpha} \right| \frac{|ds|}{|s|} \\ &\le 2||\mu - I||_{L^2(\partial\mathcal{O}_\xi)} \left|\left| \frac{v_R^{q\alpha}}{q\alpha} \right|\right|_{L^2(\partial\mathcal{O}_\xi)} \\ &\le 2c_1 ||v_R - I||_{L^2(\Sigma)} \left|\left| \frac{v_R^{q\alpha}}{q\alpha} \right|\right|_{L^2(\partial\mathcal{O}_\xi)}. \end{aligned} \tag{140}$$

Since $\beta(z) = \left(\frac{z-\xi}{z-\overline{\xi}}\right)^{1/4}$ is bounded above and below for $z \in \mathcal{O}_\xi$, $v_R^{q\alpha}(z)$ in Lemma 3.1 is bounded. Using (108) and the fact that the radius of $\mathcal{O}_\xi$ is $\delta|\xi - \overline{\xi}| = \frac{4\delta\sqrt{\gamma-1}}{\gamma}$, we have

$$\left|\left| \frac{v_R^{q\alpha}}{q\alpha} \right|\right|^2_{L^2(\partial\mathcal{O}_\xi)} \le \frac{c_2\gamma^2}{q^2(\gamma-1)^{5/2}} = \frac{c_2(2t)^2}{q^{3/2}(2t-q)^{5/2}} \tag{141}$$

for a constant $c_2 > 0$.

Write $||v_R - I||^2_{L^2(\Sigma)} = ||v_R - I||^2_{L^2(\partial\mathcal{O})} + ||v_R - I||^2_{L^2(\Sigma\setminus\partial\mathcal{O})}$. As $v_R - I = \frac{v_R^{q\alpha}}{q\alpha} + v_R^E$ in Lemma 3.1 is bounded by $\frac{c_3}{|q\alpha|}$ for a constant $c_3 > 0$, we have, as in (141),

$$||v_R - I||^2_{L^2(\partial\mathcal{O})} = 2||v_R - I||^2_{L^2(\partial\mathcal{O})_\xi} \leq \frac{c_4(2t)^2}{q^{3/2}(2t-q)^{5/2}} \tag{142}$$

for a constant $c_4 > 0$. On the other hand,

$$\begin{aligned} ||v_R - I||^2_{L^2(\Sigma\setminus\partial\mathcal{O})} &= 2||v_R - I||^2_{L^2(C_1^+)} + 2||v_R - I||^2_{L^2(C_{\text{in}}^+\cup C_{\text{out}}^+)} \\ &\leq c_3\big(||e^{-2q\alpha}||_{L^2(C_1^+)} + ||e^{2q\alpha}||_{L^2(C_{\text{in}}^+\cup C_{\text{out}}^+)}\big) \\ &\leq c_3\big(||e^{-2q\alpha}||_{L^1(C_1^+)} + ||e^{2q\alpha}||_{L^1(C_{\text{in}}^+\cup C_{\text{out}}^+)}\big) \end{aligned} \tag{143}$$

for a constant $c_3 > 0$, since $e^{-2q\alpha} \leq 1$ for $z \in C_1^+$ and $e^{2q\alpha} \leq 1$ for $z \in C_{\text{in}}^+ \cup C_{\text{out}}^+$. Hence from (124) and (130), we have

$$||v_R - I||^2_{L^2(\Sigma\setminus\partial\mathcal{O})} \leq \frac{c_4(2t)^2}{q^{3/2}(2t-q)^{5/2}} \tag{144}$$

for a constant $c_4 > 0$. By combining (141), (142), and (144), we obtain (138). □

3.6 The Airy part

From Lemmas 3.4, 3.6, 3.7, and 3.8, we find that, for $L+1 \leq q \leq [2t - Mt^{1/3} - 1]$, there is a constant $c > 0$ such that

$$\left|\sum_{i=2}^{5}(R_{22}^{(i)}(0) - \sqrt{\gamma-1}R_{21}^{(i)}(0))\right| \leq \frac{c(2t)^2}{q^{3/2}(2t-q)^{5/2}} + \frac{c(2t)^2}{q^2(2t-q)^2}. \tag{145}$$

We need the following result.

Lemma 3.9. *We have*

$$\lim_{M\to\infty}\limsup_{t\to\infty}\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\frac{(2t)^2}{q^{3/2}(2t-q)^{5/2}} = 0 \tag{146}$$

and

$$\lim_{L\to\infty}\limsup_{t\to\infty}\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\frac{(2t)^2}{q^2(2t-q)^2} = 0. \tag{147}$$

Proof. We use the following basic inequality. Let a, b be integers. Let $s(x)$ be a positive differentiable function in an interval $[a-1, b+1]$ and there is $c \in (a,b)$ such that $s'(x) < 0$ for $x \in [a-1, c)$ and $s'(x) > 0$ for $x \in (c, b+1]$. Then

$$\sum_{q=a}^{b} s(q) \leq \int_{a-1}^{b+1} s(x)dx. \tag{148}$$

As a function of $0 < q < 2t$, $\frac{(2t)^2}{q^{3/2}(2t-q)^{5/2}}$ decreases for $0 < q < \frac{3}{4}t$ and then increases for $\frac{3}{4}t < q < 2t$.

Hence

$$\begin{aligned}
&\lim_{M\to\infty}\limsup_{t\to\infty}\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\frac{(2t)^2}{q^{3/2}(2t-q)^{5/2}}\\
&\leq \lim_{M\to\infty}\limsup_{t\to\infty}\int_L^{2t-Mt^{1/3}}\frac{(2t)^2}{q^{3/2}(2t-q)^{5/2}}dq \\
&= \lim_{M\to\infty}\limsup_{t\to\infty}\left[\frac{4(-3t^2+6tq-2q^2)}{3t(2t-q)^{3/2}q^{1/2}}\right]_L^{2t-Mt^{1/3}} = 0.
\end{aligned} \tag{149}$$

As a function of $0 < q < 2t$, $\frac{(2t)^2}{q^2(2t-q)^2}$ decreases for $0 < q < t$ and then increases for $t < q < 2t$. Hence

$$\begin{aligned}
&\lim_{L\to\infty}\limsup_{t\to\infty}\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\frac{(2t)^2}{q^2(2t-q)^2}\\
&\leq \lim_{L\to\infty}\lim_{t\to\infty}\int_L^{2t-Mt^{1/3}}\frac{(2t)^2}{q^2(2t-q)^2}dq \\
&= \lim_{L\to\infty}\lim_{t\to\infty}\left[\frac{-2t+2q}{q(2t-q)}-\frac{1}{t}\log\left(\frac{2t-q}{q}\right)\right]_L^{2t-Mt^{1/3}} = 0.
\end{aligned} \tag{150}$$

□

Now we prove the main result of Section 3.

Lemma 3.10 (Airy part)**.** *We have*

$$\begin{aligned}
\lim_{L,M\to\infty}\lim_{t\to\infty}\Bigg|&\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log(\kappa_{q-1}^{-2})-\Big(t^2-2tL+\frac{1}{2}L^2\log(2t)-(\frac{1}{2}L^2-\frac{1}{12})\log L+\frac{3}{4}L^2\\
&-\frac{1}{12}M^3-\frac{1}{8}\log M+\frac{1}{24}\log 2\Big)\Bigg| = 0.
\end{aligned} \tag{151}$$

Proof. We first prove that

$$\begin{aligned}
\lim_{L,M\to\infty}\lim_{t\to\infty}\Bigg\{&\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log(\kappa_{q-1}^{-2})-\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\Big(-q\big(-\frac{2t}{q}+\log(\frac{2t}{q})+1\big)+\frac{1}{2}\log(\frac{2t}{q})\\
&+\frac{1}{8(2t-q)}+\frac{1}{12q}\Big)\Bigg\} = 0.
\end{aligned} \tag{152}$$

Using $1+R_{22}^{(1)}(0)-\sqrt{\gamma-1}R_{21}^{(1)}(0) = 1-\frac{1}{8(2t-q)}-\frac{1}{12q} \geq \frac{1}{2}$ and using Lemmas 3.4, 3.6, 3.7, and 3.8, we find that

$$\left|\frac{\sum_{i=2}^5(R_{22}^{(i)}(0)-\sqrt{\gamma-1}R_{21}^{(i)}(0))}{1+R_{22}^{(1)}(0)-\sqrt{\gamma-1}R_{21}^{(1)}(0)}\right| \leq \frac{1}{2} \tag{153}$$

for $L+1 \le q \le [2t - Mt^{1/3} - 1]$, when we take t large enough. Hence using $|\log(1+x)| \le 2|x|$ for $|x| \ge \frac{1}{2}$,

$$\begin{aligned}
&\lim_{L,M\to\infty}\lim_{t\to\infty}\left|\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log\left(1+\frac{\sum_{i=2}^{5}(R_{22}^{(i)}(0)-\sqrt{\gamma-1}R_{21}^{(i)}(0))}{1+R_{22}^{(1)}(0)-\sqrt{\gamma-1}R_{21}^{(1)}(0)}\right)\right| \\
&\le \lim_{L,M\to\infty}\lim_{t\to\infty} 4\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\left|\sum_{i=2}^{5}\left(R_{22}^{(i)}(0)-\sqrt{\gamma-1}R_{21}^{(i)}(0)\right)\right| \qquad (154)\\
&\le \lim_{L,M\to\infty}\lim_{t\to\infty} 4c\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\frac{(2t)^2}{q^{3/2}(2t-q)^{5/2}}+\frac{(2t)^2}{q^2(2t-q)^2}=0
\end{aligned}$$

using (145) and Lemma 3.9.

On the other hand, from Lemma 3.2,

$$\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log(1+R_{22}^{(1)}(0)-\sqrt{\gamma-1}R_{21}^{(1)}(0))=\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log\left(1-\frac{1}{8(2t-q)}-\frac{1}{12q}\right). \qquad (155)$$

Using $-x^2 \le \log(1-x)+x \le 0$ for $0 \le x \le \frac{1}{2}$,

$$\begin{aligned}
&\left|\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log\left(1-\frac{1}{8(2t-q)}-\frac{1}{12q}\right)+\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\left(\frac{1}{8(2t-q)}+\frac{1}{12q}\right)\right| \\
&\le \sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\left(\frac{1}{64(2t-q)^2}+\frac{1}{48(2t-q)q}+\frac{1}{144q^2}\right) \qquad (156)\\
&\le \int_{L}^{2t-Mt^{1/3}}\left(\frac{1}{64(2t-q)^2}+\frac{1}{96t}\left(\frac{1}{2t-q}+\frac{1}{q}\right)+\frac{1}{144q^2}\right)dq \to 0
\end{aligned}$$

as $t\to\infty$ by evaluating the integral explicitly.

Using (154) and (156) in (72), we obtain (152).

Now we compute each term of the sum in (152). Note that $[2t-Mt^{1/3}-1]=2t-Mt^{1/3}-1-\epsilon$ for $0\le\epsilon<1$. We have

$$\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\left(-q\left(-\frac{2t}{q}+1\right)\right)=\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}(2t-q)=\frac{1}{2}(2t-L-1)(2t-L)-\frac{1}{2}(Mt^{1/3}+\epsilon)(Mt^{1/3}+1+\epsilon) \qquad (157)$$

and

$$\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}(-q+\frac{1}{2})\log(2t)=-\frac{1}{2}((2t-Mt^{1/3}-1-\epsilon)^2-L^2)\log(2t) \qquad (158)$$

Since for positive integer m

$$\sum_{k=1}^{m-1}k\log k=m\log((m-1)!)-\sum_{k=1}^{m-1}\log(k!)=m\log(\Gamma(m))-\log G(m+1), \qquad (159)$$

we find that

$$\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\left(q-\frac{1}{2}\right)\log q=\left(2t-Mt^{1/3}-\epsilon-\frac{1}{2}\right)\log\Gamma(2t-Mt^{1/3}-\epsilon)-\log G(2t-Mt^{1/3}-\epsilon+1)$$
$$-\left(L+\frac{1}{2}\right)\log\Gamma(L+1)+\log G(L+2) \tag{160}$$

Note that from Stirling's formula for the Gamma function and the asymptotcs (48) for the Barnes G-function, as $z\to\infty$,

$$\left(z+\frac{1}{2}\right)\log\Gamma(z+1)-\log G(z+2)=\left(\frac{z^2}{2}-\frac{1}{6}\right)\log z-\frac{z^2}{4}+\frac{z}{2}-\frac{1}{4}\log(2\pi)+\frac{1}{12}-\zeta'(-1)+o(1). \tag{161}$$

Now using the fact that

$$\lim_{K\to\infty}\Big|\sum_{q=1}^{K}\frac{1}{q}-\log K-\gamma\Big|=0, \tag{162}$$

where γ is Euler's constant, we obtain

$$\lim_{t\to\infty}\left|\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\frac{1}{8(2t-q)}-\frac{1}{8}\log(2t)+\frac{1}{8}\log(Mt^{1/3})\right|=0 \tag{163}$$

and

$$\lim_{t\to\infty}\left|\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\frac{1}{12q}-\frac{1}{12}\log(2t)-\frac{1}{12}\gamma+\frac{1}{12}\sum_{q=1}^{L}\frac{1}{q}\right|=0. \tag{164}$$

Therefore, using (157), (158), (160), (163), (164), and (161), we obtain

$$\begin{aligned}\lim_{t\to\infty}\Bigg|&\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\left(-q\big(-\frac{2t}{q}+\log\big(\frac{2t}{q}\big)+1\big)+\frac{1}{2}\log\big(\frac{2t}{q}\big)+\frac{1}{8(2t-q)}+\frac{1}{12q}\right)\\&-\left(t^2-2tL+\frac{L^2}{2}\log(2t)\right)\Bigg|\\=&-\frac{1}{12}M^3-\frac{1}{8}\log M+\frac{1}{24}\log 2-\frac{1}{4}\log(2\pi)+\frac{1}{12}-\zeta'(-1)\\&-\frac{1}{2}(L+1)\left(L+\frac{1}{2}\right)\log\Gamma(L+1)+\log G(L+2)+\frac{1}{12}\left(\gamma-\sum_{q=1}^{L}\frac{1}{q}\right).\end{aligned} \tag{165}$$

Hence using (161) and (162) again, we obtain (151). □

4 Proof of (9) in Theorem 1.1: computation of $F(x)$

Recall equation (37) which we rewrite here:

$$\log(e^{-t^2}D_n)=-t^2+\underbrace{\log(D_L)}_{\text{exact part}}+\underbrace{\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log(\kappa_{q-1}^{-2})}_{\text{Airy part}}+\underbrace{\sum_{q=[2t-Mt^{1/3}]}^{[2t+xt^{1/3}]}\log(\kappa_{q-1}^{-2})}_{\text{Painlevé part}}. \tag{166}$$

For the Painlevé part, from [2],

$$
\begin{aligned}
\lim_{t\to\infty} \sum_{q=[2t-Mt^{1/3}]}^{[2t+xt^{1/3}]} \log(\kappa_{q-1}^{-2}) &= -\int_{-M}^{x} R(y)dy \\
&= -\int_{-M}^{x} \left(R(y) - \frac{1}{4}y^2 + \frac{1}{8y}\right)dy + \frac{1}{12}x^3 - \frac{1}{8}\log|x| \\
&\quad + \frac{1}{12}M^3 + \frac{1}{8}\log M.
\end{aligned} \tag{167}
$$

By combining (167), (151), and (49), we obtain

$$
\begin{aligned}
&\lim_{M,L\to\infty} \lim_{t\to\infty} \log(e^{-t^2} D_n(t)) \\
&= -\int_{-M}^{x} \left(R(y) - \frac{1}{4}y^2 + \frac{1}{8y}\right)dy + \frac{1}{12}x^3 - \frac{1}{8}\log|x| + \frac{1}{24}\log 2 + \zeta'(-1).
\end{aligned} \tag{168}
$$

5 Proof of (10) in Theorem 1.1: computation of $E(x)$

Let

$$
I_j(2t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2t\cos\theta} e^{ij\theta} d\theta. \tag{169}
$$

Set (see [4])

$$
D_\ell^{++}(t) = \det(I_{j-k}(2t) + I_{j+k+2}(2t))_{0\le j,k\le \ell-1}, \tag{170}
$$

$$
D_\ell^{-+}(t) = \det(I_{j-k}(2t) + I_{j+k+1}(2t))_{0\le j,k\le \ell-1}. \tag{171}
$$

It is shown in Corollary 7.2 of [5] that for

$$
\ell = [t + \frac{x}{2}t^{1/3}], \tag{172}
$$

where x lies in a compact subset of $\mathbb{R}$, we have

$$
\lim_{t\to\infty} |e^{-t^2/2} D_{\ell-1}^{++}(t) - F(x)E(x)| = 0, \tag{173}
$$

$$
\lim_{t\to\infty} |e^{-t^2/2-t} D_{\ell}^{-+}(t) - F(x)E(x)| = 0. \tag{174}
$$

In [5], the above results are shown (with $\widehat{x}$ in place of x) for the alternate scaling $t = \ell - \frac{\widehat{x}}{2}\ell^{1/3}$. With the above scaling $\ell = [t + \frac{x}{2}t^{1/3}]$, we find $\widehat{x} = x(1 + \frac{x}{2}t^{-2/3})^{-1/3}$. Since x is in a compact set, and $E(x)$ and $F(x)$ are continuous, the above results follow. From equations (173) and (174), for a fixed $x \in \mathbb{R}$,

$$
F(x)^2 E(x)^2 = \lim_{t\to\infty} e^{-t^2-t} D_{\ell-1}^{++} D_\ell^{-+}, \qquad \ell = [t + \frac{x}{2}t^{1/3}]. \tag{175}
$$

Let $\pi_j(z;t)$ be the monic orthogonal polynomial of degree k with respect to the weight $\frac{1}{2\pi}e^{2t\cos\theta}d\theta$ on the unit circle, as introduced in Section 1. It is shown in Corollary 2.7 of [4] that (cf. (32) above)

$$
e^{-t^2/2} D_\ell^{++}(t) = \prod_{j=\ell}^{\infty} \kappa_{2j+2}^2(t)(1 - \pi_{2j+2}(0;t)) = \prod_{j=\ell}^{\infty} \frac{\kappa_{2j+1}^2}{1 + \pi_{2j+2}(0;t)}, \tag{176}
$$

$$
e^{-t^2/2-t} D_\ell^{-+}(t) = \prod_{j=\ell}^{\infty} \kappa_{2j+1}^2(t)(1 + \pi_{2j+1}(0;t)) = \prod_{j=\ell}^{\infty} \frac{\kappa_{2j}^2}{1 - \pi_{2j+1}(0;t)}, \tag{177}
$$

where the last equalities in (176) and (177) use the basic identity (see e.g [23])

$$1-\pi_k^2(0;t)=\frac{\kappa_{k-1}^2}{\kappa_k^2}. \tag{178}$$

Using equations (176) and (177), we can write

$$D_{\ell-1}^{++}=D_{L-1}^{++}\prod_{j=L}^{\ell-1}\frac{D_j^{++}}{D_{j-1}^{++}}=D_{L-1}^{++}\prod_{j=L}^{\ell-1}\kappa_{2j-1}^{-2}(t)(1+\pi_{2j}(0;t)), \tag{179}$$

$$D_{\ell}^{-+}=D_{L}^{-+}\prod_{j=L+1}^{\ell}\frac{D_j^{-+}}{D_{j-1}^{-+}}=D_{L}^{-+}\prod_{j=L+1}^{\ell}\kappa_{2j-2}^{-2}(t)(1-\pi_{2j-1}(0;t)), \tag{180}$$

and hence we have

$$D_{\ell-1}^{++}D_{\ell}^{-+}=D_{L-1}^{++}D_{L}^{-+}\prod_{k=2L-1}^{2\ell-2}\kappa_k^{-2}(t)\cdot\prod_{j=L}^{\ell-1}\left[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))\right]. \tag{181}$$

Using (recall that $D_\ell=\det(I_{j-k}(2t))_{0\le j,k\le \ell-1}$ is the $\ell\times\ell$ Toeplitz determinant given in (27))

$$\frac{D_a}{D_b}=\prod_{j=b}^{a-1}\kappa_j^{-2}(t), \tag{182}$$

we find that

$$D_{\ell-1}^{++}D_{\ell}^{-+}=D_{L-1}^{++}D_{L}^{-+}\frac{D_{2\ell-1}}{D_{2L-1}}\prod_{j=L}^{\ell-1}[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))]. \tag{183}$$

Inserting this equation into (175), and using (29),

$$F(x)^2E(x)^2=F_2(x)\lim_{t\to\infty}e^{-t}\frac{D_{L-1}^{++}D_{L}^{-+}}{D_{2L-1}}\prod_{j=L}^{\ell-1}[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))]. \tag{184}$$

Hence we find (cf. (36))

$$E(x)^2=\lim_{t\to\infty}e^{-t}\frac{D_{L-1}^{++}D_{L}^{-+}}{D_{2L-1}}\prod_{j=L}^{\ell-1}[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))],\qquad \ell=\left[t+\frac{x}{2}t^{1/3}\right]. \tag{185}$$

In analogy to equation (37), we write

$$\begin{aligned}2\log E(x)=\lim_{L,M\to\infty}\lim_{t\to\infty}\Bigg(&-t+\underbrace{\log\frac{D_{L-1}^{++}D_{L}^{-+}}{D_{2L-1}}}_{\text{Exact part}}+\underbrace{\sum_{j=L}^{[t-\frac{M}{2}t^{1/3}]}\log[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))]}_{\text{Airy part}}\\&+\underbrace{\sum_{j=[t-\frac{M}{2}t^{1/3}+1]}^{[t+\frac{x}{2}t^{1/3}-1]}\log[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))]}_{\text{Painlevé part}}\Bigg).\end{aligned} \tag{186}$$

We compute each term as in the case of $\log F(x)$.

Lemma 5.1. (The Painlevé part) *We have for $x<0$ and $M>0$,*

$$\begin{aligned}\lim_{t\to\infty} &\sum_{j=[t-\frac{M}{2}t^{1/3}+1]}^{[t+\frac{x}{2}t^{1/3}-1]} \log[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))] \\ &= \int_{-M}^{x}\left(q(y)-\sqrt{\frac{-y}{2}}\right)dy+\frac{\sqrt{2}}{3}M^{3/2}-\frac{\sqrt{2}(-x)^{3/2}}{3}.\end{aligned} \tag{187}$$

Proof. This follows from the following results in Corollaries 7.1 of [5]: for x in a compact subset of $\mathbb{R}$,

$$\lim_{t\to\infty}\left|\sum_{j=[t+\frac{x}{2}t^{1/3}]}^{\infty}\log(1+\pi_{2j+2}(0;t))+\log E(x)\right|=0, \tag{188}$$

$$\lim_{t\to\infty}\left|\sum_{j=[t+\frac{x}{2}t^{1/3}]}^{\infty}\log(1-\pi_{2j+1}(0;t))+\log E(x)\right|=0. \tag{189}$$

We remark that in [5], the notations $v(x)=-R(x)$ and $u(x)=-q(x)$ are used. □

Lemma 5.2. (The Airy part)

$$\lim_{L,M\to\infty}\lim_{t\to\infty}\left|\sum_{j=L}^{[t-\frac{M}{2}t^{1/3}]}\log[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))]-\left(t-\frac{\sqrt{2}}{3}M^{3/2}-\left(2L-\frac{1}{2}\right)\log 2\right)\right|=0. \tag{190}$$

Proof. Using (64),

$$\sum_{j=L}^{[t-\frac{M}{2}t^{1/3}]}\log[(1+\pi_{2j}(0))(1-\pi_{2j+1}(0))]=\sum_{q=2L}^{2[t-\frac{M}{2}t^{1/3}]+1}\log\left(1+\sqrt{\frac{\gamma-1}{\gamma}}R_{11}(0;q)-\frac{1}{\sqrt{\gamma}}R_{12}(0;q)\right). \tag{191}$$

Using (70) and the fact that $\sqrt{\gamma-1}R_{11}^{(1)}(0)-R_{12}^{(1)}(0)=0$, which follows from Lemma 3.2, this equals

$$\sum_{q=2L}^{2[t-\frac{M}{2}t^{1/3}]+1}\log\left(1+\sqrt{\frac{\gamma-1}{\gamma}}\right)+\sum_{q=2L}^{2[t-\frac{M}{2}t^{1/3}]+1}\log\left(1+\frac{\sum_{i=2}^{5}(\sqrt{\gamma-1}R_{11}^{(i)}(0;q)-R_{12}^{(i)}(0;q))}{\sqrt{\gamma}+\sqrt{\gamma-1}}\right). \tag{192}$$

From Lemmas 3.4, 3.6, 3.7, and 3.8, the same estimate as in (145) holds for $\sum_{i=2}^{5}(\sqrt{\gamma-1}R_{11}^{(i)}(0;q)-R_{12}^{(i)}(0;q))$ for $L+1\le q\le[2t-Mt^{1/3}-1]$. Therefore the same argument as in (154) implies that the second sum in (192) vanishes in the limit. Therefore,

$$\lim_{L,M\to\infty}\lim_{t\to\infty}\left|\sum_{j=L}^{[t-\frac{M}{2}t^{1/3}]}\log[(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))]-\sum_{q=2L}^{2[t-\frac{M}{2}t^{1/3}]+1}\log\left(1+\sqrt{\frac{2t-q}{2t}}\right)\right|=0. \tag{193}$$

We use the Euler-Maclaurin summation formula

$$\sum_{k=a}^{b}f(k)=\int_a^b f(x)dx+\frac{f(a)+f(b)}{2}+\frac{f'(b)-b'(a)}{6\cdot 4!}+\text{Err},\qquad |\text{Err}|\le\frac{2}{(2\pi)^2}\int_a^b|f^{(3)}(x)|dx. \tag{194}$$

Set $f(q)=\log\left(1+\sqrt{\frac{2t-q}{2t}}\right)$ and write $[t-\frac{M}{2}t^{1/3}]=t-\frac{M}{2}t^{1/3}-\epsilon$ where $0\le\epsilon<1$. Then

$$\lim_{t\to\infty}\frac{f(2t-Mt^{1/3}-2\epsilon+1)+f(2L)}{2}=\frac{1}{2}\log 2, \tag{195}$$

and

$$\lim_{t\to\infty} \frac{f'(2t - Mt^{1/3} - 2\epsilon + 1) + f'(2L)}{6 \cdot 4!} = 0. \tag{196}$$

Also using $f^{(3)}(q) \geq 0$,

$$|\mathrm{Err}| \leq \frac{2}{(2\pi)^2} \int_{2L}^{2t-Mt^{1/3}-2\epsilon+1} |f^{(3)}(q)|dq = \frac{2}{(2\pi)^2}(f''(2t - Mt^{1/3} - 2\epsilon + 1) - f''(2L)) \to 0, \tag{197}$$

as $t \to \infty$. Finally, changing variables to $w = 1 + \sqrt{\frac{2t-q}{2t}}$ and setting $\delta_1 = 1 + \sqrt{\frac{t-L}{t}}$ and $\delta_2 = \sqrt{\frac{Mt^{1/3}+2\epsilon-1}{2t}}$,

$$\begin{aligned}\int_{2L}^{2t-Mt^{1/3}-2\epsilon+1} \log\left(1 + \sqrt{\frac{2t-q}{2t}}\right) dq &= 4t \int_{\delta_1}^{1+\delta_2} (1-w)\log(w)dw \\ &= 4t\left[\left(w - \frac{1}{2}w^2\right)\log w + \frac{1}{4}w^2 - w\right]_{\delta_1}^{1+\delta_2} \\ &= t - \frac{\sqrt{2}}{3}M^{3/2} - 2L\log 2 + O\left(\frac{1}{t^{1/3}}\right).\end{aligned} \tag{198}$$

Hence the result follows. □

Lemma 5.3. (The exact part) We have

$$\lim_{L\to\infty}\lim_{t\to\infty}\left[\log\frac{D_{L-1}^{++}D_L^{-+}}{D_{2L-1}} - (2L-1)\log 2\right] = 0. \tag{199}$$

Proof. Note (see [4]) that

$$\begin{aligned} D_{L-1}^{++} &= \det\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}[e^{i(j-k)\theta} - e^{i(j+k+2)\theta}]e^{2t\cos\theta}d\theta\right]_{0\leq j,k\leq L-2} \\ &= \det\left[\frac{1}{i\pi}\int_{-\pi}^{\pi} e^{i(j+1)\theta}\sin((k+1)\theta)e^{2t\cos\theta}d\theta\right]_{0\leq j,k\leq L-2} \\ &= \det\left[\frac{2}{\pi}\int_0^{\pi}\frac{\sin((j+1)\theta)}{\sin\theta}\cdot\frac{\sin((k+1)\theta)}{\sin\theta}\sin^2\theta e^{2t\cos\theta}d\theta\right]_{0\leq j,k\leq L-2}. \end{aligned} \tag{200}$$

Changing variables to $x = \cos\theta$ and noting that $\frac{\sin((j+1)\theta)}{\sin\theta}$ is a polynomial in x of the form $2^j x^j + \cdots$ (a constant multiple of the Chebyshev polynomial of the second kind), we have

$$\begin{aligned} D_{L-1}^{++} &= \det\left[\frac{2^{j+k+1}}{\pi}\int_{-1}^{1} x^{j+k}\sqrt{1-x^2}e^{2tx}dx\right]_{0\leq j,k\leq L-2} \\ &= \left(\frac{2}{\pi}\right)^{L-1}\frac{2^{(L-1)(L-2)}}{(L-1)!}\int_{[-1,1]^{L-1}}\prod_{1\leq k<\ell\leq L-1}|x_k - x_\ell|^2\prod_{j=1}^{L-1}\sqrt{1-x_j^2}e^{2tx_j}dx_j. \end{aligned} \tag{201}$$

A steepest descent analysis yields that

$$\lim_{t\to\infty} D_{L-1}^{++}\cdot\left(\frac{e^{2t(L-1)}}{t^{(L-1)^2+(L-1)/2}\pi^{(L-1)}(L-1)!}\int_{[0,\infty]^{L-1}}\prod_{1\leq k<\ell\leq L-1}|y_k - y_\ell|^2\cdot\prod_{j=1}^{L-1}\sqrt{y_j}e^{-y_j}dy_j\right)^{-1} = 1. \tag{202}$$

The multiple integral is another Selberg integral (corresponding to a Laguerre ensemble) which is evaluated explicitly as (see (17.6.5) of [22])

$$\int_{[0,\infty]^{L-1}} \prod_{1\le k<\ell\le L-1} |y_k - y_\ell|^2 \cdot \prod_{j=1}^{L-1} \sqrt{y_j} e^{-y_j} dy_j = (L-1)! \frac{G(L+\frac{1}{2})G(L)}{G(\frac{3}{2})}. \tag{203}$$

Hence

$$\lim_{t\to\infty} D_{L-1}^{++} \cdot \left(\frac{e^{2t(L-1)}}{t^{(L-1)^2+(L-1)/2}\pi^{(L-1)}} \cdot \frac{G(L+\frac{1}{2})G(L)}{G(\frac{3}{2})} \right)^{-1} = 1. \tag{204}$$

Similarly,

$$\begin{aligned} D_L^{-+} &= \det \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{i(j-k)\theta} + e^{i(j+k+1)\theta}) e^{2t\cos\theta} d\theta \right]_{0\le j,k\le L-1} \\ &= \det \left[\frac{1}{\pi} \int_{-\pi}^{\pi} e^{i(j+\frac{1}{2})\theta} \cos\left(\left(k+\frac{1}{2}\right)\theta \right) e^{2t\cos\theta} d\theta \right]_{0\le j,k\le L-1} \\ &= \det \left[\frac{1}{\pi} \int_{-\pi}^{\pi} \cos\left(\left(j+\frac{1}{2}\right)\theta \right) \cos\left(\left(k+\frac{1}{2}\right)\theta \right) e^{2t\cos\theta} d\theta \right]_{0\le j,k\le L-1} \\ &= \det \left[\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos\left((j+\frac{1}{2})\theta\right)}{\cos\frac{\theta}{2}} \frac{\cos\left((k+\frac{1}{2})\theta\right)}{\cos\frac{\theta}{2}} \cos^2\left(\frac{\theta}{2}\right) e^{2t\cos\theta} d\theta \right]_{0\le j,k\le L-1}. \end{aligned} \tag{205}$$

Setting $x = \cos\theta$ and noting that $\frac{\cos((j+\frac{1}{2})\theta)}{\cos\frac{\theta}{2}}$ is a constant multiple of a Jacobi polynomial $P_j^{(-\frac{1}{2},\frac{1}{2})}(x)$ of the form $2^j x^j + \cdots$, we have

$$\begin{aligned} D_L^{-+} &= \frac{2^{L(L-1)}}{\pi^L} \det \left[\int_{-1}^{1} x^{j+k} \sqrt{\frac{1+x}{1-x}} e^{2tx} dx \right]_{0\le j,k\le L-1} \\ &= \frac{2^{L(L-1)}}{\pi^L L!} \int_{[-1,1]^L} \prod_{1\le k<\ell\le L} |x_k - x_\ell|^2 \prod_{j=1}^{L} \sqrt{\frac{1+x_j}{1-x_j}} e^{2tx_j} dx_j. \end{aligned} \tag{206}$$

A steepest-descent analysis implies that

$$\lim_{t\to\infty} D_L^{-+} \cdot \left(\frac{e^{2tL}}{t^{L^2-L/2}\pi^L L!} \int_{[0,\infty]^L} \prod_{1\le k<\ell\le L} |y_k - y_\ell|^2 \prod_{j=1}^{L} y_j^{-1/2} e^{-y_j} dy_j \right)^{-1} = 1. \tag{207}$$

The multiple integral is also a Selberg integral (see (17.6.5) of [22]), and we obtain

$$\lim_{t\to\infty} D_L^{-+} \cdot \left(\frac{e^{2tL}}{t^{L^2-L/2}\pi^L} \cdot \frac{G(L+1)G(L+\frac{1}{2})}{G(\frac{1}{2})} \right)^{-1} = 1. \tag{208}$$

Using (204), (208), and

$$\lim_{t\to\infty} D_{2L-1} \cdot \left(\frac{e^{4tL-2t}}{(2t)^{2L^2-2L+\frac{1}{2}}(2\pi)^{L-\frac{1}{2}}} G(2L) \right)^{-1} = 1, \tag{209}$$

which follows from Section 2 (the exact part for $F_2(x)$), we obtain

$$\lim_{t\to\infty} \left[\log \frac{D_{L-1}^{++} D_L^{-+}}{D_{2L-1}} - \log \left(\frac{2^{2L^2-L}}{\pi^{L-\frac{1}{2}}} \cdot \frac{[G(L+\frac{1}{2})]^2 G(L) G(L+1)}{G(\frac{1}{2})G(\frac{3}{2})G(2L)} \right) \right] = 0. \tag{210}$$

The result is now proved using the properties (46) and (48) for the Barnes G-function. □

Combining equation (186) and Lemmas 5.1, 5.2, and 5.3, we obtain

$$2\log E(x) = \int_{-\infty}^{x} \left(q(y) - \sqrt{\frac{-y}{2}} \right) dy - \frac{\sqrt{2}(-x)^{3/2}}{3} - \frac{1}{2}\log 2. \tag{211}$$

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