Geodesic dynamo chaotic flows and non-Anosov maps in twisted magnetic flux tubes

by

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Abstract

Recently Tang and Boozer [**Phys. Plasmas (2000)**], have investigated the anisotropies in magnetic field dynamo evolution, from local Lyapunov exponents, giving rise to a metric tensor, in the Alfven twist in magnetic flux tubes (MFTs). Thiffeault and Boozer [**Chaos**(2001)] have investigated the how the vanishing of Riemann curvature constrained the Lyapunov exponential stretching of chaotic flows. In this paper, Tang-Boozer-Thiffeault differential geometric framework is used to investigate effects of twisted magnetic flux tube filled with helical chaotic flows on the Riemann curvature tensor. When Frenet torsion is positive, the Riemann curvature is unstable, while the negative torsion induces an stability when time $t \rightarrow \infty$. This enhances the dynamo action inside the MFTs. The Riemann metric, depends on the radial random flows along the poloidal and toroidal directions. The Anosov flows has been applied by Arnold, Zeldovich, Ruzmaikin and Sokoloff [**JETP (1982)**] to build a uniformly stretched dynamo flow solution, based on Arnold's Cat Map. It is easy to show that when the random radial flow vanishes, the magnetic field vanishes, since the exponential Lyapunov stretches vanishes. This is an example of the application of the Vishik's anti-fast dynamo theorem in the magnetic flux tubes. Geodesic flows of both Arnold and twisted MFT dynamos are investigated. It is shown that a constant random radial flow can be obtained from the geodesic equation. Throughout the paper one assumes, the reasonable plasma astrophysical hypothesis of the weak torsion. Pseudo-Anosov dynamo flows and maps have also been addressed by Gilbert [**Proc Roy Soc A London (1993)**].

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I Introduction

To talk about the importance of the developement of the concepts of Riemannian geometry [1] to physics, would be enough to mention the advances in Einstein's gravitational theory, more well known as the general theory of relativity [2]. More recently, uses in other parts of physics such as string theory of defects and solid state theory [3] have also experienced the same success. Yet more recently, Thiffeault , Tang and Boozer, on a series of papers [4, 5, 6, 7] have investigate the constraints experienced by chaotic flows, when the Riemann curvature tensor vanishes. In this paper, a slightly more general application of these ideas are applied to helical chaotic flows inside the twisted magnetic flux tubes [8, 9] is given. The role of random radial flows in Lyapunov exponential stretching, so fundamental to the kinematic dynamo problem, is investigated throughout the paper. The investigation of Anosov diffeomorpisms [9], an important mathematical tool from the theory of dynamical systems, has been often used, in connection with the investigation of dynamo flows and maps [9] such as the Arnold's Cat Map [9] on the torus, useful in mixing [10] problems in the physics of fluids [4]. One of the main properties of the Anosov maps is that they yield Lyapunov exponential of the chaotic exponential stretching, which are constant everywhere [5]. In their paper, Thiffeault and Boozer [5], obtained geometrical constraints in two and three dimensions, with coordinates $\mathbf{x} \in \mathbf{R}^2$ and $\mathbf{x}' \in \mathbf{R}^3$. Consequences of the vanishing of the Riemann curvature tensors, or Riemann-flat conditions in the mathematicians jargon, includes constraining the modification of the growth rate of magnetic fields in the kinematic dynamo problem or the mixing properties of chaotic advection-diffusive equation. They conclude that, the Lyapunov exponents are locally small in regions where, Riemann curvature of the stable manifold, is large. Though Euclidean metrics were used in their work to allow for the vanishing of the Riemann curvature, here this hypothesis is relaxed, which still allows one to obtain similar results for the flux tube endowed with chaotic helical flows. The Riemann and Ricci curvatures are computed in terms of the Lyapunov exponential stretching and establishing the conditions on the Frenet torsion of the tube axis to obtain positive Lyapunov exponents which leads to the fast dynamo action for chaotic flows permeatting the tube.

This last condition is fundamental for dynamo action [8]. Random radial flows [10, 11] are shown to be fundamental, for the chaotic flows to experience a non-vanishing exponential stretching. Since these MFTs are fundamental in astrophysical plasmas and solar chaotic plasmas in tokamaks and stellarators, as well in the Perm torus of liquid sodium dynamo [12], the better understanding of chaotic flow, seems to be useful to future developement of plasma physics and dynamo theory. Therefore a good understanding of the behaviour of these rotating torus flows is of utmost importance, in future dynamo experiments. This paper is organised as follows. In section 2, special types of chaotic flows in twisted MFTs are presented such as the torus automorphisms, called Arnold's Cat Map, which led Arnold [9] to build a uniform stretching dynamo solution of self-induction equation. In this case though the Jacobian differs from the Arnold, Anosov map one, the same eigenvalues are obtained, and therefore the dynamo chaotic flow is stretched along one direction and compressed along the other. Section 3 generalizes the Anosov flows to the case where the stretch along toroidal direction is not constant and therefore the Lyapunov metric of the tube leads to the computation of Riemann and Ricci curvature in the twisted MFTs metric. This kind of twisted geometry is then used to established constraints along the section to established the stability relation to the torsion of the tube, which in turn is proportional to the twist. Section 4 presents the discussion and conclusions.

II Hyperbolic dynamo maps and flows in MFTs

This section contains a brief review and some new material on hyperbolic flows and dynamo maps. While is not the main section of the paper, it is supposed to pave the way to the next section where the main results are given. Let us start by defining what exactly is an hyperbolic map. Anosov maps are actually an example of hyperbolic map with extra constraints which one shall not bother here. An hyperbolic map on a torus may be defined as an automorphism where the determinant of its Jacobian is equal to one. This constraint shall be used in this section, in analogy with Thiffeault and Boozer [4] constraints to place constraints on the dynamos and Riemannian geometry of the tubes. Let us consider the following two-dimensional twist map [13] Jacobian matrix

$$\mathbf{J_{twist}} = \begin{pmatrix} 1 & -\tau_0 \\ 0 & K_0 \end{pmatrix} \tag{II.1}$$

Here K_0 is the toroidal stretching along the tube metric [14] given by

$$dl^2 = dr^2 + r^2 d{\theta_R}^2 + K^2(r,s)ds^2 \tag{II.2}$$

where of course here, the internal cross-section radius r is constant for the two dimensional Jacobian and the twist transformation is

$$\theta(s) := \theta_R - \int \tau(s)ds \tag{II.3}$$

where the integral in this expression is the total torsion. Here, since one is considering helical chaotic flows, the Frenet torsion and curvature of the tube axis are constants and fixed to be equal. As one shall note in the next section the introduction of coordinate r into the stretching $K(r,s) = 1 - r\kappa(s)cos\theta$ allows one to has Riemann and Ricci curvature that do not vanish identically. Of course here, one have considered that K_0 is a constant or uniform stretching so important in chaos [14]. A small lemma can be proved here by constrained the Jacobian transformation to be hyperbolic. Thus

lemma: Hyperbolic maps with constant stretch in the twist MFTs, leads necessarilly to a thin tube. **Proof**: This comes immeadiatly from the determinant of matrix (II.1) equals to one

$$Det[\mathbf{J_{twist}}] = K_0 = 1 \tag{II.4}$$

which by examining the expression for $K(r,s)$ shows that $K(0,s) = 1$ is the thin tube approximation, as one wishes to prove.Let us now compute the eigenvalue problem. Just to show that the twist tube Jacobian above is a general case of the torus automorphism by Arnold, let us consider a special transformation where the twist is fixed while the radius is contracted to half his size and the stretch is double, similar to what happens in the stretch-twist-fold (STF) dynamo mechanism proposed by Vainshtein and Zeldovich [15]. Mathematically this transformation is

$$r' = \frac{1}{2}r \tag{II.5}$$

$$s' = 2s \tag{II.6}$$

while $\theta' = \theta_R$. Besides the torsion τ_0 is normalized to one, since one does not want to deal with untwisted tubes. Therefore the three-dimensional Jacobian

$$\mathbf{J_{3D}} = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & 1 & -\tau_0 \\ 0 & 0 & K_0 \end{pmatrix} \tag{II.7}$$

where K_1 is the stretching (contraction) of the radial direction, takes the form

$$\mathbf{J_{3D}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \tag{II.8}$$

By taking the determinant of this Jacobian one obtains it is equal to one, or hyperbolic, and besides the eigenvalue problem

$$Det[\mathbf{J_{3D}} - \lambda\mathbf{1}] = 0 \tag{II.9}$$

where $\mathbf{1}$ is the unity matrix

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{II.10}$$

λ being the eigenvalue. The determinant leads to the following second-order algebraic equation

$$\lambda^2 - 3\lambda + 1 = 0 \tag{II.11}$$

Note that this equation leads to the Arnold eigenvalues

$$\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2} \tag{II.12}$$

where $\lambda_+ > 0$ represents the stretching, while $\lambda_- < 0$ is the contraction of the flow. In the Arnold's case [13] the stretching is uniform. In the next section, one shall be concerned with the non-uniform stretching where the stretching Riemann metric factors, do depend upon the coordinates. Another interesting lemma can be proved for the three-dimensional matrix, as

lemma:

The hyperbolicity condition applied to twist map in $\mathbf{R}^3$, given by the above Jacobian, implies that the magnetic flux tubes stretching and contraction obeys the following conditions: $K_0 = {K_1}^{-1}$, which tell us that the stretching is the inverse of contraction as foreseen by our physical intuition.

Proof:

The proof is trivial, and obtained by applying the $Det\mathbf{J} = 1$ condition to the three-dimensional Jacobian $\mathbf{J_{3D}}$ above. This yields that $K_0 K_1 = 1$ as we wish to prove.

The domain of the flow is given by a compact Riemannian manifold represented as the product of a torus $\mathbf{T}^2$ and the closed interval $[0,1]$ of $0 \leq z \leq 1$ [9]. This results in the Arnold Riemann metric

$$ds^2 = e^{-\lambda z} dp^2 + e^{\lambda z} dq^2 + dz^2 \tag{II.13}$$

which represents the stretching and contraction in distinct Euclidean directions p and q , respectively induced by the eigenvalues $\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}$ also corresponding to magnetic eigenvectors. The stationary dynamo flow considered by Arnold was given by the simple uniformly stretching flow $\mathbf{v} = (0, 0, v)$ in (p, q, z) coordinates. More recent attempts to build a dynamo action by making use of compact Riemannian geometry includes the case of the fast dynamo of Chiconne and Latushkin [16], and the conformally stretched fast dynamo by Garcia de Andrade [17]. Anosov map in Arnold's cat map, is given by the cat dynamo Jacobian matrix [4]

$$\mathbf{J_{cat}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{II.14}$$

Note that this Jacobian is distinct to the previous Jacobians but satisfies the hyperbolicity condition and leads to the same eigenvalues as before. Let us now compute the Riemann curvature of the Arnold metric

$$R_{1212} = \lambda^2 \tag{II.15}$$

$$R_{1313} = -\lambda^2 \tag{II.16}$$

$$R_{2323} = -\lambda^2 \tag{II.17}$$

for the Riemann curvature tensor, while the Ricci tensor components are

$$R_{33} = R = 2\lambda^2 \tag{II.18}$$

the determinant of the metric $g = 1$, implies that the flow is incompressible as considered previously by Arnold. Note as well, that, as usual in Anosov systems the Riemann curvature is constant.

III Riemann curvature from Lyapunov exponents and dynamo action

In this section it is shown that in regions of weak torsion the Riemann and Ricci curvatures acquire simple forms, in terms of of exponential stretching and torsion. Negative torsion in helical chaotic flows, leads to singularities in curvatures in the $t \to \infty$ time limit. A simple examination of the Jacobians considered in the last section allows us to see that the Riemann metric of the flux tubes vanish for a constant stretch-twist and contraction. However since curvature or folding is a fundamental process in the STF dynamo mechanism, in this section it is shown that the consideration of a more general Jacobian makes the manifold acquire a curved Riemannian metric distinct from the Euclidean norm metrics considered so far. So, for a non-uniform stretching [14] usually found in general chaotic flows the Riemann and Ricci curvatures do not vanish identically. Let us start computing the curvatures corresponding to the general Riemann metric of the twisted MFT above. With the aid of an adaptation of the tensor package the computation of the curvatures is neither tedious nor long and results in

$$R_{1313} = \frac{1}{4}\frac{{\tau_0}^2 cos^2(\theta)}{(1-\tau_0 r cos\theta)} \tag{III.19}$$

$$R_{1323} = -\frac{1}{8}\frac{{\tau_0}^2 r sin2\theta}{(1-\tau_0 r cos\theta)} \tag{III.20}$$

$$R_{2323} = \frac{1}{4}\frac{{\tau_0}^2 r^2 sin2\theta}{(1-\tau_0 r cos\theta)} \tag{III.21}$$

for the Riemann curvature tensor, while the Ricci tensor components are

$$R_{11} = -\frac{1}{4}\frac{{\tau_0}^2 cos^2\theta}{(1-2\tau_0 r cos\theta + {\tau_0}^2 r cos\theta)} \tag{III.22}$$

$$R_{12} = -\frac{1}{8}\frac{{\tau_0}^2 r sin2\theta}{(1-2\tau_0 r cos\theta + {\tau_0}^2 r cos\theta)} \tag{III.23}$$

$$R_{33} = -\frac{1}{4}\frac{{\tau_0}^2}{(1-\tau_0 r cos\theta)} \tag{III.24}$$

$$R_{22} = -\frac{1}{8}\frac{{\tau_0}^2 r sin2\theta}{(1-2\tau_0 r cos\theta + {\tau_0}^2 r cos\theta)} \tag{III.25}$$

Most of these curvatures can be easily simplified in the weak torsion case. Let us now give a more dynamical character to these expressions by considering the Thiffeault-Boozer Riemann metric relation with the Lyapunov exponents of exponential stretching by expressing this metric in terms of the directions $\mathbf{t}$, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ along the curved tube as

$$g_{ij} = \Lambda_r \mathbf{e}_r \mathbf{e}_r + \Lambda_\theta \mathbf{e}_\theta \mathbf{e}_\theta + \Lambda_s \mathbf{t}\mathbf{t} \tag{III.26}$$

where $(i, j = r, \theta, s)$ and Λ_i are the Lyapunov numbers which are all positive or null. The Lyapunov exponents are given by

$$\lambda^{\infty}{}_i = lim_{t\rightarrow\infty}(\frac{ln\Lambda_i}{2t}) \tag{III.27}$$

The infinite symbol over the Lyapunov exponent indicates that this is a true Lyapunov exponent obtained as the limit of the finite-time Lyapunov exponent λ_i. Let us now compute the values of the Lyapunov exponents which are fundamental for stretching and dynamos, in terms of a random radial flow

$$\lambda_i = lim_{t\to\infty}(\frac{ln\Lambda_i}{2t}) \tag{III.28}$$

$$<r>=\int <v_r> dt \tag{III.29}$$

where here one shall consider that on a finite-time period, the random flow can be considered as approximately constant which reduces the last expression to

$$<r>=<v_r> t \tag{III.30}$$

Let us now compute the Lyapunov exponents of the curved Riemannian flux tubes in terms of the random flows as

$$\lambda_r = lim_{t\to\infty}(\frac{ln1}{2t}) = 0 \tag{III.31}$$

$$\lambda_\theta = lim_{t\to\infty}(\frac{r}{t}) =<v_r> \tag{III.32}$$

and finally

$$\lambda_s = lim_{t\to\infty}(\frac{lnK(r,s)}{2t}) \approx lim_{t\to\infty}(\frac{-\tau_0 r sin\theta}{2t}) \approx -\tau_0 <v_r> sin\theta \tag{III.33}$$

By following the computations by Tang and Boozer [5] on the magnetic fields by the self-induction equation one can say that the corresponding magnetic field components maybe written in terms of the Lyapunov exponents as

$$B_\theta \approx e^{2\lambda_\theta t} = e^{<v_r>t} \tag{III.34}$$

$$B_s \approx e^{\lambda_s t} = e^{-\tau_0 <v_r> sin\theta t} \tag{III.35}$$

while the radial magnetic field does not depend on time , which allows us to say that this is due to the confinement of the radial flow on the tube, actually in the force-free case considered by Ricca [18] the flux tube radial magnetic component B_r is assumed to vanish. From the last two expressions it is easy to observe that the magnetic field is stationary in the absence of random radial flows, and there is no fast dynamo action as well as no stretching due to the vanishing of the Lyapunov exponents. This phenomenon is actually the Vishik's anti-fast dynamo theorem [19] where no fast-dynamo action can be obtained, in non-stretching flows. Fast dynamo is obtained when the random flow has a positive average velocity. Note that in this case at least the B_s toroidal field grows in time, while the poloidal field is spatially periodic and may even grow in time in the case torsion is negative. To simplify matters let us now compute the Riemann and Ricci curvatures for the chaotic flow tube metric

$$ds_0{}^2 = dr^2 + e^{<v_r>t} d\theta^2 + e^{-\tau_0 <v_r> cos\theta t} ds^2 \tag{III.36}$$

Note that for this expression a positive torsion shows that one of the Ljapunov exponents in this Riemann metric is positive (stretching) while the other is negative, representing contraction. The only non-vanishing component for the Riemann tensor is

$$R_{2323} = -\frac{1}{4} e^{2<v_r>\tau_0(1-cos\theta)t} sin^2\theta \tag{III.37}$$

which shows that the Riemann curvature is unstable in the infinite time limit, if the torsion is positive, while it is stable if the torsion is negative. The Ricci tensor components are

$$R_{33} = e^{(1-\tau_0 cos\theta)<v_r>t} R_{22} \tag{III.38}$$

$$R_{22} = \frac{1}{2} cos\theta e^{-(1-\tau_0<v_r>t)} \tag{III.39}$$

The Ricci tensor is

$$R = \frac{1}{2} e^{-\lambda_\theta (1-\tau_0)t} sin^2\theta \tag{III.40}$$

In all these computations one has assumed the weak torsion approximation, which is very reasoble for example, in astrophysical plasmas. Note from the Riemann curvature expression that the Riemann space is flat when the random radial flow vanishes, and when the torsion $\tau_0 > 1$ the scalar curvature R is singular, or unstable, when $t \to \infty$. However, since one assumes here that the weak torsion approximation, the curvature computations are ingeneral unstable. Since the twist of plasma MFTs is proportional to torsion of the MFT axis, this is a very reasonable approximation since the twist in kink solar loops for examples is very weak of the order of $Tw \approx 10^{-10} cm^{-1}$. When torsion vanishes the stationary Riemann curvature is periodic and reduces to

$$R_{2323} = -\frac{1}{4} sin^2\theta \tag{III.41}$$

Thus when the radial random flow vanishes the Riemann curvature can be spatially periodically. Let us now compute the determinant of the Riemann metric

$$g = Det(g_{ij}) = r^2(1 - \tau_0 rcos\theta)^2 \tag{III.42}$$

This might be $g = 1$ in the case of incompressible chaotic flow as $\nabla.\mathbf{v} = 0$, which allows us to say that the chaotic helical flow inside the tube has to be compressible. Tang and Boozer [5], have also examine the problem of compressible chaotic flows. As a final observation one notes that the Frenet curvature of material lines inside the tube, increases as the Ljapunov exponents or metric decreases. This can be seen by a simple examination of the metric factor $g_{ss} = (1 - \kappa(s) rcos\theta)^2$ since the Frenet curvature $\kappa(s)$ growth implies that this stretching factor decreases, of course along the Ljapunov exponents since they depend upon the Riemann metric.

IV Geodesic flows in chaotic dynamos

Anosov [14] demonstrated that hyperbolic systems which are geodesic flows [20], are necessarilly Anosov. In this section one shows that as a consequence of the geodesic flow condition imposed the MFT twisted flows are non-Anosov since their Lyapunov exponents are not all constants. In this section two given, the first is the Anosov geodesic flow of Arnold dynamo, while the second addresses the example of geodesic flows on the twisted MFTs above. In the first case, the Arnold metric of previous section, yields the Riemann-Christoffel symbols as

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}) \tag{IV.43}$$

whose components are

$$\Gamma^1{}_{13} = -\lambda = -\Gamma^2{}_{23} \tag{IV.44}$$

$$\Gamma^3{}_{11} = \lambda e^{-2\lambda z} \tag{IV.45}$$

$$\Gamma^3{}_{22} = -\lambda e^{2\lambda z} \tag{IV.46}$$

Substitution of these symbols into the geodesic equation

$$\frac{dv^i}{dt} + \Gamma^i{}_{jk} v^j v^k = 0 \tag{IV.47}$$

yields the following equations for the geodesic flow $\mathbf{v}$ as

$$\frac{dv^1}{dt} + \Gamma^1{}_{13} v^1 v^3 = 0 \tag{IV.48}$$

which yields

$$\frac{dv^1}{dt} - \lambda v^1 v^3 = 0 \tag{IV.49}$$

Proceeding the same way with the remaining equations yields

$$\frac{dv^2}{dt} + \lambda v^2 v^3 = 0 \tag{IV.50}$$

and

$$\frac{dv^3}{dt} - \lambda[e^{-2\lambda z}(v^1)^2 - e^{2\lambda z}(v^2)^2] = 0 \tag{IV.51}$$

By assuming that the $v^3 = constant = v^3{}_0$ one obtains

$$v^1 = v^0 e^{\lambda[z + v^3{}_0 t]} \tag{IV.52}$$

$$v^2 = v^0 e^{-\lambda[z - v^3{}_0 t]} \tag{IV.53}$$

Since $v^1 = v^p = \frac{dp}{dt}$ and $v^2 = v^q = \frac{dq}{dt}$, one is able to integrate the geodesic equations to obtain the flow topology as

$$p^2 + q^2 = R^2 e^{\lambda_L t} sinh \lambda z \tag{IV.54}$$

where $\lambda_L := v_0 \lambda$ is the Lyapunov exponent for the Arnold metric, and $z := v_0 t$. This geometry represents a circle expanding on time and exponentially stretched by the action of the Lyapunov exponent. Note that the dynamo flow velocity here, is distinct from the Arnold's one, since the Arnold's dynamo flow does not possesses components v^p and v^q as in the example considered in this section. Let us now proceed computing the same geodesic flow in the case of twisted MFT metric. Their Christoffel symbols are

$$\Gamma^2{}_{33} = \frac{e^{-<v_r>(1-\tau_0)t}}{2} sin\theta \tag{IV.55}$$

$$\Gamma^3{}_{23} = \frac{e^{-<v_r>(1-\tau_0 t}}{2} sin\theta \tag{IV.56}$$

From these expressions one obtains the following geodesic equations

$$\frac{dv^1}{dt} = 0 \tag{IV.57}$$

which implies $v^r = v^1 = v^0 = constant$. This result is indeed important since, it shows that actually our hypothesis that the random radial flow is constant is feasible, and can be derived from a geodesic flow. To simplify the remaining equations one chooses a torsion $\tau_0 = \frac{1}{2}$, which yields

$$\frac{dv^2}{dt} - sin\theta(v^3)^2 = 0 \tag{IV.58}$$

and

$$\frac{dv^3}{dt} - e^{[1-\frac{1}{2}<v_r>t]}v^2v^3 = 0 \tag{IV.59}$$

These two last equations together yield

$$\frac{dv^3}{dv^2} = e^{-\frac{1}{2}<v_r>t}\frac{v^2}{v^3} \tag{IV.60}$$

Some algebra yields the following result

$$\frac{v_s}{v^\theta} = e^{(1+cos\theta)<v_r>t} \tag{IV.61}$$

Thus the toroidal flow is proportional to the random flow and to the Lyapunov exponential stretching. Note that at $t = 0$ there is an equipartition between the poloidal and toroidal flows while as time evolves the Lyapunov exponential stretching makes the toroidal flow velocity to increase without bounds with respect to the poloidal flow. This is actually analogous to the dynamo action. Equation (IV.57) also shows that one the Lyapunov exponents is constant while the other is not and therefore the flow is a non-Anosov flow.

V Conclusions

In this paper the importance of investigation the twisted torus dynamo hyperbolic chaotic maps and flows, has been stressed. The ideas developed here followed the path of realistic dynamo maps obtained from the Gilbert's [21] shearing generalization of Arnold's cat map. Actually the Tang-Boozer's [5] idea of pilling up the Alfven twist of magnetic flux tubes is fully developed geometrically by applying their Lyapunov like metric and the Riemann geometrical Ricca's model for the MFTs. Though Ricca has investigated only force-free MFTs, in this paper the fast growth of magnetic fields in MFTs, induces a dynamo action through the Lyapunov fast dynamo exponential stretching. It is easy to show that from the self-induced equation that the twisted MFTs, with zero Lyapunov exponential stretching leads to non-fast dynamo action in the tubes, a result which is nothing but the statement of anti-dynamo theorem of Vishik's [22], which has been recently discussed in the context of the MFTs [23]. Computation of geodesic flows in Arnold's fast kinematic dynamo and in twisted MFT, shows that in both case the flow depends on the exponential stretching given by the Lyapunov numbers. It is shown that at the beginning of time, there is an equipartition between chaotic toroidal and poloidal flows, while as time evolves an instability occurs due to the action of Lyapunov exponential stretching. This result was obtained from the Euler equations by Friedlander and Vishik [24].

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