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Homogeneous bundles over abelian varieties

Michel Brion

Abstract

We obtain characterizations and structure results for homogeneous principal bundles over abelian varieties, that generalize work of Miyanishi and Mukai on homogeneous vector bundles. For this, we rely on notions and methods of algebraic transformation groups, especially observable subgroups and anti-affine groups.

1 Introduction

A vector bundle E over an abelian variety A is called homogeneous (or translation-invariant), if E is isomorphic to it pull-back under any translation in A. This defines a remarkable class of vector bundles: for example, the homogeneous line bundles are exactly the algebraically trivial ones; they are parametrized by the dual abelian variety $\widehat{A}$. Also, vector bundles over an elliptic curve are close to being homogeneous; specifically, each indecomposable vector bundle of degree 0 is homogeneous (as follows from [At57a]). This result does not extend to abelian varieties of higher dimensions, but the homogeneous vector bundles still admit a nice description in that generality. Namely, by work of Miyanishi and Mukai (see [Mi73, Muk79]), the following conditions are equivalent for a vector bundle E over A:

(i) E is homogeneous.

(ii) E is an iterated extension of algebraically trivial line bundles.

(iii) $E \cong \bigoplus_i L_i \otimes U_i$, where each L_i is an algebraically trivial line bundle, and each U_i is a unipotent vector bundle, i.e., an iterated extension of trivial line bundles.

Similar results had been obtained earlier by Matsushima and Morimoto in the setting of holomorphic vector bundles over complex tori. They also showed that the homogeneous bundles are exactly those admitting a connection; then they admit an integrable connection (see [Ma59, Mo59], and [Bi04, BG08] for more recent developments).

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In the present article, we obtain somewhat analogous characterizations and structure results for those principal bundles $\pi : X \to A$ under an algebraic group G, that are homogeneous in the same sense. This holds e.g. if X is a commutative algebraic group; then π is a homomorphism with kernel G. In fact, by a classical result of Rosenlicht and Serre (see [Se59]), all principal bundles under the additive group are obtained in this way, as well as all homogeneous bundles under the multiplicative group; the latter bundles are parametrized by $\widehat{A}$.

Actually, any homogeneous bundle under an arbitrary algebraic group may be obtained from an extension of A, in view of our main result:

THEOREM 1.1. *The following assertions are equivalent for a principal bundle $\pi : X \to A$ under an algebraic group G:*

(a) *π is homogeneous.*

(b) *There exist a commutative subgroup scheme $H \subset G$ and an extension of commutative group schemes $0 \to H \to \mathcal{G} \to A \to 0$ such that π admits a reduction of structure group to the H-bundle $\mathcal{G} \to A$. Moreover, there is a smallest such subgroup scheme H, up to conjugacy in G.*

(c) *The associated bundle $X/R_u(G) \to X/G_{\mathrm{aff}}$ is homogeneous, where $R_u(G)$ denotes the unipotent radical, and G_{aff} the largest connected affine subgroup of G.*

In characteristic 0, these assertions are also equivalent to:

(d) *π admits a connection.*

(e) *π admits an integrable connection.*

When G is the general linear group GL_n, the G-bundles correspond to the vector bundles of rank n; it is easy to see that this correspondence preserves homogeneity. In this setting, Theorem 1.1 generalizes the above results of Miyanishi and Mukai. Indeed, (b) implies that any homogeneous vector bundle E of rank n is associated with a representation $\rho : H \to \mathrm{GL}_n$, where H is a commutative group scheme. Then the image of H is contained (up to conjugacy) in the subgroup of upper triangular matrices; thus, E is an iterated extension of line bundles. The latter are homogeneous by (c) and hence algebraically trivial, which yields (ii). Likewise, (iii) follows from the decomposition of ρ into a direct sum of weight spaces for the diagonalizable part of H. Conversely, if a vector bundle E satisfies (ii) or (iii), then it is homogeneous by (c).

For an arbitrary algebraic group G, Theorem 1.1 reduces somehow the study of principal G-bundles to the case that G is connected and reductive. One may obtain a further reduction to irreducible bundles in a suitable sense, and explicitly describe these bundles; this will be developed in [Br11a] for the projective linear group PGL_n (a case of special interest, that corresponds to projective bundles), and in [Br11b] for the general case.

The proof of Theorem 1.1 is based on an analysis of the group scheme of equivariant automorphisms, $\mathrm{Aut}^G(X)$. This approach, already followed by Miyanishi in [Mi73], is combined here with recent developments on the structure of (possibly non-affine) algebraic groups, see [Br09, SS09, Br10a, Br10b]. Namely, the group scheme $\mathrm{Aut}^G(X)$ is locally of finite type, and hence contains a largest subgroup scheme $\mathcal{G}$ which is anti-affine (i.e., every global regular function on $\mathcal{G}$ is constant). Then $\mathcal{G}$ is a connected commutative algebraic group that acts on A by translations via the natural homomorphism $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$. One easily shows that π is homogeneous if and only if π_* maps $\mathcal{G}$ onto A (Proposition 2.2); equivalently, $G \times \mathcal{G}$ acts transitively on X. From this, (a)$\Rightarrow$(c) and (a)$\Leftrightarrow$(b) follow without much difficulty, where H is the kernel of $\pi_{*|\mathcal{G}}$. To prove (c)$\Rightarrow$(a), we use a characterization of homogeneity in terms of associated vector bundles which is of independent interest (Theorem 2.14) together with Mukai's result that (iii)$\Rightarrow$(i). The characterization in terms of connections also follows readily from Proposition 2.2; this is carried out in Proposition 2.4.

Actually, our approach yields a classification of homogeneous G-bundles over A in terms of pairs $(H, \mathcal{G})$, where $H \subset G$ is a commutative subgroup scheme, and $\mathcal{G}$ an anti-affine extension of A by H (Theorem 3.1). Such extensions have been classified in [Br09, SS09] when H is a connected affine algebraic group; the case that H is an (affine) group scheme is obtained in Theorem 3.12 along similar lines. But classifying all commutative subgroup schemes of an algebraic group G up to conjugacy is a difficult open problem, already when G is the general linear group.

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Notation and conventions. We will consider algebraic varieties and schemes over a fixed algebraically closed field k of arbitrary characteristic. Unless explicitly mentioned, schemes under consideration are assumed to be of finite type over k; by a point of such a scheme X, we mean a closed point. A *variety* is a reduced and separated scheme; in particular, varieties need not be irreducible.

We will use the book [DG70] as a general reference for *group schemes*. Given a group scheme G, we denote by $e_G \in G(k)$ the neutral element and by G_0 the *neutral component*, i.e, the connected component of G that contains e_G. Then G_0 is a connected normal subgroup scheme of G, and the quotient G/G_0 is finite (under our assumption that G is of finite type). A homomorphism of group schemes $f : G \to H$ is *faithful* if its (scheme-theoretic) kernel is trivial.

Throughout this article, G denotes an *algebraic group*, that is, a smooth group scheme, and A an *abelian variety*, i.e., a connected complete algebraic group. Then A is commutative, hence its group law will be denoted additively, and the neutral element by 0_A. For any point $a \in A$, we denote by $\tau_a : A \to A$ the translation $x \mapsto x + a$. Our general reference for abelian varieties is the book [Mum70]; for affine or equivalently, linear algebraic groups, we refer to [Sp98].

2 Characterizations

2.1 Reduction to the case of a connected affine structure group

A variety X equipped with a G-action $G \times X \to X$, $(g, x) \mapsto g \cdot x$ and with a G-invariant morphism

$$\pi : X \longrightarrow A \tag{1}$$

is a *(principal) G-bundle*, also known as a *G-torsor*, if π is faithfully flat and the morphism $G \times X \to X \times_A X$, $(g, x) \mapsto (x, g \cdot x)$ is an isomorphism. Then the morphism π is smooth, and hence so is X. Moreover, π is locally isotrivial and quasi-projective (see e.g. [Br10b, Corollary 3.5 and Remark 3.6 (iii)]). Therefore, the variety X is quasi-projective as well. If G is connected, then X is irreducible.

Definition 2.1. The G-bundle (1) is called *homogeneous*, if the translation $\tau_a : A \to A$ lifts to a G-automorphism $\widetilde{\tau_a} : X \to X$ for any point $a \in A$.

Denote by $\pi_a : \tau_a^*(X) \to A$ the pull-back of the G-bundle (1) under τ_a, so that we have a cartesian square

$$\begin{array}{ccc} \tau_a^*(X) & \xrightarrow{\psi_a} & X \\ {\scriptstyle \pi_a}\downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\tau_a} & A. \end{array}$$

Then the G-equivariant lifts $\widetilde{\tau_a}$ of τ_a correspond bijectively to the isomorphisms of G-bundles $\varphi_a : X \to \tau_a^*(X)$. Thus, *a G-bundle is homogeneous if and only if it is isomorphic to its pull-back under any translation in A.*

We now reformulate the notion of homogeneity in terms of the *equivariant automorphism group* $\mathrm{Aut}^G(X)$. By [Br10b, Theorem 4.2], the group functor $S \mapsto \mathrm{Aut}^G_S(X \times S)$ is represented by a group scheme, locally of finite type, that we still denote by $\mathrm{Aut}^G(X)$ for simplicity. Moreover, we have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^G_A(X) \longrightarrow \mathrm{Aut}^G(X) \xrightarrow{\pi_*} \mathrm{Aut}(A) \tag{2}$$

where $\mathrm{Aut}^G_A(X)$ denotes the group scheme of *bundle automorphisms*, $S \mapsto \mathrm{Aut}^G_{A\times S}(X\times S)$, and π_* arises from the homomorphisms $\mathrm{Aut}^G_S(X\times S) \to \mathrm{Aut}_S(A\times S)$. Also, by a standard result (see e.g. [Br10b, Lemma 4.1]), the map

$$\mathrm{Hom}^G(X, G) \longrightarrow \mathrm{Aut}^G_A(X), \quad f \longmapsto (x \mapsto f(x)\cdot x)$$

is an isomorphism, where $\mathrm{Hom}^G(X, G)$ denotes the group functor $S \mapsto \mathrm{Hom}^G(X \times S, G)$ (the group of G-equivariant morphisms $X \times S \to G$ for the given G-action on X, the trivial action on S and the conjugation action on G).

If G is affine, then so is $\mathrm{Aut}^G_A(X)$; if G is the general linear group, then $\mathrm{Aut}^G_A(X)$ is a connected algebraic group. But for an arbitrary G, the group scheme $\mathrm{Aut}^G_A(X)$ need not be connected nor reduced (see Examples 3.8).

As in [Mi73], we may view $\mathrm{Aut}^G(X)$ as the scheme of pairs (φ, ψ), where $\varphi \in \mathrm{Aut}(A)$ and $\psi : X \to \varphi^*(X)$ is an isomorphism of G-bundles; then π_* is just the projection $(\varphi, \psi) \mapsto \varphi$.

The group scheme $\mathrm{Aut}(A)$ is the semi-direct product of the normal subgroup A of translations with the subgroup scheme $\mathrm{Aut}_{\mathrm{gp}}(A)$ of automorphisms of the algebraic group A. Moreover, $A = \mathrm{Aut}_0(A)$, and $\mathrm{Aut}_{\mathrm{gp}}(A)$ is the constant group scheme associated with a subgroup of $\mathrm{GL}_N(\mathbb{Z})$ for some integer $N \geq 1$. Clearly, *(1) is homogeneous if and only if the image of π_* contains A.*

Next, recall that the group scheme $\mathrm{Aut}^G(X)$ admits a largest subgroup scheme $\mathcal{G}$ which is *anti-affine*, i.e., $\mathcal{O}(\mathcal{G}) = k$. Moreover, $\mathcal{G}$ is a normal connected commutative algebraic subgroup, and the quotient $\mathrm{Aut}^G(X)/\mathcal{G}$ is the *affinization* of $\mathrm{Aut}^G(X)$ (the largest affine quotient group scheme), see [DG70, III, §3, no. 8]. We set

$$\mathcal{G} = \mathrm{Aut}^G(X)_{\mathrm{ant}}.$$

Note that $\mathcal{G}$ is mapped to a subgroup of A under $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$.

Also, recall that X admits a largest anti-affine group of automorphisms, that we denote by $\mathrm{Aut}(X)_{\mathrm{ant}}$; moreover, this subgroup centralizes any connected algebraic group of automorphisms (see [Br10a, Lemma 3.1] when X is irreducible; the case of an arbitary variety X follows readily, since any irreducible component of X is stable under a connected group of automorphisms). In particular, $\mathcal{G} \subset \mathrm{Aut}(X)_{\mathrm{ant}}$ with equality if G is connected.

We may now state our key observation:

PROPOSITION 2.2. *With the above notation, the G-bundle (1) is homogeneous if and only if π_* maps $\mathcal{G}$ onto A.*

If in addition G is affine, then $\mathcal{G}$ is the smallest subgroup scheme of $\mathrm{Aut}^G(X)$ that is mapped onto A via π_*.

PROOF. Assume that (1) is homogeneous. Then π_* restricts to a surjective morphism of group schemes $\pi_*^{-1}(A) \to A$. Let $H \subset \mathrm{Aut}^G(X)$ denote the reduced neutral component of $\pi_*^{-1}(A)$. Then π_* still maps H onto A, and H contains $\mathcal{G}$ as a normal subgroup such that $H/\mathcal{G}$ is affine. It follows that $A/\pi_*(\mathcal{G})$, a quotient of $H/\mathcal{G}$, is affine as well. But $A/\pi_*(\mathcal{G})$ is complete as a quotient of A. We conclude that $\pi_*(\mathcal{G}) = A$. The converse is obvious.

Next, assume that G is affine. Let $\mathcal{H} \subset \mathrm{Aut}^G(X)$ be a subgroup scheme such that $\pi_*(\mathcal{H}) = A$ and consider the largest anti-affine subgroup $\mathcal{H}_{\mathrm{ant}} \subset \mathcal{H}$. Then, as above, $A/\pi_*(\mathcal{H}_{\mathrm{ant}})$ is a quotient of the affinization $\mathcal{H}/\mathcal{H}_{\mathrm{ant}}$ and hence $\pi_*(\mathcal{H}_{\mathrm{ant}}) = A$. Since $\mathcal{H}_{\mathrm{ant}} \subset \mathcal{G}$, it follows that $\mathcal{G} = \mathcal{K}\mathcal{H}_{\mathrm{ant}}$ where $\mathcal{K} := \mathcal{G} \cap \mathrm{Aut}^G_A(X)$. Then $\mathcal{K}$ is affine; thus, $\mathcal{G}/\mathcal{H}_{\mathrm{ant}}$ is affine (as a quotient of $\mathcal{K}$) and anti-affine (as a quotient of $\mathcal{G}$), and hence trivial. We conclude that $\mathcal{G} = \mathcal{H}_{\mathrm{ant}} \subset \mathcal{H}$. □

REMARK 2.3. For an arbitrary G-bundle (1), the central subgroup $G_{\mathrm{ant}} \subset G$ acts on X by bundle automorphisms. In fact, *the induced map $G_{\mathrm{ant}} \to \mathrm{Aut}^G_A(X)_{\mathrm{ant}}$ is an isomorphism.*

Indeed, for any $x \in X$, the evaluation homomorphism

$$\mathrm{Aut}^G_A(X) \cong \mathrm{Hom}^G(X, G) \longrightarrow G, \quad u \longmapsto u(x)$$

sends $\mathrm{Hom}^G(X, G)_{\mathrm{ant}}$ to G_{ant}. Thus,

$$\mathrm{Aut}^G_A(X)_{\mathrm{ant}} \cong \mathrm{Hom}^G(X, G)_{\mathrm{ant}} \subset \mathrm{Hom}^G(X, G_{\mathrm{ant}}) = \mathrm{Hom}(A, G_{\mathrm{ant}})$$

where the latter equality holds since G acts trivially on G_{ant}. Moreover, for any morphism $f : A \to G_{\mathrm{ant}}$, the map $a \mapsto f(a)f(0)^{-1}$ is a group homomorphism, and hence

$$\mathrm{Hom}(A, G_{\mathrm{ant}}) \cong G_{\mathrm{ant}} \ltimes \mathrm{Hom}_{\mathrm{gp}}(A, G_{\mathrm{ant}}).$$

Also, one easily shows that $\mathrm{Hom}_{\mathrm{gp}}(A, G_{\mathrm{ant}})$ is a free abelian group of finite rank. Thus, $\mathrm{Hom}(A, G_{\mathrm{ant}})_0 = G_{\mathrm{ant}}$ which yields our claim.

If the bundle (1) is trivial, then the exact sequence (2) splits. Hence the group scheme $\mathrm{Aut}^G(X)$ is the semi-direct product of $\mathrm{Aut}(A)$ with the normal subgroup scheme

$$\mathrm{Aut}^G_A(X) \cong \mathrm{Hom}^G(G \times A, G) \cong \mathrm{Hom}(A, G).$$

Moreover, as above,

$$\mathrm{Hom}(A, G) \cong G \ltimes \mathrm{Hom}_{\mathrm{gp}}(A, G) \cong G \ltimes \mathrm{Hom}_{\mathrm{gp}}(A, G_{\mathrm{ant}}).$$

It follows that $\mathcal{G} \cong G_{\mathrm{ant}} \times A$.

As a first application of Proposition 2.2, we reduce the study of homogeneous torsors to the case that X is *irreducible*. For an arbitrary G-torsor (1), note that G acts transitively on the set of irreducible components of X. Choose such a component X_1 and denote by $G_1 \subset G$ its stabilizer. Then G_1 is a closed subgroup containing G_0, and $X \cong G \times^{G_1} X_1$, i.e., the map $G \times X_1 \to X$, $(g, x_1) \mapsto g \cdot x_1$ is a G_1-bundle for the action defined by $g_1 \cdot (g, x_1) = (gg_1^{-1}, g_1 \cdot x_1)$; moreover, the restriction $\pi_1 := \pi_{|X_1} : X_1 \to A$ is a G_1-bundle as well.

PROPOSITION 2.4. *With the above notation, $\pi : X \to A$ is homogeneous if and only if so is $\pi_1 : X_1 \to A$.*

PROOF. The connected group $\mathcal{G}$ stabilizes every component of X, and hence restricts to an anti-affine subgroup of $\mathrm{Aut}^{G_1}(X_1)$. In view of Proposition 2.2, it follows that π_1 is homogeneous if so is π. For the converse, note that there is a natural homomorphism $\Phi : \mathrm{Aut}^{G_1}(X_1) \to \mathrm{Aut}^{G}(X)$ which is compatible with the homomorphisms to $\mathrm{Aut}(A)$ (see Proposition 2.8 for the construction of Φ in a more general setting). □

Next, we assume that X is irreducible, and reduce to G being *connected*. For this, we consider the factorization

$$X \xrightarrow{\pi_0} Y = X/G_0 \xrightarrow{\varphi} A \tag{3}$$

of the G-bundle (1), where $G_0 \subset G$ stands for the neutral component, π_0 is a G_0-bundle, and φ is a G/G_0-bundle. (The existence of (3) follows e.g. from [MFK94, Proposition 7.1]). Then Y is an irreducible variety and φ is a finite étale covering. By the Serre-Lang theorem (see e.g. [Mum70, IV.18]), Y is an abelian variety, once a base point 0_Y is chosen in the fiber of φ at 0_A; moreover, φ is a separable isogeny.

THEOREM 2.5. *With the above notation and assumptions, π is homogeneous if and only if so is π_0.*

PROOF. If π is homogeneous, then $\pi_*(\mathcal{G}) = A$ by Proposition 2.2. Moreover, $\mathcal{G} \subset \mathrm{Aut}^{G_0}(X)$ acts on Y via $(\pi_0)_*$. As $\mathcal{G}$ is a connected algebraic group, it follows that $\mathcal{G}$ is mapped onto Y by $(\pi_0)_*$, i.e., π_0 is homogeneous.

For the converse, consider the group

$$\mathcal{G}_o := \mathrm{Aut}^{G_0}(X)_{\mathrm{ant}}$$

on which G acts by conjugation via its finite quotient group $F := G/G_0$. By assumption, $(\pi_0)_*$ maps $\mathcal{G}_o$ onto Y, and hence π_* maps $\mathcal{G}_o$ onto A; moreover, π_* is invariant under the action of F. Given $a \in A$, let $u \in \mathcal{G}_o$ such that $\pi_*(u) = a$; then $\pi_*(\prod_{\gamma \in F} \gamma \cdot u) = Na$

where N denotes the order of F. Thus, Na admits a lift in the fixed point subgroup $\mathcal{G}_o^F$. But $\mathcal{G}_o^F \subset \mathrm{Aut}^G(X)$ and the multiplication $N : A \to A$ is surjective; hence π is homogeneous. □

We now assume that G is connected, and reduce to G being *affine*. By Chevalley's structure theorem (see [Co02] for a modern proof), G admits a largest connected affine subgroup G_{aff}; moreover, G_{aff} is a normal subgroup of G and G/G_{aff} is an abelian variety. Note that $G = G_{\mathrm{aff}}G_{\mathrm{ant}}$, where G_{ant} denotes the largest anti-affine subgroup of G (indeed, $G_{\mathrm{aff}}G_{\mathrm{ant}}$ is a closed normal subgroup of G, and the quotient $G/G_{\mathrm{aff}}G_{\mathrm{ant}}$ is both affine and complete). Also, recall from [Br10b, Corollary 3.5] that the G-bundle (1) has a unique factorization

$$X \xrightarrow{p} Y = X/G_{\mathrm{aff}} \xrightarrow{q} A = X/G \tag{4}$$

where p is a G_{aff}-bundle and q is a G/G_{aff}-bundle. Then Y *is an abelian variety and* q *is a homomorphism of algebraic groups*, again once an origin 0_Y is chosen in the fiber of q at 0_A. Indeed, we have the following result, which is well-known in characteristic 0 (see [Se58, Corollaire 1]):

Proposition 2.6. *Let $\pi : X \to A$ be a principal bundle under an abelian variety G. Then X is an abelian variety as well. Moreover, G acts on X via an exact sequence of abelian varieties*

$$0 \longrightarrow G \longrightarrow X \longrightarrow A \longrightarrow 0.$$

Proof. Recall that $X \cong G \times^H Y$ for some finite subgroup scheme $H \subset G$ and some closed H-stable subscheme $Y \subset X$; moreover, the restriction $\pi_{|Y} : Y \to A$ is an H-bundle (see e.g. [Br10b, Corollary 3.5]). By [No83, Remark 2]), we have $Y \cong H \times^K B$ for some closed subgroup scheme $K \subset H$ and some abelian variety B fitting into an exact sequence

$$0 \longrightarrow K \longrightarrow B \longrightarrow A \longrightarrow 0.$$

Thus,

$$X \cong G \times^H (H \times^K B) \cong G \times^K B$$

and π is identified with the morphism $p : G \times^K B \to B/K = A$. Moreover, $G \times^K B = (G \times B)/K$ is an abelian variety, and p is a homomorphism with kernel $G \times^K K \cong G$. □

Theorem 2.7. *The following assertions are equivalent for a principal bundle $\pi : X \to A$ under a connected algebraic group G:*

(i) *π is homogeneous.*

(ii) *The associated G_{aff}-bundle $p : X \to Y$ is homogeneous (where Y is an abelian variety by Proposition 2.6).*

PROOF. The group $\mathcal{G} \subset \mathrm{Aut}^{G_{\mathrm{aff}}}(X)$ acts on Y via $p_* : \mathrm{Aut}^{G_{\mathrm{aff}}}(X) \to \mathrm{Aut}(Y)$. If π is homogeneous, then this action is transitive. Indeed, $\mathcal{G}$ acts transitively on A by Proposition 2.2; moreover, $\mathcal{G}$ contains the largest anti-affine subgroup $G_{\mathrm{ant}} \subset G$ which acts trivially on A, and transitively on the fibers of the G/G_{aff}-bundle $q : Y \to A$, since the natural homomorphism $G_{\mathrm{ant}} \to G/G_{\mathrm{aff}}$ is surjective. As $\mathcal{G}$ is connected and Y is an abelian variety, it follows that p_* induces a surjective homomorphism $\mathcal{G} \to Y$. Thus, p is homogeneous if so is π.

The converse implication is obvious, since every translation of A lifts to a translation of Y. □

2.2 Change of structure group

Given a G-bundle $\pi : X \to S$ over an arbitrary base, recall that a *reduction* to a subgroup scheme $H \subset G$ is a closed H-stable subscheme $Y \subset X$ such that the natural map $G \times^H Y \to X$ is an isomorphism. Then Y is the fiber of the associated G-equivariant morphism $f : X \to G/H$ at the base point of G/H. In fact, the data of Y and of f are equivalent, and π restricts to an H-bundle $Y \to S$ (see e.g. [Br10b, Lemma 3.2]).

Also, given a homomorphism of algebraic groups $f : G \to G'$, we may consider the G'-bundle $X' := G' \times^G X \to S$ obtained by *extension of structure group*. This bundle might not always exist in this generality (i.e., X' may not be a scheme), but it does exist in our setting, and this construction preserves homogeneity:

PROPOSITION 2.8. *Let $\pi : X \to A$ be a G-bundle, and $f : G \to G'$ a homomorphism of algebraic groups.*

(i) *The G'-bundle*

$$\pi' : X' := G' \times^G X \longrightarrow A$$

exists, and there is a natural homomorphism

$$\Phi : \mathrm{Aut}^G(X) \longrightarrow \mathrm{Aut}^{G'}(X'),$$

compatible with $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$ and with $\pi'_* : \mathrm{Aut}^{G'}(X') \to \mathrm{Aut}(A)$. In particular, π' is homogeneous if so is π.

(ii) *Conversely, if π' is homogeneous, then so is π under the additional assumptions that f is faithful and $G'/f(G)$ is quasi-affine.*

PROOF. (i) Consider the neutral component $G_0 \subset G$ and the (finite) quotient group $F := G/G_0$. Then the G'-bundle $G' \times^{G_0} X \to X/G_0$ exists by [Br10b, Corollary 3.4]; moreover, X/G_0 is a finite cover of A, and hence is projective. Thus, the variety $G' \times^{G_0} X$ is quasi-projective (see [Br10b, Corollary 3.5]; it carries an action of F arising from the

diagonal action of G on $G' \times X$. The quotient $(G' \times^{G_0} X)/F$ exists by [Mum70], and yields the desired G-bundle π'.

To construct Φ, note that any $u \in \mathrm{Aut}^G(X)$ yields an automorphism

$$\mathrm{id} \times u : G' \times X \longrightarrow G' \times X, \quad (g', x) \longmapsto (g', u(x))$$

which is $G' \times G$-equivariant, and hence descends to $u' \in \mathrm{Aut}^{G'}(X')$. Moreover, the assignement $u \mapsto u' =: \Phi(u)$ readily extends to a homomorphism of group functors, and hence yields the desired homomorphism of group schemes.

(ii) Let $\mathcal{G}' := \mathrm{Aut}^{G'}(X')_{\mathrm{ant}}$. Then $\pi'_* : \mathcal{G}' \to A$ is surjective by Proposition 2.2. Also, the natural morphism

$$p : X' = G' \times^G X \to G'/f(G)$$

is invariant under $\mathcal{G}'$ by Lemma 2.9. Thus, $\mathcal{G}'$ acts on X (the fiber of p at the base point of $G'/f(G)$), and this action lifts the A-action on itself by translations. Hence π is homogeneous. □

LEMMA 2.9. *Let Z be a variety, $\mathcal{H}$ its largest anti-affine group of automorphisms, and $f : Z \to W$ a morphism to a quasi-affine variety. Then f is $\mathcal{H}$-invariant.*

PROOF. The action of $\mathcal{H}$ on Z yields a structure of rational $\mathcal{H}$-module on the algebra of regular functions $\mathcal{O}(Z)$. Since $\mathcal{H}$ is anti-affine, it must act trivially on $\mathcal{O}(Z)$ and hence on the image of the algebra homomorphism $f^\# : \mathcal{O}(W) \longrightarrow \mathcal{O}(Z)$. As the regular functions on W separate points, this means that f is $\mathcal{H}$-invariant. □

REMARKS 2.10. (i) With the notation of the above proposition, the homogeneity of π' does not always imply that of π. For instance, if f is constant, then π' is the trivial bundle $G' \times A \to A$, and hence is homogeneous even if π is not. For less trivial instances, where f is faithful, see Examples 2.17 and 3.8 (ii).

(ii) Recall that a closed subgroup H of an affine algebraic group G is *observable in G*, if the homogeneous space G/H is quasi-affine; we refer to [Gr97, Theorem 2.1] for further characterizations of observable subgroups. Also, recall that H is observable in G if and only if so is the radical $R(H)$ (the largest connected solvable normal subgroup of H), see [loc. cit., Theorem 2.7]. Moreover, any closed commutative subgroup is observable by [loc. cit., Corollary 2.9]. As a consequence, *H is observable in G if $R(H)$ is commutative.*

We will encounter quasi-affine homogeneous spaces G/H where $H \subset G$ is a possibly non-reduced subgroup scheme. This presents no additional difficulty in view of the following result:

LEMMA 2.11. *Let G be an algebraic group, H a closed subgroup scheme, and $H_{\mathrm{red}} \subset H$ the associated reduced subgroup scheme (i.e., the largest algebraic subgroup).*

(i) *If G/H is quasi-affine, then H contains G_{ant}. In other words, G acts on G/H via its affinization G/G_{ant}.*

(ii) *G/H is quasi-affine if and only if so is $G_0/(G_0 \cap H)$.*

(iii) *G/H is quasi-affine if and only if so is G/H_{red}.*

PROOF. (i) The anti-affine group G_{ant} acts trivially on the rational G-module $\mathcal{O}(G/H)$, and hence on G/H as the latter is quasi-affine.

(ii) follows from the fact that G_0 acts on G/H with finitely many orbits, all of them being closed and isomorphic as varieties to $G_0/(G_0 \cap H)$. Indeed, for any $g \in G$, the G_0-orbit of $gH \in G/H$ is isomorphic to $G_0/(G_0 \cap \mathrm{Int}(g)H) = G_0/\mathrm{Int}(g)(G_0 \cap H)$, where $\mathrm{Int}(g) : G \to G$ denotes the conjugation by g. Moreover, $\mathrm{Int}(g)$ induces an isomorphism of varieties $G_0/(G_0 \cap H) \to G_0/\mathrm{Int}(g)(G_0 \cap H)$.

(iii) By (ii) and the equality $(G_0 \cap H)_{\mathrm{red}} = G_0 \cap H_{\mathrm{red}}$, we may assume that H is connected.

If G/H is quasi-affine, then we may find a finite-dimensional vector space $V \subset \mathcal{O}(G/H)$ such that the associated morphism $G/H \to V^*$ is a locally closed immersion. Then V is contained in a finite-dimensional G-submodule of $\mathcal{O}(G/H)$, and hence we may assume that V itself is G-stable. Let $R \subset \mathcal{O}(G/H)$ denote the subalgebra generated by V, and S the integral closure of R in the field of rational functions $k(G/H_{\mathrm{red}})$. Then S is a G-stable, finitely generated algebra; thus, we may find a finite-dimensional G-submodule $W \subset S$ that generates S. Moreover, since the fraction field of R is $k(G/H)$, we see that $k(G/H_{\mathrm{red}})$ is the fraction field of S. Thus, the natural map $G/H_{\mathrm{red}} \to W^*$ is birational; since it is G-equivariant, it is a locally closed immersion. Hence G/H_{red} is quasi-affine.

Conversely, assume that G/H_{red} is quasi-affine and choose a finite-dimensional G-submodule $W \subset \mathcal{O}(G/H_{\mathrm{red}})$ that yields a locally closed immersion $G/H_{\mathrm{red}} \to W^*$. Since the natural morphism $G/H_{\mathrm{red}} \to G/H$ is purely inseparable, there exists a positive integer q (a power of the characteristic exponent of k) such that $W^q \subset \mathcal{O}(G/H)$. Let $R \subset \mathcal{O}(G/H)$ denote the integral closure of the subalgebra generated by W^q. Then, as in the above step, R is generated by a finite-dimensional G-submodule V, and the corresponding map $G/H \to V^*$ is a locally closed immersion. □

2.3 Associated vector bundles

A vector bundle

$$p : E \longrightarrow A \tag{5}$$

is *homogeneous*, if $E \cong \tau_a^*(E)$ for all $a \in A$. As for principal bundles, the isomorphisms of vector bundles $E \to \tau_a^*(E)$ correspond bijectively to those lifts $\widetilde{\tau}_a : E \to E$ of $\tau_a : A \to A$ that are linear on fibers, i.e., that commute with the action of the multiplicative group $\mathbb{G}_m$ by scalar multiplication on fibers. In other words, *E is homogeneous if and only if each translation τ_a lifts to some $\widetilde{\tau}_a \in \mathrm{Aut}^{\mathbb{G}_m}(E)$* (the equivariant automorphism group of E viewed as a variety with $\mathbb{G}_m$-action).

Also, the group functor $S \mapsto \mathrm{Aut}_S^{\mathbb{G}_m}(E \times S)$ is represented by a group scheme, locally of finite type, that we still denote by $\mathrm{Aut}^{\mathbb{G}_m}(E)$. (This may be deduced from the analogous statement for principal bundles by using Lemma 2.12 (ii); alternatively, one may identify $\mathrm{Aut}^{\mathbb{G}_m}(E)$ with a closed subfunctor of $\mathrm{Aut}(\mathbb{P}(E \oplus O_A))$, where $\mathbb{P}(E \oplus O_A)$ denotes the projective variety obtained by completing E with its "hyperplane at infinity" $\mathbb{P}(E)$). Moreover, we have an exact sequence of group schemes

$$1 \longrightarrow \mathrm{Aut}_A^{\mathbb{G}_m}(E) \longrightarrow \mathrm{Aut}^{\mathbb{G}_m}(E) \xrightarrow{p_*} \mathrm{Aut}(A),$$

where $\mathrm{Aut}_A^{\mathbb{G}_m}(E)$ denotes the group scheme of automorphisms of the vector bundle E over A, and p_* assigns to each $\mathbb{G}_m$-automorphism of E, the induced automorphism of A viewed as the categorical quotient $E//\mathbb{G}_m$. Note that $\mathrm{Aut}_A^{\mathbb{G}_m}(E)$ is a connected affine algebraic group, since it is open in the affine space $\mathrm{End}_A^{\mathbb{G}_m}(E) = H^0(A, End(E))$. Thus, $\mathrm{Aut}^{\mathbb{G}_m}(E)$ is an algebraic group (locally of finite type).

We now relate the equivariant automorphism groups and notions of homogeneity for vector bundles and principal bundles:

LEMMA 2.12. *Let $\pi : X \to A$ be a G-bundle, and $\rho : G \to \mathrm{GL}(V)$ a finite-dimensional representation.*

(i) *The associated vector bundle*

$$p : E := X \times^G V \longrightarrow A$$

exists, and there is a natural homomorphism

$$\Phi : \mathrm{Aut}^G(X) \longrightarrow \mathrm{Aut}^{\mathbb{G}_m}(E)$$

compatible with $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$ and $p_* : \mathrm{Aut}^{\mathbb{G}_m}(E) \to \mathrm{Aut}(A)$.

(ii) *If $G = \mathrm{GL}(V)$ and $\rho = \mathrm{id}$, then Φ is an isomorphism.*

(iii) *If the principal bundle π is homogeneous, then so is the vector bundle p. The converse holds if $G = \mathrm{GL}(V)$ and $\rho = \mathrm{id}$.*

PROOF. (i) The existence of p follows from faithfully flat descent. The construction of Φ is similar to that in the proof of Proposition 2.8 (ii).

(ii) follows from the correspondence between vector bundles with fiber V and principal bundles under $\mathrm{GL}(V)$. Specifically, we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Aut}_A^{\mathrm{GL}(V)}(X) & \longrightarrow & \mathrm{Aut}^{\mathrm{GL}(V)}(X) & \xrightarrow{\pi_*} & \mathrm{Aut}(Y) \\ & & \Psi\downarrow & & \Phi\downarrow & & =\downarrow \\ 1 & \longrightarrow & \mathrm{Aut}_A^{\mathbb{G}_m}(E) & \longrightarrow & \mathrm{Aut}^{\mathbb{G}_m}(E) & \xrightarrow{p_*} & \mathrm{Aut}(Y), \end{array}$$

where $\Psi : \mathrm{Aut}_A^{\mathrm{GL}(V)}(X) = \mathrm{Hom}^{\mathrm{GL}(V)}(X, \mathrm{GL}(V)) \to \mathrm{Aut}_A^{\mathbb{G}_m}(E) \subset H^0(A, End(E))$ associates with each equivariant morphism $f : X \to \mathrm{GL}(V) \subset \mathrm{End}(V)$, the same f viewed as a section of the associated bundle $\mathrm{End}(E)$. Thus, Ψ is an isomorphism. Moreover, π_* and p_* have the same image, since for a given $u \in \mathrm{Aut}(A)$, the condition that $u^*(X) \cong X$ as $\mathrm{GL}(V)$-bundles is equivalent to $u^*(E) \cong E$ as vector bundles.

(iii) is a direct consequence of (i) and (ii). □

Next, we record the following preliminary result:

LEMMA 2.13. *Any affine algebraic group G admits a faithful finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ such that the homogeneous space $\mathrm{GL}(V)/\rho(G)$ is quasi-affine; in other words, $\rho(G)$ is observable in $\mathrm{GL}(V)$.*

PROOF. Choose a faithful finite-dimensional representation

$$\iota : G \longrightarrow \mathrm{GL}(M).$$

By a theorem of Chevalley, there exist a $\mathrm{GL}(M)$-module N and a line $\ell \subset N$ such that G is the isotropy subgroup scheme of ℓ in $\mathrm{GL}(M)$. Then G acts on ℓ via a character $\chi : G \to \mathbb{G}_m$. This yields a faithful homomorphism

$$(\iota, \chi) : G \longrightarrow \mathrm{GL}(M) \times \mathbb{G}_m.$$

Moreover, $\mathrm{GL}(M) \times \mathbb{G}_m$ acts linearly on N via

$$(g, t) \cdot x := t^{-1} g \cdot x,$$

and the isotropy subgroup scheme of any point $x \in \ell$ for this action is exactly $(\iota \times \chi)(G)$. It follows that the homogeneous space

$$(\mathrm{GL}(M) \times \mathbb{G}_m)/(\iota, \chi)(G) \cong (\mathrm{GL}(M) \times \mathbb{G}_m) \cdot x$$

is a locally closed subvariety of N, and hence is quasi-affine. Next, the natural embedding

$$\mathrm{GL}(M) \times \mathbb{G}_m = \mathrm{GL}(M) \times \mathrm{GL}(1) \subset \mathrm{GL}(M \oplus k)$$

yields a faithful representation $\rho : G \to \mathrm{GL}(M \oplus k)$, and the homogeneous space

$$\mathrm{GL}(M \oplus k)/\rho(G) = \mathrm{GL}(M \oplus k) \times^{\mathrm{GL}(M)\times\mathbb{G}_m} (\mathrm{GL}(M) \times \mathbb{G}_m)/(\iota, \chi)(G)$$

is a locally closed subvariety of $\mathrm{GL}(M \oplus k) \times^{\mathrm{GL}(M)\times\mathbb{G}_m} N$. The latter is the total space of a vector bundle over the affine variety $\mathrm{GL}(M \oplus k)/(\mathrm{GL}(M) \times \mathbb{G}_m)$, and hence is affine as well. Thus, $\mathrm{GL}(M \oplus k)/\rho(G)$ is quasi-affine. □

We may now state the main result of this subsection:

THEOREM 2.14. *Given a G-bundle (1) where G is affine, the following conditions are equivalent:*

(i) *π is homogeneous.*

(ii) *All vector bundles associated with irreducible representations are homogeneous.*

(iii) *All associated vector bundles are homogeneous.*

(iv) *The associated vector bundle (5) is homogeneous for some faithful finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ such that $\rho(G)$ is observable in $\mathrm{GL}(V)$. (Such a representation exists by Lemma 2.13).*

PROOF. (i)⇒(ii) follows from Lemma 2.12 (iii).

(ii)⇒(iii) holds since any extension of homogeneous vector bundles is homogeneous, as follows from [Muk79, Theorem 4.19] that characterizes homogeneity in terms of the Fourier-Mukai transform.

We sketch an alternative proof of this fact via methods of algebraic groups: given two homogeneous vector bundles E_1, E_2 over A with respective anti-affine groups of automorphisms $\mathcal{G}_1$, $\mathcal{G}_2$, the fibered product $\mathcal{G}_1 \times_A \mathcal{G}_2$ acts linearly on $\mathrm{Ext}^1_A(E_2, E_1)$ and its largest anti-affine subgroup $\mathcal{G}$ acts trivially. Thus, for any extension $0 \to E_1 \to E \to E_2 \to 0$, we see that $\mathcal{G}$ acts on E by lifting the translations in A.

(iii)⇒(iv) is obvious.

(iv)⇒(i) The $\mathrm{GL}(V)$-bundle $\mathrm{GL}(V) \times^G X \to A$ is homogeneous by Lemma 2.12 (iii). Since $\mathrm{GL}(V)/\rho(G)$ is quasi-affine, it follows that π is homogeneous, in view of Proposition 2.8 (iii). □

As a direct consequence, we obtain:

COROLLARY 2.15. *Let V be a finite-dimensional vector space, and $G \subset \mathrm{GL}(V)$ a reductive subgroup. Then a G-bundle $\pi : X \to A$ is homogeneous if and only if the associated vector bundle $p : X \times^G V \to A$ is homogeneous.*

(Indeed, the variety $\mathrm{GL}(V)/G$ is affine).

Note that the same statement holds, with a completely different proof, for G-bundles over complete rational homogeneous varieties in characteristic 0; see [BT10, Theorem 1.1].

Another straightforward consequence of Theorem 2.14 is the following result:

COROLLARY 2.16. *Let G be an affine algebraic group, $U \subset G$ a normal unipotent subgroup scheme, $\pi : X \to A$ a G-bundle, and $\bar{\pi} : \bar{X} := X/U \to A$ the associated principal bundle under $\bar{G} := G/U$. Then π is homogeneous if and only if so is $\bar{\pi}$.*

In particular, any principal bundle under a unipotent group is homogeneous.

(Indeed, any irreducible representation $\rho : G \to \mathrm{GL}(V)$ factors through $\bar{G}$).

Combining Corollary 2.16 with Proposition 2.4, Theorem 2.5 and Theorem 2.7, we obtain the equivalence (a)⇔(c) in Theorem 1.1. The equivalences (a)⇔(d)⇔(e) will be obtained in Subsection 2.4, and (a)⇔(b) in Subsection 3.1.

To conclude this subsection, we provide the example announced in Remark 2.10 (i):

EXAMPLE 2.17. Consider a non-trivial line bundle L over A, generated by n global sections. Then L fits into an exact sequence of vector bundles over A

$$0 \longrightarrow E \longrightarrow O_A^n \longrightarrow L \longrightarrow 0$$

where O_A denotes the trivial line bundle. This corresponds to a reduction of the trivial GL_n-bundle $\pi' : \mathrm{GL}_n \times A \to A$ to a principal P-bundle $\pi : X \to A$, where $P \subset \mathrm{GL}_n$ denotes the parabolic subgroup that stabilizes a hyperplane. Taking for U the unipotent radical of P, we see that $\bar{P} \cong \mathrm{GL}_{n-1} \times \mathbb{G}_m$ and the associated bundle $\bar{\pi} : \bar{X} \to A$ is the product over A of the principal bundles associated with E and L. Since L is not homogeneous, the same holds for $\bar{\pi}$, and hence π is not homogeneous as well; but π' is obviously homogeneous.

2.4 Connections

We first recall some classical constructions associated with principal bundles (see [At57b]) in our special setting. For a G-bundle $\pi : X \to A$, we have an exact sequence of tangent sheaves on X

$$0 \longrightarrow T_\pi \longrightarrow T_X \xrightarrow{d\pi} \pi^* T_A \longrightarrow 0$$

and isomorphisms of sheaves of Lie algebras

$$\mathcal{O}_X \otimes \mathrm{Lie}(G) \xrightarrow{\cong} T_\pi, \quad \mathcal{O}_A \otimes \mathrm{Lie}(A) \xrightarrow{\cong} T_A$$

with an obvious notation. This yields an exact sequence of sheaves of Lie algebras on X

$$0 \longrightarrow \mathcal{O}_X \otimes \mathrm{Lie}(G) \longrightarrow T_X \xrightarrow{d\pi} \mathcal{O}_X \otimes \mathrm{Lie}(A) \longrightarrow 0. \tag{6}$$

Also, all these sheaves are G-linearized, where G acts on $\mathrm{Lie}(G)$ via its adjoint representation, and G acts trivially on $\mathrm{Lie}(A)$; moreover, the morphisms in (6) preserve the linearizations.

Now the category of G-linearized locally free sheaves on X is equivalent to that of locally free sheaves on A, via the invariant direct image π_*^G and the pull-back π^*. Thus, (6) corresponds to an exact sequence of sheaves of Lie algebras on A, the *Atiyah sequence*

$$0 \longrightarrow \pi_*(\mathcal{O}_X \otimes \mathrm{Lie}(G))^G \longrightarrow \mathrm{At}(X) \xrightarrow{d\pi} \mathcal{O}_A \otimes \mathrm{Lie}(A) \longrightarrow 0 \tag{7}$$

where the left-hand side is the sheaf of local sections of the associated vector bundle $X \times^G \mathrm{Lie}(G) \to A$ (the *adjoint bundle*), and $\mathrm{At}(X) := \pi_*(T_X)^G$.

Taking global sections in (7) yields an exact sequence of Lie algebras

$$0 \longrightarrow \mathrm{Hom}^G(X, \mathrm{Lie}(G)) \longrightarrow H^0(X, T_X)^G \xrightarrow{d\pi} \mathrm{Lie}(A) \tag{8}$$

which may also be obtained from the exact sequence of group schemes (2) by taking their Lie algebras (see e.g. [Br10b, Section 4]).

Moreover, the G-equivariant splittings of the exact sequence (6) correspond to the splittings of (7), i.e., to the sections of $d\pi : H^0(X, T_X)^G \to \mathrm{Lie}(A)$. Such a section

$$D : \mathrm{Lie}(A) \longrightarrow H^0(X, T_X)^G$$

is called a *connection*; D is *integrable* (or *flat*) if it is a homomorphism of Lie algebras, i.e., its image is an abelian Lie subalgebra.

We may now state the following algebraic version of a homogeneity criterion for holomorphic bundles over complex tori, due to Matsushima (see [Ma59, Théorème 1]) and Morimoto ([Mo59, Théorème 1]):

PROPOSITION 2.18. *In characteristic* 0*, the following conditions are equivalent for a* G*-bundle* $\pi : X \to A$*:*

(i) π *is homogeneous.*

(ii) π *admits an integrable connection.*

(iii) π *admits a connection.*

PROOF. (i)⇒(ii) The restriction $\pi_{*|\mathcal{G}} : \mathcal{G} \to A$ is surjective by Proposition 2.2, and hence so is $d\pi_{*|\mathcal{G}} : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(A)$. Any splitting of the latter map yields an integrable connection, since $\mathcal{G}$ is commutative.

(ii)⇒(iii) is left to the reader.

(iii)⇒(i) The map $d\pi : H^0(X, T_X)^G \to \mathrm{Lie}(A)$ is surjective, and hence $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$ is surjective on neutral components. □

When $k = \mathbb{C}$ and G is a connected reductive group, the above statement has also been proved by Biswas (see [Bi04, Theorem 3.1]), via very different arguments based on semistability. The case of holomorphic principal bundles over complex tori is treated by Biswas and Gomez (see [BG08, Theorem 3.1]) via the theory of Higgs G-bundles; in this setting, they also characterize homogeneous bundles in terms of pseudostability and vanishing of characteristic classes.

On the other hand, in positive characteristics, the group scheme $\mathrm{Aut}^G(X)$ need not be reduced (see Example 3.8 (iii)) and hence characterizing homogeneous bundles in terms of connections would make little sense.

3 Structure

3.1 Homogeneous bundles as homogeneous spaces

By a refinement of Proposition 2.2, we obtain our main structure result:

THEOREM 3.1. *Let $\pi : X \to A$ be a principal bundle under an algebraic group G and consider the $G \times \mathcal{G}$-action on X, where $\mathcal{G} \subset \mathrm{Aut}^G(X)$ denotes the largest anti-affine subgroup. Then the following conditions are equivalent:*

(i) *π is homogeneous.*

(ii) *There is a $G \times \mathcal{G}$-equivariant isomorphism*

$$X \cong G \times^H \mathcal{G}, \tag{9}$$

where $H \subset G$ is a closed subgroup scheme that also fits into an extension of commutative group schemes

$$0 \longrightarrow H \longrightarrow \mathcal{G} \xrightarrow{\pi_*} A \longrightarrow 0 \tag{10}$$

via $\pi_* : \mathcal{G} \to A$. (In particular, H is commutative).

Moreover, the assignement $X \mapsto (\mathcal{G}, H)$ yields a bijection between the isomorphism classes of G-bundles over A, and the equivalence classes of pairs consisting of a commutative subgroup scheme $H \subset G$ and an extension (10), where $\mathcal{G}$ is anti-affine. Here two pairs are equivalent if the corresponding subgroups H, H' are conjugate in G, and if in addition the resulting isomorphism $H \to H'$ lifts to an isomorphism of extensions.

PROOF. Assume that π is homogeneous and consider the action of $G \times \mathcal{G}$ on X. This action is transitive, since G acts transitively on the fibers of π while $\mathcal{G}$ acts transitively on A (Proposition 2.2). Thus, $X \cong (G \times \mathcal{G})/H$ as $G \times \mathcal{G}$-varieties, where $H \subset G \times \mathcal{G}$ denotes the isotropy subgroup scheme of some point $x \in X$; this isomorphism maps x to the base point $(e_G, e_{\mathcal{G}})H$ of the homogeneous space $(G \times \mathcal{G})/H$. Denote by $p : H \to G$

and $q : H \to \mathcal{G}$ the projections. Then the (scheme-theoretic) kernel of q is identified with a subgroup scheme of G that fixes x. Since π is a G-bundle, it follows that $\mathrm{Ker}(q)$ is trivial. Likewise, the kernel of p is identified with a subgroup scheme of $\mathcal{G}$ that fixes x. Since $\mathcal{G}$ centralizes $G \times \mathcal{G}$, it follows that $\mathrm{Ker}(p)$ acts trivially on X, and hence is trivial as $\mathcal{G}$ acts faithfully. This yields the assertions of (ii).

Conversely, if (ii) holds, then $\mathcal{G}$ acts on X by G-automorphisms that lift the translations of A. Thus, π is homogeneous.

This proves the equivalence of (i) and (ii). The final assertion follows from the fact that the isotropy subgroup scheme $H = (G \times \mathcal{G})_x$ is uniquely determined up to conjugacy in $G \times \mathcal{G}$ and hence in G. □

COROLLARY 3.2. *Given an abelian variety A and a connected commutative algebraic group G, the isomorphism classes of homogeneous G-bundles over A correspond bijectively to the isomorphism classes of extensions of connected algebraic groups*

$$1 \longrightarrow G \longrightarrow X \longrightarrow A \longrightarrow 1. \tag{11}$$

Moreover, any such extension is commutative.

PROOF. For any homogeneous G-bundle (1), the variety $X := (G \times \mathcal{G})/H$ has a structure of commutative algebraic group such that the projection $\pi : X \to A$ is a homomorphism with kernel G. This group structure is unique up to the choice of the neutral element in the fiber of π at 0_A, i.e., up to a translation in G.

Conversely, any extension (11) obviously yields a homogeneous G-bundle. Moreover, $X = X_{\mathrm{aff}} X_{\mathrm{ant}}$, and $X_{\mathrm{aff}} \subset G$ is commutative. Since X_{ant} is central in X, it follows that X is commutative.

Also, note that the automorphism group of a G-bundle $\pi : X \to A$ is

$$\mathrm{Aut}^G_A(X) \cong \mathrm{Hom}^G(X, G) = \mathrm{Hom}(A, G)$$

as seen in Remark 2.3, and the automorphism group of an extension (11) is $\mathrm{Hom}_{\mathrm{gp}}(A, G)$. Moreover, $\mathrm{Hom}(A, G)$ is the semi-direct product of $\mathrm{Hom}_{\mathrm{gp}}(A, G)$ with the normal subgroup G of translations. Putting these facts together yields our assertion. □

REMARKS 3.3. (i) If k is the algebraic closure of a finite field, then every anti-affine group is an abelian variety (see [Br09, Proposition 2.2]). Thus, $\pi_* : \mathcal{G} \to A$ is an isogeny for any homogeneous G-bundle (1). In other words, the group scheme H is finite. This does not extend to any other algebraically closed field, in view of [Br09, Example 3.11].

(ii) With the notation of Theorem 3.1, the subgroup scheme H always contains G_{ant}, since by Remark 2.3,

$$G_{\mathrm{ant}} = \mathrm{Aut}^G_A(X)_{\mathrm{ant}} \subset \mathrm{Aut}^G(X)_{\mathrm{ant}} \cap \mathrm{Aut}_A(X) = \mathrm{Ker}(\pi_* : \mathcal{G} \to A).$$

Moreover, we have for the trivial bundle: $H = G_{\mathrm{ant}}$ and $\mathcal{G} = G_{\mathrm{ant}} \times A$. So Theorem 3.1 (stated for an arbitrary algebraic group G) yields a "natural" classification if and only if G is *affine*; then the trivial bundle does correspond to the trivial subgroup scheme H.

(iii) With the notation of Theorem 3.1 again, the projection $G \times^H \mathcal{G} \to G/H$ yields a G-equivariant morphism

$$\varphi : X \longrightarrow G/H$$

which is a principal bundle under the anti-affine group $\mathcal{G}$. We may view φ as a reduction of structure group to the H-bundle $\pi_* : \mathcal{G} \to A$. If G is affine, then the variety G/H is quasi-affine, as follows from [Gr97, Corollary 1.2.9] together with Lemma 2.11. Moreover, the morphism $\varphi^\# : \mathcal{O}(G/H) \to \mathcal{O}(X)$ is an isomorphism, since $\mathcal{G}$ is anti-affine. Thus, φ may be viewed as the canonical morphism $X \to \operatorname{Spec} \mathcal{O}(X)$ and hence depends only on the abstract variety X.

If in addition G is connected, then $\pi : X \to A$ is the Albanese morphism (the universal morphism to an abelian variety). Thus, π also depends only on the abstract variety X.

In fact, the above reduction of structure group is minimal in the following sense:

PROPOSITION 3.4. *Let $\pi : X \to A$ be a homogeneous principal bundle under an affine algebraic group G and let $H \subset G$ be a subgroup scheme as in Theorem 3.1. Then π admits a reduction to an observable subgroup scheme $K \subset G$ if and only if K contains a conjugate of H in G.*

PROOF. Consider a reduction of structure group $q : X \to G/K$. If K is observable in G, then q is $\mathcal{G}$-invariant by Lemma 2.9 and hence is $(G \times \mathcal{G})$-equivariant for the trivial action of $\mathcal{G}$ on G/K. Thus, q maps the base point of $X = (G \times \mathcal{G})/H$ to an H-fixed point. In other words, H is contained in a conjugate of K. The converse implication is obvious. □

In the above statement, the assumption that K is observable cannot be omitted, as shown by Example 3.8 (ii).

3.2 Equivariant automorphism groups

The structure theorem 3.1 yields a further characterization of homogeneity:

PROPOSITION 3.5. *The following conditions are equivalent for a principal bundle $\pi : X \to A$ under an affine algebraic group G:*

(i) *π is homogeneous.*

(ii) *π is trivialized by an anti-affine extension $\varphi : \mathcal{G} \to A$.*

PROOF. (i)⇒(ii) Theorem 3.1 yields a cartesian square

$$\begin{array}{ccc} G \times \mathcal{G} & \xrightarrow{p} & \mathcal{G} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ X & \xrightarrow{\pi} & A, \end{array}$$

where p denotes the second projection, and q the quotient morphism by H.

(ii)⇒(i) By assumption, we have a similar cartesian square with π_* replaced by φ. Thus, the group scheme $H := \mathrm{Ker}(\varphi)$ acts on $G \times \mathcal{G}$ by G-equivariant automorphisms that lift the H-action on $\mathcal{G}$ by multiplication: this action must be of the form

$$h \cdot (g, \gamma) = (gf(h, \gamma), h\gamma)$$

for some morphism $f : H \times \mathcal{G} \to G$. Since $\mathcal{G}$ is anti-affine and G is affine, f factors through a morphism $\psi : H \to G$. Moreover, ψ is a homomorphism of group schemes, since H acts on $G \times \mathcal{G}$. Thus, $X \cong G \times^H \mathcal{G}$ is homogeneous. □

REMARK 3.6. In particular, a G-bundle is homogeneous if it is trivialized by an isogeny, or equivalently, by the multiplication map $n_A : A \to A$ for some positive integer n.

Also, given a homogeneous bundle $X \cong G \times^H \mathcal{G}$, one easily checks that the pull-back $n_A^*(X)$ is the G-bundle $G \times^{H/H_n} \mathcal{G}/H_n$, where $H_n \subset H$ denotes the kernel of n_H, and H/H_n is viewed as a subgroup of G via the isomorphism $n_H : H/H_n \to H$. In particular, X is trivialized by n_A if and only if H is killed by n.

Theorem 3.1 also yields a description of bundle automorphisms:

PROPOSITION 3.7. *Consider a homogeneous bundle $\pi : X \to A$ under an affine algebraic group G. Write $X = G \times^H \mathcal{G}$ as in Theorem 3.1 and denote by $C_G(H)$ the centralizer of H in G. Then the group scheme $C_G(H)$ acts on X by bundle automorphisms, via $z \cdot (g, \gamma)H := (gz^{-1}, \gamma)H$. Moreover, this action yields an isomorphism*

$$F : C_G(H) \xrightarrow{\cong} \mathrm{Aut}_A^G(X). \tag{12}$$

As a consequence, the group scheme $\pi_*^{-1}(A)$ (consisting of those G-automorphisms of X that lift the translations of A) is isomorphic to $C_G(H) \times^H \mathcal{G} = (C_G(H) \times \mathcal{G})/H$.

PROOF. Given a scheme S and a morphism $\psi : S \to C_G(H)$, we obtain a morphism

$$G \times \mathcal{G} \times S \to G \times \mathcal{G} \times S, \quad (g, \gamma, s) \mapsto (g\psi(s)^{-1}, \gamma, s)$$

which is $G \times H$-equivariant and hence descends to a G-equivariant morphism $F(\psi) : X \times S \to X \times S$. Clearly, $F(\psi)$ is compatible with $\pi \times \mathrm{id} : X \times S \to A \times S$; in

other words, $F(\psi) \in \mathrm{Aut}^G_A(X)(S)$. This yields a homomorphism of group functors $F : C_G(H) \to \mathrm{Aut}_A(X)$.

To show that F is an isomorphism, note that

$$\mathrm{Aut}^G_A(X)(S) = \mathrm{Aut}^G_{A\times S}(X \times S) \cong \mathrm{Hom}^G(G \times^H \mathcal{G} \times S, G) \cong \mathrm{Hom}^H(\mathcal{G} \times S, G)$$

where G, and hence H, acts trivially on S. Now each $f \in \mathrm{Hom}(\mathcal{G} \times S, G)$ factors through a morphism $\varphi : S \to G$, as in the proof of Proposition 3.5. Moreover, f is H-equivariant if and only if φ is H-equivariant, i.e., φ factors through a morphism $\psi : S \to C_G(H)$. Tracing through the above isomorphisms, one checks that F and the assignement $f \mapsto \psi$ are mutually inverse.

For the final assertion, note that $\pi_*^{-1}(A) = \mathrm{Aut}^G_A(X)\,\mathcal{G} = C_G(H)\,\mathcal{G}$. Moreover, the scheme-theoretic intersection $\mathrm{Aut}^G_A(X) \cap \mathcal{G}$ is the kernel of $\pi_* : \mathcal{G} \to A$, that is, H. □

Examples 3.8. (i) If G is the general linear group $\mathrm{GL}(V)$, then $C_G(H)$ is the group of invertible elements of a finite-dimensional associative algebra: the centralizer of H in $\mathrm{End}(V)$. In particular, $C_G(H)$ is a connected algebraic group, and hence $\mathrm{Aut}^G(X)$ is an algebraic group (locally of finite type). This also follows from the well-known structure of automorphisms of vector bundles, in view of Lemma 2.12.

(ii) If $G = \mathrm{SO}_3$ is the special orthogonal group associated to the quadratic form $x^2 + y^2 + z^2$ in characteristic $\neq 2$, then the subgroup $H \subset G$ consisting of diagonal matrices is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and satisfies $H = C_G(H)$. Let $\mathcal{G}$ be an elliptic curve; then the kernel of the multiplication map $2_{\mathcal{G}}$ is isomorphic to H. Then $X := G \times^H \mathcal{G}$ is a homogeneous G-bundle over the elliptic curve $A := \mathcal{G}/H$, and $\mathrm{Aut}^G_A(X) \cong H$ is not connected.

In fact, $\mathrm{Aut}^G(X)$ is not connected either: since any $h \in H$ satisfies $h = h^{-1}$, the G-automorphism of $G \times \mathcal{G}$ given by $(g, \gamma) \mapsto (g, -\gamma)$ descends to a G-automorphism of X that lifts $-\mathrm{id}_A \in \mathrm{Aut}_{\mathrm{gp}}(A)$.

Next, we claim that π admits a reduction of structure group to a principal bundle under a Borel subgroup $B \subset G$, but no such reduction is homogeneous.

Indeed, π is locally trivial for the Zariski topology in view of [Se58, Section 5.6]. Equivalently, there exists a H-equivariant rational map $\sigma : \mathcal{G} - \to G$. Composing with the projection $G \to G/B$ yields an H-equivariant rational map $\tau : \mathcal{G} - \to G/B$ which is in fact regular, since $\mathcal{G}$ is a smooth complete curve. This yields in turn a G-equivariant morphism $f : X = G \times^H \mathcal{G} \to G/B$, i.e., a reduction of structure group to B.

Assume that such a reduction is homogeneous. Then we have a G-isomorphism $X \cong G \times^B X'$ where $X' \subset X$ is a closed B-stable subvariety, and $\pi : X' \to A$ is a B-bundle; moreover, $Y \cong B \times^{H'} \mathcal{G}'$ where $\mathcal{G}'$ is an anti-affine extension of A by H', and $H' \subset B$ is a commutative subgroup. Therefore, $X \cong G \times^{H'} \mathcal{G}'$; moreover, the resulting projection

$\varphi' : X \to G/H'$ is the canonical morphism $X \to \operatorname{Spec} \mathcal{O}(X)$, by Remark 3.3 (iii). In particular, the G-equivariant morphism φ' is the same as $\varphi : X \to G/H$. Thus, H is conjugate to a subgroup of B. As a consequence, H stabilizes a line in k^3 that is isotropic for the quadratic form $x^2 + y^2 + z^2$. But this contradicts the fact that the H-stable lines are exactly the coordinate lines.

(iii) If $G = \mathrm{SL}_2$ in characteristic 2 and H is the subgroup generated by a non-trivial unipotent element, then $H \cong \mathbb{Z}/2\mathbb{Z}$ and $C_G(H) \cong \mu_2 \times \mathbb{G}_a$, where μ_2 is the scheme-theoretic center of SL_2, and $\mathbb{G}_a$ the subgroup of upper triangular unipotent matrices. In particular, H is reduced whereas $C_G(H)$ is not. Let $\mathcal{G}$ be an ordinary elliptic curve, so that H is the set-theoretic kernel of the multiplication map $2_{\mathcal{G}}$. Then again, $X := G \times^H \mathcal{G}$ is a homogeneous bundle over the elliptic curve $A := \mathcal{G}/H$; but now $\mathrm{Aut}^G_A(X)$ is not reduced, nor is $\mathrm{Aut}^G(X)$.

3.3 Anti-affine extensions

By Theorem 3.1, the classification of homogeneous G-bundles over A reduces to those of commutative subgroup schemes $H \subset G$ up to conjugacy, and of anti-affine extensions $\mathcal{G}$ of A by H. We now determine these extensions under the assumption that H is affine; this holds of course if G is affine. For this, we adapt the arguments of [Br09, Section 2], which treats the case that H is a connected (affine) algebraic group over an arbitrary field.

Given an abelian variety A and a commutative affine group scheme H, consider an extension (10) where $\mathcal{G}$ is a connected commutative algebraic group. We have

$$H = H_u \times H_s$$

where H_u (resp. H_s) denotes the largest unipotent (resp. diagonalizable) subgroup scheme of H. Thus, setting $\mathcal{G}_s := \mathcal{G}/H_u$ and $\mathcal{G}_u := \mathcal{G}/H_s$, we obtain extensions

$$0 \longrightarrow H_s \longrightarrow \mathcal{G}_s \xrightarrow{\pi_s} A \longrightarrow 0, \tag{13}$$

$$0 \longrightarrow H_u \longrightarrow \mathcal{G}_u \xrightarrow{\pi_u} A \longrightarrow 0, \tag{14}$$

and an isomorphism

$$\mathcal{G} \cong \mathcal{G}_u \times_A \mathcal{G}_s$$

arising from the natural homomorphisms $\mathcal{G} \to \mathcal{G}_u$, $\mathcal{G} \to \mathcal{G}_s$ over A. We may now state the following result, that generalizes [Br09, Proposition 2.5] with a simpler proof:

Lemma 3.9. *With the above notation, $\mathcal{G}$ is anti-affine if and only if so are $\mathcal{G}_u$ and $\mathcal{G}_s$.*

PROOF. If $\mathcal{G}$ is anti-affine, then so are its quotient groups $\mathcal{G}_u$ and $\mathcal{G}_s$. Conversely, assume that $\mathcal{G}_u$ and $\mathcal{G}_s$ are anti-affine and consider a surjective homomorphism $f : \mathcal{G} \to G$ where G denotes a (connected, commutative) affine algebraic group. Then f induces a surjective homomorphism $f_u : \mathcal{G}_u = \mathcal{G}/H_s \to G/G_s$ which must be trivial, since $\mathcal{G}_u$ is anti-affine and G/G_s is affine. Thus, $G = G_s$ is diagonalizable. Likewise, G is unipotent, and hence is trivial. This shows that any affine quotient of $\mathcal{G}$ is trivial, i.e., $\mathcal{G}$ is anti-affine. □

By that lemma, we may treat the unipotent and diagonalizable cases separately; we begin with the former case. In characteristic 0, the commutative unipotent group H_u is a *vector group*, that is, the additive group of a finite-dimensional vector space. Moreover, there is a universal extension of A by a vector group,

$$0 \longrightarrow H(A) \longrightarrow E(A) \longrightarrow A \longrightarrow 0,$$

where we set

$$H(A) := H^1(A, \mathcal{O}_A)^*.$$

Also, recall that $H^1(A, \mathcal{O}_A)$ is the Lie algebra of the dual abelian variety $\widehat{A}$; in particular, $\dim H(A) = \dim(A)$. Thus, the extension (14) is classified by a linear map

$$\gamma : H(A) \longrightarrow H_u.$$

By [Br09, Proposition 2.3], $\mathcal{G}_u$ is anti-affine if and only if γ is surjective.

The situation in characteristic $p > 0$ is quite different:

LEMMA 3.10. *In positive characteristics, the extension (14) is anti-affine if and only if $\mathcal{G}_u$ is an abelian variety. Equivalently, H_u is finite and its Cartier dual $\widehat{H_u}$ is a local subgroup scheme of $\widehat{A}$, the kernel of the isogeny $\widehat{A} \to \widehat{\mathcal{G}_u}$ dual of the isogeny $\pi_u : \mathcal{G}_u \to A$.*

PROOF. If $\mathcal{G}_u$ is anti-affine, then it is a semi-abelian variety by [Br09, Proposition 2.2]. In other words, $(\mathcal{G}_u)_{\mathrm{aff}}$ is a torus. But $(\mathcal{G}_u)_{\mathrm{aff}}$ is contained in H_u, and hence is trivial: $\mathcal{G}_u$ is an abelian variety. The remaining assertion follows by duality for isogenies of abelian varieties. □

Next, we treat the diagonalizable case in arbitrary characteristic. Denote by $\widehat{H_s}$ the character group of H_s. Then any $\chi \in \widehat{H_s}$ yields an extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}_s \times^{H_s} \mathbb{G}_m \longrightarrow A \longrightarrow 0,$$

the pushout of the extension (13) by $\chi : H_s \to \mathbb{G}_m$. The line bundle on A associated with this $\mathbb{G}_m$-bundle on A is algebraically trivial (see e.g. [Se59, VII Théorème 6]) and hence yields a point $c(\chi) \in \widehat{A}(k)$. The so defined map

$$c : \widehat{H_s} \longrightarrow \widehat{A}(k)$$

is easily seen to be a homomorphism of abstract groups, that classifies the extension (13).

LEMMA 3.11. *With the above notation and assumptions, $\mathcal{G}_s$ is anti-affine if and only if c is injective.*

PROOF. This statement is exactly [Br09, Proposition 2.1 (i)] in the case that H_s is a torus. The general case reduces to that one as follows: H_s is an extension of a finite diagonalizable group scheme F by a torus T, which yields an exact sequence of character groups

$$0 \longrightarrow \widehat{F} \longrightarrow \widehat{H_s} \longrightarrow \widehat{T} \longrightarrow 0.$$

Also, the quotient $B := \mathcal{G}_s/T$ is an abelian variety, and hence $\mathcal{G}_s$ is a semi-abelian variety. Moreover, we have an exact sequence

$$0 \longrightarrow F \longrightarrow B \longrightarrow A \longrightarrow 0,$$

and the dual exact sequence fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{F} & \longrightarrow & \widehat{H_s} & \longrightarrow & \widehat{T} & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle c_0}\downarrow & & \\ 0 & \longrightarrow & \widehat{F} & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{B} & \longrightarrow & 0, \end{array}$$

where c_0 is the classifying homomorphism for the extension $0 \to T \to \mathcal{G}_s \to B \to 0$. Now $\mathcal{G}_s$ is anti-affine if and only if c_0 is injective; this is equivalent to c being injective. □

Combining the above results yields the desired classification:

THEOREM 3.12. *In characteristic 0, the anti-affine extensions of A by H are classified by the pairs (δ, c), where $\delta : H_u^* \to H^1(A, \mathcal{O}_A)$ is an injective linear map, and $c : \widehat{H_s} \to \widehat{A}(k)$ is an injective homomorphism.*

In positive characteristics, the anti-affine extensions of A by H exist only if H_u is finite; then they are classified by the pairs (δ, c), where $\delta : \widehat{H_u} \to \widehat{A}$ is a faithful homomorphism of group schemes, and $c : \widehat{H_s} \to \widehat{A}(k)$ is an injective homomorphism.

REMARKS 3.13. (i) The above theorem yields a description of the image of the homomorphism $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$ for a homogeneous G-bundle $\pi : X \to A$. Indeed, since that image contains A, it suffices to determine its intersection with $\mathrm{Aut}_{\mathrm{gp}}(A)$. Let X be given by a commutative subgroup scheme $H \subset G$ and data δ, c as above. Then $\varphi \in \mathrm{Aut}_{\mathrm{gp}}(A)$ admits a lift in $\mathrm{Aut}^G(X)$ if and only if there exists $z \in N_G(H)$ (the normalizer of H in G) such that the square

$$\begin{array}{ccc} \widehat{H_s} & \xrightarrow{c} & \widehat{A} \\ {\scriptstyle \mathrm{Int}(z)}\downarrow & & {\scriptstyle \widehat{\varphi}}\downarrow \\ \widehat{H_s} & \xrightarrow{c} & \widehat{A} \end{array}$$

commutes, and the analogous square with δ commutes as well. Here $\mathrm{Int}(z)$ denotes the conjugation by z, that acts on H and hence on H_s, H_u and their Cartier duals.

For example, the multiplication by -1 in A lifts to a G-automorphism of X if and only if there exists $z \in N_G(H)$ such that $\mathrm{Int}(z)$ acts on H by the inverse $h \mapsto h^{-1}$.

(ii) In characteristic 0, any unipotent vector bundle over A can be written as $E(A) \times^H V$ for some representation V of the vector group $H(A)$, uniquely determined up to conjugacy. Moreover, given two unipotent vector bundles E_1, E_2 and denoting by V_1, V_2 the associated representations, we have

$$\mathrm{Hom}_A(E_1, E_2) = H^0(A, E_1^* \otimes E_2) \cong (\mathcal{O}(E(A)) \otimes V_1^* \otimes V_2)^{H(A)} \cong \mathrm{Hom}^{H(A)}(V_1, V_2).$$

Thus, *the assignement* $V \mapsto E(A) \times^{H(A)} V$ *yields an equivalence of categories from finite-dimensional representations of* $H(A)$ *to unipotent vector bundles.*

Since the category of finite-dimensional $H(A)$-modules is equivalent to that of modules of finite length over the local ring $\mathcal{O}_{\widehat{A},0}$, we recover Mukai's result that *the category of unipotent bundles is equivalent to that of coherent sheaves over* $\widehat{A}$ *supported at* 0 (see [Muk79, Theorem 4.12]). Note however that Mukai's proof, via the Fourier transform, is valid in all characteristics.

(iii) Consider homogeneous G-bundles over an elliptic curve E in characteristic 0. Since $H^1(E, \mathcal{O}_E) \cong k$, the data of H_u and δ as above are equivalent to that of a unipotent element of G which generates H_u. Also, $E \cong \widehat{E}$. It follows that *the homogeneous G-bundles over E are classified by the triples (u, H_s, c) where $u \in G$ is unipotent, H_s is a diagonalizable subgroup of the centralizer $C_G(u)$, and $c : \widehat{H_s} \to E(k)$ is an injective homomorphism, up to simultaneous conjugacy of u and H_s in G.*

This classification still holds for homogeneous G-bundles over an *ordinary* elliptic curve E in characteristic $p > 0$. Indeed, $\widehat{H_u}$ is then a local subgroup scheme of E of order some power p^r, and hence is isomorphic to μ_{p^r}; equivalently, $H_u \cong \mathbb{Z}/p^r\mathbb{Z}$. So the data of H_u and δ are again equivalent to that of a unipotent element of G.

As a consequence, in either case, the unipotent vector bundles of rank n over E are classified by the unipotent conjugacy classes in GL_n. In particular, there is a unique indecomposable such bundle up to isomorphism: the Atiyah bundle, associated with the unipotent class having one Jordan block. But this indecomposable bundle, viewed as a homogeneous vector bundle, has very different structures in characteristic 0 and in positive characteristics. We refer to [Sc10] for further results on this topic, and to [FMW98, La98, FM00] for developments on (possibly non-homogeneous) holomorphic principal bundles over complex elliptic curves.

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Homogeneous bundles over abelian varieties

Michel Brion

Abstract

We obtain characterizations and structure results for homogeneous principal bundles over abelian varieties, that generalize work of Miyanishi and Mukai on homogeneous vector bundles. For this, we rely on notions and methods of algebraic transformation groups, especially observable subgroups and anti-affine groups.

1 Introduction

A vector bundle E over an abelian variety A is called homogeneous (or translation-invariant), if E is isomorphic to it pull-back under any translation in A. This defines a remarkable class of vector bundles: for example, the homogeneous line bundles are exactly the algebraically trivial ones; they are parametrized by the dual abelian variety $\widehat{A}$. Also, vector bundles over an elliptic curve are close to being homogeneous; specifically, each indecomposable vector bundle of degree 0 is homogeneous (as follows from [At57a]). This result does not extend to abelian varieties of higher dimensions, but the homogeneous vector bundles still admit a nice description in that generality. Namely, by work of Miyanishi and Mukai (see [Mi73, Muk79]), the following conditions are equivalent for a vector bundle E over A:

(i) E is homogeneous.

(ii) E is an iterated extension of algebraically trivial line bundles.

(iii) $E \cong \bigoplus_i L_i \otimes U_i$, where each L_i is an algebraically trivial line bundle, and each U_i is a unipotent vector bundle, i.e., an iterated extension of trivial line bundles.

Similar results had been obtained earlier by Matsushima and Morimoto in the setting of holomorphic vector bundles over complex tori. They also showed that the homogeneous bundles are exactly those admitting a connection; then they admit an integrable connection (see [Ma59, Mo59], and [Bi04, BG08] for more recent developments).

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In the present article, we obtain somewhat analogous characterizations and structure results for those principal bundles $\pi : X \to A$ under an algebraic group G, that are homogeneous in the same sense. This holds e.g. if X is a commutative algebraic group; then π is a homomorphism with kernel G. In fact, by a classical result of Rosenlicht and Serre (see [Se59]), all principal bundles under the additive group are obtained in this way, as well as all homogeneous bundles under the multiplicative group; the latter bundles are parametrized by $\widehat{A}$.

Actually, any homogeneous bundle under an arbitrary algebraic group may be obtained from an extension of A, in view of our main result:

THEOREM 1.1. *The following assertions are equivalent for a principal bundle $\pi : X \to A$ under an algebraic group G:*

(a) *π is homogeneous.*

(b) *There exist a commutative subgroup scheme $H \subset G$ and an extension of commutative group schemes $0 \to H \to \mathcal{G} \to A \to 0$ such that π admits a reduction of structure group to the H-bundle $\mathcal{G} \to A$. Moreover, there is a smallest such subgroup scheme H, up to conjugacy in G.*

(c) *The associated bundle $X/R_u(G) \to X/G_{\mathrm{aff}}$ is homogeneous, where $R_u(G)$ denotes the unipotent radical, and G_{aff} the largest connected affine subgroup of G.*

In characteristic 0, these assertions are also equivalent to:

(d) *π admits a connection.*

(e) *π admits an integrable connection.*

When G is the general linear group GL_n, the G-bundles correspond to the vector bundles of rank n; it is easy to see that this correspondence preserves homogeneity. In this setting, Theorem 1.1 generalizes the above results of Miyanishi and Mukai. Indeed, (b) implies that any homogeneous vector bundle E of rank n is associated with a representation $\rho : H \to \mathrm{GL}_n$, where H is a commutative group scheme. Then the image of H is contained (up to conjugacy) in the subgroup of upper triangular matrices; thus, E is an iterated extension of line bundles. The latter are homogeneous by (c) and hence algebraically trivial, which yields (ii). Likewise, (iii) follows from the decomposition of ρ into a direct sum of weight spaces for the diagonalizable part of H. Conversely, if a vector bundle E satisfies (ii) or (iii), then it is homogeneous by (c).

For an arbitrary algebraic group G, Theorem 1.1 reduces somehow the study of principal G-bundles to the case that G is connected and reductive. One may obtain a further reduction to irreducible bundles in a suitable sense, and explicitly describe these bundles; this is developed in [Br11a] for the projective linear group (equivalently, for projective bundles), and in [Br11b] for the projective orthogonal and symplectic groups.

The proof of Theorem 1.1 is based on an analysis of the group scheme of equivariant automorphisms, $\mathrm{Aut}^G(X)$. This approach, already followed by Miyanishi in [Mi73], is combined here with recent developments on the structure of (possibly non-affine) algebraic groups, see [Br09, SS09, Br10a, Br10b]. Namely, the group scheme $\mathrm{Aut}^G(X)$ is locally of finite type, and hence contains a largest subgroup scheme $\mathcal{G}$ which is anti-affine (i.e., every global regular function on $\mathcal{G}$ is constant). Then $\mathcal{G}$ is a connected commutative algebraic group that acts on A by translations via the natural homomorphism $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$. One easily shows that π is homogeneous if and only if π_* maps $\mathcal{G}$ onto A (Proposition 2.2); equivalently, $G \times \mathcal{G}$ acts transitively on X. From this, (a)$\Rightarrow$(c) and (a)$\Leftrightarrow$(b) follow without much difficulty, where H is the kernel of $\pi_{*|\mathcal{G}}$. To prove (c)$\Rightarrow$(a), we use a characterization of homogeneity in terms of associated vector bundles which is of independent interest (Theorem 2.14) together with Mukai's result that (iii)$\Rightarrow$(i). The characterization in terms of connections also follows readily from Proposition 2.2; this is carried out in Proposition 2.4.

Actually, our approach yields a classification of homogeneous G-bundles over A in terms of pairs $(H, \mathcal{G})$, where $H \subset G$ is a commutative subgroup scheme, and $\mathcal{G}$ an anti-affine extension of A by H (Theorem 3.1). Such extensions have been classified in [Br09, SS09] when H is a connected affine algebraic group; the case that H is an (affine) group scheme is obtained in Theorem 3.12 along similar lines. But classifying all commutative subgroup schemes of an algebraic group G up to conjugacy is a difficult open problem, already when G is the general linear group.

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Notation and conventions. We will consider algebraic varieties and schemes over a fixed algebraically closed field k of arbitrary characteristic. Unless explicitly mentioned, schemes under consideration are assumed to be of finite type over k; by a point of such a scheme X, we mean a closed point. A *variety* is a reduced and separated scheme; in particular, varieties need not be irreducible.

We will use the book [DG70] as a general reference for *group schemes*. Given a group scheme G, we denote by $e_G \in G(k)$ the neutral element and by G_0 the *neutral component*, i.e, the connected component of G that contains e_G. Then G_0 is a connected normal subgroup scheme of G, and the quotient G/G_0 is finite (under our assumption that G is of finite type). A homomorphism of group schemes $f : G \to H$ is *faithful* if its (scheme-theoretic) kernel is trivial.

Throughout this article, G denotes an *algebraic group*, that is, a smooth group scheme, and A an *abelian variety*, i.e., a connected complete algebraic group. Then A is commutative, hence its group law will be denoted additively, and the neutral element by 0_A. For any point $a \in A$, we denote by $\tau_a : A \to A$ the translation $x \mapsto x + a$. Our general reference for abelian varieties is the book [Mum70]; for affine or equivalently, linear algebraic groups, we refer to [Sp98].

2 Characterizations

2.1 Reduction to the case of a connected affine structure group

A variety X equipped with a G-action $G \times X \to X$, $(g, x) \mapsto g \cdot x$ and with a G-invariant morphism

$$\pi : X \longrightarrow A \tag{1}$$

is a *(principal) G-bundle*, also known as a *G-torsor*, if π is faithfully flat and the morphism $G \times X \to X \times_A X$, $(g, x) \mapsto (x, g \cdot x)$ is an isomorphism. Then the morphism π is smooth, and hence so is X. Moreover, π is locally isotrivial and quasi-projective (see e.g. [Br10b, Corollary 3.5 and Remark 3.6 (iii)]). Therefore, the variety X is quasi-projective as well. If G is connected, then X is irreducible.

Definition 2.1. The G-bundle (1) is called *homogeneous*, if the translation $\tau_a : A \to A$ lifts to a G-automorphism $\widetilde{\tau_a} : X \to X$ for any point $a \in A$.

Denote by $\pi_a : \tau_a^*(X) \to A$ the pull-back of the G-bundle (1) under τ_a, so that we have a cartesian square

$$\begin{array}{ccc} \tau_a^*(X) & \xrightarrow{\psi_a} & X \\ {\scriptstyle \pi_a}\downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\tau_a} & A. \end{array}$$

Then the G-equivariant lifts $\widetilde{\tau_a}$ of τ_a correspond bijectively to the isomorphisms of G-bundles $\varphi_a : X \to \tau_a^*(X)$. Thus, *a G-bundle is homogeneous if and only if it is isomorphic to its pull-back under any translation in A.*

We now reformulate the notion of homogeneity in terms of the *equivariant automorphism group* $\mathrm{Aut}^G(X)$. By [Br10b, Theorem 4.2], the group functor $S \mapsto \mathrm{Aut}^G_S(X \times S)$ is represented by a group scheme, locally of finite type, that we still denote by $\mathrm{Aut}^G(X)$ for simplicity. Moreover, we have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^G_A(X) \longrightarrow \mathrm{Aut}^G(X) \xrightarrow{\pi_*} \mathrm{Aut}(A) \tag{2}$$

where $\mathrm{Aut}^G_A(X)$ denotes the group scheme of *bundle automorphisms*, $S \mapsto \mathrm{Aut}^G_{A\times S}(X\times S)$, and π_* arises from the homomorphisms $\mathrm{Aut}^G_S(X\times S) \to \mathrm{Aut}_S(A\times S)$. Also, by a standard result (see e.g. [Br10b, Lemma 4.1]), the map

$$\mathrm{Hom}^G(X, G) \longrightarrow \mathrm{Aut}^G_A(X), \quad f \longmapsto (x \mapsto f(x)\cdot x)$$

is an isomorphism, where $\mathrm{Hom}^G(X, G)$ denotes the group functor $S \mapsto \mathrm{Hom}^G(X \times S, G)$ (the group of G-equivariant morphisms $X \times S \to G$ for the given G-action on X, the trivial action on S and the conjugation action on G).

If G is affine, then so is $\mathrm{Aut}^G_A(X)$; if G is the general linear group, then $\mathrm{Aut}^G_A(X)$ is a connected algebraic group. But for an arbitrary G, the group scheme $\mathrm{Aut}^G_A(X)$ need not be connected nor reduced (see Examples 3.8).

As in [Mi73], we may view $\mathrm{Aut}^G(X)$ as the scheme of pairs (φ, ψ), where $\varphi \in \mathrm{Aut}(A)$ and $\psi : X \to \varphi^*(X)$ is an isomorphism of G-bundles; then π_* is just the projection $(\varphi, \psi) \mapsto \varphi$.

The group scheme $\mathrm{Aut}(A)$ is the semi-direct product of the normal subgroup A of translations with the subgroup scheme $\mathrm{Aut}_{\mathrm{gp}}(A)$ of automorphisms of the algebraic group A. Moreover, $A = \mathrm{Aut}_0(A)$, and $\mathrm{Aut}_{\mathrm{gp}}(A)$ is the constant group scheme associated with a subgroup of $\mathrm{GL}_N(\mathbb{Z})$ for some integer $N \geq 1$. Clearly, *(1) is homogeneous if and only if the image of π_* contains A.*

Next, recall that the group scheme $\mathrm{Aut}^G(X)$ admits a largest subgroup scheme $\mathcal{G}$ which is *anti-affine*, i.e., $\mathcal{O}(\mathcal{G}) = k$. Moreover, $\mathcal{G}$ is a normal connected commutative algebraic subgroup, and the quotient $\mathrm{Aut}^G(X)/\mathcal{G}$ is the *affinization* of $\mathrm{Aut}^G(X)$ (the largest affine quotient group scheme), see [DG70, III, §3, no. 8]. We set

$$\mathcal{G} = \mathrm{Aut}^G(X)_{\mathrm{ant}}.$$

Note that $\mathcal{G}$ is mapped to a subgroup of A under $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$.

Also, recall that X admits a largest anti-affine group of automorphisms, that we denote by $\mathrm{Aut}(X)_{\mathrm{ant}}$; moreover, this subgroup centralizes any connected algebraic group of automorphisms (see [Br10a, Lemma 3.1] when X is irreducible; the case of an arbitary variety X follows readily, since any irreducible component of X is stable under a connected group of automorphisms). In particular, $\mathcal{G} \subset \mathrm{Aut}(X)_{\mathrm{ant}}$ with equality if G is connected.

We may now state our key observation:

PROPOSITION 2.2. *With the above notation, the G-bundle (1) is homogeneous if and only if π_* maps $\mathcal{G}$ onto A.*

If in addition G is affine, then $\mathcal{G}$ is the smallest subgroup scheme of $\mathrm{Aut}^G(X)$ that is mapped onto A via π_*.

PROOF. Assume that (1) is homogeneous. Then π_* restricts to a surjective morphism of group schemes $\pi_*^{-1}(A) \to A$. Let $H \subset \mathrm{Aut}^G(X)$ denote the reduced neutral component of $\pi_*^{-1}(A)$. Then π_* still maps H onto A, and H contains $\mathcal{G}$ as a normal subgroup such that $H/\mathcal{G}$ is affine. It follows that $A/\pi_*(\mathcal{G})$, a quotient of $H/\mathcal{G}$, is affine as well. But $A/\pi_*(\mathcal{G})$ is complete as a quotient of A. We conclude that $\pi_*(\mathcal{G}) = A$. The converse is obvious.

Next, assume that G is affine. Let $\mathcal{H} \subset \mathrm{Aut}^G(X)$ be a subgroup scheme such that $\pi_*(\mathcal{H}) = A$ and consider the largest anti-affine subgroup $\mathcal{H}_{\mathrm{ant}} \subset \mathcal{H}$. Then, as above, $A/\pi_*(\mathcal{H}_{\mathrm{ant}})$ is a quotient of the affinization $\mathcal{H}/\mathcal{H}_{\mathrm{ant}}$ and hence $\pi_*(\mathcal{H}_{\mathrm{ant}}) = A$. Since $\mathcal{H}_{\mathrm{ant}} \subset \mathcal{G}$, it follows that $\mathcal{G} = \mathcal{K}\mathcal{H}_{\mathrm{ant}}$ where $\mathcal{K} := \mathcal{G} \cap \mathrm{Aut}^G_A(X)$. Then $\mathcal{K}$ is affine; thus, $\mathcal{G}/\mathcal{H}_{\mathrm{ant}}$ is affine (as a quotient of $\mathcal{K}$) and anti-affine (as a quotient of $\mathcal{G}$), and hence trivial. We conclude that $\mathcal{G} = \mathcal{H}_{\mathrm{ant}} \subset \mathcal{H}$. □

REMARK 2.3. For an arbitrary G-bundle (1), the central subgroup $G_{\mathrm{ant}} \subset G$ acts on X by bundle automorphisms. In fact, *the induced map $G_{\mathrm{ant}} \to \mathrm{Aut}^G_A(X)_{\mathrm{ant}}$ is an isomorphism.*

Indeed, for any $x \in X$, the evaluation homomorphism

$$\mathrm{Aut}^G_A(X) \cong \mathrm{Hom}^G(X, G) \longrightarrow G, \quad u \longmapsto u(x)$$

sends $\mathrm{Hom}^G(X, G)_{\mathrm{ant}}$ to G_{ant}. Thus,

$$\mathrm{Aut}^G_A(X)_{\mathrm{ant}} \cong \mathrm{Hom}^G(X, G)_{\mathrm{ant}} \subset \mathrm{Hom}^G(X, G_{\mathrm{ant}}) = \mathrm{Hom}(A, G_{\mathrm{ant}})$$

where the latter equality holds since G acts trivially on G_{ant}. Moreover, for any morphism $f : A \to G_{\mathrm{ant}}$, the map $a \mapsto f(a)f(0)^{-1}$ is a group homomorphism, and hence

$$\mathrm{Hom}(A, G_{\mathrm{ant}}) \cong G_{\mathrm{ant}} \ltimes \mathrm{Hom}_{\mathrm{gp}}(A, G_{\mathrm{ant}}).$$

Also, one easily shows that $\mathrm{Hom}_{\mathrm{gp}}(A, G_{\mathrm{ant}})$ is a free abelian group of finite rank. Thus, $\mathrm{Hom}(A, G_{\mathrm{ant}})_0 = G_{\mathrm{ant}}$ which yields our claim.

If the bundle (1) is trivial, then the exact sequence (2) splits. Hence the group scheme $\mathrm{Aut}^G(X)$ is the semi-direct product of $\mathrm{Aut}(A)$ with the normal subgroup scheme

$$\mathrm{Aut}^G_A(X) \cong \mathrm{Hom}^G(G \times A, G) \cong \mathrm{Hom}(A, G).$$

Moreover, as above,

$$\mathrm{Hom}(A, G) \cong G \ltimes \mathrm{Hom}_{\mathrm{gp}}(A, G) \cong G \ltimes \mathrm{Hom}_{\mathrm{gp}}(A, G_{\mathrm{ant}}).$$

It follows that $\mathcal{G} \cong G_{\mathrm{ant}} \times A$.

As a first application of Proposition 2.2, we reduce the study of homogeneous torsors to the case that X is *irreducible*. For an arbitrary G-torsor (1), note that G acts transitively on the set of irreducible components of X. Choose such a component X_1 and denote by $G_1 \subset G$ its stabilizer. Then G_1 is a closed subgroup containing G_0, and $X \cong G \times^{G_1} X_1$, i.e., the map $G \times X_1 \to X$, $(g, x_1) \mapsto g \cdot x_1$ is a G_1-bundle for the action defined by $g_1 \cdot (g, x_1) = (gg_1^{-1}, g_1 \cdot x_1)$; moreover, the restriction $\pi_1 := \pi_{|X_1} : X_1 \to A$ is a G_1-bundle as well.

PROPOSITION 2.4. *With the above notation, $\pi : X \to A$ is homogeneous if and only if so is $\pi_1 : X_1 \to A$.*

PROOF. The connected group $\mathcal{G}$ stabilizes every component of X, and hence restricts to an anti-affine subgroup of $\mathrm{Aut}^{G_1}(X_1)$. In view of Proposition 2.2, it follows that π_1 is homogeneous if so is π. For the converse, note that there is a natural homomorphism $\Phi : \mathrm{Aut}^{G_1}(X_1) \to \mathrm{Aut}^{G}(X)$ which is compatible with the homomorphisms to $\mathrm{Aut}(A)$ (see Proposition 2.8 for the construction of Φ in a more general setting). □

Next, we assume that X is irreducible, and reduce to G being *connected*. For this, we consider the factorization

$$X \xrightarrow{\pi_0} Y = X/G_0 \xrightarrow{\varphi} A \tag{3}$$

of the G-bundle (1), where $G_0 \subset G$ stands for the neutral component, π_0 is a G_0-bundle, and φ is a G/G_0-bundle. (The existence of (3) follows e.g. from [MFK94, Proposition 7.1]). Then Y is an irreducible variety and φ is a finite étale covering. By the Serre-Lang theorem (see e.g. [Mum70, IV.18]), Y is an abelian variety, once a base point 0_Y is chosen in the fiber of φ at 0_A; moreover, φ is a separable isogeny.

THEOREM 2.5. *With the above notation and assumptions, π is homogeneous if and only if so is π_0.*

PROOF. If π is homogeneous, then $\pi_*(\mathcal{G}) = A$ by Proposition 2.2. Moreover, $\mathcal{G} \subset \mathrm{Aut}^{G_0}(X)$ acts on Y via $(\pi_0)_*$. As $\mathcal{G}$ is a connected algebraic group, it follows that $\mathcal{G}$ is mapped onto Y by $(\pi_0)_*$, i.e., π_0 is homogeneous.

For the converse, consider the group

$$\mathcal{G}_o := \mathrm{Aut}^{G_0}(X)_{\mathrm{ant}}$$

on which G acts by conjugation via its finite quotient group $F := G/G_0$. By assumption, $(\pi_0)_*$ maps $\mathcal{G}_o$ onto Y, and hence π_* maps $\mathcal{G}_o$ onto A; moreover, π_* is invariant under the action of F. Given $a \in A$, let $u \in \mathcal{G}_o$ such that $\pi_*(u) = a$; then $\pi_*(\prod_{\gamma \in F} \gamma \cdot u) = Na$

where N denotes the order of F. Thus, Na admits a lift in the fixed point subgroup $\mathcal{G}_o^F$. But $\mathcal{G}_o^F \subset \mathrm{Aut}^G(X)$ and the multiplication $N : A \to A$ is surjective; hence π is homogeneous. □

We now assume that G is connected, and reduce to G being *affine*. By Chevalley's structure theorem (see [Co02] for a modern proof), G admits a largest connected affine subgroup G_{aff}; moreover, G_{aff} is a normal subgroup of G and G/G_{aff} is an abelian variety. Note that $G = G_{\mathrm{aff}}G_{\mathrm{ant}}$, where G_{ant} denotes the largest anti-affine subgroup of G (indeed, $G_{\mathrm{aff}}G_{\mathrm{ant}}$ is a closed normal subgroup of G, and the quotient $G/G_{\mathrm{aff}}G_{\mathrm{ant}}$ is both affine and complete). Also, recall from [Br10b, Corollary 3.5] that the G-bundle (1) has a unique factorization

$$X \xrightarrow{p} Y = X/G_{\mathrm{aff}} \xrightarrow{q} A = X/G \tag{4}$$

where p is a G_{aff}-bundle and q is a G/G_{aff}-bundle. Then Y *is an abelian variety and* q *is a homomorphism of algebraic groups*, again once an origin 0_Y is chosen in the fiber of q at 0_A. Indeed, we have the following result, which is well-known in characteristic 0 (see [Se58, Corollaire 1]):

Proposition 2.6. *Let $\pi : X \to A$ be a principal bundle under an abelian variety G. Then X is an abelian variety as well. Moreover, G acts on X via an exact sequence of abelian varieties*

$$0 \longrightarrow G \longrightarrow X \longrightarrow A \longrightarrow 0.$$

Proof. Recall that $X \cong G \times^H Y$ for some finite subgroup scheme $H \subset G$ and some closed H-stable subscheme $Y \subset X$; moreover, the restriction $\pi_{|Y} : Y \to A$ is an H-bundle (see e.g. [Br10b, Corollary 3.5]). By [No83, Remark 2]), we have $Y \cong H \times^K B$ for some closed subgroup scheme $K \subset H$ and some abelian variety B fitting into an exact sequence

$$0 \longrightarrow K \longrightarrow B \longrightarrow A \longrightarrow 0.$$

Thus,

$$X \cong G \times^H (H \times^K B) \cong G \times^K B$$

and π is identified with the morphism $p : G \times^K B \to B/K = A$. Moreover, $G \times^K B = (G \times B)/K$ is an abelian variety, and p is a homomorphism with kernel $G \times^K K \cong G$. □

Theorem 2.7. *The following assertions are equivalent for a principal bundle $\pi : X \to A$ under a connected algebraic group G:*

(i) *π is homogeneous.*

(ii) *The associated G_{aff}-bundle $p : X \to Y$ is homogeneous (where Y is an abelian variety by Proposition 2.6).*

PROOF. The group $\mathcal{G} \subset \mathrm{Aut}^{G_{\mathrm{aff}}}(X)$ acts on Y via $p_* : \mathrm{Aut}^{G_{\mathrm{aff}}}(X) \to \mathrm{Aut}(Y)$. If π is homogeneous, then this action is transitive. Indeed, $\mathcal{G}$ acts transitively on A by Proposition 2.2. Also, $\mathcal{G}$ contains the largest anti-affine subgroup $G_{\mathrm{ant}} \subset G$; moreover, G_{ant} acts trivially on A and transitively on the fibers of the G/G_{aff}-bundle $q : Y \to A$, since the natural homomorphism $G_{\mathrm{ant}} \to G/G_{\mathrm{aff}}$ is surjective. As $\mathcal{G}$ is connected and Y is an abelian variety, it follows that p_* induces a surjective homomorphism $\mathcal{G} \to Y$. Thus, p is homogeneous if so is π.

The converse implication is obvious, since every translation of A lifts to a translation of Y. □

2.2 Change of structure group

Given a G-bundle $\pi : X \to S$ over an arbitrary base, recall that a *reduction* to a subgroup scheme $H \subset G$ is a closed H-stable subscheme $Y \subset X$ such that the natural map $G \times^H Y \to X$ is an isomorphism. Then Y is the fiber of the associated G-equivariant morphism $f : X \to G/H$ at the base point of G/H. In fact, the data of Y and of f are equivalent, and π restricts to an H-bundle $Y \to S$ (see e.g. [Br10b, Lemma 3.2]).

Also, given a homomorphism of algebraic groups $f : G \to G'$, we may consider the G'-bundle $X' := G' \times^G X \to S$ obtained by *extension of structure group*. This bundle might not always exist in this generality (i.e., X' may not be a scheme), but it does exist in our setting, and this construction preserves homogeneity:

PROPOSITION 2.8. *Let $\pi : X \to A$ be a G-bundle, and $f : G \to G'$ a homomorphism of algebraic groups.*

(i) *The G'-bundle*

$$\pi' : X' := G' \times^G X \longrightarrow A$$

exists, and there is a natural homomorphism

$$\Phi : \mathrm{Aut}^G(X) \longrightarrow \mathrm{Aut}^{G'}(X'),$$

compatible with $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$ and with $\pi'_* : \mathrm{Aut}^{G'}(X') \to \mathrm{Aut}(A)$. In particular, π' is homogeneous if so is π.

(ii) *Conversely, if π' is homogeneous, then so is π under the additional assumptions that f is faithful and $G'/f(G)$ is quasi-affine.*

PROOF. (i) Consider the neutral component $G_0 \subset G$ and the (finite) quotient group $F := G/G_0$. Then the G'-bundle $G' \times^{G_0} X \to X/G_0$ exists by [Br10b, Corollary 3.4]; moreover, X/G_0 is a finite cover of A, and hence is projective. Thus, the variety $G' \times^{G_0} X$ is quasi-projective (see [Br10b, Corollary 3.5]; it carries an action of F arising from the

diagonal action of G on $G' \times X$. The quotient $(G' \times^{G_0} X)/F$ exists by [Mum70], and yields the desired G-bundle π'.

To construct Φ, note that any $u \in \mathrm{Aut}^G(X)$ yields an automorphism

$$\mathrm{id} \times u : G' \times X \longrightarrow G' \times X, \quad (g', x) \longmapsto (g', u(x))$$

which is $G' \times G$-equivariant, and hence descends to $u' \in \mathrm{Aut}^{G'}(X')$. Moreover, the assignement $u \mapsto u' =: \Phi(u)$ readily extends to a homomorphism of group functors, and hence yields the desired homomorphism of group schemes.

(ii) Let $\mathcal{G}' := \mathrm{Aut}^{G'}(X')_{\mathrm{ant}}$. Then $\pi'_* : \mathcal{G}' \to A$ is surjective by Proposition 2.2. Also, the natural morphism

$$p : X' = G' \times^G X \to G'/f(G)$$

is invariant under $\mathcal{G}'$ by Lemma 2.9. Thus, $\mathcal{G}'$ acts on X (the fiber of p at the base point of $G'/f(G)$), and this action lifts the A-action on itself by translations. Hence π is homogeneous. □

LEMMA 2.9. *Let Z be a variety, $\mathcal{H}$ its largest anti-affine group of automorphisms, and $f : Z \to W$ a morphism to a quasi-affine variety. Then f is $\mathcal{H}$-invariant.*

PROOF. The action of $\mathcal{H}$ on Z yields a structure of rational $\mathcal{H}$-module on the algebra of regular functions $\mathcal{O}(Z)$. Since $\mathcal{H}$ is anti-affine, it must act trivially on $\mathcal{O}(Z)$ and hence on the image of the algebra homomorphism $f^\# : \mathcal{O}(W) \longrightarrow \mathcal{O}(Z)$. As the regular functions on W separate points, this means that f is $\mathcal{H}$-invariant. □

REMARKS 2.10. (i) With the notation of the above proposition, the homogeneity of π' does not always imply that of π. For instance, if f is constant, then π' is the trivial bundle $G' \times A \to A$, and hence is homogeneous even if π is not. For less trivial instances, where f is faithful, see Examples 2.17 and 3.8 (ii).

(ii) Recall that a closed subgroup H of an affine algebraic group G is *observable in G*, if the homogeneous space G/H is quasi-affine; we refer to [Gr97, Theorem 2.1] for further characterizations of observable subgroups. Also, recall that H is observable in G if and only if so is the radical $R(H)$ (the largest connected solvable normal subgroup of H), see [loc. cit., Theorem 2.7]. Moreover, any closed commutative subgroup is observable by [loc. cit., Corollary 2.9]. As a consequence, *H is observable in G if $R(H)$ is commutative.*

We will encounter quasi-affine homogeneous spaces G/H where $H \subset G$ is a possibly non-reduced subgroup scheme. This presents no additional difficulty in view of the following result:

LEMMA 2.11. *Let G be an algebraic group, H a closed subgroup scheme, and $H_{\mathrm{red}} \subset H$ the associated reduced subgroup scheme (i.e., the largest algebraic subgroup).*

(i) *If G/H is quasi-affine, then H contains G_{ant}. In other words, G acts on G/H via its affinization G/G_{ant}.*

(ii) *G/H is quasi-affine if and only if so is $G_0/(G_0 \cap H)$.*

(iii) *G/H is quasi-affine if and only if so is G/H_{red}.*

PROOF. (i) The anti-affine group G_{ant} acts trivially on the rational G-module $\mathcal{O}(G/H)$, and hence on G/H as the latter is quasi-affine.

(ii) follows from the fact that G_0 acts on G/H with finitely many orbits, all of them being closed and isomorphic as varieties to $G_0/(G_0 \cap H)$. Indeed, for any $g \in G$, the G_0-orbit of $gH \in G/H$ is isomorphic to $G_0/(G_0 \cap \mathrm{Int}(g)H) = G_0/\mathrm{Int}(g)(G_0 \cap H)$, where $\mathrm{Int}(g) : G \to G$ denotes the conjugation by g. Moreover, $\mathrm{Int}(g)$ induces an isomorphism of varieties $G_0/(G_0 \cap H) \to G_0/\mathrm{Int}(g)(G_0 \cap H)$.

(iii) By (ii) and the equality $(G_0 \cap H)_{\mathrm{red}} = G_0 \cap H_{\mathrm{red}}$, we may assume that H is connected.

If G/H is quasi-affine, then we may find a finite-dimensional vector space $V \subset \mathcal{O}(G/H)$ such that the associated morphism $G/H \to V^*$ is a locally closed immersion. Then V is contained in a finite-dimensional G-submodule of $\mathcal{O}(G/H)$, and hence we may assume that V itself is G-stable. Let $R \subset \mathcal{O}(G/H)$ denote the subalgebra generated by V, and S the integral closure of R in the field of rational functions $k(G/H_{\mathrm{red}})$. Then S is a G-stable, finitely generated algebra; thus, we may find a finite-dimensional G-submodule $W \subset S$ that generates S. Moreover, since the fraction field of R is $k(G/H)$, we see that $k(G/H_{\mathrm{red}})$ is the fraction field of S. Thus, the natural map $G/H_{\mathrm{red}} \to W^*$ is birational; since it is G-equivariant, it is a locally closed immersion. Hence G/H_{red} is quasi-affine.

Conversely, assume that G/H_{red} is quasi-affine and choose a finite-dimensional G-submodule $W \subset \mathcal{O}(G/H_{\mathrm{red}})$ that yields a locally closed immersion $G/H_{\mathrm{red}} \to W^*$. Since the natural morphism $G/H_{\mathrm{red}} \to G/H$ is purely inseparable, there exists a positive integer q (a power of the characteristic exponent of k) such that $W^q \subset \mathcal{O}(G/H)$. Let $R \subset \mathcal{O}(G/H)$ denote the integral closure of the subalgebra generated by W^q. Then, as in the above step, R is generated by a finite-dimensional G-submodule V, and the corresponding map $G/H \to V^*$ is a locally closed immersion. □

2.3 Associated vector bundles

A vector bundle

$$p : E \longrightarrow A \tag{5}$$

is *homogeneous*, if $E \cong \tau_a^*(E)$ for all $a \in A$. As for principal bundles, the isomorphisms of vector bundles $E \to \tau_a^*(E)$ correspond bijectively to those lifts $\widetilde{\tau}_a : E \to E$ of $\tau_a : A \to A$ that are linear on fibers, i.e., that commute with the action of the multiplicative group $\mathbb{G}_m$ by scalar multiplication on fibers. In other words, *E is homogeneous if and only if each translation τ_a lifts to some $\widetilde{\tau}_a \in \mathrm{Aut}^{\mathbb{G}_m}(E)$* (the equivariant automorphism group of E viewed as a variety with $\mathbb{G}_m$-action).

Also, the group functor $S \mapsto \mathrm{Aut}_S^{\mathbb{G}_m}(E \times S)$ is represented by a group scheme, locally of finite type, that we still denote by $\mathrm{Aut}^{\mathbb{G}_m}(E)$. (This may be deduced from the analogous statement for principal bundles by using Lemma 2.12 (ii); alternatively, one may identify $\mathrm{Aut}^{\mathbb{G}_m}(E)$ with a closed subfunctor of $\mathrm{Aut}(\mathbb{P}(E \oplus O_A))$, where $\mathbb{P}(E \oplus O_A)$ denotes the projective variety obtained by completing E with its "hyperplane at infinity" $\mathbb{P}(E)$). Moreover, we have an exact sequence of group schemes

$$1 \longrightarrow \mathrm{Aut}_A^{\mathbb{G}_m}(E) \longrightarrow \mathrm{Aut}^{\mathbb{G}_m}(E) \xrightarrow{p_*} \mathrm{Aut}(A),$$

where $\mathrm{Aut}_A^{\mathbb{G}_m}(E)$ denotes the group scheme of automorphisms of the vector bundle E over A, and p_* assigns to each $\mathbb{G}_m$-automorphism of E, the induced automorphism of A viewed as the categorical quotient $E//\mathbb{G}_m$. Note that $\mathrm{Aut}_A^{\mathbb{G}_m}(E)$ is a connected affine algebraic group, since it is open in the affine space $\mathrm{End}_A^{\mathbb{G}_m}(E) = H^0(A, End(E))$. Thus, $\mathrm{Aut}^{\mathbb{G}_m}(E)$ is an algebraic group (locally of finite type).

We now relate the equivariant automorphism groups and notions of homogeneity for vector bundles and principal bundles:

LEMMA 2.12. *Let $\pi : X \to A$ be a G-bundle, and $\rho : G \to \mathrm{GL}(V)$ a finite-dimensional representation.*

(i) *The associated vector bundle*

$$p : E := X \times^G V \longrightarrow A$$

exists, and there is a natural homomorphism

$$\Phi : \mathrm{Aut}^G(X) \longrightarrow \mathrm{Aut}^{\mathbb{G}_m}(E)$$

compatible with $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$ and $p_* : \mathrm{Aut}^{\mathbb{G}_m}(E) \to \mathrm{Aut}(A)$.

(ii) *If $G = \mathrm{GL}(V)$ and $\rho = \mathrm{id}$, then Φ is an isomorphism.*

(iii) *If the principal bundle π is homogeneous, then so is the vector bundle p. The converse holds if $G = \mathrm{GL}(V)$ and $\rho = \mathrm{id}$.*

PROOF. (i) The existence of p follows from faithfully flat descent. The construction of Φ is similar to that in the proof of Proposition 2.8 (ii).

(ii) follows from the correspondence between vector bundles with fiber V and principal bundles under $\mathrm{GL}(V)$. Specifically, we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Aut}_A^{\mathrm{GL}(V)}(X) & \longrightarrow & \mathrm{Aut}^{\mathrm{GL}(V)}(X) & \xrightarrow{\pi_*} & \mathrm{Aut}(A) \\ & & \Psi\downarrow & & \Phi\downarrow & & =\downarrow \\ 1 & \longrightarrow & \mathrm{Aut}_A^{\mathbb{G}_m}(E) & \longrightarrow & \mathrm{Aut}^{\mathbb{G}_m}(E) & \xrightarrow{p_*} & \mathrm{Aut}(A), \end{array}$$

where $\Psi : \mathrm{Aut}_A^{\mathrm{GL}(V)}(X) = \mathrm{Hom}^{\mathrm{GL}(V)}(X, \mathrm{GL}(V)) \to \mathrm{Aut}_A^{\mathbb{G}_m}(E) \subset H^0(A, End(E))$ associates with each equivariant morphism $f : X \to \mathrm{GL}(V) \subset \mathrm{End}(V)$, the same f viewed as a section of the associated bundle $\mathrm{End}(E)$. Thus, Ψ is an isomorphism. Moreover, π_* and p_* have the same image, since for a given $u \in \mathrm{Aut}(A)$, the condition that $u^*(X) \cong X$ as $\mathrm{GL}(V)$-bundles is equivalent to $u^*(E) \cong E$ as vector bundles.

(iii) is a direct consequence of (i) and (ii). □

Next, we record the following preliminary result:

LEMMA 2.13. *Any affine algebraic group G admits a faithful finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ such that the homogeneous space $\mathrm{GL}(V)/\rho(G)$ is quasi-affine; in other words, $\rho(G)$ is observable in $\mathrm{GL}(V)$.*

PROOF. Choose a faithful finite-dimensional representation

$$\iota : G \longrightarrow \mathrm{GL}(M).$$

By a theorem of Chevalley, there exist a $\mathrm{GL}(M)$-module N and a line $\ell \subset N$ such that G is the isotropy subgroup scheme of ℓ in $\mathrm{GL}(M)$. Then G acts on ℓ via a character $\chi : G \to \mathbb{G}_m$. This yields a faithful homomorphism

$$(\iota, \chi) : G \longrightarrow \mathrm{GL}(M) \times \mathbb{G}_m.$$

Moreover, $\mathrm{GL}(M) \times \mathbb{G}_m$ acts linearly on N via

$$(g, t) \cdot x := t^{-1} g \cdot x,$$

and the isotropy subgroup scheme of any point $x \in \ell$ for this action is exactly $(\iota \times \chi)(G)$. It follows that the homogeneous space

$$(\mathrm{GL}(M) \times \mathbb{G}_m)/(\iota, \chi)(G) \cong (\mathrm{GL}(M) \times \mathbb{G}_m) \cdot x$$

is a locally closed subvariety of N, and hence is quasi-affine. Next, the natural embedding

$$\mathrm{GL}(M) \times \mathbb{G}_m = \mathrm{GL}(M) \times \mathrm{GL}(1) \subset \mathrm{GL}(M \oplus k)$$

yields a faithful representation $\rho : G \to \mathrm{GL}(M \oplus k)$, and the homogeneous space

$$\mathrm{GL}(M \oplus k)/\rho(G) = \mathrm{GL}(M \oplus k) \times^{\mathrm{GL}(M)\times\mathbb{G}_m} (\mathrm{GL}(M) \times \mathbb{G}_m)/(\iota, \chi)(G)$$

is a locally closed subvariety of $\mathrm{GL}(M \oplus k) \times^{\mathrm{GL}(M)\times\mathbb{G}_m} N$. The latter is the total space of a vector bundle over the affine variety $\mathrm{GL}(M \oplus k)/(\mathrm{GL}(M) \times \mathbb{G}_m)$, and hence is affine as well. Thus, $\mathrm{GL}(M \oplus k)/\rho(G)$ is quasi-affine. □

We may now state the main result of this subsection:

THEOREM 2.14. *Given a G-bundle (1) where G is affine, the following conditions are equivalent:*

(i) *π is homogeneous.*

(ii) *All vector bundles associated with irreducible representations are homogeneous.*

(iii) *All associated vector bundles are homogeneous.*

(iv) *The associated vector bundle (5) is homogeneous for some faithful finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ such that $\rho(G)$ is observable in $\mathrm{GL}(V)$. (Such a representation exists by Lemma 2.13).*

PROOF. (i)⇒(ii) follows from Lemma 2.12 (iii).

(ii)⇒(iii) holds since any extension of homogeneous vector bundles is homogeneous, as follows from [Muk79, Theorem 4.19] that characterizes homogeneity in terms of the Fourier-Mukai transform.

We sketch an alternative proof of this fact via methods of algebraic groups: given two homogeneous vector bundles E_1, E_2 over A with respective anti-affine groups of automorphisms $\mathcal{G}_1$, $\mathcal{G}_2$, the fibered product $\mathcal{G}_1 \times_A \mathcal{G}_2$ acts linearly on $\mathrm{Ext}^1_A(E_2, E_1)$ and its largest anti-affine subgroup $\mathcal{G}$ acts trivially. Thus, for any extension $0 \to E_1 \to E \to E_2 \to 0$, we see that $\mathcal{G}$ acts on E by lifting the translations in A.

(iii)⇒(iv) is obvious.

(iv)⇒(i) The $\mathrm{GL}(V)$-bundle $\mathrm{GL}(V) \times^G X \to A$ is homogeneous by Lemma 2.12 (iii). Since $\mathrm{GL}(V)/\rho(G)$ is quasi-affine, it follows that π is homogeneous, in view of Proposition 2.8 (ii). □

As a direct consequence, we obtain:

COROLLARY 2.15. *Let V be a finite-dimensional vector space, and $G \subset \mathrm{GL}(V)$ a reductive subgroup. Then a G-bundle $\pi : X \to A$ is homogeneous if and only if the associated vector bundle $p : X \times^G V \to A$ is homogeneous.*

(Indeed, the variety $\mathrm{GL}(V)/G$ is affine).

Note that the same statement holds, with a completely different proof, for G-bundles over complete rational homogeneous varieties in characteristic 0; see [BT10, Theorem 1.1].

Another straightforward consequence of Theorem 2.14 is the following result:

COROLLARY 2.16. *Let G be an affine algebraic group, $U \subset G$ a normal unipotent subgroup scheme, $\pi : X \to A$ a G-bundle, and $\bar{\pi} : \bar{X} := X/U \to A$ the associated principal bundle under $\bar{G} := G/U$. Then π is homogeneous if and only if so is $\bar{\pi}$.*

In particular, any principal bundle under a unipotent group is homogeneous.

(Indeed, any irreducible representation $\rho : G \to \mathrm{GL}(V)$ factors through $\bar{G}$).

Combining Corollary 2.16 with Proposition 2.4, Theorem 2.5 and Theorem 2.7, we obtain the equivalence (a)⇔(c) in Theorem 1.1. The equivalences (a)⇔(d)⇔(e) will be obtained in Subsection 2.4, and (a)⇔(b) in Subsection 3.1.

To conclude this subsection, we provide the example announced in Remark 2.10 (i):

EXAMPLE 2.17. Consider a non-trivial line bundle L over A, generated by n global sections. Then L fits into an exact sequence of vector bundles over A

$$0 \longrightarrow E \longrightarrow O_A^n \longrightarrow L \longrightarrow 0$$

where O_A denotes the trivial line bundle. This corresponds to a reduction of the trivial GL_n-bundle $\pi' : \mathrm{GL}_n \times A \to A$ to a principal P-bundle $\pi : X \to A$, where $P \subset \mathrm{GL}_n$ denotes the parabolic subgroup that stabilizes a hyperplane. Taking for U the unipotent radical of P, we see that $\bar{P} \cong \mathrm{GL}_{n-1} \times \mathbb{G}_m$ and the associated bundle $\bar{\pi} : \bar{X} \to A$ is the product over A of the principal bundles associated with E and L. Since L is not homogeneous, the same holds for $\bar{\pi}$, and hence π is not homogeneous as well; but π' is obviously homogeneous.

2.4 Connections

We first recall some classical constructions associated with principal bundles (see [At57b]) in our special setting. For a G-bundle $\pi : X \to A$, we have an exact sequence of tangent sheaves on X

$$0 \longrightarrow T_\pi \longrightarrow T_X \xrightarrow{d\pi} \pi^* T_A \longrightarrow 0$$

and isomorphisms of sheaves of Lie algebras

$$\mathcal{O}_X \otimes \mathrm{Lie}(G) \xrightarrow{\cong} T_\pi, \quad \mathcal{O}_A \otimes \mathrm{Lie}(A) \xrightarrow{\cong} T_A$$

with an obvious notation. This yields an exact sequence of sheaves of Lie algebras on X

$$0 \longrightarrow \mathcal{O}_X \otimes \mathrm{Lie}(G) \longrightarrow T_X \xrightarrow{d\pi} \mathcal{O}_X \otimes \mathrm{Lie}(A) \longrightarrow 0. \tag{6}$$

Also, all these sheaves are G-linearized, where G acts on $\mathrm{Lie}(G)$ via its adjoint representation, and G acts trivially on $\mathrm{Lie}(A)$; moreover, the morphisms in (6) preserve the linearizations.

Now the category of G-linearized locally free sheaves on X is equivalent to that of locally free sheaves on A, via the invariant direct image π_*^G and the pull-back π^*. Thus, (6) corresponds to an exact sequence of sheaves of Lie algebras on A, the *Atiyah sequence*

$$0 \longrightarrow \pi_*(\mathcal{O}_X \otimes \mathrm{Lie}(G))^G \longrightarrow \mathrm{At}(X) \xrightarrow{d\pi} \mathcal{O}_A \otimes \mathrm{Lie}(A) \longrightarrow 0 \tag{7}$$

where the left-hand side is the sheaf of local sections of the associated vector bundle $X \times^G \mathrm{Lie}(G) \to A$ (the *adjoint bundle*), and $\mathrm{At}(X) := \pi_*(T_X)^G$.

Taking global sections in (7) yields an exact sequence of Lie algebras

$$0 \longrightarrow \mathrm{Hom}^G(X, \mathrm{Lie}(G)) \longrightarrow H^0(X, T_X)^G \xrightarrow{d\pi} \mathrm{Lie}(A) \tag{8}$$

which may also be obtained from the exact sequence of group schemes (2) by taking their Lie algebras (see e.g. [Br10b, Section 4]).

Moreover, the G-equivariant splittings of the exact sequence (6) correspond to the splittings of (7), i.e., to the sections of $d\pi : H^0(X, T_X)^G \to \mathrm{Lie}(A)$. Such a section

$$D : \mathrm{Lie}(A) \longrightarrow H^0(X, T_X)^G$$

is called a *connection*; D is *integrable* (or *flat*) if it is a homomorphism of Lie algebras, i.e., its image is an abelian Lie subalgebra.

We may now state the following algebraic version of a homogeneity criterion for holomorphic bundles over complex tori, due to Matsushima (see [Ma59, Théorème 1]) and Morimoto ([Mo59, Théorème 1]):

PROPOSITION 2.18. *In characteristic* 0*, the following conditions are equivalent for a* G*-bundle* $\pi : X \to A$*:*

(i) π *is homogeneous.*

(ii) π *admits an integrable connection.*

(iii) π *admits a connection.*

PROOF. (i)⇒(ii) The restriction $\pi_{*|\mathcal{G}} : \mathcal{G} \to A$ is surjective by Proposition 2.2, and hence so is $d\pi_{*|\mathcal{G}} : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(A)$. Any splitting of the latter map yields an integrable connection, since $\mathcal{G}$ is commutative.

(ii)⇒(iii) is left to the reader.

(iii)⇒(i) The map $d\pi : H^0(X, T_X)^G \to \mathrm{Lie}(A)$ is surjective, and hence $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$ is surjective on neutral components. □

When $k = \mathbb{C}$ and G is a connected reductive group, the above statement has also been proved by Biswas (see [Bi04, Theorem 3.1]), via very different arguments based on semistability. The case of holomorphic principal bundles over complex tori is treated by Biswas and Gomez (see [BG08, Theorem 3.1]) via the theory of Higgs G-bundles; in this setting, they also characterize homogeneous bundles in terms of pseudostability and vanishing of characteristic classes.

On the other hand, in positive characteristics, the group scheme $\mathrm{Aut}^G(X)$ need not be reduced (see Example 3.8 (iii)) and hence characterizing homogeneous bundles in terms of connections would make little sense.

3 Structure

3.1 Homogeneous bundles as homogeneous spaces

By a refinement of Proposition 2.2, we obtain our main structure result:

THEOREM 3.1. *Let $\pi : X \to A$ be a principal bundle under an algebraic group G and consider the $G \times \mathcal{G}$-action on X, where $\mathcal{G} \subset \mathrm{Aut}^G(X)$ denotes the largest anti-affine subgroup. Then the following conditions are equivalent:*

(i) *π is homogeneous.*

(ii) *There is a $G \times \mathcal{G}$-equivariant isomorphism*

$$X \cong G \times^H \mathcal{G}, \tag{9}$$

where $H \subset G$ is a closed subgroup scheme that also fits into an extension of commutative group schemes

$$0 \longrightarrow H \longrightarrow \mathcal{G} \xrightarrow{\pi_*} A \longrightarrow 0 \tag{10}$$

via $\pi_* : \mathcal{G} \to A$. (In particular, H is commutative).

Moreover, the assignement $X \mapsto (\mathcal{G}, H)$ yields a bijection between the isomorphism classes of G-bundles over A, and the equivalence classes of pairs consisting of a commutative subgroup scheme $H \subset G$ and an extension (10), where $\mathcal{G}$ is anti-affine. Here two pairs are equivalent if the corresponding subgroups H, H' are conjugate in G, and if in addition the resulting isomorphism $H \to H'$ lifts to an isomorphism of extensions.

PROOF. Assume that π is homogeneous and consider the action of $G \times \mathcal{G}$ on X. This action is transitive, since G acts transitively on the fibers of π while $\mathcal{G}$ acts transitively on A (Proposition 2.2). Thus, $X \cong (G \times \mathcal{G})/H$ as $G \times \mathcal{G}$-varieties, where $H \subset G \times \mathcal{G}$ denotes the isotropy subgroup scheme of some point $x \in X$; this isomorphism maps x to the base point $(e_G, e_{\mathcal{G}})H$ of the homogeneous space $(G \times \mathcal{G})/H$. Denote by $p : H \to G$

and $q : H \to \mathcal{G}$ the projections. Then the (scheme-theoretic) kernel of q is identified with a subgroup scheme of G that fixes x. Since π is a G-bundle, it follows that $\mathrm{Ker}(q)$ is trivial. Likewise, the kernel of p is identified with a subgroup scheme of $\mathcal{G}$ that fixes x. Since $\mathcal{G}$ centralizes $G \times \mathcal{G}$, it follows that $\mathrm{Ker}(p)$ acts trivially on X, and hence is trivial as $\mathcal{G}$ acts faithfully. This yields the assertions of (ii).

Conversely, if (ii) holds, then $\mathcal{G}$ acts on X by G-automorphisms that lift the translations of A. Thus, π is homogeneous.

This proves the equivalence of (i) and (ii). The final assertion follows from the fact that the isotropy subgroup scheme $H = (G \times \mathcal{G})_x$ is uniquely determined up to conjugacy in $G \times \mathcal{G}$ and hence in G. □

COROLLARY 3.2. *Given an abelian variety A and a connected commutative algebraic group G, the isomorphism classes of homogeneous G-bundles over A correspond bijectively to the isomorphism classes of extensions of connected algebraic groups*

$$1 \longrightarrow G \longrightarrow X \longrightarrow A \longrightarrow 1. \tag{11}$$

Moreover, any such extension is commutative.

PROOF. For any homogeneous G-bundle (1), the variety $X := (G \times \mathcal{G})/H$ has a structure of commutative algebraic group such that the projection $\pi : X \to A$ is a homomorphism with kernel G. This group structure is unique up to the choice of the neutral element in the fiber of π at 0_A, i.e., up to a translation in G.

Conversely, any extension (11) obviously yields a homogeneous G-bundle. Moreover, $X = X_{\mathrm{aff}} X_{\mathrm{ant}}$, and $X_{\mathrm{aff}} \subset G$ is commutative. Since X_{ant} is central in X, it follows that X is commutative.

Also, note that the automorphism group of a G-bundle $\pi : X \to A$ is

$$\mathrm{Aut}^G_A(X) \cong \mathrm{Hom}^G(X, G) = \mathrm{Hom}(A, G)$$

as seen in Remark 2.3, and the automorphism group of an extension (11) is $\mathrm{Hom}_{\mathrm{gp}}(A, G)$. Moreover, $\mathrm{Hom}(A, G)$ is the semi-direct product of $\mathrm{Hom}_{\mathrm{gp}}(A, G)$ with the normal subgroup G of translations. Putting these facts together yields our assertion. □

REMARKS 3.3. (i) If k is the algebraic closure of a finite field, then every anti-affine group is an abelian variety (see [Br09, Proposition 2.2]). Thus, $\pi_* : \mathcal{G} \to A$ is an isogeny for any homogeneous G-bundle (1). In other words, the group scheme H is finite. This does not extend to any other algebraically closed field, in view of [Br09, Example 3.11].

(ii) With the notation of Theorem 3.1, the subgroup scheme H always contains G_{ant}, since by Remark 2.3,

$$G_{\mathrm{ant}} = \mathrm{Aut}^G_A(X)_{\mathrm{ant}} \subset \mathrm{Aut}^G(X)_{\mathrm{ant}} \cap \mathrm{Aut}_A(X) = \mathrm{Ker}(\pi_* : \mathcal{G} \to A).$$

Moreover, we have for the trivial bundle: $H = G_{\mathrm{ant}}$ and $\mathcal{G} = G_{\mathrm{ant}} \times A$. So Theorem 3.1 (stated for an arbitrary algebraic group G) yields a "natural" classification if and only if G is *affine*; then the trivial bundle does correspond to the trivial subgroup scheme H.

(iii) With the notation of Theorem 3.1 again, the projection $G \times^H \mathcal{G} \to G/H$ yields a G-equivariant morphism

$$\varphi : X \longrightarrow G/H$$

which is a principal bundle under the anti-affine group $\mathcal{G}$. We may view φ as a reduction of structure group to the H-bundle $\pi_* : \mathcal{G} \to A$. If G is affine, then the variety G/H is quasi-affine, as follows from [Gr97, Corollary 1.2.9] together with Lemma 2.11. Moreover, the morphism $\varphi^\# : \mathcal{O}(G/H) \to \mathcal{O}(X)$ is an isomorphism, since $\mathcal{G}$ is anti-affine. Thus, φ may be viewed as the canonical morphism $X \to \operatorname{Spec} \mathcal{O}(X)$ and hence depends only on the abstract variety X.

If in addition G is connected, then $\pi : X \to A$ is the Albanese morphism (the universal morphism to an abelian variety). Thus, π also depends only on the abstract variety X.

In fact, the above reduction of structure group is minimal in the following sense:

PROPOSITION 3.4. *Let $\pi : X \to A$ be a homogeneous principal bundle under an affine algebraic group G and let $H \subset G$ be a subgroup scheme as in Theorem 3.1. Then π admits a reduction to an observable subgroup scheme $K \subset G$ if and only if K contains a conjugate of H in G.*

PROOF. Consider a reduction of structure group $q : X \to G/K$. If K is observable in G, then q is $\mathcal{G}$-invariant by Lemma 2.9 and hence is $(G \times \mathcal{G})$-equivariant for the trivial action of $\mathcal{G}$ on G/K. Thus, q maps the base point of $X = (G \times \mathcal{G})/H$ to an H-fixed point. In other words, H is contained in a conjugate of K. The converse implication is obvious. □

In the above statement, the assumption that K is observable cannot be omitted, as shown by Example 3.8 (ii).

3.2 Equivariant automorphism groups

The structure theorem 3.1 yields a further characterization of homogeneity:

PROPOSITION 3.5. *The following conditions are equivalent for a principal bundle $\pi : X \to A$ under an affine algebraic group G:*

(i) *π is homogeneous.*

(ii) *π is trivialized by an anti-affine extension $\varphi : \mathcal{G} \to A$.*

PROOF. (i)⇒(ii) Theorem 3.1 yields a cartesian square

$$\begin{array}{ccc} G \times \mathcal{G} & \xrightarrow{p} & \mathcal{G} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ X & \xrightarrow{\pi} & A, \end{array}$$

where p denotes the second projection, and q the quotient morphism by H.

(ii)⇒(i) By assumption, we have a similar cartesian square with π_* replaced by φ. Thus, the group scheme $H := \mathrm{Ker}(\varphi)$ acts on $G \times \mathcal{G}$ by G-equivariant automorphisms that lift the H-action on $\mathcal{G}$ by multiplication: this action must be of the form

$$h \cdot (g, \gamma) = (gf(h, \gamma), h\gamma)$$

for some morphism $f : H \times \mathcal{G} \to G$. Since $\mathcal{G}$ is anti-affine and G is affine, f factors through a morphism $\psi : H \to G$. Moreover, ψ is a homomorphism of group schemes, since H acts on $G \times \mathcal{G}$. Thus, $X \cong G \times^H \mathcal{G}$ is homogeneous. □

REMARK 3.6. In particular, a G-bundle is homogeneous if it is trivialized by an isogeny, or equivalently, by the multiplication map $n_A : A \to A$ for some positive integer n.

Also, given a homogeneous bundle $X \cong G \times^H \mathcal{G}$, one easily checks that the pull-back $n_A^*(X)$ is the G-bundle $G \times^{H/H_n} \mathcal{G}/H_n$, where $H_n \subset H$ denotes the kernel of n_H, and H/H_n is viewed as a subgroup of G via the isomorphism $n_H : H/H_n \to H$. In particular, *X is trivialized by n_A if and only if H is killed by n.*

Theorem 3.1 also yields a description of bundle automorphisms:

PROPOSITION 3.7. *Consider a homogeneous bundle $\pi : X \to A$ under an affine algebraic group G. Write $X = G \times^H \mathcal{G}$ as in Theorem 3.1 and denote by $C_G(H)$ the centralizer of H in G. Then the group scheme $C_G(H)$ acts on X by bundle automorphisms, via $z \cdot (g, \gamma)H := (gz^{-1}, \gamma)H$. Moreover, this action yields an isomorphism*

$$F : C_G(H) \xrightarrow{\cong} \mathrm{Aut}_A^G(X). \tag{12}$$

As a consequence, the group scheme $\pi_*^{-1}(A)$ (consisting of those G-automorphisms of X that lift the translations of A) is isomorphic to $C_G(H) \times^H \mathcal{G} = (C_G(H) \times \mathcal{G})/H$.

PROOF. Given a scheme S and a morphism $\psi : S \to C_G(H)$, we obtain a morphism

$$G \times \mathcal{G} \times S \to G \times \mathcal{G} \times S, \quad (g, \gamma, s) \mapsto (g\psi(s)^{-1}, \gamma, s)$$

which is $G \times H$-equivariant and hence descends to a G-equivariant morphism $F(\psi) : X \times S \to X \times S$. Clearly, $F(\psi)$ is compatible with $\pi \times \mathrm{id} : X \times S \to A \times S$; in

other words, $F(\psi) \in \mathrm{Aut}^G_A(X)(S)$. This yields a homomorphism of group functors $F : C_G(H) \to \mathrm{Aut}_A(X)$.

To show that F is an isomorphism, note that

$$\mathrm{Aut}^G_A(X)(S) = \mathrm{Aut}^G_{A\times S}(X \times S) \cong \mathrm{Hom}^G(G \times^H \mathcal{G} \times S, G) \cong \mathrm{Hom}^H(\mathcal{G} \times S, G)$$

where G, and hence H, acts trivially on S. Now each $f \in \mathrm{Hom}(\mathcal{G} \times S, G)$ factors through a morphism $\varphi : S \to G$, as in the proof of Proposition 3.5. Moreover, f is H-equivariant if and only if φ is H-equivariant, i.e., φ factors through a morphism $\psi : S \to C_G(H)$. Tracing through the above isomorphisms, one checks that F and the assignement $f \mapsto \psi$ are mutually inverse.

For the final assertion, note that $\pi_*^{-1}(A) = \mathrm{Aut}^G_A(X)\,\mathcal{G} = C_G(H)\,\mathcal{G}$. Moreover, the scheme-theoretic intersection $\mathrm{Aut}^G_A(X) \cap \mathcal{G}$ is the kernel of $\pi_* : \mathcal{G} \to A$, that is, H. □

Examples 3.8. (i) If G is the general linear group $\mathrm{GL}(V)$, then $C_G(H)$ is the group of invertible elements of a finite-dimensional associative algebra: the centralizer of H in $\mathrm{End}(V)$. In particular, $C_G(H)$ is a connected algebraic group, and hence $\mathrm{Aut}^G(X)$ is an algebraic group (locally of finite type). This also follows from the well-known structure of automorphisms of vector bundles, in view of Lemma 2.12.

(ii) If $G = \mathrm{SO}_3$ is the special orthogonal group associated to the quadratic form $x^2 + y^2 + z^2$ in characteristic $\neq 2$, then the subgroup $H \subset G$ consisting of diagonal matrices is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and satisfies $H = C_G(H)$. Let $\mathcal{G}$ be an elliptic curve; then the kernel of the multiplication map $2_{\mathcal{G}}$ is isomorphic to H. Then $X := G \times^H \mathcal{G}$ is a homogeneous G-bundle over the elliptic curve $A := \mathcal{G}/H$, and $\mathrm{Aut}^G_A(X) \cong H$ is not connected.

In fact, $\mathrm{Aut}^G(X)$ is not connected either: since any $h \in H$ satisfies $h = h^{-1}$, the G-automorphism of $G \times \mathcal{G}$ given by $(g, \gamma) \mapsto (g, -\gamma)$ descends to a G-automorphism of X that lifts $-\mathrm{id}_A \in \mathrm{Aut}_{\mathrm{gp}}(A)$.

Next, we claim that π admits a reduction of structure group to a principal bundle under a Borel subgroup $B \subset G$, but no such reduction is homogeneous.

Indeed, π is locally trivial for the Zariski topology in view of [Se58, Section 5.6]. Equivalently, there exists a H-equivariant rational map $\sigma : \mathcal{G} - \to G$. Composing with the projection $G \to G/B$ yields an H-equivariant rational map $\tau : \mathcal{G} - \to G/B$ which is in fact regular, since $\mathcal{G}$ is a smooth complete curve. This yields in turn a G-equivariant morphism $f : X = G \times^H \mathcal{G} \to G/B$, i.e., a reduction of structure group to B.

Assume that such a reduction is homogeneous. Then we have a G-isomorphism $X \cong G \times^B X'$ where $X' \subset X$ is a closed B-stable subvariety, and $\pi : X' \to A$ is a B-bundle; moreover, $Y \cong B \times^{H'} \mathcal{G}'$ where $\mathcal{G}'$ is an anti-affine extension of A by H', and $H' \subset B$ is a commutative subgroup. Therefore, $X \cong G \times^{H'} \mathcal{G}'$; moreover, the resulting projection

$\varphi' : X \to G/H'$ is the canonical morphism $X \to \operatorname{Spec} \mathcal{O}(X)$, by Remark 3.3 (iii). In particular, the G-equivariant morphism φ' is the same as $\varphi : X \to G/H$. Thus, H is conjugate to a subgroup of B. As a consequence, H stabilizes a line in k^3 that is isotropic for the quadratic form $x^2 + y^2 + z^2$. But this contradicts the fact that the H-stable lines are exactly the coordinate lines.

(iii) If $G = \mathrm{SL}_2$ in characteristic 2 and H is the subgroup generated by a non-trivial unipotent element, then $H \cong \mathbb{Z}/2\mathbb{Z}$ and $C_G(H) \cong \mu_2 \times \mathbb{G}_a$, where μ_2 is the scheme-theoretic center of SL_2, and $\mathbb{G}_a$ the subgroup of upper triangular unipotent matrices. In particular, H is reduced whereas $C_G(H)$ is not. Let $\mathcal{G}$ be an ordinary elliptic curve, so that H is the set-theoretic kernel of the multiplication map $2_{\mathcal{G}}$. Then again, $X := G \times^H \mathcal{G}$ is a homogeneous bundle over the elliptic curve $A := \mathcal{G}/H$; but now $\mathrm{Aut}^G_A(X)$ is not reduced, nor is $\mathrm{Aut}^G(X)$.

3.3 Anti-affine extensions

By Theorem 3.1, the classification of homogeneous G-bundles over A reduces to those of commutative subgroup schemes $H \subset G$ up to conjugacy, and of anti-affine extensions $\mathcal{G}$ of A by H. We now determine these extensions under the assumption that H is affine; this holds of course if G is affine. For this, we adapt the arguments of [Br09, Section 2], which treats the case that H is a connected (affine) algebraic group over an arbitrary field.

Given an abelian variety A and a commutative affine group scheme H, consider an extension (10) where $\mathcal{G}$ is a connected commutative algebraic group. We have

$$H = H_u \times H_s$$

where H_u (resp. H_s) denotes the largest unipotent (resp. diagonalizable) subgroup scheme of H. Thus, setting $\mathcal{G}_s := \mathcal{G}/H_u$ and $\mathcal{G}_u := \mathcal{G}/H_s$, we obtain extensions

$$0 \longrightarrow H_s \longrightarrow \mathcal{G}_s \xrightarrow{\pi_s} A \longrightarrow 0, \tag{13}$$

$$0 \longrightarrow H_u \longrightarrow \mathcal{G}_u \xrightarrow{\pi_u} A \longrightarrow 0, \tag{14}$$

and an isomorphism

$$\mathcal{G} \cong \mathcal{G}_u \times_A \mathcal{G}_s$$

arising from the natural homomorphisms $\mathcal{G} \to \mathcal{G}_u$, $\mathcal{G} \to \mathcal{G}_s$ over A. We may now state the following result, that generalizes [Br09, Proposition 2.5] with a simpler proof:

Lemma 3.9. *With the above notation, $\mathcal{G}$ is anti-affine if and only if so are $\mathcal{G}_u$ and $\mathcal{G}_s$.*

PROOF. If $\mathcal{G}$ is anti-affine, then so are its quotient groups $\mathcal{G}_u$ and $\mathcal{G}_s$. Conversely, assume that $\mathcal{G}_u$ and $\mathcal{G}_s$ are anti-affine and consider a surjective homomorphism $f : \mathcal{G} \to G$ where G denotes a (connected, commutative) affine algebraic group. Then f induces a surjective homomorphism $f_u : \mathcal{G}_u = \mathcal{G}/H_s \to G/G_s$ which must be trivial, since $\mathcal{G}_u$ is anti-affine and G/G_s is affine. Thus, $G = G_s$ is diagonalizable. Likewise, G is unipotent, and hence is trivial. This shows that any affine quotient of $\mathcal{G}$ is trivial, i.e., $\mathcal{G}$ is anti-affine. □

By that lemma, we may treat the unipotent and diagonalizable cases separately; we begin with the former case. In characteristic 0, the commutative unipotent group H_u is a *vector group*, that is, the additive group of a finite-dimensional vector space. Moreover, there is a universal extension of A by a vector group,

$$0 \longrightarrow H(A) \longrightarrow E(A) \longrightarrow A \longrightarrow 0,$$

where we set

$$H(A) := H^1(A, \mathcal{O}_A)^*.$$

Also, recall that $H^1(A, \mathcal{O}_A)$ is the Lie algebra of the dual abelian variety $\widehat{A}$; in particular, $\dim H(A) = \dim(A)$. Thus, the extension (14) is classified by a linear map

$$\gamma : H(A) \longrightarrow H_u.$$

By [Br09, Proposition 2.3], $\mathcal{G}_u$ is anti-affine if and only if γ is surjective.

The situation in characteristic $p > 0$ is quite different:

LEMMA 3.10. *In positive characteristics, the extension (14) is anti-affine if and only if $\mathcal{G}_u$ is an abelian variety. Equivalently, H_u is finite and its Cartier dual $\widehat{H_u}$ is a local subgroup scheme of $\widehat{A}$, the kernel of the isogeny $\widehat{A} \to \widehat{\mathcal{G}_u}$ dual of the isogeny $\pi_u : \mathcal{G}_u \to A$.*

PROOF. If $\mathcal{G}_u$ is anti-affine, then it is a semi-abelian variety by [Br09, Proposition 2.2]. In other words, $(\mathcal{G}_u)_{\mathrm{aff}}$ is a torus. But $(\mathcal{G}_u)_{\mathrm{aff}}$ is contained in H_u, and hence is trivial: $\mathcal{G}_u$ is an abelian variety. The remaining assertion follows by duality for isogenies of abelian varieties. □

Next, we treat the diagonalizable case in arbitrary characteristic. Denote by $\widehat{H_s}$ the character group of H_s. Then any $\chi \in \widehat{H_s}$ yields an extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}_s \times^{H_s} \mathbb{G}_m \longrightarrow A \longrightarrow 0,$$

the pushout of the extension (13) by $\chi : H_s \to \mathbb{G}_m$. The line bundle on A associated with this $\mathbb{G}_m$-bundle on A is algebraically trivial (see e.g. [Se59, VII Théorème 6]) and hence yields a point $c(\chi) \in \widehat{A}(k)$. The so defined map

$$c : \widehat{H_s} \longrightarrow \widehat{A}(k)$$

is easily seen to be a homomorphism of abstract groups, that classifies the extension (13).

LEMMA 3.11. *With the above notation and assumptions, $\mathcal{G}_s$ is anti-affine if and only if c is injective.*

PROOF. This statement is exactly [Br09, Proposition 2.1 (i)] in the case that H_s is a torus. The general case reduces to that one as follows: H_s is an extension of a finite diagonalizable group scheme F by a torus T, which yields an exact sequence of character groups

$$0 \longrightarrow \widehat{F} \longrightarrow \widehat{H_s} \longrightarrow \widehat{T} \longrightarrow 0.$$

Also, the quotient $B := \mathcal{G}_s/T$ is an abelian variety, and hence $\mathcal{G}_s$ is a semi-abelian variety. Moreover, we have an exact sequence

$$0 \longrightarrow F \longrightarrow B \longrightarrow A \longrightarrow 0,$$

and the dual exact sequence fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{F} & \longrightarrow & \widehat{H_s} & \longrightarrow & \widehat{T} & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle c_0}\downarrow & & \\ 0 & \longrightarrow & \widehat{F} & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{B} & \longrightarrow & 0, \end{array}$$

where c_0 is the classifying homomorphism for the extension $0 \to T \to \mathcal{G}_s \to B \to 0$. Now $\mathcal{G}_s$ is anti-affine if and only if c_0 is injective; this is equivalent to c being injective. □

Combining the above results yields the desired classification:

THEOREM 3.12. *In characteristic 0, the anti-affine extensions of A by H are classified by the pairs (δ, c), where $\delta : H_u^* \to H^1(A, \mathcal{O}_A)$ is an injective linear map, and $c : \widehat{H_s} \to \widehat{A}(k)$ is an injective homomorphism.*

In positive characteristics, the anti-affine extensions of A by H exist only if H_u is finite; then they are classified by the pairs (δ, c), where $\delta : \widehat{H_u} \to \widehat{A}$ is a faithful homomorphism of group schemes, and $c : \widehat{H_s} \to \widehat{A}(k)$ is an injective homomorphism.

REMARKS 3.13. (i) The above theorem yields a description of the image of the homomorphism $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(A)$ for a homogeneous G-bundle $\pi : X \to A$. Indeed, since that image contains A, it suffices to determine its intersection with $\mathrm{Aut}_{\mathrm{gp}}(A)$. Let X be given by a commutative subgroup scheme $H \subset G$ and data δ, c as above. Then $\varphi \in \mathrm{Aut}_{\mathrm{gp}}(A)$ admits a lift in $\mathrm{Aut}^G(X)$ if and only if there exists $z \in N_G(H)$ (the normalizer of H in G) such that the square

$$\begin{array}{ccc} \widehat{H_s} & \xrightarrow{c} & \widehat{A} \\ {\scriptstyle \mathrm{Int}(z)}\downarrow & & {\scriptstyle \widehat{\varphi}}\downarrow \\ \widehat{H_s} & \xrightarrow{c} & \widehat{A} \end{array}$$

commutes, and the analogous square with δ commutes as well. Here $\mathrm{Int}(z)$ denotes the conjugation by z, that acts on H and hence on H_s, H_u and their Cartier duals.

For example, the multiplication by -1 in A lifts to a G-automorphism of X if and only if there exists $z \in N_G(H)$ such that $\mathrm{Int}(z)$ acts on H by the inverse $h \mapsto h^{-1}$.

(ii) In characteristic 0, any unipotent vector bundle over A can be written as $E(A) \times^H V$ for some representation V of the vector group $H(A)$, uniquely determined up to conjugacy. Moreover, given two unipotent vector bundles E_1, E_2 and denoting by V_1, V_2 the associated representations, we have

$$\mathrm{Hom}_A(E_1, E_2) = H^0(A, E_1^* \otimes E_2) \cong (\mathcal{O}(E(A)) \otimes V_1^* \otimes V_2)^{H(A)} \cong \mathrm{Hom}^{H(A)}(V_1, V_2).$$

Thus, *the assignement $V \mapsto E(A) \times^{H(A)} V$ yields an equivalence of categories from finite-dimensional representations of $H(A)$ to unipotent vector bundles.*

Since the category of finite-dimensional $H(A)$-modules is equivalent to that of modules of finite length over the local ring $\mathcal{O}_{\widehat{A},0}$, we recover Mukai's result that *the category of unipotent vector bundles is equivalent to that of coherent sheaves over $\widehat{A}$ supported at* 0 (see [Muk79, Theorem 4.12]). Note however that Mukai's proof, via the Fourier transform, is valid in all characteristics.

(iii) Consider homogeneous G-bundles over an elliptic curve E in characteristic 0. Since $H^1(E, \mathcal{O}_E) \cong k$, the data of H_u and δ as above are equivalent to that of a unipotent element of G which generates H_u. Also, $E \cong \widehat{E}$. It follows that *the homogeneous G-bundles over E are classified by the triples (u, H_s, c) where $u \in G$ is unipotent, H_s is a diagonalizable subgroup of the centralizer $C_G(u)$, and $c : \widehat{H_s} \to E(k)$ is an injective homomorphism, up to simultaneous conjugacy of u and H_s in G.*

This classification still holds for homogeneous G-bundles over an *ordinary* elliptic curve E in characteristic $p > 0$. Indeed, $\widehat{H_u}$ is then a local subgroup scheme of E of order some power p^r, and hence is isomorphic to μ_{p^r}; equivalently, $H_u \cong \mathbb{Z}/p^r\mathbb{Z}$. So the data of H_u and δ are again equivalent to that of a unipotent element of G.

As a consequence, in either case, the unipotent vector bundles of rank n over E are classified by the unipotent conjugacy classes in GL_n. In particular, there is a unique indecomposable such bundle up to isomorphism: the Atiyah bundle, associated with the unipotent class having one Jordan block. But this indecomposable bundle, viewed as a homogeneous vector bundle, has very different structures in characteristic 0 and in positive characteristics. We refer to [Sc09] for further results on this topic, and to [FMW98, La98, FM00] for developments on (possibly non-homogeneous) holomorphic principal bundles over complex elliptic curves.

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Université Grenoble I
Institut Fourier
UMR 5582, BP 74
38402 Saint-Martin d'Hères Cedex
France

E-mail address: Michel.Brion@ujf-grenoble.fr