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有限长压电层合简支板自由振动的三维精确解 [1)]

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摘要 基于三维弹性理论和压电理论，导出了有限长矩形压电层合简支板的动力学方程及相应的边界条件，给出了一种求解压电层合板自由振动三维精确解的方法；分析了正、逆向压电效应对层合板振动频率的影响. 本文所述的方法和结果对于求解其他三维动态问题，验证、比较其他简化模型、有限元计算结果以及工程应用都有指导意义.

关键词 三维精确解，有限长压电层合板，自由振动

引 言

由于压电材料具有正、逆向压电效应，并且有质量小、价格低、技术成熟、应用方便等优点，目前压电材料是智能结构中应用最多的一种材料，被广泛地用作感知和作动元件，应用于结构的形状控制、振动和噪声控制等许多领域. 复合材料层合板具有高的刚度比和强度比，因而工程中一般将压电材料与层合板结合在一起应用. 对于压电层合梁、板，已建立了多种简化计算模型[1～3]和有限元计算模型[4]，同时对其精确解的分析也引起人们的广泛注意. Ray[5,6]等首先分析了无限长、有限长压电层合板的静态精确解，但采用的是各向同性且仅有横向和面内之间机电耦合的 PVDF 压电材料；Heyliger 和 Brooks[7]，以及 Bisegna[8]等也分别得到了压电层合板的静态二维、三维精确解. Heyliger[9]等给出无限长压电层合板的自由振动二维精确解，Batra[10,11]等分析了有限长压电层合板的振动，但将压电层看作是一层薄膜，作了线性化处理，对电势的分布也作了线性假设，因此实际上这不能看作是精确解.

上面提到的文章，所用到的求精确解的方法基本上都是基于 Pagano[12]提出的求解一般层合板精确解方法，这种方法用于求解压电层合板的静态问题是适宜的. 但对于动态问题，特别是有限长层合板三维动态问题，就会变的很复杂. 本文基于三维弹性理论，未作任何简化处理，导出了有限长矩形含压电层复合材料层合简支板的动力学方程及相应的边界条件，运用幂级数展开方法得到了这种情况下压电层合板自由振动的三维精确解. 本文所用方法，避免了 Pagano 方法求解动态问题所遇到的困难. 通过实例与其他精确解结果进行了比较，证实了本文方法的正确性并得出了一些结果. 分析了正、逆向压电效应对层合板振动频率的影响. 本文所述的方法对于求解其他三维动态问题，验证、比较其他简化模型、有限元计算结果以及工程应用都有指导意义.

1 基本方程

设有限长层合板有 N 层，边长为 a，b (如图 1). 压电层可放置在层合板的表面，也可在层间. 设层合板材料是正交各向异性的，建立整体坐标系 xyz.

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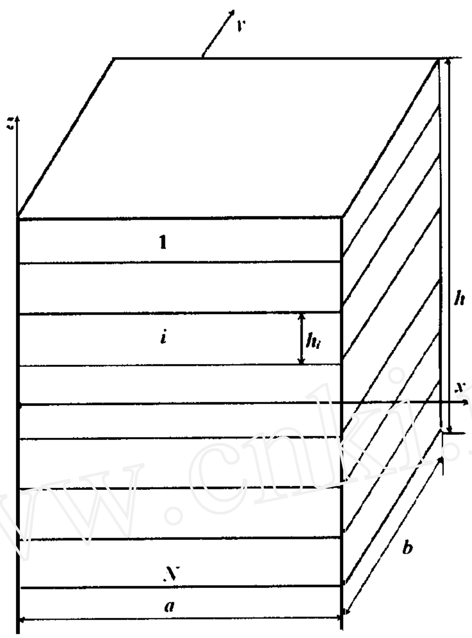

图 1　压电层合板示意图

Fig. 1　A N-layer rectangular piezoelectric laminate

线性压电材料的本构方程是

$$
\left.\begin{aligned}
\sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e_{kij}E_k \\
D_i &= e_{ikl}\varepsilon_{kl} + \varepsilon_{ik}E_k
\end{aligned}\right\}
$$

)

有 限　　板 三　　能 看　　l瑞 l解　*e*

几何方程

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

电场强度 E **可用电势表示为**

$$E_i = -\frac{\partial \Phi}{\partial x_i} \tag{3}$$

将 (2), (3) **代入** (1) **得**

$$\left.\begin{aligned}
\sigma_x &= C_{11}u_{,x} + C_{12}v_{,y} + C_{13}w_{,z} + e_{31}\Phi_{,z}\\
\sigma_y &= C_{12}u_{,x} + C_{22}v_{,y} + C_{32}w_{,z} + e_{32}\Phi_{,z}\\
\sigma_z &= C_{13}u_{,x} + C_{23}v_{,y} + C_{33}w_{,z} + e_{33}\Phi_{,z}\\
\tau_{yz} &= C_{44}(v_{,z} + w_{,y}) + e_{24}\Phi_{,y}\\
\tau_{xz} &= C_{55}(u_{,z} + w_{,x}) + e_{15}\Phi_{,x}\\
\tau_{xy} &= C_{66}(u_{,y} + v_{,x})\\
D_x &= e_{15}(u_{,z} + w_{,x}) - \varepsilon_{11}\Phi_{,x}\\
D_y &= e_{24}(v_{,z} + w_{,y}) - \varepsilon_{22}\Phi_{,y}\\
D_z &= e_{31}u_{,z} + e_{32}v_{,y} + e_{33}w_{,z} - \varepsilon_{33}\Phi_{,z}
\end{aligned}\right\} \tag{5}$$

合板求解 层

边界条件及层间连续条件: **设层合板的边界是简支的, 有**

$$\left.\begin{aligned}
&x = 0, a \text{ 时}, \quad v = w = 0, \quad \sigma_x = 0, \quad \Phi = 0\\
&y = 0, b \text{ 时}, \quad u = w = 0, \quad \sigma_y = 0, \quad \Phi = 0
\end{aligned}\right\} \tag{8}$$

p an

设层合板有 N 层, 则有 $N-1$ 个界面, 在界面上位移、应力、电学量应满足连续条件

$$\left.\begin{array}{lll} u^i = u^{i+1}, & v^i = v^{i+1}, & w^i = w^{i+1} \\ \sigma_z^i = \sigma_z^{i+1}, & \tau_{yz}^i = \tau_{yz}^{i+1}, & \tau_{xz}^i = \tau_{xz}^{i+1} \\ D_z^i = D_z^{i+1}, & \Phi^i = \Phi^{i+1} & \end{array}\right\} \tag{10a}$$

$$\left.\begin{array}{c} (C_{13}+C_{55})u_{,xz} + (C_{23}+C_{44})v_{,yz} + C_{55}w_{,xx} + C_{44}w_{,yy} + \\ C_{33}w_{,zz} + e_{15}\Phi_{,xx} + e_{24}\Phi_{,yy} + e_{33}\Phi_{,zz} = \rho w_{,tt} \\ (e_{15}+e_{31})u_{,xz} + (e_{24}+e_{32})v_{,yz} + e_{15}w_{,xx} + e_{24}w_{,yy} + \\ e_{33}w_{,zz} - \varepsilon_{11}\Phi_{,xx} - \varepsilon_{22}\Phi_{,yy} - \varepsilon_{33}\Phi_{,zz} = 0 \end{array}\right\} \tag{11}$$

ur

其中 $p = m\pi/a$, $q = n\pi/b$, m, $n = 1, 2, \cdots, \infty$, ω是振动频率.

将 (11) 代入 (10) 得 $A_x(k)$, $A_y(k)$, $A_z(k)$, $A_\Phi(k)$ 的递推关系为

$$
\left.
\begin{aligned}
& C_{55}(k+2)(k+1)A_x(k+2) = (C_{11}p^2 + C_{66}q^2 - \rho\omega^2)A_x(k) + (C_{12}+C_{66})pqA_y(k) - \\
&\quad (C_{13}+C_{55})p(k+1)A_z(k+1) - (e_{31}+e_{15})p(k+1)A_\Phi(k+1) \\
& C_{44}(k+2)(k+1)A_y(k+2) = (C_{66}p^2 + C_{22}q^2 - \rho\omega^2)A_y(k) + (C_{12}+C_{66})pqA_x(k) - \\
&\quad (C_{23}+C_{44})q(k+1)A_z(k+1) - (e_{32}+e_{24})q(k+1)A_\Phi(k+1) \\
& (e_{33}^2 + C_{33}\varepsilon_{33})(k+2)(k+1)A_z(k+2) = \\
&\quad [\varepsilon_{33}(C_{13}+C_{55}) + e_{33}(e_{15}+e_{31})]p(k+1)A_x(k+1) + \\
&\quad [\varepsilon_{33}(C_{23}+C_{44}) + e_{33}(e_{24}+e_{32})]q(k+1)A_y(k+1) + \\
&\quad [\varepsilon_{33}(C_{55}p^2 + C_{44}q^2 - \rho\omega^2) + e_{33}(e_{15}p^2 + e_{24}q^2)]A_z(k) + \\
&\quad [\varepsilon_{33}(e_{15}p^2 + e_{24}q^2) - e_{33}(\varepsilon_{11}p^2 + \varepsilon_{22}q^2)]A_\Phi(k) \\
& (e_{33}^2 + C_{33}\varepsilon_{33})(k+2)(k+1)A_\Phi(k+2) = \\
&\quad [\varepsilon_{33}(C_{13}+C_{55}) - C_{33}(e_{15}+e_{31})]p(k+1)A_x(k+1) + \\
&\quad [e_{33}(C_{23}+C_{44}) - C_{33}(e_{24}+e_{32})]q(k+1)A_y(k+1) + \\
&\quad [e_{33}(C_{55}p^2 + C_{44}q^2 - \rho\omega^2) - C_{33}(e_{15}p^2 + e_{24}q^2)]A_z(k) + \\
&\quad [e_{33}(e_{15}p^2 + e_{24}q^2) + C_{33}(\varepsilon_{11}p^2 + \varepsilon_{22}q^2)]A_\Phi(k)
\end{aligned}
\right\}
$$

2 电层

将(13)代入(12)式,并利用 H 的初值就可得到 $H_x(k,l)$, $H_y(k,l)$, $H_z(k,l)$, $H\Phi(k,l)$ 的递推关系公式,其中 $k=0,1,\cdots,\infty$; $l=1,2,\cdots,8$. 因其递推公式与(12)中 A_x, A_y, A_z, $A\Phi$ 的递推公式类似,只需将 A 的有关项换成 H 的有关项即可,这里将其递推公式省略.

将(11),(12)代入(9)式得

$$\left.\begin{aligned}
\sigma_z &= \sum_{k=0}^{\infty}[-C_{13}pA_x(k) - C_{23}qA_y(k) + C_{33}(k+1)A_z(k+1) + e_{33}(k+1)A\Phi(k+1)]\times \\
&\quad z^k e^{i\omega t}\sin px\sin qy \\
\tau_{yz} &= \sum_{k=0}^{\infty}[C_{44}(k+1)A_y(k+1) + C_{44}qA_z(k) + e_{24}qA\Phi(k)]\times z^k e^{i\omega t}\sin px\cos qy \\
\tau_{xz} &= \sum_{k=0}^{\infty}[C_{55}(k+1)A_x(k+1)) + C_{55}pA_z(k) + e_{15}pA\Phi(k)]\times z^k e^{i\omega t}\cos px\sin qy \\
D_z &= \sum_{k=0}^{\infty}[-e_{31}pA_x(k) - e_{32}qA_y(k) + e_{33}(k+1)A_z(k+1) - \varepsilon_{33}(k+1)A\Phi(k+1)]\times \\
&\quad z^k e^{i\omega t}\sin px\sin qy
\end{aligned}\right\}$$

间的关系. 材料参数如下

$$[C] = \begin{bmatrix} 81.3 & 0.329 & 0.432 & 0 & 0 & 0 \\ 0.329 & 81.3 & 0.432 & 0 & 0 & 0 \\ 0.432 & 0.432 & 64.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30.6 \end{bmatrix} \times 10^{9}\ \text{Pa}$$

其他省略.

$$e_{31} = e_{32} = -5.20\ \text{c/m}^2, \quad e_{33} = 15.08\ \text{c/m}^2, \quad e_{15} = e_{24} = 12.72\ \text{c/m}^2$$

$$\varepsilon_{11} = \varepsilon_{22} = 1475\ \varepsilon_0, \quad \varepsilon_{33} = 1300\ \varepsilon_0, \quad \varepsilon_0 = 8.854 \times 10^{-12}\ \text{F/m}$$

$$\rho = 7.5 \times 10^3 \text{kg/m}^3, \quad 板厚\ h = 0.01\ \text{m}.$$

首先讨论无限长、有限长板计算结果的比较.

对于厚板,取 $a = 4h$, 对于薄板取 $a = 50h$,改变 b 的大小,取 $b = 1a, 2a, \cdots, Ma$,分析有限长至无限长情况.

取 $m = n = 1$, 并计算第一阶模态振动频率. 表 1 给出了计算结果.

表 1 压电板自由振动频率(1/s)

Table 1 Natural frequencies of the piezoelectric plate at first mode (1/s)

$b = Ma$	$a/h = 4$		$a/h = 50$	
	closed	open	closed	open
1a	94 490.36	96 300.14	711.18	711.65
2 a	63 471.12	64 440.08	455.37	455.64
4 a	56 001.34	56 818.81	397.40	397.62
6 a	54 661.71	5 545.15	387.14	387.35
8 a	54 197.07	54 982.46	383.59	383.80
10 a	53 982.81	54 764.53	381.96	382.16
12 a	53 866.63	54 646.53	381.07	381.27
14 a	53 796.65	54 575.21	380.54	380.74
100 a	53 606.96	54 382.32	379.09	379.29
	(52 580.67)	(53046.76)	(373.65)	(373.68)

上表括号中的数据是 Heyliger[9]给出的结果,本文得到的结果与 Heyliger 给出的结果非常接近,两种计算结果之间的相对误差小于 2.5 %. 说明本文方法是正确的,同时可看到开路情况(此时压电层相当于感知片),比闭路情况(此时压电层相当于作动片)的振动频率要大. 但在薄板情况下,这个差别变得很小. 另外可得一个重要结论,当矩形板的一条边是另一条边的 8～10 倍以上,就可看作是无限长板的情况,计算误差在 2 %以内.

表 2 给出了不同 m, n 值时,厚板、薄板的不同模态下振动频率值.

表 2　不同模态下压电板振动频率值 (1/ s)

Table 2　Natural frequencies of the piezoelectric plate at different mode (1/ s)

m , n	mode	$a = 4h,\ b = 20a$		$a = 50h,\ b = 10a$	
		closed	open	closed	open
1 , 1	*I*	53 697.82	54 474.82	381.96	382.16
1 , 2	*I*	53 982.80	54 764.53	390.73	390.94
1 , 3	*I*	54 459.64	55 249.56	405.64	405.87
2 , 1	*I*	176 135.77	181 466.30	1 517.74	1 517.81
2 , 2	*I*	176 345.84	181 680.00	1 524.40	1 525.76
2 , 3	*I*	176 696.25	182 036.00	1 538.90	1 540.28

例 2　三层压电层合板

从上面算起,第一层是压电层,压电参数同前例. 第二、三层是纤维层,按(0/ 90) 正交铺设, 材料常数按 Pagano 给出的数值.

$E_1 = 25\ E_0$, $E_2 = E_3 = E_0$, $G_{12} = G_{13} = 0.5\ E_0$, $G_{23} = 0.2\ E_0$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$, $E_0 = 6.849\,76 \times 10^9$ Pa, 设三层的厚度是 h_1, h_2, h_3 (此例 $h_1 = h_2 = h_3 = 0.01$ m)

表 3 给出了薄层合板情况下,从有限长至无限长层合板第一阶模态频率的变化.

表 3　薄层合板一阶模态频率 (1/ s)

Table 3　Natural frequencies of the piezoelectric laminnate at first mode (1/ s)

b = Ma	$a = 20(h_1 + h_2 + h_3)$　$m = 1$　$n = 1$	
	closed	open
1a	1 877.4	1 891.6
2 *a*	1 034.8	1 041.9
4 *a*	849.1	854.0
6 *a*	817.3	821.9
8 *a*	806.1	810.5
10 *a*	800.5	801.4
12 *a*	797.2	799.2
14 *a*	794.9	797.4
100 *a*	784.8	788.8

由表 3 可看出,对于压电层合板可得到与单层压电板相类似的结论.

表 4 给出了一组层合板振动频率(1/ s) .

表 4　层合板不同模态振动频率 (1/ s)

Table 4　Natural frequencies of the piezoelectric laminnate at different mode (1/ s)

m , n	mode	$a = 20(h_1 + h_2 + h_3)$,　$b = 10a$	
		closed	*open*
1 , 1	I	800.5	804.8
1 , 2	I	828.4	833.2
1 , 3	I	874.4	880.1
2 , 1	I	3 107.4	3 122.8
2 , 2	I	3 143.9	3 159.9
2 , 3	I	3 185.9	3 202.3
3 , 1	I	6 747.9	6 777.1
3 , 2	I	6 808.0	6 838.2
3 , 3	I	6 853.9	6 884.6

4 结 论

本文基于三维弹性理论,未作任何简化处理,导出了有限长矩形含压电层复合材料层合简支板的动力学方程及相应的边界条件,运用幂级数展开方法得到了这种情况下压电层合板自由振动的三维精确解. 本文所用方法,避免了 *Pagano* 方法求解动态问题所遇到的困难. 通过实例与其他精确解结果进行了比较,证实了本文方法的正确性并得出了一些结果. 分析了正、逆向压电效应对层合板振动频率的影响. 本文所述的方法对于求解其他三维动态问题,验证、比较其他简化模型、有限元计算结果以及工程应用都有指导意义.

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THREE-DIMENSIONAL EXACT ANALYSIS FOR FREE VIBRATION OF SIMPLY-SUPPORTED PIEZOELECTRIC COMPOSITE LAMINATES [1)]

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Abstract　In recent years, piezoelectric materials have been widely used as sensors and actuators in structural shape control, vibration suppression, noise attenuation and damage monitoring. Exact study can provide better understanding of the mechanical and electric behaviors of piezoelectric structures. An analysis for the three-dimensional solutions of piezoelectric laminated plates has been conducted by several researchers. The exact solution methods, used in the papers before, mostly follow the strategy of Pagano. This approach has been proven effective to static problems, but it is very complicated for the three-dimension dynamic problems of finite long piezoelectric laminated plates.

In this paper, free vibration of a finite long simply-supported rectangular piezoelectric composite laminates has been studied on the basis of three dimensional linear elasticity and piezoelectricity without any simplification. The laminates can be composed of an arbitrary number of elastic and piezoelectric layers of orthotropic materials. The solution to the derived governing differential equations is obtained by the power series expansion method. The results show that the present approach is more effective than Pagan method mentioned above. Different boundary conditions are investigated to model the direct and inverse piezoelectric effects. The natural frequencies of free vibration are investigated, also. The obtained results can not only be used to assess various approximate theories, but also enhance the understanding of the dynamic behavior of piezoelectric structures.

Key words　three-dimensional exact solution, finite length laminates, free vibration

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