

Spectral Universality of Real Chiral Random Matrix Ensembles

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Abstract

We investigate the universality of microscopic eigenvalue correlations for Random Matrix Theories with the global symmetries of the QCD partition function. In this article we analyze the case of real valued chiral Random Matrix Theories ($\beta = 1$) by relating the kernel of the correlations functions for $\beta = 1$ to the kernel of chiral Random Matrix Theories with complex matrix elements ($\beta = 2$), which is already known to be universal. Our proof is based on a novel asymptotic property of the skew-orthogonal polynomials: an integral over the corresponding wavefunctions oscillates about half its asymptotic value in the region of the bulk of the zeros. This results solves the puzzle that microscopic universality persists in spite of contributions to the microscopic correlators from the region near the largest zero of the skew-orthogonal polynomials. Our analytical results are illustrated by the numerical construction of the skew-orthogonal polynomials for an x^4 probability potential.

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1 Introduction

Since its first application to the spacing of nuclear resonances [1], Random Matrix Theories (RMT) have been very successful in explaining the statistical properties of spectra. Originally, the so called Wigner-Dyson ensembles of matrices with independently distributed Gaussian matrix elements were introduced to replace the unknown nuclear Hamiltonian. More recently, the applicability of RMT has been related to the chaotic dynamics of the corresponding classical system and spectra of many chaotic systems with only a few degrees of freedom have been successfully described by RMT [2, 3, 4]. The successes of RMT has raised the question whether the statistical properties of spectra are universal. This question has been investigated in great detail within the context of RMT. The idea is to show that large deformations of the probability distribution of the random matrix elements leave the properly rescaled spectral correlations unaffected, whereas the average spectral density changes on a macroscopic scale. This program has been carried out most completely for the Hermitian random matrix ensembles (denoted by the Dyson index $\beta = 2$; for recent reviews see [4, 5]) which are mathematically much simpler than real or quaternion-real random matrix ensembles (with Dyson index $\beta = 1$ and $\beta = 4$, respectively). Nevertheless, several universality proofs are available for these ensembles as well [6, 7, 8, 9, 10, 11, 12, 13, 14].

In addition to the Wigner-Dyson ensembles there are seven other classes of Random Matrix Theories. They can be classified according to the Cartan classification of symmetric spaces [15]. In this article we are interested in chiral Random Matrix Theories (chRMT). These are ensembles of random matrices with the chiral symmetry of the QCD Dirac operator [16, 17]. The nonzero eigenvalues of these ensembles occur in pairs $\pm\lambda$. Therefore, $\lambda = 0$ is a special point, and the average spectral density on the scale of the average level spacing shows universal properties. With the average spacing of the eigenvalues given by $\pi/\Sigma N$ (with N the total number of eigenvalues and Σ a parameter known as the chiral condensate), the microscopic spectral density is defined as [16]

$$\rho_s(u) = \lim_{N \rightarrow \infty} \frac{1}{\Sigma N} \langle \rho(\frac{u}{\Sigma N}) \rangle. \quad (1)$$

Both $\rho_s(u)$ and the microscopic k -point correlation functions are universal. This has been shown in great detail for the chiral Unitary Ensemble (chUE), which is the ensemble of Hermitian chiral random matrices with no anti-unitary symmetries [18, 19, 20, 21, 5, 22, 23, 24, 25, 26, 27, 28] [29, 12, 30, 31, 32, 33]. The chiral ensembles with real or quaternion real matrix elements (known as the chiral Orthogonal Ensemble (chOE) and the chiral Symplectic Ensemble (chSE), respectively) are mathematically much more complicated. The general result for the microscopic spectral density of the chiral Gaussian Orthogonal Ensemble (chGOE) [34] and the of the chiral Gaussian Symplectic Ensemble (chGSE) [35] was first obtained by an explicit construction of the corresponding skew-orthogonal

polynomials [36, 6, 37] (several special cases were analyzed in [38, 20, 39]). This method does not seem to be easy generalizable to non-Gaussian probability potentials.

Recent progress was made by relating the kernel for the correlation functions of the chOE to the universal kernel of the chUE [10]. This method was based on a generalization of an operator construction of skew-orthogonal polynomials for the Wigner-Dyson ensembles [7] to the chiral ensembles. Remarkably, the skew-orthogonal polynomials do not enter in the relation between the kernels. Indeed, an elegant construction of the correlation functions relying only on operator relations was recently given in [10, 11]. The operator method was successfully applied to Gaussian Orthogonal and Gaussian Symplectic Ensembles with an additional fermion determinant [40]. Universality for the so called massive chiral ensembles for $\beta = 1$ and $\beta = 4$ was shown by relating them to the corresponding massless ensembles [14]. In this way analytical results for the massive spectral correlators could be obtained [14, 41].

The universality proof based on the relations between kernels given in [10] is incomplete for $\beta = 1$. The reason is that the correlation functions for $\beta = 1$ depend on an integral over the positive real axis of a derivative of the kernel for $\beta = 2$. Naively, one would expect that the presence of such nonlocal contributions would lead to nonuniversal behavior. The resolution of this puzzle, and thus the completion of the universality proof for $\beta = 1$, is the primary objective of this paper. As was already discussed in [10], for $\beta = 4$ there is no such problem. As secondary objective, we illustrate some of the arguments given in [10] by the analysis of the asymptotic behavior of the skew-orthogonal polynomials for a quartic probability potential.

The results of this article are relevant for Dirac spectra of QCD with two colors in the fundamental representations (or for Dirac spectra of QCD with staggered fermions in the adjoint representation). Indeed, the microscopic spectral density of the chGOE has been observed in lattice QCD [42, 43] and in instanton liquid simulations [44]. Many more results justifying the chiral Random Matrix description of the microscopic spectral density in lattice QCD have been obtained for $\beta = 2$ and $\beta = 4$ [45, 46, 47, 48, 49] (for a recent review and a complete list of references see [50]). Our results also apply to the superconducting ensembles with Dyson index $\beta = 1$ (with joint eigenvalue density given by a special case of the chiral ensembles) as well as to two-sublattice models [51, 20].

This paper is organized as follows. Chiral Random Matrix Theory is introduced in section 2. In section 3 we discuss the relation between the kernels for $\beta = 1$ and $\beta = 2$. Most of this section already appeared in [10]. Two examples, the Gaussian case and the quartic probability potential are worked out in detail in section 4. In section 5 we derive a novel asymptotic property of the skew-orthogonal polynomials, allowing us to complete the proof of [10]. This is the most important result of this paper. Universality of the microscopic correlations is shown in section 6. In this section we also give the explicit

universal expressions for the microscopic spectral density and the microscopic kernel which allows to calculate all correlation functions. In section 7 the skew-orthogonal polynomials for a quartic probability potential are studied numerically and concluding remarks are made in section 8.

2 Chiral Random Matrix Theory

In this section we introduce a Random Matrix Theory with the global symmetries of the QCD partition function. We first define the Dyson index β of a Dirac operator. Its value is equal to the number independent degrees of freedom per matrix element and is determined by the anti-unitary symmetries of the Dirac operator. Any anti-unitary symmetry operator can be written as $U = AK$ with A unitary and K the complex conjugation operator. The operator U^2 is unitary and is thus proportional to the identity in an irreducible subspace of the unitary symmetries of the Dirac operator. One can easily convince one-self that the only two possibilities are $A^2 = 1$ and $A^2 = -1$ in this subspace. These two cases are denoted by the Dyson index $\beta = 1$ and $\beta = 4$, respectively. If there are no anti-unitary symmetries, the Dyson index of the Dirac operator is $\beta = 2$. For $\beta = 1$ it is possible to find a basis for which the Dirac matrix is real for *all* gauge field configurations. For $\beta = 4$ it is possible to construct a basis for which the Dirac matrix is quaternion real for *all* gauge field configurations. For $\beta = 2$ the matrix elements of the Dirac operator do not have any reality properties and are given by complex numbers. The Dyson index of the Dirac operator depends on the representation of the gauge fields and may be different for the discretized and continuum versions of the Dirac operator [17, 52]. For example, the continuum Dirac operator for $N_c = 2$ and fermions in the fundamental representation is in the class $\beta = 1$, whereas the corresponding staggered Dirac operator is in the class $\beta = 4$. For gauge fields in the adjoint representation the Dyson index of the continuum Dirac operator is $\beta = 4$, whereas the Dyson index of the staggered lattice discretization is $\beta = 1$.

In a chiral basis in the sector of topological charge ν the Dirac matrix has the block structure

$$D = \begin{pmatrix} 0 & C \\ C^\dagger & 0 \end{pmatrix}, \quad (2)$$

where C is an $n \times (n + \nu)$ matrix with reality properties given by the Dyson index. For generic values of its matrix elements D has exactly ν zero eigenvalues. In QCD, the matrix elements of D depend in a complicated way on the gauge fields, which are distributed according to the QCD partition function. In chiral Random Matrix Theory (chRMT) we replace the matrix elements of the Dirac operator by space-time independent random

numbers with probability distribution given by the partition function

$$Z(m) = m^\nu \int dC \det^{N_f}(D + m) e^{-\frac{n\beta}{2} \text{Tr} V(C^\dagger C)}. \quad (3)$$

Here, N_f is the number of quark flavors with mass m (below we only consider the massless case $m = 0$). The integral is over the independent degrees of freedom of the matrix elements of C . The probability potential $V(x)$ is in general an arbitrary polynomial $V(x) = \sum_{k=1}^p a_k x^k$ (with $a_p > 0$). For the Gaussian chiral ensembles, which are mathematically much simpler, the potential is $V(x) = \Sigma^2 x$. In that case the parameter Σ is related to the average spectral density

$$\rho(\lambda) = \left\langle \sum_{k=1}^N \delta(\lambda - \lambda_k) \right\rangle \quad (4)$$

by the Banks-Casher formula [53]

$$\Sigma = \lim_{N \rightarrow \infty} \frac{\pi \rho(0')}{N}. \quad (5)$$

The prime indicates that the argument of $\rho(\lambda)$ should be near zero but much larger than the smallest eigenvalue. For this reason Σ is interpreted as the chiral condensate, the order parameter of the chiral phase transition. The integrals in (3) can be easily evaluated in the thermodynamic limit. The result coincides with so called finite volume partition functions which were first derived on the basis of chiral symmetry [54, 55].

In this paper we will study the chiral Orthogonal Ensemble (chOE) and show that the microscopic spectral density does not depend on the coefficients of the probability potential. Our starting point is the joint probability distribution of the eigenvalues of the Dirac operator. It can be obtained from the partition function by making a polar decomposition of C

$$C = U \Lambda V^{-1}, \quad (6)$$

with U and V orthogonal, unitary or symplectic matrices for $\beta = 1$, $\beta = 2$ and $\beta = 4$, respectively, and Λ a semi-positive definite diagonal matrix. For the massless case the joint probability distribution in terms of the squares of the eigenvalues, $x_k = \Lambda_k^2$, is given by

$$\rho(x_1, \dots, x_n) = |\Delta(\{x_i\})|^\beta \prod_k x_k^{N_f - 1 + \beta|\nu|/2 + \beta/2} e^{-\frac{n\beta}{2} \sum_k V(x_k)} \quad (7)$$

where the Vandermonde determinant is defined by

$$\Delta(\{x_i\}) = \prod_{k < l} (x_k - x_l). \quad (8)$$

The exponent of the x_k for $\beta = 1$ will be denoted by

$$a = N_f - 1 + |\nu|/2 + 1/2, \quad (9)$$

and below we consider the joint probability density

$$\rho(x_1, \dots, x_n) = |\Delta(\{x_i\})|^\beta \prod_k e^{-\beta\phi_a(x_k)}. \quad (10)$$

with probability potential given by

$$\phi_a = \frac{n}{2}V(x) - a \log x. \quad (11)$$

For technical reason we will restrict ourselves to even n and use the notation $n = 2\bar{n}$.

The spectral density, and in general the k -point correlation functions, can be obtained from the joint eigenvalue density by integrating over all but k eigenvalues. For $\beta = 2$ this can be simply done by exploiting the orthogonality of the polynomials [56]

$$\int_0^\infty dx e^{-2\phi_a(x)} P_k^{2a}(x) P_l^{2a}(x) = \delta_{kl}. \quad (12)$$

The resulting spectral correlation functions can be expressed in terms of the kernel¹

$$K_n^{2a}(x, y) = \sum_{k=0}^{n-1} P_k^{2a}(x) P_k^{2a}(y). \quad (13)$$

Universality can be established by showing that in the microscopic limit the so called wave functions $P_k^{2a}(x) \exp(-\phi_a(x))$ depend only on the potential through a scale factor determined by the average spectral density [19]. With $\Sigma = \pi\rho(0)/2n$ the microscopic limit of the kernel for the chUE is given by (a factor $2\sqrt{uv}$ from the integration measure has been included)

$$\begin{aligned} \lim_{N=2n \rightarrow \infty} \frac{2}{\Sigma N} \left(\frac{uv}{\Sigma^2 N^2} \right)^{2a+1/2} K_n^{2a} \left(\frac{u^2}{\Sigma^2 N^2}, \frac{v^2}{\Sigma^2 N^2} \right) &= \sqrt{uv} \frac{u J_{2a+1}(u) J_{2a}(v) - v J_{2a}(u) J_{2a+1}(v)}{u^2 - v^2} \\ &\equiv B^{2a}(u, v). \end{aligned} \quad (14)$$

This kernel is known as the Bessel kernel [57]. Sometimes it is simpler to use an integral representation of the Bessel kernel given by

$$B^{2a}(u, v) = \sqrt{uv} \int_0^1 t dt J_{2a}(ut) J_{2a}(vt). \quad (15)$$

¹For later convenience we do not follow the usual convention to include the weight functions in the kernel.

3 Relation between the kernel for $\beta = 1$ and $\beta = 2$

For the orthogonal ensembles the integrations over the eigenvalues can be performed by means of orthogonality relations for the skew-orthogonal polynomials of the second kind [36, 6]. These polynomials are defined by the scalar products

$$\langle R_k, R_l \rangle_R = J_{kl}, \quad (16)$$

with the nonzero matrix elements of J_{kl} given by $J_{2k,2k+1} = -J_{2k+1,2k} = -1$. The skew-scalar product is defined by

$$\langle f, g \rangle_R = \int_0^\infty dx e^{-2\phi_a(x)} f(x) \hat{Z}g(x), \quad (17)$$

where we have introduced the operator \hat{Z} by [7, 10],

$$\hat{Z}g(x) = \int_0^\infty dy e^{\phi_a(x)} \epsilon(x-y) e^{-\phi_a(y)} g(y). \quad (18)$$

As usual, $\epsilon(x) = x/2|x|$. All correlation functions can be expressed in terms of the kernel [6]

$$K_R^a(x, y) = \int_0^x dz e^{-\phi_a(z)} k_R^a(y, z) e^{-\phi_a(y)}, \quad (19)$$

with the pre-kernel defined by

$$k_R^a(y, z) = \sum_{i,j=0}^{2\bar{n}-1} R_i^a(y) J_{ij} R_j^a(z). \quad (20)$$

For example, the spectral density is given by

$$\rho(x) = K_R^a(x, x) - \frac{1}{2} K_R^a(\infty, x). \quad (21)$$

We construct the skew-orthogonal polynomials for the chUE by a generalization of an operator method [10] introduced by Brézin and Neuberger [7]. The skew-orthogonal polynomials are expressed in terms of orthogonal polynomials P_k^{2a+b} with weight function $x^{2a+b} \exp(-nV(x)/2)$,

$$R_i^a(x) = \sum_{j=0}^i T_{ij} P_j^{2a+b}(x). \quad (22)$$

We will derive recursion relations for the expansion coefficients T_{ij} . It is useful to introduce the operators

$$\hat{X}, \quad \hat{X}^b \hat{\partial}, \quad \hat{X}^{-b} \hat{Z}, \quad (23)$$

and the operator

$$\hat{L} = \hat{X}^b \hat{\partial} - \hat{X}^b \phi'(\hat{X}) + b\hat{X}^{b-1}. \quad (24)$$

The coordinate operator and the derivative operator are defined by $\hat{X}f(x) = xf(x)$ and $\hat{\partial}f(x) = f'(x)$, respectively, and the operator \hat{L} is the inverse of $\hat{X}^{-b}\hat{Z}$, i.e.,

$$\hat{L}\hat{X}^{-b}\hat{Z}g(x) = g(x). \quad (25)$$

The matrix elements of the operators in (23) and (24) with respect to the basis $P_k^{2a+b}(x)$ will be denoted by X_{kl} , D_{kl} , Y_{kl} and L_{kl} , in this order. For an operator \hat{A} the matrix elements are defined by

$$A_{kl} = \int_0^\infty dx e^{-2\phi_a(x)+b\log x} P_l^{2a+b}(x) \hat{A} P_k^{2a+b}(x). \quad (26)$$

By partial integration it can be shown that L and D are related by

$$L_{kl} = \frac{1}{2}(D_{kl} - D_{lk}). \quad (27)$$

For integer values of $b \geq 1$ the matrix elements of L thus vanish for $|k - l| > \max(b - 1, b + p - 1)$ (where p is the order of the polynomial probability potential). The optimum value of b is thus $b = 1$ which will be our choice in the remainder of this article.

The orthogonality relation (16) can be written as

$$TYT^T = -J. \quad (28)$$

By acting with L and $T^T J T$ on YT^T it follows that

$$L = T^T J T. \quad (29)$$

In matrix form this equation can be rewritten as

$$L_{kl} = \sum_{p=0}^{\infty} [T_{2p+1,k} T_{2p,l} - T_{2p,k} T_{2p+1,l}]. \quad (30)$$

For known L_{kl} the coefficients T_{ik} can be determined recursively from this relation. If the L_{kl} vanish outside a band $|k - l| > p$ (as is the case for $V(x)$ given by a polynomial of order p), the coefficients T_{ik} are nonzero only inside a band $i - k < M < 2p$.

The pre-kernel can be written as

$$\begin{aligned} k_R^a(x, y) &= \sum_{i,j=0}^{2\bar{n}-1} \sum_{k \leq i} \sum_{l \leq j} P_k^{2a+1}(x) T_{ki}^T J_{ij} T_{jl} P_l^{2a+1}(y) \\ &= \sum_{k,l=0}^{2\bar{n}-1} \sum_{i,j} P_k^{2a+1}(x) T_{ki}^T J_{ij} T_{jl} P_l^{2a+1}(y) - R(x, y), \end{aligned} \quad (31)$$

²For $2a = 0$ the matrix L remains a band matrix even for $b = 0$.

where remainder term is given by

$$R(x, y) = \sum_{i,j=2\bar{n}}^{2\bar{n}+M-2} \sum_{k \leq i} \sum_{l \leq j} P_k^{2a+1}(x) T_{ki}^T J_{ij} T_{jl} P_l^{2a+1}(y). \quad (32)$$

There are now no restrictions on the summation over i and j in the last line of (31) and the relation (29) can be used to simplify the expressions. For finite M and a smooth dependence of the coefficients T_{ij} on the order of the skew-orthogonal polynomials the number of terms in $R(x, y)$, and thus the contribution of $R(x, y)$ to the pre-kernel, is subleading in $1/n$. In the next section, this will be shown explicitly both for a Gaussian and a quartic probability potential. To leading order in $1/n$ we thus find

$$\begin{aligned} k_R^a(x, y) &= \sum_{k,l=0}^{2\bar{n}-1} P_k^{2a+1}(x) L_{kl} P_l^{2a+1}(y) \\ &= \frac{1}{2} \sum_{k,l=0}^{2\bar{n}-1} P_k^{2a+1}(x) [D_{kl} - D_{lk}] P_l^{2a+1}(y). \end{aligned} \quad (33)$$

By re-expressing the matrix elements of D in terms of the operators $x\partial_x$ and $y\partial_y$ we find the following relation between the kernel for the chiral Orthogonal Ensemble and the kernel for the chiral Unitary Ensemble [10]

$$k_R(x, y) = \frac{1}{2} (y\partial_y - x\partial_x) K_n^{2a+1}(x, y). \quad (34)$$

This relation was first obtained for the Gaussian case in [34]. Since it has been shown that $K_n^{2a+1}(x, y)$ is universal [19], we thus have proved that the pre-kernel is universal [10]. The only problem is that an integral of the pre-kernel over the complete spectrum contributes to the spectral density and the spectral correlators. Even in the microscopic limit this results in contributions from non-universal regions. Before going to the general case, we first analyze in detail the chGOE and the case of a quartic probability potential.

4 Two Examples

In this section we study the quadratic and the quartic probability potential. In the first case, the matrix elements of L and the skew-orthogonal polynomials will be derived exactly, whereas in the second case only asymptotic results for large order polynomials will be obtained.

4.1 The chiral Gaussian Orthogonal Ensemble

In this section we study the chiral Gaussian Orthogonal Ensemble by means of the operator construction discussed in the previous section. The weight function is given by

$w(x) = x^{2a+1}e^{-nx}$, and the corresponding orthonormal polynomials are the Laguerre polynomials,

$$P_k^{2a+1}(x) = \frac{1}{\sqrt{s_k^{2a+1}}} L_k^{2a+1}(nx), \quad (35)$$

with normalization constants

$$s_k^\alpha = \frac{h_k^\alpha}{n^{\alpha+1}} \quad \text{and} \quad h_k^\alpha = \frac{\Gamma(k + \alpha + 1)}{k!}. \quad (36)$$

The matrix elements of L follow immediately from the recursion relation

$$x\partial_x L_k^{2a+1}(x) = kL_k^{2a+1}(x) - (k + 2a + 1)L_{k-1}^{2a+1}(x), \quad (37)$$

and are given by

$$L_{kl} = \frac{1}{2}[(l + 2a + 1)\sqrt{\frac{h_k^{2a+1}}{h_l^{2a+1}}}\delta_{k,l-1} - (k + 2a + 1)\sqrt{\frac{h_l^{2a+1}}{h_k^{2a+1}}}\delta_{l,k-1}]. \quad (38)$$

One easily verifies that the recursion relation (30) does not have a solution for diagonal matrices T_{kl} . A solution is obtained by taking $T_{2k,2k}$, $T_{2k+1,2k+1}$, $T_{2k+1,2k}$ and $T_{2k+1,2k-1}$ as the only nonzero coefficients. In the normalization $R_{2k}(x) = L_{2k}^{2a+1}(nx)/\sqrt{s_{2k}^{2a+1}}$ (i.e. $T_{2k,2k} = 1$) the recursion relations (29) simply read

$$T_{2k+1,2k+1} = L_{2k+1,2k}, \quad T_{2k+1,2k-1} = L_{2k,2k-1}, \quad (39)$$

and the skew-orthogonal polynomials are thus given by

$$\begin{aligned} R_{2k}^a(x) &= \frac{1}{\sqrt{s_{2k}^{2a+1}}} L_{2k}^{2a+1}(nx), \\ R_{2k+1}^a(x) &= -\frac{(2k + 2a + 2)}{2} \frac{\sqrt{s_{2k}^{2a+1}}}{s_{2k+1}^{2a+1}} L_{2k+1}^{2a+1}(nx) + \frac{(2k + 1 + 2a)}{2} \frac{1}{\sqrt{s_{2k}^{2a+1}}} L_{2k-1}^{2a+1}(nx) \\ &+ T_{2k+1,2k} L_{2k}^{2a+1}(x). \end{aligned} \quad (40)$$

The coefficients $T_{2k+1,2k}$ are not fixed by the orthogonality relations. Indeed, this is the well-known property that the odd order skew-orthogonal polynomials are only determined up to a multiple of the even order polynomials of one degree lower. One can verify that these polynomials are normalized according to $\langle R_{2k+1}^{2a+1}, R_{2k}^{2a+1} \rangle = 1$, and, with an adjustment of the normalization, they coincide with the polynomials obtained in [34] for a specific choice of coefficient $T_{2k+1,2k}$.

In fact, we can calculate the pre-kernel directly from (31) using the matrix elements of $T^T J T$ without relying on explicit expressions for the skew-orthogonal polynomials.

Because only one term contributes to $R_{2k}(x)$, there are no terms in (30) that are outside the summation range in (31). We thus have

$$k_R(x, y)^a = \sum_{k,l=0}^{2\bar{n}-1} \frac{1}{\sqrt{s_k^{2a+1} s_l^{2a+1}}} L_k^{2a+1}(nx) L_{kl} L_l^{2a+1}(ny) = \frac{1}{2} (y\partial_y - x\partial_x) K_n^{2a+1}(x, y), \quad (41)$$

which was first obtained in [34] from the explicit properties of the skew-orthogonal polynomials. Using recursion relations for the Laguerre polynomials the pre-kernel can be rewritten as

$$\begin{aligned} \frac{1}{2} (y\partial_y - x\partial_x) K_n^\alpha(x, y) &= \sum_{k=0}^{2\bar{n}-1} \frac{(k + \alpha)}{2s_k^\alpha} (L_{k-1}^\alpha(nx) L_k^{\alpha-1}(ny) - L_{k-1}^\alpha(ny) L_k^{\alpha-1}(nx)) \\ &= \frac{1}{2} (\partial_y - \partial_x) K_n^{\alpha-1}(x, y), \end{aligned} \quad (42)$$

where the factor $k + \alpha$ has been absorbed in the normalization of the orthogonal polynomials.

According to (21), the spectral density is given by

$$\frac{1}{2} \int_0^x dz e^{-\phi_a(z) - \phi_a(x)} (\partial_z - \partial_x) K_n^{2a}(x, z) - \frac{1}{4} \int_0^\infty dz e^{-\phi_a(z) - \phi_a(x)} (\partial_z - \partial_x) K_n^{2a}(x, z) \quad (43)$$

The microscopic limit of the first term results in an integral over the universal Bessel kernel. However, it is not possible to interchange the integral and the microscopic limit in the second term. To see this we return to the definition of the pre-kernel. Then the second term is given by

$$- \frac{1}{4} \int_0^\infty dz e^{-\phi_a(z)} \sum_{i,j=0}^{2\bar{n}-1} R_i^a(x) J_{ij} R_j^a(z) e^{-\phi_a(x)}. \quad (44)$$

We thus consider the integral

$$\int_0^\infty dz z^a e^{-\phi_a(z)} R_i^a(z) = \int_0^\infty dz z^a e^{-nz/2} R_i^a(z). \quad (45)$$

By using the explicit expressions for the skew-orthogonal polynomials and the relation

$$L_n^{2a+1}(2x) = \sum_{m=0}^n L_{n-m}^a(x) L_m^a(x), \quad (46)$$

it follows that the integral over the odd order skew-orthogonal polynomials vanishes. The integral over the even order skew-orthogonal polynomials is up to a normalization constant given by

$$\frac{(n/2)^{a+1}}{h_p^a} \int_0^\infty dz z^a e^{-nz/2} L_{2p}^{2a+1}(nz) = 1. \quad (47)$$

Let us now calculate the integral by interchanging the integration and the microscopic limit. Using the asymptotic relation between Laguerre polynomials and Bessel functions

$$L_n^\alpha(nx) \sim x^{-\alpha/2} J_\alpha(2n\sqrt{x}) \quad (48)$$

the microscopic limit of the integral is given by

$$\int_0^\infty \lim_{\substack{p \rightarrow \infty \\ p/n = \text{fixed}}} dz z^a e^{-nz/2} \frac{(n/2)^{a+1}}{h_p^a} L_{2p}^{2a+1}(nz) = \int_0^\infty dw J_{2a+1}(2w) = \frac{1}{2}. \quad (49)$$

Exactly half of the integral is missing. In the next section we will argue that this is a general feature of the skew-orthogonal polynomials. However, the integral over the odd-order polynomials does not vanish in general. The result for the microscopic spectral density is thus given by

$$\rho_s(u) = \frac{1}{2} \int_0^u dw (\partial_w - \partial_u) B^{2a}(u, w) - \frac{1}{2} \int_0^\infty dw (\partial_w - \partial_u) B^{2a}(u, w), \quad (50)$$

where an additional factor of 2 has been included in the second term and the Bessel kernel is defined in (14).

4.2 Chiral Orthogonal Ensemble with Quartic Potential

In order to construct the skew-orthogonal polynomials using the Brézin-Neuberger formalism [7], we need an expression for the derivative of orthogonal polynomials. To derive such relation we start from the recursion relation [19]

$$xP_k(x) = -r_k(P_{k+1} - P_k) + s_k(P_k - P_{k-1}). \quad (51)$$

The coefficients r_k and s_k are related by

$$s_k = \frac{h_k r_{k-1}}{h_{k-1}}, \quad (52)$$

and $h_k = \int_0^\infty dx w(x) P_k^2(x)$ is the normalization integral. The recursion relation (51) is valid for orthogonal polynomials normalized according to $P_k(0) = 1$. To make contact with the analysis of [19] we will use this normalization in this section.

The derivative of the polynomials for arbitrary weight function is given by [5]

$$\begin{aligned} yP_k'(y) &= \int_0^\infty dx w(x) \sum_{l=0}^k \frac{P_l(x)P_l(y)}{h_l} xP_k'(x) \\ &= kP_k(y) + \int_0^\infty dx w(x) \sum_{l=0}^{k-1} \frac{P_l(x)P_l(y)}{h_l} xP_k'(x) \\ &= kP_k(y) - \int_0^\infty dx w'(x) \sum_{l=0}^{k-1} \frac{P_l(x)P_l(y)}{h_l} xP_k(x). \end{aligned} \quad (53)$$

where the terms following the last equal sign have been obtained by partial integration.

Next we derive the asymptotic form of large order skew-orthogonal polynomials for a quartic probability potential. In terms of the $x_k = \lambda_k^2$ the weight function is given by $w(x) = x^{2a+1}e^{-nx^2/2}$. Only the terms $l = k - 1$ and $l = k - 2$ are nonvanishing in the last sum in (53). Using the recursion relation (51) to calculate the integrals we find

$$yP'_k(y) = kP_k(y) - ns_k[r_k + s_k + r_{k-1} + s_{k-1}]P_{k-1}(y) + ns_k s_{k-1}P_{k-2}(y). \quad (54)$$

Taking into account the normalization of the orthogonal polynomials the matrix elements of L are given by

$$\begin{aligned} L_{kl} &= \frac{1}{2} \int_0^\infty dx w(x) \frac{1}{\sqrt{h_k h_l}} [P_l(x) x P'_k(x) - P_k(x) x P'_l(x)] \\ &= \delta_{l,k-2} \frac{n}{2} \sqrt{\frac{h_{k-2}}{h_k}} s_k s_{k-1} - \delta_{l,k-1} \frac{n}{2} \sqrt{\frac{h_{k-1}}{h_k}} s_k [r_k + s_k + r_{k-1} + s_{k-1}] \\ &\quad + \delta_{k,l-1} \frac{n}{2} \sqrt{\frac{h_{l-1}}{h_l}} s_l [r_l + s_l + r_{l-1} + s_{l-1}] - \delta_{k,l-2} \frac{n}{2} \sqrt{\frac{h_{l-2}}{h_l}} s_l s_{l-1}. \end{aligned} \quad (55)$$

As expected, the L_{kl} vanish for $|k - l| > 2$.

In the limit $n \rightarrow \infty$ the leading order contributions to the kernel are for terms with large values of k and l . The large- k asymptotic behavior of the coefficients in the recursion relation (51) was obtained in [19] for an arbitrary polynomial probability potential (but for integer values of $2a$). For fixed $t \equiv k/n$, the continuum limit of the coefficients can be parameterized as

$$h_k = \frac{1}{n^{4a+3}} h(t), \quad s_k = s(t), \quad \text{and} \quad r_k = r(t). \quad (56)$$

For $k \rightarrow \infty$, only s_k and r_k enter in the matrix elements of L . They are given by [19]

$$r(t) = s(t) = \sqrt{\frac{t}{3}}. \quad (57)$$

The leading order asymptotic result for the matrix elements of L thus reads

$$L_{kl} = \delta_{l,k-2} \frac{nt}{6} - \delta_{l,k-1} \frac{4nt}{6} + \delta_{k,l-1} \frac{4nt}{6} - \delta_{k,l-2} \frac{nt}{6}. \quad (58)$$

By inspection of the recursion relation (29) one easily finds that for a quartic potential (i.e. $p = 2$) the skew-orthogonal polynomials can be expressed as

$$R_{2k}(x) = \tilde{P}_{2k}(x) + T_{2k,2k-1} \tilde{P}_{2k-1}(x), \quad (59)$$

$$R_{2k+1}(x) = T_{2k+1,2k+1} \tilde{P}_{2k+1}(x) + T_{2k+1,2k} \tilde{P}_{2k}(x) + T_{2k+1,2k-1} \tilde{P}_{2k-1}(x) + T_{2k+1,2k-2} \tilde{P}_{2k-2}(x), \quad (60)$$

where the orthonormal polynomials have been denoted by $\tilde{P}_k(x) \equiv P_k(x)/\sqrt{h_k}$. The coefficients $T_{i,k}$ obey the recursion relations

$$\begin{aligned} T_{2k-1,2k-1} - T_{2k,2k-1}T_{2k+1,2k-2} &= L_{2k-1,2k-2}, & T_{2k+1,2k-2} &= -L_{2k,2k-2}, \\ T_{2k,2k-1}T_{2k+1,2k} - T_{2k+1,2k-1} &= L_{2k,2k-1}, & T_{2k,2k-1}T_{2k+1,2k+1} &= L_{2k+1,2k-1}. \end{aligned} \quad (61)$$

Using the asymptotic values for the matrix elements of L we find a recursive equation for the coefficients $T_{2p,2p-1}$,

$$T_{2k,2k-1}T_{2k-2,2k-3} + 4T_{2k-2,2k-3} + 1 = 0. \quad (62)$$

This recursion relation has two fixed points given by the roots of

$$x^2 + 4x + 1 = 0. \quad (63)$$

For large k the coefficients $T_{2k,2k-1}$ should depend smoothly on k and are thus given by one of the fixed points. The polynomial corresponding to the stable fixed point, $x = -2 - \sqrt{3}$, given by $\tilde{P}_{2k}(x) - (2 + \sqrt{3})\tilde{P}_{2k-1}(x)$, is negative for $x = 0$ (we work in the convention $P_l(0) = 1$, and $\tilde{P}_{2k}(0)$ and $\tilde{P}_{2k-1}(0)$ are equal to leading order in $1/k$) and positive for $x \rightarrow \infty$ and thus has an odd number of zeros. Since even orthogonal polynomials should have an even number of zeros, the relevant solution is thus given by the unstable fixed point $x = -2 + \sqrt{3}$. This counter-intuitive result is a reflection of the numerical instability of the iterative construction of orthogonal polynomials. A numerical confirmation will be given in section 7. All other coefficients simply follow from the relations (61) resulting in the polynomials

$$\begin{aligned} R_{2k}(x) &= \tilde{P}_{2k}(x) - (2 - \sqrt{3})\tilde{P}_{2k-1}(x), \\ R_{2k+1}(x) &= \frac{-nt}{6(2 - \sqrt{3})}\tilde{P}_{2k+1}(x) + \frac{4nt}{6}\tilde{P}_{2k-1}(x) - \frac{nt}{6}\tilde{P}_{2k-2}(x) \\ &\quad + T_{2k+1,2k}(\tilde{P}_{2k}(x) - (2 - \sqrt{3})\tilde{P}_{2k-1}(x)). \end{aligned} \quad (64)$$

However, we do not need the explicit expressions for the skew-orthogonal polynomials. The pre-kernel can be expressed directly in the matrix elements of L and two additional terms that are outside the summation range in (31),

$$\begin{aligned} k_R(x, y) &= \sum_{k,l=0}^{2\bar{n}-1} \frac{1}{\sqrt{h_k h_l}} P_k(x) L_{kl} P_l(y) \\ &\quad + \frac{T_{2\bar{n},2\bar{n}-1}T_{2\bar{n}+1,2\bar{n}-2}}{\sqrt{h_{2\bar{n}-1}h_{2\bar{n}-2}}} [P_{2\bar{n}-1}(x)P_{2\bar{n}-2}(y) - P_{2\bar{n}-1}(y)P_{2\bar{n}-2}(x)]. \end{aligned} \quad (65)$$

Because $\lim_{n \rightarrow \infty} T_{2n,2n-1} = -2 + \sqrt{3}$, the additional terms are of the same order of magnitude as each of the terms in the sum. Therefore, to leading order in $1/n$ the relation

between the pre-kernel for $\beta = 1$ and the kernel for $\beta = 2$ is the same as for the Gaussian case. However, it is not clear whether the interchange of the integral and the microscopic limit in $K_R(\infty, x)$ also proceeds in the same way. This question will be analyzed in the next section for an arbitrary probability potential.

5 A property of large order skew-orthogonal polynomials

In this section we will derive an asymptotic relation for large order skew-orthogonal polynomials. Our starting point is the skew-orthogonality relation

$$\langle R_k, R_0 \rangle_R = \int_0^\infty dx \int_0^\infty dy R_k(x) e^{-\phi_a(x) - \phi_a(y)} \epsilon(x - y) = 0. \quad (66)$$

For asymptotically large k we can distinguish three different domains in the integration over x . The region near $x = 0$, the region around the largest zero of $R_k(x)$, and the oscillatory region which is in between these two regions. We split the integration over x into two parts separated by M chosen to be inside the oscillatory region. In our normalization the spacing of the smallest eigenvalues scales as $\Delta x \sim 1/n^2$. The main contribution to the integral over $[0, M]$ is from the region around the smallest eigenvalues $x \sim 1/n^2$, whereas the y -integral has contributions up to $nV(y) \sim 1$. If the potential behaves as y^p with $p > 1/2$ near $y = 0$, to leading order in $1/n$ the contribution to the y -integral is from the region with $y > x$ and $\epsilon(x - y) = -1/2$. The main contribution to the integral over $[M, \infty)$ is from the region near the largest zero of $R_k(x)$. To leading order in $1/n$ we can then replace $\epsilon(x - y) \rightarrow 1/2$. The integrals over x and y factorize and we obtain the following asymptotic relation

$$\int_0^M dx R_k^a(x) e^{-\phi_a(x)} dx = \int_M^\infty dx R_k^a(x) e^{-\phi_a(x)} dx. \quad (67)$$

This implies that

$$\int_0^\infty dx R_k^a(x) e^{-\phi_a(x)} dx = 2 \int_0^M dx R_k^a(x) e^{-\phi_a(x)} dx, \quad (68)$$

which is valid for $k \rightarrow \infty$ provided that the r.h.s. is independent of M in the oscillatory region.

In our derivation we have made the assumption that the skew-orthogonal polynomials show the same oscillatory behavior as the regular orthogonal polynomials. This is certainly true if they can be expressed in a finite number of regular orthogonal polynomials which is the case for a finite order polynomial probability potential [10]. Of course, a different oscillatory behavior is possible if the leading order asymptotic terms in the skew-orthogonal polynomials cancel. This results in contributions that are subleading

in $1/n$. A priori, it cannot be excluded that the skew-orthogonal polynomials near zero and the l.h.s. of (68) are of the same order in $1/n$ and thus both contribute to the kernel. For example, this is the case for the odd-order skew-Laguerre polynomials (40) with $T_{2k+1,2k} = 0$. However, in this case the l.h.s. of (68) vanishes, and we do not have to worry about the asymptotic behavior of the r.h.s. of (68). In general, there is no reason to expect that the leading order asymptotic expansion of the P_k^{2a+1} cancels. For example, the asymptotic behavior of P_k^{2a+1} in the oscillatory region depends smoothly on k , and from the explicit expressions for the even order skew-orthogonal polynomials in (64) it then follows that the leading order asymptotic behavior does not cancel. For the odd order skew-orthogonal polynomials a cancellation can only be achieved by fine tuning the coefficient $T_{2k+1,2k}$.

For a finite order probability potential, the generic asymptotic behavior of the $R_k^a(x)$ is thus the same as that of $e^{-\phi_a(x)}P_k^{2a+1}(x)$. In the region near zero and in the oscillatory region, it is given by $J_{2a+1}(c\sqrt{kx})/\sqrt{x}$ (with c a constant that can be obtained from the recursion relations). If the integral (68) is nonvanishing to leading order in $1/n$, we thus find from the asymptotic behavior of $e^{-\phi_a(x)}P_k^{2a+1}(x)$ that the integral converges for $k \rightarrow \infty$ and M inside the oscillatory region.

As will be shown below, the even order skew-orthogonal polynomials are determined up to a multiplicative constant. From the orthogonality relations it is clear that the odd order polynomials $R_{2k+1}(x)$ are only determined up to the addition of a multiple of $R_{2k}(x)$. We may use this freedom to chose a normalization such that

$$\int_0^\infty dx e^{-\phi_a(x)} R_{2k+1}^a(x) = 0. \quad (69)$$

In this way we avoid the ambiguity in the asymptotic behavior of the odd order skew-orthogonal polynomials.

The main contributions to the integral in the r.h.s. of (68) are from the region close to $x = 0$ and the integrand can be replaced by its microscopic limit. Our final result is

$$\lim_{\substack{n \rightarrow \infty \\ k/n = \text{const.}}} \int_0^\infty R_{2k}^a(x) e^{-\phi_a(x)} dx = 2 \int_0^\infty \lim_{\substack{n \rightarrow \infty \\ xn^2 = z^2 \\ k/n = \text{const.}}} R_{2k}^a(x) e^{-\phi_a(x)} dx. \quad (70)$$

Interchanging the microscopic limit and the integral gives rise to an extra factor two. If the integrand in the kernel $K_R^a(\infty, x)$ is replaced by its microscopic limit, the same extra factor two has to be included,

$$\lim_{\substack{n \rightarrow \infty \\ xn^2 = z}} K_R^a(\infty, x) = 2 \int_0^\infty \lim_{\substack{n \rightarrow \infty \\ xn^2 = z}} dw e^{-\phi_a(w/n^2) - \phi_a(z/n^2)} \frac{1}{n^2} k_R^a\left(\frac{w}{n^2}, \frac{z}{n^2}\right). \quad (71)$$

We emphasize that this relation is based on the asymptotic properties of the even order skew-orthogonal polynomials only.

6 Universality of the Chiral Orthogonal Ensemble

The proof of universality of microscopic spectral correlation functions for the chiral orthogonal ensemble for a finite polynomial probability potential is now straightforward. It is an immediate consequence of the following three results. i) The relation between the pre-kernel for the chOE and the chUE kernel is independent of the probability potential to leading order in $1/n$ [10]. ii) The microscopic limit of the chUE kernel is universal if the eigenvalues are expressed in units of the average level spacing [19]. iii) Because of a novel asymptotic property of large order skew-orthogonal polynomials, the microscopic limit and the integrals that occur in the spectral correlation functions can be interchanged at the expense of a factor of 2.

All k -point correlation functions of the chOE can be expressed compactly as quaternion determinants of quaternions [6]

$$S(x_k, x_l) = \begin{pmatrix} K_{R,n}^a(x_k, x_l) - \frac{1}{2}K_{R,n}^a(\infty, x_l) & \int_{x_l}^{x_k} dz [K_{R,n}^a(x_k, z) - \frac{1}{2}K_{R,n}^a(\infty, z)] - \epsilon(x - y) \\ \partial_{x_k} K_{R,n}^a(x_k, x_l) & K_{R,n}^a(x_l, x_k) - \frac{1}{2}K_{R,n}^a(\infty, x_k) \end{pmatrix} \quad (72)$$

where $K_R^a(x, y)$ is defined in (19). We already noticed that the spectral density is given by $\rho(x) = K_{R,n}^a(x, x) - \frac{1}{2}K_{R,n}^a(\infty, x)$. The two-point cluster function is given by [6]

$$T(x, y) = \frac{1}{2} \text{Tr} S(x, y) S(y, x). \quad (73)$$

The universal microscopic kernel for the chOE is obtained by taking the microscopic limit of the integrands and replacing the factors $1/2 \rightarrow 1$,

$$\begin{aligned} S(u, v) &= \lim_{N=2n \rightarrow \infty} \frac{2\sqrt{uv}}{\Sigma^2 N^2} S\left(\frac{u^2}{\Sigma^2 N^2}, \frac{v^2}{\Sigma^2 N^2}\right) \\ &= 2\sqrt{uv} \begin{pmatrix} Q^a(u, v) - Q^a(\infty, v) & \Sigma^2 N^2 \int_{v^2}^{u^2} dw^2 [Q^a(u, w) - Q^a(\infty, w)] \\ \frac{1}{\Sigma^2 N^2} \partial_{u^2} Q^a(u, v) & Q^a(v, u) - Q^a(\infty, u) \end{pmatrix}, \end{aligned} \quad (74)$$

where $Q^a(u, v)$ is the microscopic limit of the kernel $K_{R,n}^a(x, y)$. Universality then follows from the relation between the microscopic kernels for $\beta = 1$ and $\beta = 2$ and the universality of the microscopic limit of the kernel for $\beta = 2$. The universal microscopic result for $Q^a(u, v)$ is given by

$$\begin{aligned} Q^a(u, v) &\equiv \lim_{N=2n \rightarrow \infty} \frac{1}{\Sigma^2 N^2} K_{R,n}^{2a}\left(\frac{u^2}{\Sigma^2 N^2}, \frac{v^2}{\Sigma^2 N^2}\right) \\ &= \frac{1}{4} \int_0^{u^2} d[w^2] (wv)^{2a} (\partial_{w^2} - \partial_{v^2}) (wv)^{-2a-1/2} B^{2a}(v, w). \end{aligned} \quad (75)$$

By using the integral representation (15) of the Bessel kernel we find the following explicit representation of the kernel

$$Q^a(u, v) = \frac{u}{4} \int_0^1 dw \int_0^1 t^2 dt \left[\frac{uw}{v} J_{2a}(uwt) J_{2a+1}(vt) - J_{2a+1}(uwt) J_{2a}(vt) \right]. \quad (76)$$

One of the integrals can be performed analytically. By using identities for Bessel functions one ultimately finds the result

$$\begin{aligned} 2\sqrt{uv}Q^a(u, v) &= B^{2a+1}(u, v) - \sqrt{uv} \frac{J_{2a+1}(v)}{2v} \left(\int_0^u dw J_{2a+1}(w) - 1 \right) \\ &= B^{2a}(u, v) - \sqrt{uv} \frac{J_{2a+1}(v)}{2v} \left(\int_0^u dw J_{2a-1}(w) - 1 \right). \end{aligned} \quad (77)$$

The microscopic spectral density for the chOE is given by $\rho_s(u) = 2u(Q^a(u, u) - Q^a(\infty, u))$. For its universal form we thus obtain [34]

$$\rho_s(u) = \frac{u^2}{2} \int_0^1 t^2 dt \int_0^1 dw [w J_{2a}(uwt) J_{2a+1}(ut) - J_{2a+1}(uwt) J_{2a}(ut)] + \frac{1}{2} J_{2a+1}(u). \quad (78)$$

This expression for the microscopic spectral density can be simplified to

$$\begin{aligned} \rho_s(u) &= \frac{u}{2} [J_{2a+1}^2(u) - J_{2a+2}(u) J_{2a}(u)] - \frac{1}{2} J_{2a+1}(u) \left(\int_0^u dw J_{2a+1}(w) - 1 \right) \\ &= \frac{u}{2} [J_{2a}^2(u) - J_{2a+1}(u) J_{2a-1}(u)] - \frac{1}{2} J_{2a+1}(u) \left(\int_0^u dw J_{2a-1}(w) - 1 \right). \end{aligned} \quad (79)$$

The simplified results for $\rho_s(u)$ and the kernel were first obtained in [58] and was derived independently in [59] (with a typo, see also [11, 41, 14]). The first term can be recognized as the microscopic spectral density for the chGUE [61, 44]. The microscopic limit of the two-point correlation function follows immediately from (73) and the microscopic limit of the kernel.

Recently, universal results for massive spectral correlators at $\beta = 1$ and $\beta = 4$ have been obtained by relating the kernels for the massive correlators to the corresponding massless kernel [14]. Similar relations have been derived for the Gaussian case [41].

7 Numerical study of the x^4 -potential

In this section we explicitly construct the skew-orthogonal polynomials for an x^4 potential and test the asymptotic results obtained in previous sections. We consider the distribution of the squared eigenvalues $x_k = \lambda_k^2$ on $[0, \infty[$ with weight function $\phi_a(x) = x^2/2 - a \log x$. For reasons of numerical accuracy we only consider integer values of a .

The skew-orthogonal polynomials can be expanded in terms of monomials x^k as

$$R_k^a(x) = r_{kk}^a x^k + r_{k,k-1}^a x^{k-1} + \dots + r_{k0}^a. \quad (80)$$

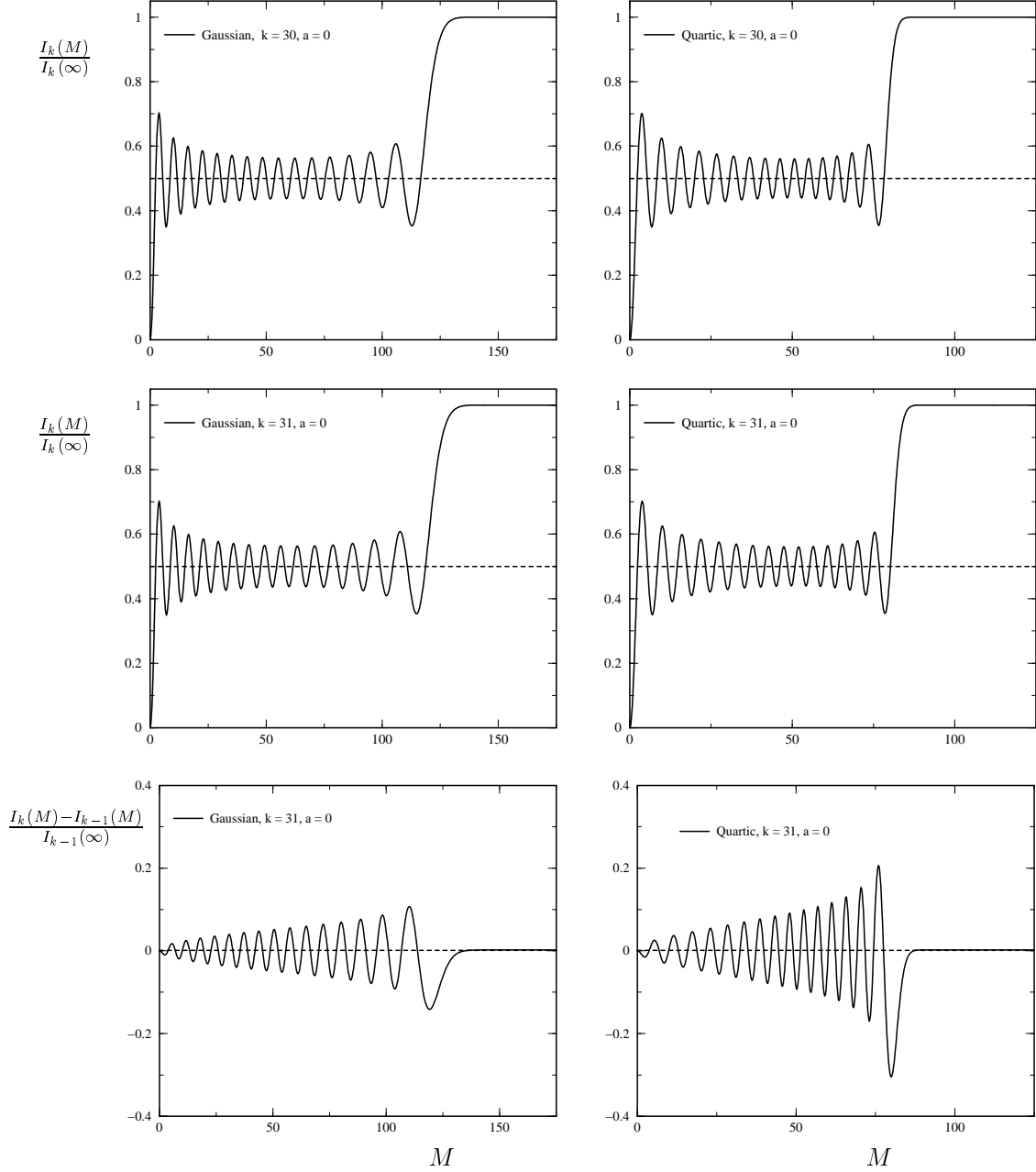


Figure 1: Ratios of the integral $I_k(M) = \int_0^{(M/\pi\rho(0))^2} dx e^{-\phi_a(x)} R_k(x)$ versus M for a Gaussian (left) and a quartic (right) probability potential. The parameter a and the value of k are given in the label of the figures.

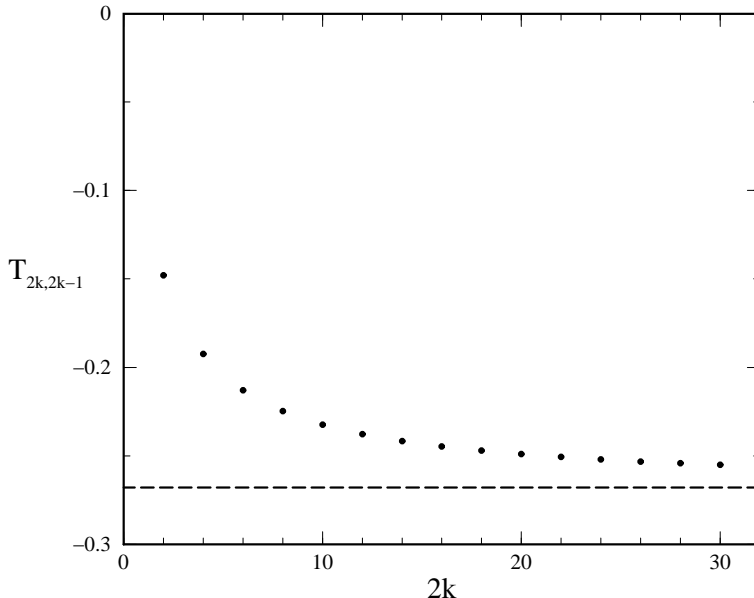


Figure 2: The asymptotic behavior of the coefficients $T_{2k,2k-1}$ versus k for a quartic probability potential. Its asymptotic value of $-2 + \sqrt{3}$ is depicted by the dashed curve.

The weight function is given by $\phi_a(x) = x/2$ for the Gaussian case and by $\phi_a(x) = x^2/2$ for the quartic case. We have redefined M in units of $\pi\rho(0)$, with $\rho(0)$ the the average spectral density near zero (notice that with our convention for the weight function $\rho(0)$ has a nontrivial k -dependence). This figure shows that in the intermediate domain the integral $I_k(M)$ oscillates around $I_k(\infty)/2$. For even k , the integral appears to converge in the oscillatory region. For odd k we show results for monic polynomials with normalizations $r_{2k+1,2k} = 0$ (middle figures) and by $R_{2k+1}(0) = 0$ (lower figures). In the first case, the odd order polynomials behave similarly to the even order ones, whereas in the second case the behavior is quite different, because the leading order asymptotic contributions cancel. The oscillations in the lower figures are still exactly about $I_k(\infty)/2$ which in this case is close to zero. The even order skew-orthogonal polynomials always have as many positive zeros as the order of the polynomial. The odd order polynomials in the normalization $r_{2k+1,2k} = 0$ are not very different from the preceding even order polynomials (see middle figure). However, typically one of their zeros is located on the negative real axis. For the normalization $R_{2k+1}(0) = 0$, the total number of zeros is equal to $2k + 1$, with one zero at $x = 0$. Fortunately, as we have seen in the previous section, the integral (69) over the odd order polynomials can always be tuned to zero so that we do not have to worry about the ambiguity of the asymptotic properties of the odd order skew-orthogonal polynomials.

In Figure 2, we show the k -dependence of the coefficients $T_{2k,2k-1}$ as defined in equation (64). Also shown is the analytical result for the asymptotic value of $-2 + \sqrt{3}$. The convergence to the asymptotic result is better illustrated by extrapolating to second order

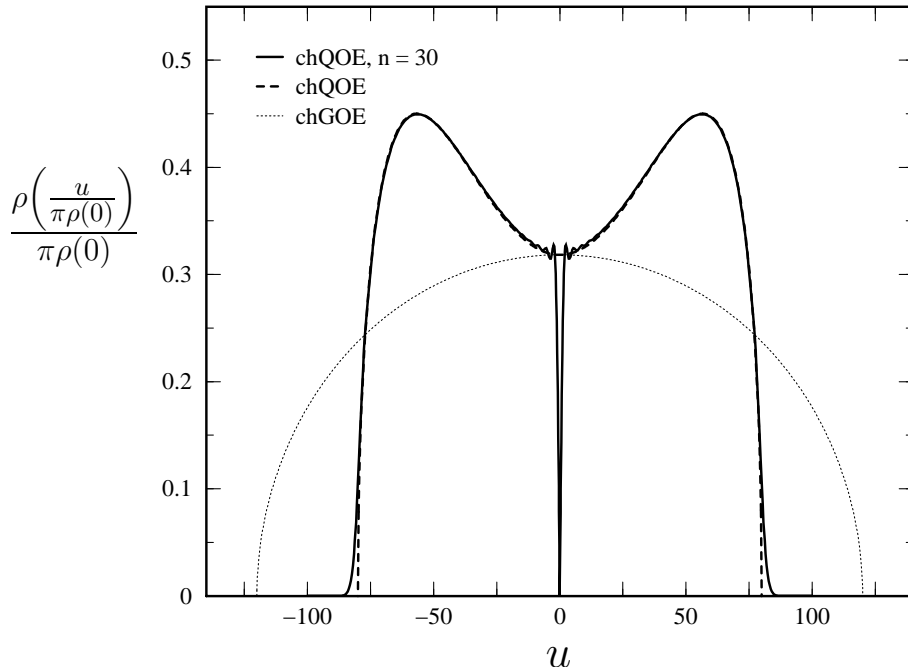


Figure 3: The average spectral density $\rho(u/\pi\rho(0))/\pi\rho(0)$ versus u . For a quartic probability potential we show results obtained from the 30 lowest order skew-orthogonal polynomials (full curve) and the large- n analytical result (dashed curve). The semicircular distribution (dotted curve) represents the large- n average spectral density for a Gaussian probability potential. All results are for $a = 0$.

in $1/k$. The values of $T_{2k,2k-1}$ at $k = 10, 20$ and 30 extrapolate to -0.26787 to be compared to $-2 + \sqrt{3} = -0.26795$.

In Figure 3, we show the average spectral density calculated from the quartic skew-orthogonal polynomials (full curve) and the analytical result (dashed curve) given by

$$\frac{\rho(u/\pi\rho(0))}{\pi\rho(0)} = \frac{1}{\pi} \left(1 + \frac{3}{4n}u^2\right) \left(1 - \frac{3}{8n}u^2\right)^{1/2}. \quad (85)$$

For comparison we also show the semicircular distribution obtained for a Gaussian potential (dotted curve).

In Fig. 4 we show the microscopic spectral density calculated from the first 30 skew-orthogonal polynomials for a quartic potential and $a = 0$. In the same figure we also show the result for a Gaussian potential. Clearly, the microscopic spectral density converges to the asymptotic result for $n \rightarrow \infty$. Both in Fig. 3 and Fig. 4, the average spectral density $\rho(0)$ depends on n because of the normalization of our weight function.

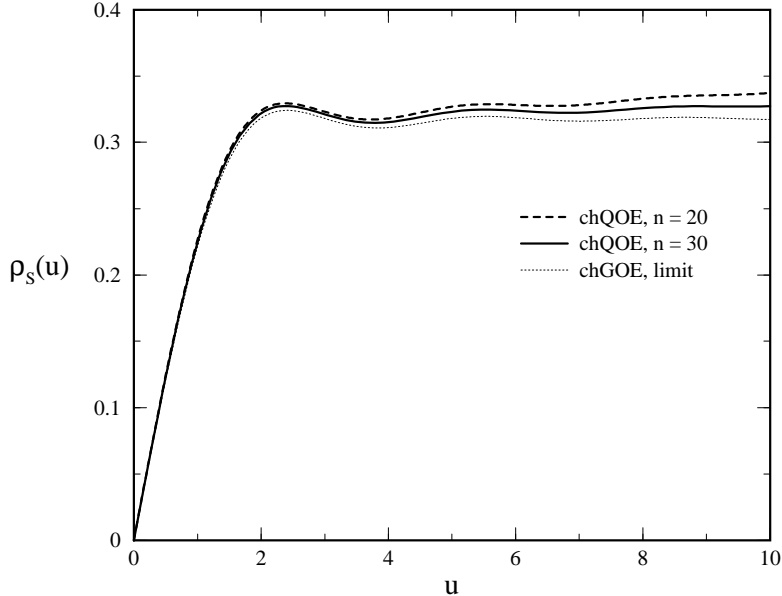


Figure 4: Microscopic spectral density $\rho_s(u)$ for a Gaussian and a quartic probability potential.

8 Conclusions

We have shown universality for the chiral Orthogonal Ensembles. Our proof is based on a relation between the kernels for $\beta = 1$ and $\beta = 2$ and the universality of the kernel for $\beta = 2$. In this article we have completed the proof outlined in [10] by deriving an asymptotic property of the skew-orthogonal polynomials which relates an integral over the region near the largest zero to an integral in the microscopic region. Universality now has been shown for all three chiral ensembles.

An alternative method for ensembles with $\beta = 1$ and $\beta = 4$ was recently proposed in [11]. This method does not rely on the construction of the skew-orthogonal polynomials at all, but it is our point of view is that both methods are equivalent. For example, the matrix elements of some of the operators in our construction are also required in the method proposed by Widom. The advantage of our method is best illustrated by the result of this article in which universality has been proved by means of an asymptotic relation of skew-orthogonal polynomials. It would be interesting to identify this relation within Widom's approach.

Finally, we wish to mention an alternative way of looking at universality. For theories with broken chiral symmetry and a mass gap we have two types of modes, the soft modes and the hard modes. An effective partition function is obtained by integrating out the hard modes. If the Goldstone bosons corresponding to the spontaneous breaking of chiral symmetry are the only soft modes, there is no need to do this calculation. The effective partition function can be written down solely on the basis of the symmetries of the theory

and is thus the same for all partition functions with the same global symmetries. The equivalence of the effective theory for the Goldstone modes and chiral Random Matrix Theory has been demonstrated nonperturbatively for the chiral Unitary Ensemble [45, 60, 31, 32]. For the the other two values of the Dyson index this has only been shown perturbatively [13].

The mass of the relevant Goldstone modes in the generating function of the spectrum is proportional to the square root of the distance to $\lambda = 0$. Universal behavior is thus immanent in the microscopic limit. However, proving universality is equivalent to showing that the theory has a mass gap and that the Goldstone modes are the only soft modes. In QCD this is equivalent to proving confinement.

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References

- [1] E.P. Wigner, *Ann. Math.* 53 (1951) 36.
- [2] O. Bohigas, M. Giannoni, *Lecture notes in Physics* 209 (1984) 1; O. Bohigas, M. Giannoni and C. Schmit, *Phys. Rev. Lett.* 52 (1984) 1.
- [3] T.H. Seligman, J.J.M. Verbaarschot, and M. Zirnbauer, *Phys. Rev. Lett.* 53 (1984) 215; T.H. Seligman and J.J.M. Verbaarschot, *Phys. Lett. A* 108 (1985) 183.
- [4] T. Guhr, A. Müller-Groeling and H.A. Weidenmüller, *Phys. Rept.* 299 (1998) 189.
- [5] E. Kanzieper and V. Freilikher, *cond-mat/9809365*.
- [6] G. Mahoux and M. Mehta, *Comm. Math. Phys.* 79, (1981) 327; *Indian J. Pure Appl. Math.* 22 (1991) 531; *J. Phys. A*14 (1981) 579; *J. Phys. France I* (1991) 1093.
- [7] E. Brézin and H. Neuberger, *Nucl. Phys. B*350, 513 (1991).
- [8] C.W.J. Beenakker, *Phys. Rev. Lett.* 70 (1993) 1155.
- [9] G. Hackenbroich and H.A. Weidenmüller, *Phys. Rev. Lett.* 74 (1995) 4118.
- [10] M.K. Şener and J.J.M. Verbaarschot, *Phys. Rev. Lett.* 81 (1998) 248.

- [11] H. Widom, *J. Stat. Phys.* 94 (1999) 347.
- [12] G. Akemann and P.H. Damgaard, *Nucl. Phys.* B528 (1998) 411; G. Akemann and P.H. Damgaard, *Phys. Lett.* B432 (1998) 390; G. Akemann and P.H. Damgaard, hep-th/9910190.
- [13] D. Toublan and J.J.M. Verbaarschot, *Nucl. Phys. B* 560 (1999) 259.
- [14] G. Akemann and E. Kanzieper, hep-th/0001188.
- [15] M.R. Zirnbauer, *J. Math. Phys.* 37 (1996) 4986; F.J. Dyson, *Commun. Math. Phys.* 19 (1970) 235.
- [16] E.V. Shuryak and J.J.M. Verbaarschot, *Nucl. Phys.* A560 (1993) 306.
- [17] J. Verbaarschot, *Phys. Rev. Lett.* 72 (1994) 2531.
- [18] J. Ambjørn, J. Jurkiewicz, and Y.M. Makeenko, *Phys. Lett. B* 251 (1990) 517.
- [19] G. Akemann, P. Damgaard, U. Magnea and S. Nishigaki, *Nucl. Phys. B* 487[FS] (1997) 721.
- [20] E. Brézin, S. Hikami and A. Zee, *Nucl. Phys.* B464 (1996) 411.
- [21] E. Kanzieper and V. Freilikher, cond-mat/9704149.
- [22] A.D. Jackson, M.K. Sener and J.J.M. Verbaarschot, *Nucl. Phys.* B479 (1996) 707.
- [23] T. Guhr and T. Wettig, *Nucl. Phys.* B506 (1997) 589.
- [24] A.D. Jackson, M.K. Sener and J.J.M. Verbaarschot, *Nucl. Phys.* B506 (1997) 612.
- [25] K. Splittorff, *Nucl. Phys. B* 548 (1999) 613.
- [26] K. Splittorff and A.D. Jackson, hep-lat/9805018.
- [27] P.H. Damgaard and S.M. Nishigaki, *Nucl. Phys. B* 518 (1998) 495.
- [28] S.M. Nishigaki, P.H. Damgaard, and T. Wettig, *Phys. Rev. D* 58 (1998) 087704.
- [29] P.H. Damgaard and S.M. Nishigaki, *Phys. Rev. D* 57 (1998) 5299.
- [30] P.H. Damgaard, *Phys. Lett. B* 424 (1998) 322.
- [31] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, *Nucl. Phys. B* 540 (1999) 317.
- [32] P.H. Damgaard, J.C. Osborn, D. Toublan, and J.J.M. Verbaarschot, *Nucl. Phys. B* 547 (1999) 305.

- [33] B. Seif, T. Wettig, and T. Guhr, Nucl. Phys. B548 (1999) 475.
- [34] J.J.M. Verbaarschot, Nucl. Phys. B426[FS] (1994) 559.
- [35] T. Nagao and P.J. Forrester, Nucl. Phys. B 435 (1995) 401.
- [36] F. Dyson, J. Math. Phys. 13 (1972) 90.
- [37] T. Nagao and M. Wadati, J. Phys. Soc. Jap. 60 (1991) 3298; J. Phys. Soc. Jap. 61 (1992) 78, 1910.
- [38] T. Nagao and M. Wadati, J. Phys. Soc. Jap. 62 (1993) 3845.
- [39] A. Altland and M.R. Zirnbauer, Phys. Rev. Lett. 76 (1996) 3420;
- [40] C. Hilmoine and R. Niclasen, hep-th/0004081.
- [41] T. Nagao and S.M. Nishigaki, hep-th/0001137; hep-th/0003009.
- [42] R.G. Edwards, U.M. Heller, and R. Narayanan, Phys. Rev. D 60 (1999) 077502.
- [43] R.G. Edwards, U.M. Heller, J. Kiskis, and R. Narayanan, Phys. Rev. Lett. 82 (1999) 4188.
- [44] J.J.M. Verbaarschot, Nucl. Phys. B427 (1994) 434.
- [45] J.J.M. Verbaarschot, Phys. Lett. B368 (1996) 137.
- [46] M.E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J.J.M. Verbaarschot and T. Wettig, Phys. Rev. Lett. 80 (1998) 1146.
- [47] P.H. Damgaard, U.M. Heller, and A. Krasnitz, Phys. Lett. B 445 (1999) 366.
- [48] M. Göckeler, H. Hehl, P.E.L. Rakow, A. Schäfer, and T. Wettig, Phys. Rev. D 59 (1999) 094503.
- [49] F. Farchioni, I. Hip, and C.B. Lang, Phys. Lett. B 471 (1999) 58.
- [50] J.J.M. Verbaarschot and T. Wettig, hep-ph/0003017, to be published in Ann. Rev. Nucl. Part. Sci. (2000).
- [51] R. Gade and F. Wegner, Nucl. Phys. B 360 (1991) 213; R. Gade, Nucl. Phys. B 398 (1993) 499.
- [52] M.A. Halasz and J.J.M. Verbaarschot, Phys. Rev. Lett. 74 (1995) 3920; M.A. Halasz, T. Kalkreuter and J.J.M. Verbaarschot, Nucl. Phys. Proc. Suppl. 53 (1997) 266.

- [53] T. Banks and A. Casher, Nucl. Phys. B 169 (1980) 103.
- [54] H. Leutwyler and A. Smilga, Phys. Rev. D 46 (1992) 5607.
- [55] A. Smilga and J.J.M. Verbaarschot, Phys. Rev. D 51 (1995) 829.
- [56] M. L. Mehta, Random Matrices, Academic Press, San Diego, 1991.
- [57] C.A. Tracy and H. Widom, Commun. Math. Phys. 161 (1994) 289.
- [58] T. Wettig, Habilitationsschrift, University of Munich, April 1998.
- [59] P.J. Forrester, T. Nagao, and G. Honner, Nucl. Phys. B 553 (1999) 601.
- [60] J.C. Osborn and J.J.M. Verbaarschot, J.C. Osborn and J.J.M. Verbaarschot, Phys. Rev. Lett. 81 (1998) 268; Nucl. Phys. B 525 (1998) 738.
- [61] J. Verbaarschot and I. Zahed, Phys. Rev. Lett. 70, 3852 (1993).