Cauchy-Riemann Geometry and Contact Topology in Three Dimensions

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Abstract

We introduce a global Cauchy-Riemann(CR)-invariant and discuss its behavior on the moduli space of CR-structures. We argue that this study is related to the Smale conjecture in 3-topology and the problem of counting complex structures. Furthermore, we propose a contact-analogue of Ray-Singer's analytic torsion. This "contact torsion" is expected to be able to distinguish among "contact lens" spaces. We also propose the study of a certain kind of monopole equation associated with a contact structure.

Key Words: Cauchy-Riemann geometry, contact structure, contact torsion, monopole equation, Smale conjecture

I Introduction

We study low-dimensional problems in topology and geometry via a study of contact and Cauchy-Riemann (CR) structures. Let us start with a closed (compact without boundary) oriented three-manifold M. A contact structure (or bundle) ξ on M is a completely non-integrable rank 2 subbundle of TM. It is well known that there are no local invariants for contact structures according to a classical theorem of Darboux. Also, two nearby contact structures on a closed manifold are isotopy-equivalent by Gray's theorem (Gray, 1959; Hamilton, 1982). Therefore, a contact structure has no continuous moduli. In this sense, it is a kind of geometric structure even softer than a complex structure. The isotopy classes are distinguished by so-called tight or overtwisted contact structures (Eliashberg, 1992). The existence of contact structures on a closed oriented three-manifold is known from the work of Martinet (1971) and Lutz (1977). (See also Altschuler (1995) for an analytic proof using the so-called linear contact flow.)

Given a contact structure, we can consider a CR-structure, i.e., a "complex structure" defined on a contact bundle. Different from the usual complex structure, a CR-structure does have local invariants. Thus, analysis is needed. In Section II, we give a brief introduction to CR-geometry and an application in K$\ddot{a}$hler geometry. In Section III, we introduce a global CR-invariant μ_ξ and discuss its behavior on the moduli space of CR-structures. Also, we argue that the contractibility of our CR moduli space for S^3 confirms the so-called Smale conjecture.

In Section IV, we discuss spherical CR-structures: the critical points of μ_ξ. To distinguish among "CR lens" spaces, we propose a possible CR-invariant defined for spherical CR-structures, which is a contact-analogue of Ray-Singer's analytic torsion. In Section V, we give a heuristic argument for how our understanding of μ_ξ can be applied to the problem of counting the number of complex structures on a closed four-manifold. In Section VI, we propose the study of a certain kind of monopole equation for contact three-manifolds.

II Basics in CR-Geometry

A CR-structure J compatible with the contact structure ξ is a complex structure on ξ, i.e., a bundle endomorphism $J : \xi \rightarrow \xi$, such that $J^2 = -Identity$. Natural examples come from boundaries of strictly pseudoconvex domains D in C^2. Let J_{C^2} denote the multiplication by i in C^2. Let our three-manifold $M = \partial D$, the boundary of D. The contact structure ξ is considered to be the intersection of TM and $J_{C^2}TM$, the tangent subspaces invariant under J_{C^2}. In addition, our CR-structure is taken to be a restriction of J_{C^2} on ξ. This CR-structure is usually called the CR-structure induced from C^2.

In his famous theorem, Fefferman (1974) asserts that two strictly pseudoconvex domains with smooth boundaries in C^{n+1} are biholomorphic to each other if and only if their boundaries are CR-equivalent. Therefore the CR-structure on the boundary reflects the complex structure of the inside domain. It is well known that we have the Riemann mapping theorem in C^1. However, this theorem is no longer true for higher dimensions. Indeed, we do have local invariants for our CR manifold (M, ξ, J) (e.g., Cartan (1932) and Chern and Moser (1974)).

First, choose eigenvectors Z_1, $Z_{\bar{1}}$ of J with eigenvalues i, $-i$, respectively. Let $\{\theta^1, \theta^{\bar{1}}\}$ be a set of complex one-forms dual to $\{Z_1, Z_{\bar{1}}\}$. Then, choose a local one-form θ annihilating ξ (called contact form) so that

$$d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + \theta \wedge \phi$$

for some real one-form ϕ and positive $h_{1\bar{1}}$. (We will use $h_{1\bar{1}}$ and $h^{1\bar{1}} = (h_{1\bar{1}})^{-1}$

to raise or lower indices.) Now, for a different choice of coframe $(\tilde{\theta}, \tilde{\theta}^1, \tilde{\theta}^{\bar{1}}; \tilde{\phi})$ satisfying the above equation, we have the following transformation relation:

$$\begin{cases} \tilde{\theta} = u\theta \\ \tilde{\theta}^1 = u_1{}^1\theta^1 + v^1\theta \\ \tilde{\phi} = -\frac{du}{u} + \phi + 2Re(iu^{-1}v^1u_{\bar{1}1}\theta^{\bar{1}}) + s\theta \end{cases} \tag{1}$$

for positive u and some real function s. Differentiating θ^1, ϕ gives the first structural equations:

$$\begin{cases} d\theta^1 = \theta^1\wedge\phi_1{}^1 + \theta\wedge\phi^1 \\ d\phi = 2Re(i\theta_{\bar{1}}\wedge\phi^{\bar{1}}) + \theta\wedge\psi \end{cases} \tag{2}$$

for the connection forms $\phi_1{}^1, \phi^1, \psi$. Differentiating the connection forms again and requiring certain trace conditions (e.g., Chern and Moser (1974)), we obtain the second set of structural equations:

$$\begin{cases} d\phi_1{}^1 - i\theta_1\wedge\phi^1 + 2i\phi_1\wedge\theta^1 + \frac{1}{2}\psi\wedge\theta = 0 \\ d\phi^1 - \phi\wedge\phi^1 - \phi^1\wedge\phi_1{}^1 + \frac{1}{2}\psi\wedge\theta^1 = Q^1{}_{\bar{1}}\theta^{\bar{1}}\wedge\theta \\ d\psi - \phi\wedge\psi - 2i\phi^1\wedge\phi_1 = (R_1\theta^1 + R_{\bar{1}}\theta^{\bar{1}})\wedge\theta, \end{cases} \tag{3}$$

in which $Q^1{}_{\bar{1}}$ or Q_{11} is called the Cartan (curvature) tensor, and $R_1, R_{\bar{1}}$ are determined by means of suitable covariant derivatives of $Q^1{}_{\bar{1}}$ (Cheng, 1987). The normalization condition: $\phi - \phi_1{}^1 - \phi_{\bar{1}}{}^{\bar{1}} = 0$ and the above structural equations Eqs.(2) and (3) uniquely determine the connection forms $\phi_1{}^1, \phi^1, \psi$. Under the change of coframe Eq.(1), the Cartan tensor is transformed as follows:

$$Q_{11} = \tilde{Q}_{11}u(u_1{}^1)^2.$$

The fundamental theorem of 3-dimensional CR-geometry due to Cartan (1932a,1932b) asserts that $Q_{11} = 0$ if and only if (M, ξ, J) is locally CR-equivalent to $(S^3, \hat{\xi}, \hat{J})$, where $(\hat{\xi}, \hat{J})$ denotes the standard CR-structure on the unit 3-sphere S^3, induced from C^2.

Definition. We call a CR manifold (M, ξ, J) or just J spherical if it is locally CR-equivalent to $(S^3, \hat{\xi}, \hat{J})$. Quantitatively, a CR-structure is spherical if $Q_{11} = 0$ according to Cartan's theorem.

An Application in Kähler Geometry

Let N be an n-dimensional Kähler manifold. Suppose we have a holomorphic line bundle L with the first Chern class being the Kähler class so that a suitable circle bundle $M \subset L$ with the induced CR-structure is closely related to the Kähler geometry of N (Webster, 1977). It turns out that we can identify (up to a constant) the Cartan tensor Q_{11} of M with $R_{,11}$, the covariant derivative of the scalar curvature R of N in the (1,0)-direction twice. When $n \geq 2$, we can identify the Chern tensor (Chern and Moser, 1974, 1983) in higher dimensional CR-geometry with the Bochner tensor of N. In 1977, Sid Webster applied CR-geometry to obtain the following result:

> Let N be a simply-connected closed Kähler manifold of dimension n. Suppose that N admits a Hodge metric for which the Bochner tensor vanishes if $n \geq 2$ or for which $R_{,11}$ vanishes if $n = 1$. Then, N is holomorphically isometric to complex projective space CP^n with a standard Fubini-Study metric. (Webster, 1977)

Next, relative to a special coframe $(\theta, \theta^1, \theta^{\bar{1}}; \phi = 0)$ satisfying $d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$, we can define the so-called pseudohermitian connection ${\omega_1}^1$, torsion A_{11}, and curvature $\mathcal{W}$, called the Tanaka-Webster curvature (Tanaka, 1975; Webster, 1977). These data are uniquely determined by the following equations:

$$\begin{cases} d\theta^1 = \theta^1 \wedge {\omega_1}^1 + {A^1}_{\bar{1}}\theta \wedge \theta^{\bar{1}} \\ d{\omega_1}^1 = \mathcal{W}\theta^1 \wedge \theta^{\bar{1}} \; (mod\, \theta) \\ {\omega_1}^1 + {\omega_{\bar{1}}}^{\bar{1}} = h^{1\bar{1}} dh_{1\bar{1}}. \end{cases}$$

The torsion A_{11} and the Tanaka-Webster curvature $\mathcal{W}$ are not "tensorial" under the change of contact form $\tilde{\theta} = u\theta$ (Lee, 1986), but are "tensorial" under the change $\tilde{\theta}^1 = {u_1}^1\theta^1$. The Cartan tensor can be expressed in terms of these data (Cheng and Lee, 1990):

$$Q_{11} = \frac{1}{6}\mathcal{W}_{,11} + \frac{i}{2}\mathcal{W}A_{11} - A_{11,0} - \frac{2i}{3}A_{11,\bar{1}1}.$$

Here, covariant derivatives are taken with respect to the pseudohermitian connection ${\omega_1}^1$, and "0" means the T- direction. (The tangent vector field T is uniquely determined by $\theta(T) = 1$ and $L_T\theta = 0$.) Before going on, another result should be noted:

The boundary of a circular domain in C^{n+1} is CR-equivalent to the unit sphere $S^{2n+1} \subset C^{n+1}$ with the standard induced CR-structure if and only if the Tanaka-Webster curvature $\mathcal{W} \equiv constant$ (with respect to a suitable choice of contact form)(Unpublished paper by J. Bland and P. M. Wang).

The proof of the above result in the original draft contains a gap which can be remedied by the following result:

> Let N be a closed complex manifold with two Kähler metrics g, $\tilde{g}$. Suppose the Bochner tensor of g vanishes and the scalar curvature of $\tilde{g}$ is a constant. Then, the fact that the Kähler class of g is cohomologous to the Kähler class of $\tilde{g}$ implies that (N, g) and $(N, \tilde{g})$ are isometric to each other (Chen and Lue, 1981).

III The μ_ξ-Invariant and the Moduli Space

First, we will construct an energy functional on the space of CR-structures so that the critical points consist of spherical CR-structures. Let Π denote the $su(2,1)$-valued Cartan connection form defined by

$$\Pi = \begin{pmatrix} -\frac{1}{3}({\phi_1}^1 + \phi) & \theta^1 & 2\theta \\ -i\phi_1 & \frac{1}{3}(2{\phi_1}^1 - \phi) & 2i\theta_1 \\ -\frac{1}{4}\psi & \frac{1}{2}\phi^1 & \frac{1}{3}(\phi + {\phi_{\bar{1}}}^{\bar{1}}) \end{pmatrix}.$$

The curvature form Ω is defined as usual by $\Omega = d\Pi - \Pi \wedge \Pi$. The transgression $TC_2(\Pi)$ of the second Chern form is given by

$$\begin{aligned} TC_2(\Pi) &= \frac{1}{8\pi^2}[tr(\Pi \wedge \Omega + \frac{1}{3}tr(\Pi \wedge \Pi \wedge \Pi)] \\ &= \frac{1}{24\pi^2} tr(\Pi \wedge \Pi \wedge \Pi) \\ &\quad (since\, tr(\Pi \wedge \Omega) = 0). \end{aligned}$$

We can verify that the 3-form $TC_2(\Pi)$ is invariant under the change of contact form and invariant up to an exact form under the coframe change

Eq.(1). In the late 1980's, Burns and Epstein (1988)(also Cheng and Lee (1990)) discovered that the integral of $TC_2(\Pi)$, denoted as μ_ξ, is a global CR-invariant (assuming trivial holomorphic tangent bundle as in Burns and Epstein (1988); extended to arbitrary M by a relative version of the invariant in Cheng and Lee (1990)):

$$\begin{aligned}
\mu_\xi(J) &= \frac{1}{24\pi^2}\int_M tr(\Pi\wedge\Pi\wedge\Pi) \\
&= \frac{1}{8\pi^2}\int_M [2Re(i\theta^1\wedge\phi^{\bar{1}}\wedge{\phi_1}^1) + \frac{1}{2}\theta\wedge\psi\wedge\phi - 2i\theta\wedge\phi^1\wedge\phi^{\bar{1}} - \frac{1}{2}d(\theta\wedge\psi)] \\
&= \frac{1}{8\pi^2}\int_M [(\frac{1}{6}\mathcal{W}^2 + 2|A_{11}|^2)\theta\wedge d\theta + \frac{2}{3}{\omega_1}^1\wedge d{\omega_1}^1] \\
&\quad (in\ terms\ of\ pseudohermitian\ geometry).
\end{aligned}$$

It is remarkable that the above integral is independent of the choice of contact form, and that the integrand involves only the second and lower-order derivatives (relative to a coframe field) while the lowest order of local invariants is of order 4 as indicated by the Cartan tensor Q_{11}.

Next, we will discuss the moduli space of CR-structures. Let $\mathcal{J}_\xi$ denote the space of all CR-structures compatible with ξ. Let $\mathcal{C}_\xi$ denote the group of contact diffeomorphisms with respect to ξ. Clearly, $\mathcal{C}_\xi$ acts on $\mathcal{J}_\xi$ by pulling back. The invariant μ_ξ is actually defined on the moduli space $\mathcal{J}_\xi/\mathcal{C}_\xi$.

Given a CR-structure J in $\mathcal{J}_\xi$, we call a "submanifold" S passing through J a local slice if it is transverse to the orbit of $\mathcal{C}_\xi$-action, so that any element in $\mathcal{J}_\xi$ near J can be pulled back to an element of S by means of a certain contact diffeomorphism. In the early 1990's, Jack Lee and the author proved the following:

Local slices always exist for all cases (Cheng and Lee, 1995).

As a corollary, the standard spherical CR-structure $[\hat{J}]$ in $\mathcal{J}_{\hat{\xi}}/\mathcal{C}_{\hat{\xi}}$ for S^3 is a strict local minimum for $\mu_{\hat{\xi}}$ (Cheng and Lee, 1995).

Let $Q_J = 2Re[iQ_1{}^{\bar{1}}\theta^1 \otimes Z_{\bar{1}}]$. It is a straightforward computation to obtain the first variation formula: $\delta\mu_\xi(J) = -\frac{1}{8\pi^2}Q_J$. Consider the downward gradient flow for μ_ξ:

$$\partial_t J_{(t)} = Q_{J_{(t)}}. \tag{4}$$

Since δQ_J is subelliptic modulo the action of our symmetry group $\mathcal{C}_\xi$, we can play a suitable "De-Turck trick" to break the symmetry and imitate the usual L^2-theory for elliptic operators to obtain the short time solution of Eq.(4)(Cheng and Lee, 1990). However, we can not prove the long term solution and convergence even for $M = S^3$. This is related to the so-called Smale conjecture as first pointed out by Eliashberg.

The Smale conjecture asserts that the diffeomorphism group of S^3 is homotopy-equivalent to the orthogonal group $O(4)$. Suppose we have the long term solution and convergence of Eq.(4) for $M = S^3$. Then, any starting J must converge to $\hat{J}$, the unique spherical CR-structure on S^3 (up to symmetry). Therefore, the (certain marked) CR moduli space $\mathcal{J}'_{\hat{\xi}}/\mathcal{C}'_{\hat{\xi}}$ is contractible. But $\mathcal{J}'_{\hat{\xi}}$ is contractible, too. It follows that $\mathcal{C}'_{\hat{\xi}}$ is contractible. Then, with the aid of contact geometry, we can confirm the Smale conjecture.

To learn more analytic techniques which can be used to tackle Eq.(4), we have been working on some comparatively easier flows like the CR Calabi flow and the CR Yamabe flow. For the CR Yamabe flow, S.-C. Chang and the author deformed a contact form in the direction of the Tanaka-Webster curvature:

$$\partial_t \theta_{(t)} = \mathcal{W} \theta_{(t)}. \tag{5}$$

In their present work, Chang and Cheng obtain a Harnack estimate and (possibly) the long term solution for Eq.(5).

IV The Moduli Space of Spherical CR-Structures

Let $\mathcal{S}_\xi$ denote the space of all spherical CR-structures compatible with ξ. Since the linearization of the Cartan tensor is subelliptic modulo the action of $\mathcal{C}_\xi$, the virtual dimension of $\mathcal{S}_\xi/\mathcal{C}_\xi$: the moduli space of spherical CR-structures is finite. Let M be a circle bundle over a closed surface of genus $g > 1$ with the Euler class $e(M) < 0$. Let $Pic(g, c_1)$, the universal Picard variety, denote the space of all pairs (L, N) in which L is a holomorphic line bundle over a Riemann surface N of genus $g > 1$ with $c_1(L) = e(M)$ modulo an equivalence relation defined by diffeomorphisms. In 1996 and 1997, I-Hsun Tsai and the author studied the relation between $\mathcal{S}_\xi/\mathcal{C}_\xi$ and $Pic(g, c_1)$. We found the following:

For an above-mentioned circle bundle M, there is a diffeomorphism between $\mathcal{S}_\xi/'\mathcal{C}_\xi$ and

$Pic(g, c_1)'$. (The prime means a suitably modified version.) Moreover, $Pic(g, c_1)'$ is a complex manifold of dimension $4g - 3$ (Cheng and Tsai, 2000).

Our above result is similar to describing a Teichmuller space by means of conformal classes. It is known in Teichmuller theory that we can pick up a unique hyperbolic metric as a representative for each conformal class. A similar situation occurs for our spherical CR manifolds. In fact, our theory for the universal Picard variety has counterparts in Teichmuller theory as shown in Table 1.

Table 1. Comparison of two theories

Teichmuller space	universal Picard variety
conformal classes	spherical CR circle bundles
Riemannian hyperbolic metrics	pseudohermitian hyperbolic geometries

Local Rigidity of Spherical CR – Structures
(**Discrete Moduli** :$dim\mathcal{S}_\xi/\mathcal{C}_\xi = 0$)

Let $Aut_{CR}(S^3)$ denote the CR-automorphism group of $(S^3, \hat{\xi}, \hat{J})$, which is known to be isomorphic to $SU(2,1)/center$. Let Γ denote a fixed point free finite subgroup of $Aut_{CR}(S^3)$. Then, $\Gamma\backslash S^3$ inherits both contact and (spherical) CR-structures from $(S^3, \hat{\xi}, \hat{J})$. This induced spherical CR-structure on $\Gamma\backslash S^3$ is locally rigid; i.e. it has no nontrivial deformation. (The algebraic reason is that $H^1(\Gamma, su(2,1)) = 0$, in which the group cohomology has coefficients in the holonomy representation: developing map composed with the adjoint representation) (Burns and Shnider, 1976). On the other hand, note that $\Gamma\backslash S^3$ has positive constant Tanaka-Webster curvature and zero torsion. Now, generalizing using an analytical method, we obtain the following:

Let (M, J) be a closed spherical CR three-manifold. Suppose there is a contact form such that the torsion $A_{11} = 0$ and $\mathcal{W} > 0$, $4\mathcal{W}(5\mathcal{W}^2 + 3\Delta_b\mathcal{W}) - 3|\nabla_b\mathcal{W}|^2_\theta > 0$. Then, J is locally rigid (Cheng, 1999).

Next, we want to compare two $\Gamma\backslash S^3$. Suppose $\Gamma_1\backslash S^3$ and $\Gamma_2\backslash S^3$ are diffeomorphic. How can we distinguish one spherical CR-structure from the other one? (They have the same μ_ξ-value.) To deal with this problem, we borrow ideas from quantum physics. If we view μ_ξ as a Lagrangian (action, more accurately) in $2+1$ dimensions, spherical CR-structures are just classical fields. Therefore, "quantum fluctuations" should give us refined invariants. In practice, we compute the partition function heuristically:

$$\begin{aligned}\mathcal{Z}_k &= \int_{\mathcal{J}_\xi/\mathcal{C}_\xi} \mathcal{D}[J] e^{ik\mu_\xi([J])} \\ &= k^{-\frac{dim}{2}}(\mathcal{Z}_{sc} + O(k^{-1}))\ (k\ large),\end{aligned}$$

in which $\mathcal{Z}_{sc}$ is called the semi-classical approximation. Note that only classical fields make contributions to $\mathcal{Z}_{sc}$. By imitating the finite dimensional case, we can compute the modulus of $\mathcal{Z}_{sc}$ (Cheng, 1995):

$$\begin{aligned}|\mathcal{Z}_{sc}| &= lim_{k\to\infty} k^{\frac{dim}{2}} |\mathcal{Z}_k| \\ &= \Sigma_{J:spherical} \left|\frac{det\Box_J}{det'\delta Q_J}\right|^{\frac{1}{2}},\end{aligned}$$

in which $\Box_J$ is a fourth-order subelliptic self-adjoint operator related to the $\mathcal{C}_\xi$-action, and δQ_J, the second variation of μ_ξ, is also a fourth-order subelliptic self-adjoint operator modulo the $\mathcal{C}_\xi$-action. We can regularize two determinants via zeta functions. (det' means taking a regularized determinant under a certain gauge-fixing condition.)

Conjecture: If J is spherical,

$$Tor(J) \stackrel{def}{=} \left| \frac{det \Box_J}{det' \delta Q_J} \right|^{\frac{1}{2}}$$

is independent of any choice of contact form, i.e., a CR invariant.

We expect to use $Tor(J)$ to distinguish among "contact lens" (or "CR lens") spaces $\{\Gamma \backslash S^3\}$. Also, we note that $Tor(J)$ is a contact-analogue of Ray-Singer's analytic torsion while no contact-analogue is known for the Reidemeister torsion.

V Counting the Number of Complex Structures

This is another "quantum level" problem in our ongoing project. We will discuss the problem of counting the number of complex structures on a closed (compact without boundary) four-manifold. We hope to view this number as the partition function of a certain 3+1 quantum field theory (QFT in short).

Let us begin with a 0+1 theory, i.e., a particle moving in a closed manifold N. The Hamiltonian of such a theory with supersymmetry is the Laplace-Beltrami operator Δ. All quantum ground states or vacua are cohomology classes of N, represented by harmonic forms (=zero eigenforms of Δ). Now suppose f is a Morse function on N. Consider Δ_{tf} in which d is replaced by $e^{-tf}de^{tf}$. When $t \rightarrow \infty$, the harmonic forms of Δ_{tf} are concentrated near the critical points of f. These are the classical ground states (Witten, 1982).

The harmonic form corresponding to a critical point P has a small correction due to another critical point Q via the trajectories of ∇f from P to Q. This is quantum mechanical tunnelling, which describes the probability of the transition $P \rightarrow Q$. The boundary operator of Witten's chain complex (See Witten (1982) or Atiyah (1988) for a clear explanation.) is interpreted in terms of such tunnelling. (The homology of Witten's chain complex can be shown to identify with the homology of N.) Witten's idea was later adopted by Floer (1989) and applied to the infinite-dimensional case of the manifold of connections.

Next, we will give a brief introduction to the Donaldson-Floer theory. It is a 3+1 QFT. A "field" when restricted to the three-space M in this theory is a connection (or gauge field) of a certain, say, $SU(2)$ bundle over M. The Morse function as mentioned above is the Chern-Simons functional defined on the space of connections in this case. The critical points consist of flat connections which are the classical ground states. Through consideration of the associated Witten complex, we obtain the so-called Floer homology or cohomology group $HF(M)$. This is the space of quantum ground states or vacua for this theory. Now, suppose we decompose a closed 4-manifold X along M (say, a homology 3-sphere) as shown in Fig.1.

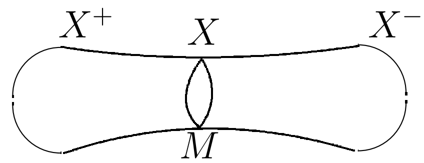

Fig.1. Decomposing X along M

where $X = X^+ \cup_M X^-$. Let Σ^+(Σ^-, respectively) denote the set of restrictions on M of all instantons on X^+(X^-, respectively). Then, Σ^+, Σ^- form cycles in $HF(M)$. The intersection number represents the algebraic number of instantons on X, (assuming it is finite) the Donaldson invariant, denoted as $Z(X)$. We can write

$$Z(X) =< vac(X^+)|\, vac(X^-) >,$$

in which the vacuum $vac(X^+) = [\Sigma^+]$ and the vacuum $vac(X^-) = [\Sigma^-]$ are both elements of $HF(M)$. Also $< | >$ denotes the middle-dmension intersection number. In Witten (1988), Witten presented a Lagrangian for this theory so that $Z(X)$ identifies with its partition function.

Now, we can describe our 3+1 QFT. We put an auxiliary contact structure ξ on our closed oriented three-manifold M. A "field" is a complex structure with the restriction on M being a CR-structure compatible with ξ. Our Morse function is the μ_ξ which we introduce in §3. Spherical CR-structures which are critical points of μ_ξ are our classical ground states in this theory.

Let Σ^+(Σ^-, respectively) denote the set of all CR-structures compatible with ξ on M, which can be extended to a complex structure on X^+(X^-, respectively). Now, what is the associated "Floer" homology group $HF(M, \xi)$, i.e., the space of quantum vacua, for this theory? Since the Hessian $\delta^2\mu_\xi$ at a spherical J is subelliptic modulo $\mathcal{C}_\xi$, the dimension of its negative eigenspace is finite. Therefore, the Morse index is well defined. (We do not need the relative Morse index as in the case of the Donaldson-Floer theory.) As usual, $\Sigma^\pm$ form cycles $[\Sigma^\pm]$ in $HF(M,\xi)$ by pushing along the gradient flow of μ_ξ and seeing which critical points they "hang" on (Atiyah, 1988). The vacuum $vac(X^+)(vac(X^-)$, respectively) is defined as the homology class $[\Sigma^+]$ ($[\Sigma^-]$, respectively) in $HF(M,\xi)$. Moreover, we define the quantity $Z_\xi(X)$ as

$$\begin{aligned} Z_\xi(X) &\stackrel{def}{=} < vac(X^+)|vac(X^-) > \\ &\stackrel{def}{=} intersection\ number\ of\ [\Sigma^+]\ and\ [\Sigma^-]. \end{aligned}$$

The sum of $Z_\xi(X)$ over the isomorphism classes of tight contact structures, denoted as $Z(X)$, can be interpreted as the (algebraic) number of complex structures on X. We propose the following "physical" problem:

Problem 1. Find a Lagrangian for the above theory so that its partition function identifies with $Z(X)$.

There are topological obstructions for M to admit spherical CR-structures (Goldman, 1983). For instance, the three-torus T^3 does not admit any spherical CR-structure (compatible with any given contact structure ξ). Therefore, $HF_\star(T^3,\xi)=0$ for any ξ, and we can propose the following problem for "nonexistence":

Problem 2. Suppose $X = X^+ \cup_{T^3} X^-$. Find conditions on X and, perhaps, $X^\pm$ such that $Z(X)=0$.

We still need to investigate the relation between $Z(X) = 0$ and the nonexistence of complex structures. Another situation occurs when M is the standard contact 3-sphere $(S^3, \hat{\xi})$. This admits only one compatible spherical CR-structure, namely, the standard one $\hat{J}$, which is a strict local minimum for $\mu_{\hat{\xi}}$ modulo symmetry as mentioned in Section III. It follows that $HF_0(S^3, \hat{\xi}) = Z$ and $HF_k(S^3, \hat{\xi}) = 0$ for $k \neq 0$. Therefore, we can propose the following problem concerning "global rigidity":

Problem 3. Suppose $X = X^+ \cup_{S^3} X^-$. Find conditions on X and, perhaps, $X^\pm$ such that $Z(X) = 1$.

Note that any tight contact structure on S^3 is isotopy-equivalent to $\hat{\xi}$ according to Eliashberg (1992). Therefore, $Z(X)$ in Problem 3 is just $Z_{\hat{\xi}}(X)$.

VI Monopoles and Contact Structures

Recently, Kronheimer and Mrowka (1997) studied contact structures on 3-manifolds via the 4-dimensional Seiberg-Witten monopole theory. Here, we will outline another approach by Cheng and Chiu (1999).

Given a contact 3-manifold (M, ξ) and a background pseudohermitian structure (J, θ), we can discuss a canonical *spin*^{c}-structure c_ξ on $\xi^\star$. With respect to c_ξ, we will consider the equations for our "monopole" Φ coupled to the "gauge field" A. Here, A, the *spin*^{c}-connection, is required to be compatible with the pseudohermitian connection on M. The Dirac operator D_ξ relative to A is identified with a certain boundary $\bar{\partial}$-operator $\sqrt{2}(\bar{\partial}_b^a + (\bar{\partial}_b^a)^\star)$. In terms of the components (α, β) of Φ, our equations read as

$$\begin{cases} (\bar{\partial}_b^a + (\bar{\partial}_b^a)^\star)(\alpha + \beta) = 0 \\ (or\ \alpha^a_{,\bar{1}} = 0,\ \beta^a_{\bar{1},1} = 0) \\ da(e_1, e_2) - \mathcal{W} = |\alpha|^2 - |\beta_{\bar{1}}|^2, \end{cases} \tag{6}$$

where $A = A_{can} + iaI$ and $\mathcal{W}$ denotes the Tanaka-Webster curvature. Our first step in understanding Eq.(6) is as follows:

> Suppose the torsion $A_{11} = 0$. Also, suppose ξ is symplectically semifillable, and that the Euler class $e(\xi)$ is not a torsion class. Then, Eq.(6) has nontrivial solutions (i.e., α and β are not identically zero simultaneously)(Cheng and Chiu, 1999).

On the other hand, the Weitzenbock-type formula gives a nonexistence result for $\mathcal{W} > 0$. Together with the above existence result, we can conclude the following:

> Suppose the torsion $A_{11} = 0$ and the Tanaka-Webster curvature $\mathcal{W} > 0$. Then, either ξ is not symplectically semifillable, or $e(\xi)$ is a torsion class (Cheng and Chiu, 1999).

We note that Rumin (1994) proved that M must be a rational homology sphere under the conditions given above using a different method. Also, we do not know how to deal with the solution space of Eq.(6) in general although we hope that further study of Eq.(6) will produce invariants of contact structures.

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References

Altschuler, S. J. (1995) A geometric heat flow for one-forms on three-dimensional manifolds. *Illinois J. of Math.*, **39**, 98-118.

Atiyah, M. F. (1988) New invariants of three and four dimensional manifolds. *The Mathematical Heritage of Herman Weyl, Proc. Symp. Pure Math.*, **48**, 285-299.

Burns, D. and C. Epstein (1988) A global invariant for three dimensional CR-manifolds. *Invent. Math.*, **92**, 333-348.

Burns, D. and C. Epstein (1990) Characteristic numbers of bounded domains. *Acta Math.*, **164**, 29-71.

Burns, D. and S. Shnider (1976) Spherical hypersurfaces in complex manifolds. *Invent. Math.*, **33**, 223-246.

Cartan, E. (1932a) Sur la géometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexe I. *Ann. Mat.*, **11**, 17-90.

Cartan, E. (1932b) Sur la géometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexe II. *Ann. Sc. Norm. Sup. Pisa*, **1**, 333-354.

Cheng, J. H. and H. L. Chiu (1999) *Monopoles and contact 3-manifolds*. Part of NSC Report for Project NSC 88-2115-M-001-015, National Science Council, R.O.C., Taipei, Taiwan, R.O.C..

Cheng, J. H. (1987) On the curvature of CR-structures and its covariant derivatives. *Math. Z.*, **196**, 203-209.

Cheng, J. H. (1999) Rigidity of automorphisms and spherical CR structures. *Proc. A.M.S.*, (to appear).

Cheng, J. H. (1995) The geometry of Cauchy-Riemann manifolds in dimension three. *Proceedings of the Second Asian Mathematics Conference*, Nakhon Ratchasima, Thailand.

Cheng, J. H. and J. M. Lee (1990) The Burns-Epstein invariant and deformation of CR structures. *Duke Math. J.*, **60**, 221-254.

Cheng, J. H. and J. M. Lee (1995) A local slice theorem for 3-dimensional CR structures. *Amer. J. Math.*, **117**, 1249-1298.

Chen, B. Y. and H. S. Lue (1981) Chern classes and Bochner-Kaehler metrics. *Coll. Math.*, **44**, 203-208.

Chern, S. S. and J. K. Moser (1974) Real hypersurfaces in complex manifolds. *Acta Math.*, **133**, 219-271.

Chern, S. S. and J. K. Moser (1983) Erratum: Real hypersurfaces in complex manifolds. *Acta Math.*, **150**, 297.

Cheng, J. H. and I-H. Tsai (2000) Deformation of spherical CR structures and the universal Picard variety. *Commun. Anal. and Geom.*, (to appear)

Eliashberg, Y. (1992) Contact 3-manifolds twenty years since J. Martinet's work. *Ann. Inst. Fourier*, **42**, 165-192.

Fefferman, C. (1974) The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *Invent. Math.*, **26**, 1-65.

Floer, A. (1989) An instanton invariant for 3-manifolds. *Commun. in Math. Phys.*, **118**, 215-240.

Goldman, W. (1983) Conformally flat manifolds with nilpotent holonomy and the uniformization problem for 3-manifolds. *Trans. Amer. Math. Soc.*, **278**, 573-583.

Gray, J. W. (1959) Some global properties of contact structures. *Ann. Math.*, **69**, 421-450.

Hamilton, R. S. (1982) The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc.*, **7**, 65-222.

Kronheimer, P. B. and T. S. Mrowka (1997) Monopoles and contact structures. *Invent. Math.*, **130**, 209-255.

Lee, J. M. (1986) The Fefferman metric and pseudohermitian invariants. *Trans. Amer. Math. Soc.*, **296**, 411-429.

Lutz, R. (1977) Structures de contact sur les fibre's principaux en cercles de dimension 3. *Ann. Inst. Fourier*, **3**, 1-15.

Martinet, J. (1971) Formes de contact sur les variétiés de dimension 3. *Lect. Notes in Math.*, **209**, 142-163.

Rumin, M. (1994) Formes différentielles sur les variétés de contact. *J. Diff. Geom.*, **39**, 281-330.

Tanaka, N. (1975) *A Differential Geometric Study on Strongly Pseudo-Convex Manifolds*, Kinokuniya Co. Ltd., Tokyo

Webster, S. M. (1977) On the pseudo-conformal geometry of a Kähler manifold. *Math. Z.*, **157**, 265-270.

Webster, S. M. (1978) Pseudohermitian structures on a real hypersurface. *J. Diff. Geom.*, **13**, 25-41.

Witten, E. (1982) Supersymmetry and Morse theory. *J. Diff. Geom.*, **17**, 661-692.

Witten, E. (1988) Topological quantum field theory. *Commun. in Math. Phys.*, **117**, 353-386.