Counterexamples in the Levin-Wen model, group categories, and Turaev unimodality

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We remark on the claim that the string-net model of Levin and Wen[1] is a microscopic Hamiltonian formulation of the Turaev-Viro topological quantum field theory[2]. Using simple counterexamples we indicate where interesting extra structure may be needed in the Levin-Wen model for this to hold (however we believe that some form of the correspondence is true).

In order to be accessible to the condensed matter community we provide a very brief and gentle introduction to the relevant concepts in category theory (relying heavily on analogy with ordinary group representation theory). Likewise, some physical ideas are briefly surveyed for the benefit of the more mathematical reader.

The main feature of group categories under consideration is Turaev's unimodality. We pinpoint where unimodality should fit into the Levin-Wen construction, and show that the simplest example[1] fails to be unimodal. Unimodality is straightforward to compute for group categories, and we provide a complete classification at the end of the paper.

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I. INTRODUCTION

In this note we briefly consider some aspects in the relationship between the Levin-Wen string-net model[1] and the Turaev-Viro/Barrett-Westbury state sum model[3,4]. Although some work towards a correspondence proof has been provided[5,6], we emphasize an important extra structure appears necessary in the Levin-Wen model. This structure may have physical implications.

Although the details come from category theory, for the physics community we use simple analogies with ordinary group representation theory. Also, our discussion of the physics is kept brief and simple so that interested mathematicians may have access.

The key issue is that 6j-symbols with *tetrahedral symmetry* play a defining role in both models. In order to build 6j-symbols with tetrahedral symmetry, Turaev's construction uses the *unimodality* condition. In particular unimodality assures that the "band-breaking" maneuver depicted in Fig (5) is well-defined.

However, in the Levin-Wen model the analogous maneuver (see Fig (4)) is implicitly allowed without restriction. To highlight how this may be problematic, we show here that the first example computed by Levin-Wen is, in fact, **not** unimodal. This implies that the Turaev construction cannot be applied.

This example, along with most of the examples computed by Levin-Wen, are *group categories*[7]. At the end we formulate a theorem that clarifies conditions when a group category is unimodal.

Before proceeding into the heart of the examples, we mention the recent work of Hong[8] that (partially) generalizes the Levin-Wen model to unitary spherical categories using so-called *mirror conjugate symmetry* rather than tetrahedral symmetry. It would be interesting to explore the examples considered here in this more general context.

II. LEVIN-WEN MODEL

Levin-Wen[1] consider a model on a fixed trivalent graph (with oriented edges) embedded in 2d. A typical configuration is pictured in Fig (1) where each edge is labelled by a **string type** j. There are finitely-many string types, hence we simply refer to them by number $\{0, 1, \ldots, N\}$.

There is a duality $j \mapsto j^*$ that satisfies $()^{**} = ()$, and the 0 label means "no string", hence we require $0^* = 0$. We are allowed to reverse the orientation of an edge if we reverse its label $j \mapsto j^*$.

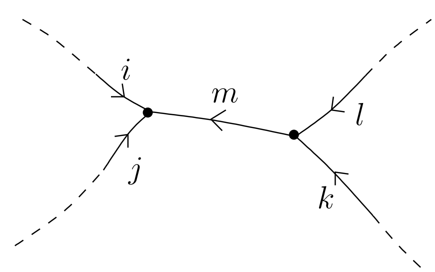

FIG. 1: A sample string-net configuration: a trivalent lattice in 2-dimensions with strings "colored" by integer string types i. In category language the labels are simple objects V_i. The dashed lines indicate that the string-net continues outside of the picture.

In order to quantize this Levin and Wen introduce a Hilbert space. It is defined by promoting every configuration X to be an orthonormal basis vector $|X\rangle$. An arbitrary state in the Hilbert space is a formal linear combinations of these. Given any string-net configuration $|X\rangle$, an arbitrary state $|\Phi\rangle$ has an associated probability amplitude $\Phi(X) := \langle X|\Phi\rangle$ of being in that particular configuration. From now on we do not differentiate between states $|\Phi\rangle$ and their wavefunctions Φ.

We are interested in a subspace of states $\{\Phi\}$ that encodes the topological information of the phase. One of their main results[1] is that $\{\Phi\}$ can be realized as the ground state subspace of an exactly-soluble Hamiltonian.

Rather than write down an explicit Hamiltonian, it is instructive to review the strategy that Levin-Wen use to find the topological subspace $\{\Phi\}$. The idea is based on *renormalization semigroup (RG)* flow. For concreteness suppose that we start with a (possibly too complicated) Hamiltonian H.

Usually H has some parameters that can be adjusted (e.g. coupling constants), and as we adjust these parameters we also deform its ground state(s) Φ (see Fig (2)). Hence we actually have a family $\{(H_\alpha, \Phi_\alpha)\}_\alpha$ of different (but related) theories where α denotes all of the adjustable parameters.

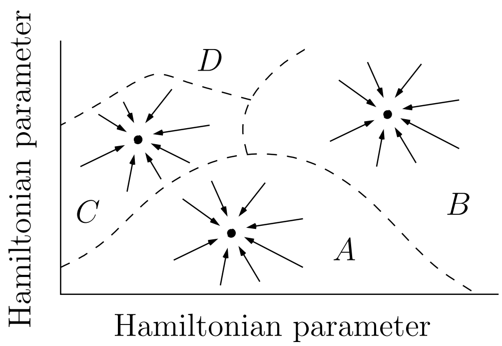

FIG. 2: Cartoon phase diagram

On the other hand, often we do not know the exact Hamiltonian H (nor the true parameters α), or if we do then its complexity is intractable. Fortunately, for a fixed set of parameters α (say in the "A" region in Fig (2)) the Hamiltonian H_α is usually approximated by a much simpler *effective* Hamiltonian $H_{A(\alpha)}$ (equipped with its own adjustable parameters $A(\alpha)$ that are in principle derived from the original parameters α).

If we adjust the original parameters α for H_α slightly, then the effective parameters $A(\alpha)$ adjust slightly.

However, if we make a large adjustment to the parameters α (e.g. α is adjusted into the "B" region) then we cannot expect that the same approximations remain valid. Instead a different set of approximations may be appropriate, yielding a different form for the effective Hamiltonian $H_{B(\alpha)}$.

In this way the family of theories $\{(H_\alpha, \Phi_\alpha)\}_\alpha$ is carved into **phases**. A phase A is a region of parameters α where the Hamiltonians can all can be approximated similarly. Although the effective Hamiltonians still depend on adjustable parameters $A(\alpha)$, the Hamiltonians are of the same *form*.

Assume that we pick a Hamiltonian H_A in the A phase (we no longer refer to the original parameters α). The RG flow procedure involves performing an iterated *scaling out* while averaging the details. From experience we expect that this preserves the form of the Hamiltonian, but may adjust the coupling constants, etc. In other words, scaling is just a particular recipe to flow to a new Hamiltonian $H_{A'}$ that is in the same phase. The new Hamiltonian is simpler because at each scaling iteration we average out the details.

Eventually scaling produces a system that is minimally simplistic, and further scaling does not lead to further simplification. This produces a scale-invariant fixed point theory $(H, \Phi)_{\text{fixed}}$, i.e. a conformal field theory (CFT).

Since Φ_{fixed} is scale independent the long-range properties of any other groundstate Φ_A in the phase are well-approximated by Φ_{fixed}. Hence in the following we concentrate only on Φ_{fixed}, and we drop the "fixed" subscript.

6j-symbols

Since the scale-invariant fixed point is somehow the "simplest" system, we have the best chance of identifying the topological subspace there. On the other hand, Levin-Wen make the ansatz that the topological phase is completely determined by the data from a **modular tensor category**[3].

Combining these ideas, they propose a set of rules at the fixed point that determines how any state Φ in the topological subspace transforms under graphical deformations.

Before describing the rules, it is useful to discuss the origins of the basic data in category theory. Actually, they really only need some of the data, starting with the list of string types $\{0, \ldots, N\}$ described above. In category language the string types are **simple objects** $\{V_0, V_1, \ldots, V_N\}$, and they can be viewed as generalized irreducible group representations. They satisfy fusion rules (generalized Clebsch-Gordan rules)

$$V_i \otimes V_j = \oplus N_{ij}^k \, V_k \quad N_{ij}^k \in \mathbb{Z} \tag{II.1}$$

Order is important in the generalized tensor product that we encounter in category theory (i.e. $V_i \otimes V_j \neq V_j \otimes V_i$). From now on we denote the integers $\{0, \ldots, N\}$ by I.

For simplicity, Levin and Wen restrict themselves to fusion rules satisfying

$$N_{ij}^k = 0 \text{ or } 1 \tag{II.2}$$

in which case the fusion rules are called **branching rules**.

The dual representation $(V_i)^*$ of an irrep is also an irrep, hence we denote it by V_{i^*} with the convention that $i^{**} = i$. We shall see that this harmless-looking notation is at the core of the problem, i.e. in a category this rule is subject to some conditions that are automatically satisfied in ordinary group representation theory.

Given the simple objects and branching rules $\delta_{ijm} := N_{ij}^{m^*} = 0$ or 1, the necessary data is provided by a system of tensors

$$(F_{kln}^{ijm}, d_i) \tag{II.3}$$

satisfying some consistency equations (there may be many solutions)

$$F^{ijm}_{j^*i^*0} = \frac{\sqrt{d_m}}{\sqrt{d_i}\sqrt{d_j}}\delta_{ijm} \tag{II.4}$$
$$F^{ijm}_{kln} = F^{lkm^*}_{jin} = F^{jim}_{lkn^*} = F^{imj}_{k^*nl}\frac{\sqrt{d_m}\sqrt{d_n}}{\sqrt{d_j}\sqrt{d_l}}$$
$$\sum_{n\in I} F^{mlq}_{kp^*n}F^{jip}_{mns^*}F^{js^*n}_{lkr^*} = F^{jip}_{q^*kr^*}F^{riq^*}_{mls^*}$$

The first line is a normalization condition. The second is the tetrahedral symmetry, and the third is the Biedenharn-Elliott identity.

If we know the underlying modular tensor category then we can find these tensors explicitly. $F^{ijm}_{kln} \in \mathbb{C}$ is the **quantum 6j-symbol** and each $d_i \in \mathbb{C}$ is the **quantum dimension** of V_i.[15] These notions are generalizations of the ordinary 6j-symbols and vector space dimensions associated to irreps.

Returning to a wavefunction Φ in the topological subspace, at the fixed point Levin-Wen propose transformation rules as in Fig (3) (this list is not complete). Since any graph can be popped down to nothing using these rules, Φ is determined uniquely (however if we embed the graph in a 2d surface with nontrivial topology then Φ is not unique).

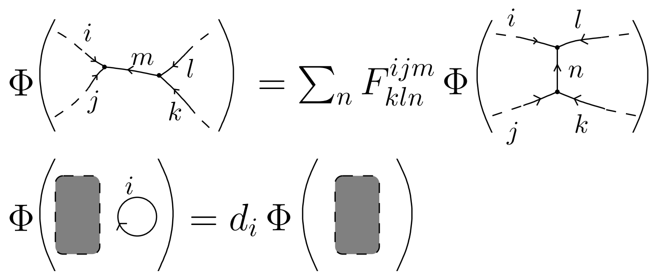

FIG. 3: Conditions to determine fixed-point ground state Φ

III. GROUP CATEGORIES

Group categories (or *pointed* categories) have been studied by a variety of authors in a variety of contexts (e.g.[9,10,11]). We recommend[7] for further details and compatible notation.

Braiding

Ordinary group representation theory is a reasonable prototype to model modular tensor categories.[16] Consider a many-body bosonic or fermionic system in (2+1)-dimensions. Each elementary particle is a copy of an irreducible representation V_i. If we have two particles then we have the tensor product (again ordering is important)

$$V_i \otimes V_j \tag{III.1}$$

(which, of course, can be decomposed using the fusion rules in equation (II.1)). For bosons we have an exchange rule

$$\begin{aligned} \text{Perm}: &V_i\otimes V_j \xrightarrow{\sim} V_j \otimes V_i \\ &v\otimes w \mapsto w\otimes v \qquad v\in V_i, w\in V_j \end{aligned} \tag{III.2}$$

and for fermions we have a different exchange rule

$$\begin{aligned} \text{Perm}: &V_i\otimes V_j \xrightarrow{\sim} V_j \otimes V_i \\ &v\otimes w \mapsto -w\otimes v \qquad v\in V_i, w\in V_j \end{aligned} \tag{III.3}$$

In both cases we have

$$\text{Perm}^2 = I \qquad \text{(the identity matrix)} \tag{III.4}$$

In this way both bosonic and fermionic systems are considered "commutative" since we know what happens when we exchange the particles $V_i \otimes V_j \xrightarrow{\sim} V_j \otimes V_i$ (and two permutations is equivalent to doing nothing). Confusingly, in category language they are both called **symmetric** theories (despite the usual physical nomenclature "antisymmetric" for fermions).

Categories allow a much more interesting *weakened* form of commutativity: **braiding**. Here we have invertible braiding matrices

$$c_{V_i,V_j} : V_i \otimes V_j \xrightarrow{\sim} V_j \otimes V_i \tag{III.5}$$

that do *not* satisfy $c_{V_j,V_i}\circ c_{V_i,V_j} = I$, but rather a more elaborate set of conditions - the **hexagon relations**.

For details concerning the hexagon relations we refer the reader to standard references[3],[12],[13],[11]. It turns out that the hexagon relations come from an obvious picture.

Braiding for group categories

The hexagon relations for group categories are exhaustively solved[7]. In fact the structure is much richer than will be apparent here.

Every group category $\mathscr{C}_{(\mathcal{D},q)}$ is constructed from some fundamental data $(\mathcal{D}, q)$. $\mathcal{D}$ is any finite abelian group, and q is a function on $\mathcal{D}$ that returns a complex phase $\exp(2\pi i q)$. Since q is a phase it is only well-defined up to the interval $[0,1]$. In fact q takes values in $\mathbb{Q}/\mathbb{Z}$. The function q must also be a *pure quadratic form*.

To understand the quadratic form the reader should think about quadratic functions on real numbers $q(x) := \frac{1}{2}x^2$. We can easily define an induced bilinear function: $b(x,y) := q(x+y) - q(x) - q(y) = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x)^2 - \frac{1}{2}(y)^2 = xy$.

Likewise, a pure quadratic form is a function such that the induced function

$$\begin{aligned} &b : \mathcal{D}\otimes\mathcal{D} \to \mathbb{Q}/\mathbb{Z} \\ &b(x,y) := q(x+y) - q(x) - q(y) \pmod 1 \end{aligned} \tag{III.6}$$

is bilinear. The adjective "pure" means

$$q(nx) \equiv n^2 q(x) \pmod 1 \qquad n \in \mathbb{Z} \tag{III.7}$$

Example III.8. The simplest example is when $\mathcal{D} = \mathbb{Z}_2 = \{\hat{0}, \hat{1}\}_+$, i.e. $\hat{1} + \hat{1} = \hat{2} \equiv \hat{0} \pmod 2$ (we use the hat to differentiate elements of the group from numbers in $\mathbb{Q}/\mathbb{Z}$). Then, since

$$2b(\hat{1}, \hat{1}) \equiv b(2\hat{1}, \hat{1}) \equiv b(\hat{2}, \hat{1}) \equiv b(\hat{0}, \hat{1}) \equiv 0 \pmod 1 \tag{III.9}$$

(bilinearity is used in every step, and b is valued in $\mathbb{Q}/\mathbb{Z}$) we have two possibilities for b:

$$b(\hat{1}, \hat{1}) \equiv 0 \pmod 1 \quad \text{or} \quad b(\hat{1}, \hat{1}) \equiv \frac{1}{2} \pmod 1 \tag{III.10}$$

Because of bilinearity the function b is fully determined by its values on the generator $\hat{1}$ of the group $\mathcal{D}$. The same statement also holds for the quadratic form q by using the purity condition in equation (III.7)).

Suppose we consider the first case, i.e. $b(\hat{1}, \hat{1}) \equiv 0 \pmod 1$. Then there are two possible pure quadratic forms that produce this b:

$$q(\hat{1}) \equiv 0 \pmod 1 \quad \text{or} \quad q(\hat{1}) \equiv \frac{1}{2} \pmod 1 \tag{III.11}$$

It turns out that *both* group categories $\mathscr{C}_{(\mathcal{D},q)}$ (defined below) constructed from these two quadratic forms produce the same data (F^{ijm}_{kln}, d_i) and hence are the same (from the Levin-Wen perspective). They both correspond to $\mathbb{Z}_2$-lattice gauge theory.

More interesting examples occur when $b(\hat{1}, \hat{1}) \equiv \frac{1}{2} \pmod 1$. Then there are two possible pure quadratic forms that produce this b:

$$q(\hat{1}) \equiv \frac{1}{4} \pmod 1 \quad \text{or} \quad q(\hat{1}) \equiv \frac{3}{4} \pmod 1 \tag{III.12}$$

The group category $\mathscr{C}_{(\mathcal{D},q)}$ from the LHS corresponds[7] to $U(1)$ Chern-Simons at level 2. The other one corresponds to $U(1)$ Chern-Simons at level -2.

On the other hand, both theories produce the same data (F^{ijm}_{kln}, d_i) and hence (from the Levin-Wen perspective) both produce the same *doubled* Chern-Simons theory. We shall show below that **neither example is unimodal**.

Since this is the first example considered by Levin and Wen[1], we already encounter the example promised in the introduction.

We continue our construction of the group category $\mathscr{C}_{(\mathcal{D},q)}$ given the data $(\mathcal{D}, q)$. The simple objects (string types) and fusion rules (branching rules) are easy to define. The list of string types $\{0, \ldots, N\}$ is replaced by $\mathcal{D}$. Hence we use string labels like $i, j, k \in I$ interchangeably with group elements $x, y, z \in \mathcal{D}$ often here.

For every $x \in \mathcal{D}$ we define a simple object (these were denoted V_i above)

$$\mathbb{C}_x \quad x \in \mathcal{D} \tag{III.13}$$

which is a 1-dimensional complex vector space *graded* by the group element x.

The fusion rules are also easy to define, and indeed satisfy the *branching rule* condition $N^z_{xy} = 0$ or 1:

$$\mathbb{C}_x \otimes \mathbb{C}_y = \delta_{z,x+y}\mathbb{C}_z \tag{III.14}$$

Here δ is the Kronecker delta.

The braiding is slightly more subtle. First, given the simple fusion rules, it is clear that the braiding matrix

$$\begin{aligned} c_{x,y} : &\mathbb{C}_x \otimes \mathbb{C}_y \xrightarrow{\sim} \mathbb{C}_y \otimes \mathbb{C}_x \\ &\mathbb{C}_{x+y} \xrightarrow{\sim} \mathbb{C}_{x+y} \end{aligned} \tag{III.15}$$

must be multiplication by a complex number. For group categories this complex number is a phase:

$$\begin{aligned} c_{x,y} : &\mathbb{C}_{x+y} \xrightarrow{\sim} \mathbb{C}_{x+y} \\ &v_{x+y} \mapsto \exp(2\pi i s(x,y)) v_{x+y} \end{aligned} \tag{III.16}$$

where $v_{x+y} \in \mathbb{C}_{x+y}$ and $s(x,y) \in \mathbb{Q}/\mathbb{Z}$.

It remains to specify the phase angle $s(x,y) \in \mathbb{Q}/\mathbb{Z}$. Every finite abelian group $\mathcal{D}$ can be decomposed (non-uniquely) as a direct product of cyclic groups $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots$. Pick a generator $\hat{1}_s$ for each cyclic group, and denote the order of that cyclic group by n_s. Then an arbitrary element $x \in \mathcal{D}$ can be written uniquely (once generators are picked) as

$$x = \sum_s x_s \hat{1}_s \qquad 0 \leq x_s \leq n_s \tag{III.17}$$

Pick an arbitrary ordering $\hat{1}_1 < \hat{1}_2 < \ldots$ on the various generators. Define for convenience

$$\begin{aligned} q_s &:= q(\hat{1}_s) \\ b_{st} &:= b(\hat{1}_s, \hat{1}_t) \end{aligned} \tag{III.18}$$

(we refer the reader to[7] for discussion concerning how these results change for different choices of generators and orderings). Then if we write arbitrary elements $x, y \in \mathcal{D}$ in terms of the generators

$$\begin{aligned} x &= \sum_s x_s \hat{1}_s \qquad 0 \leq x_s \leq n_s \\ y &= \sum_s y_s \hat{1}_s \qquad 0 \leq y_s \leq n_s \end{aligned} \tag{III.19}$$

then we define the braiding phase angle by

$$s(x,y) := \sum_{s<t} x_s y_t b_{st} + \sum_s x_s y_s q_s \tag{III.20}$$

Example III.21. We revisit example (III.8) in the case

$$b(\hat{1}, \hat{1}) \equiv \frac{1}{2} \pmod 1 \qquad q(\hat{1}) \equiv \frac{1}{4} \pmod 1 \tag{III.22}$$

Then it is easy to compute the braiding matrices (here just phases)

$$c_{\hat{0},\hat{0}} = \exp\left(2\pi i\left(0\cdot 0\cdot\frac{1}{4}\right)\right) = 1 \tag{III.23}$$
$$c_{\hat{0},\hat{1}} = \exp\left(2\pi i\left(0\cdot 1\cdot\frac{1}{4}\right)\right) = 1$$
$$c_{\hat{1},\hat{0}} = \exp\left(2\pi i\left(1\cdot 0\cdot\frac{1}{4}\right)\right) = 1$$
$$c_{\hat{1},\hat{1}} = \exp\left(2\pi i\left(1\cdot 1\cdot\frac{1}{4}\right)\right) = i$$

Twists

For bosons and fermions in $(3+1)$-dimensions we have the *spin-statistics theorem*. This relates the effect of exchanging two particles (the exchange statistics when the Perm operation is applied) to their individual spins.

On the other hand, the spin for an individual particle determines how it is affected under 3-dimensional rotations. Hence we may view the spin-statistics theorem as a relationship between multi-particle Perm operations and single-particle rotation operations.

In $(2+1)$-dimensions we might imagine a similar story, except here the analogue of "spin" must describe what happens to an irrep (simple object) under SO(2) rotations. This is the **twist** matrix

$$\theta_i : V_i \xrightarrow{\sim} V_i \tag{III.24}$$

Furthermore, we have seen that in $(2+1)$-dimensions Perm can be generalized to braiding c_{V_i,V_j}. In categories with *both* braiding and twist, it is reasonable to assume that a compatibility relationship exists analogous to the spin-statistics theorem.

Indeed there is a necessary compatibility between braiding and twists (with an easy geometric interpretation in terms of ribbons, see e.g.[14]) called **balancing**. Balancing gives certain restrictions on twists (much like fermions have half-integer spin).

In the spirit of this paper we merely give a formula for the twist matrices for group categories. Given a simple object $\mathbb{C}_x$ where $x \in \mathcal{D}$ the twist matrix is just multiplication by a phase

$$\theta_x : \mathbb{C}_x \xrightarrow{\sim} \mathbb{C}_x \tag{III.25}$$
$$v_x \mapsto \exp(2\pi i q(x)) v_x \qquad v_x \in \mathbb{C}_x$$

Example III.26. We revisit the same example (III.8) in the case

$$b(\hat{1},\hat{1}) \equiv \frac{1}{2} \pmod 1 \qquad q(\hat{1}) \equiv \frac{1}{4} \pmod 1 \tag{III.27}$$

Then it is easy to compute the twist matrices (here just phases)

$$\theta_{\hat{0}} = \exp(2\pi i(0)) = 1 \tag{III.28}$$
$$\theta_{\hat{1}} = \exp\left(2\pi i\left(\frac{1}{4}\right)\right) = i$$

Duality and ribbon categories

Duality is a fundamental property of representation theory. Given an irreducible representation V_i we can always consider the dual representation $(V_i)^*$, which we dangerously denoted V_{i^*} above.

In category theory **rigidity** is the appropriate generalization, however the details can be rather different than those in ordinary group representation theory. For now, each simple object (string type) V_i has another simple object associated to it: its *right* dual $(V_i)^*$. Furthermore, we have a categorical notion of pair creation and annihilation. These are the **birth** and **death** matrices

$$b_{V_i} : \mathbb{1} \to V_i \otimes (V_i)^* \tag{III.29}$$
$$d_{V_i} : (V_i)^* \otimes V_i \to \mathbb{1}$$

($\mathbb{1}$ is interpreted as the vacuum).

Notice that the ordering of the objects is exactly opposite for the birth and death matrices, hence we *cannot* simply birth a pair and then annihilate it without performing some intermediate moves (such as braiding and twisting).[17]

In a category with braiding, twists, *and* rigidity we may desire some compatibility between all three structures (we already mentioned the "balancing" condition between braiding and twists for example). Again in the spirit of this paper, rather than discuss this point further we mention that a **ribbon category** is a category with braiding, twists, and rigidity such that all three structures interact appropriately.

Even for group categories rigidity is slightly subtle (again see[7]). However, for this paper it suffices to merely define when objects are dual to each other. The reader can guess that each simple object $\mathbb{C}_x$ has a right dual

$$\mathbb{C}_{-x} \tag{III.30}$$

IV. UNIMODALITY

We have specified enough information about $\mathscr{C}_{(\mathcal{D},q)}$ in order to decide when a given group category is unimodal (we provide a classification theorem below). Before doing this, however, let us motivate why unimodality is important.

In the Levin-Wen model each *directed* edge is labelled by a string type i (simple object V_i). Because of the dangerous identification $(V_i)^* = V_{i^*}$ we can always use simplified labels such as i or j or m^*, and we may always perform a "string-breaking" move such as in figure (4).

On the other hand, in Turaev's formulation[3] there are non-trivial invertible matrices

$$w_{i^*} : V_{i^*} \xrightarrow{\sim} (V_i)^* \tag{IV.1}$$

that form part of the defining structure of the category. They also provide a "band-breaking" maneuver

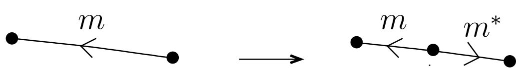

FIG. 4: String-breaking maneuver is implicit in Levin-Wen model, but non-trivial in Turaev's construction.

as in figure (5), however the procedure may not be self-consistent. It turns out that an inconsistency may arise when simple objects are self-dual (i.e. when $V_i = V_{i^*}$), and hence we must enforce the extra unimodality condition.

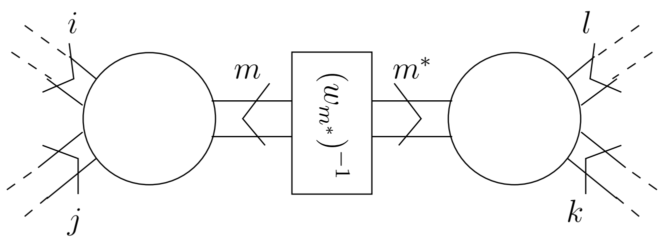

FIG. 5: Band-breaking maneuver in Turaev's construction.

Rather than go into details, we give the unimodality condition for group categories: suppose $\mathbb{C}_x$ is self-dual

$$\mathbb{C}_x = \mathbb{C}_{-x} \tag{IV.2}$$

(in other words $x = -x \in \mathcal{D}$). Then unimodality is a condition on the twist and braiding:

$$\theta_x \cdot c_{x,x} \overset{?}{=} 1 \qquad \text{whenever } x = -x \tag{IV.3}$$

If this is true for all $x = -x$ then the category is **unimodal**.

Counterexample

Again we revisit the case $\mathcal{D} = \mathbb{Z}_2$ in example (III.8) (however *not* $\mathbb{Z}_2$-lattice gauge theory). Given the braiding and twist tables provided in equations (III.23) and (III.28) we compute that for the self-dual object $\hat{1}$

$$\theta_{\hat{1}} \cdot c_{\hat{1},\hat{1}} = i \cdot i = -1 \tag{IV.4}$$

This example is **not unimodal** and Turaev's machinery does not apply. However, as already discussed this is $U(1)$ Chern-Simons at level 2 and is the first example considered by Levin-Wen[1].

Alternatively, we can suspect a discrepancy by examining the quantum dimension $d_{\hat{1}}$ computed in[1]. There they assert that

$$d_{\hat{1}} = -1 \tag{IV.5}$$

On the other hand, it is proven in [7] that for group categories $d_x = 1$ for every simple object $\mathbb{C}_x$.[18]

V. UNIMODALITY AND GROUP CATEGORIES

It is straightforward to classify conditions when a group category is unimodal:

Theorem V.1. *Let $\mathscr{C}_{(\mathcal{D},q)}$ be a group category. Arbitrarily decompose the finite abelian group $\mathcal{D}$ into cyclic groups of orders n_s and generators $\hat{1}_s$ (s indexes the cyclic factors). Pick an arbitrary ordering $\hat{1}_1 < \hat{1}_2 < \ldots$ for the generators. Then we have the following cases:*

1. *If $\mathcal{D}$ contains a cyclic factor of odd order (i.e. if one of the n_s is odd) then $\mathscr{C}_{(\mathcal{D},q)}$ is unimodal.*
2. *If $\mathcal{D}$ contains no odd-order cyclic factors then $\mathscr{C}_{(\mathcal{D},q)}$ is unimodal if and only if*

$$\sum_s \frac{1}{2}(n_s)^2 q_s \in \mathbb{Z} \tag{V.2}$$

Proof. Suppose $x = -x$. Decompose x and $-x$ using the generators

$$x = \sum_s x_s \hat{1}_s \qquad 0 \le x_s \le n_s \tag{V.3}$$
$$-x = \sum_s (n_s - x_s)\hat{1}_s$$

The condition that $x = -x$ implies

$$n_s - x_s = x_s \quad \forall s \tag{V.4}$$

which implies that $n_s = 2x_s$. In particular this implies that every n_s must be even, hence if there exists any n_s odd then $x \neq -x$ for every $x \in \mathcal{D}$.

Now suppose that every n_s is even. Then the braiding and twist formulas imply

$$\theta_x \cdot c_{x,x} = \exp(2\pi i q(x)) \exp(2\pi i s(x,x)) \tag{V.5}$$

Equation (III.20) says

$$s(x,x) := \sum_{s<t} x_s x_t b_{st} + \sum_s (x_s)^2 q_s \tag{V.6}$$

The quadratic form $q(x) = q\left(\sum_s x_s \hat{1}_s\right)$ can be decomposed using successive iterations of the defining formula

$$q(x+y) - q(x) - q(y) = b(x,y) \tag{V.7}$$

rearranged as $q(x+y) = b(x,y) + q(x) + q(y)$. For example the first iteration is

$$q\left(\sum_s x_s \hat{1}_s\right) = q\left(x_1\hat{1}_1 + \sum_{s>1} x_s \hat{1}_s\right) \tag{V.8}$$
$$= b\left(x_1\hat{1}_1, \sum_{s>1} x_s\hat{1}_s\right) + q(x_1\hat{1}_1) \tag{V.9}$$
$$+ q\left(\sum_{s>1} x_s \hat{1}_s\right)$$
$$= \sum_{s>1} x_1 x_s b_{1s} + (x_1)^2 q_1 + q\left(\sum_{s>1} x_s \hat{1}_s\right)$$

We used bilinearity of b and the fact that q is pure in the last equality. Now, concentrating on the last term, we repeat the process. Iterating we finally obtain

$$q(x) = \sum_{s<t} x_s x_t b_{st} + \sum_s (x_s)^2 q_s \tag{V.10}$$

which is precisely the formula for $s(x,x)$.

Hence

$$\theta_x \cdot c_{x,x} = \exp(2\pi i(q(x) + s(x,x)) = \exp(2\pi i 2q(x)) \tag{V.11}$$

Recall that we are only considering $x \in \mathcal{D}$ such that $x = -x$. Also recall from that beginning of the proof that then $n_s = 2x_s$. Hence we substitute $x_s = \frac{1}{2}n_s$ into the expression for $2q(x)$ and obtain

$$\begin{aligned} 2q(x) &= \sum_{s<t} 2\frac{n_s}{2} x_t b_{st} + \sum_s 2(\frac{n_s}{2})^2 q_s \qquad \text{(V.12)} \\ &= \sum_{s<t} x_t n_s b_{st} + \sum_s \frac{1}{2}(n_s)^2 q_s \end{aligned}$$

But $x_t n_s b_{st} = x_t b(n_s \hat{1}_s, \hat{1}_t) = x_t b(0, \hat{1}_t) \equiv 0 \pmod 1$, hence the first sum is always an integer and does not play a role in the exponent.

Viewing the second factor we arrive at the desired result, i.e.

$$\theta_x \cdot c_{x,x} = \exp(2\pi i 2q(x)) = 1 \Leftrightarrow \sum_s \frac{1}{2}(n_s)^2 q_s \in \mathbb{Z} \tag{V.13}$$

□

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[15] More appropriately these should be called "braided 6j-symbol" and "braided dimensions" since ordinary group representation theory is also "quantum", i.e. bosons and fermions.
[16] Actually we neither require nor desire the full structure of modular tensor categories, hence instead we can consider the simpler structure of **semisimple ribbon Ab-categories**. It happens that ordinary group representation theory is a closer prototype for semisimple ribbon categories anyway.
[17] This is how the **quantum dimension** d_i of a simple object V_i is computed, for example.
[18] Our result also agrees with the quantum dimension computed in the quantum group $\mathrm{SU}_q(2)$ at level 1. This is relevant since rank-level duality asserts that $\mathrm{SU}_q(n)$ at level k is isomorphic to $\mathrm{SU}_q(k)$ at level n whenever $n, k > 1$. If $n = 1$ (or $k = 1$) then rank-level duality is slightly modified: U(1) Chern-Simons at level k is isomorphic to $\mathrm{SU}_q(k)$ at level 1.