Longitudinal and transverse velocity scaling exponents from merging of the Vortex filament and Multifractal models.

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We suggest a simple explanation of the difference between transverse and longitudinal scaling exponents observed in experiments and simulations. Based on the Vortex filament model and Multifractal conjecture, we calculate both scaling exponents for any n without any fitting parameters and ESS anzatz. The results are in very good agreement with the data of simulations.

Numerous direct numerical simulations (DNS) of hydrodynamical turbulent flow, as well as high-resolution experiments, have been performed during last twenty years [1]. The positive result of these investigations is that numerical and experimental approaches demonstrate a very good agreement. In particular, velocity statistics calculated from DNS practically coincides with that obtained from experiments [1]. This justifies the accuracy of both kinds of results. But there still is a lack of physical understanding of the processes occuring in a turbulent flow and contributing to statistics. Which of the infinite number of solutions to the NSE are responsible for the observed intermittent behavior, and why?

Okamoto et al. [2] claimed that the most part of the flow's helicity and, in some sense, the most part of information about the flow are contained in the same small part of the liquid volume spread more or less uniformly throughout the liquid. These structures are stable, their lifetime exceeds many times the largest eddy turnover time in the flow.

This seems us to be a very important step. Probably, these elongated regions of high vorticity are just the structures that determine the small-scale statistics of a turbulent flow. Thus, it would be useful to elaborate some physical understanding of these objects.

Such an attempt was performed in [3–10]. The main idea is that in the regions where vorticity is high (vortex filaments), the vorticity itself stabilizes the motion. So, despite the stochastic large-scale forces that give energy to a filament, the motion inside the filament has an essential non-stochastic component. This may be the reason of stability of filaments. The random large-scale forces then act, on average, as a stretching force. Stretching (not breaking!) of vortices is the mechanism that provides the observed energy flux from larger to smaller scales, and observed statistical properties of turbulence. This is the main difference of our approach from the Kolmogorov's approach developed in the K41 theory [11].

The relation between longitudinal and transversal Euler velocity structure functions is one of the problems that have had no answer up to now. These functions are

defined by

$$S_n^\parallel(l) = \left\langle \left| \Delta \mathbf{v} \cdot \frac{\mathbf{l}}{l} \right|^n \right\rangle , \quad S_n^\perp(l) = \left\langle \left| \Delta \mathbf{v} \times \frac{\mathbf{l}}{l} \right|^n \right\rangle$$

Here $\Delta \mathbf{v} = \mathbf{v}(\mathbf{r} + \mathbf{l}) - \mathbf{v}(\mathbf{r})$ is velocity difference between two near points separated by $\mathbf{l}$, and the average is taken over all pairs of points separated by given l.

Experiments and DNS show that inside the inertial range the structure functions obey scaling laws,

$$S_n^\perp(l) \propto l^{\zeta_n^\perp} , \quad S_n^\parallel(l) \propto l^{\zeta_n^\parallel} .$$

The scaling exponents $\zeta_n^\perp$, $\zeta_n^\parallel$ are believed to be independent on conditions of the experiment. They are intermittent, i.e., $\zeta_n^\perp/n$ and $\zeta_n^\parallel/n$ are decreasing functions of n.

The question whether the two scaling exponents coincide in isotropic and homogeneous turbulence or not, is open. There is an exact statement that $\zeta_n^\parallel = \zeta_n^\perp$ for $n = 2$ and 3 [11]. On the other hand, modern experiments [12, 13] and numerical simulations [14–17] show some significant difference between $\zeta_n^\parallel$ and $\zeta_n^\perp$ at higher n. But the proponents of the equality argue that the difference may result from finite Reynolds number effects [18–20] or anisotropy [21].

The aim of the paper is to analyze the question by means of the Vortex filament model. We show that the divergence of the scaling exponents can be understood and calculated from general physical considerations using the tools provided by the Vortex filament model and the well-known Multifractal (MF) conjecture.

Before we consider the problem from the viewpoint of the Vortex filament model, we note that isotropic and homogeneous medium does not necessarily imply $\zeta_n^\perp = \zeta_n^\parallel$. For a simple illustration, consider a gas of tops, each rotating around its own axis, the axes directions distributed randomly. Although locally a strong asymmetry could be found near each top, the whole picture remains isotropic. The Vortex filament model provides analogous situation that might be in real turbulence.

In [10] we suggested to join our model of vortex filaments with the MF model to get velocity scaling exponents. We now remind briefly some ideas of the two models and the results of [10]. Then we proceed to the

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difference between the longitudinal and transverse scaling exponents.

The MF model implies that the determinative contribution to velocity structure functions is given by the solutions (regions) where

$$\Delta v(l) = |\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})| \sim l^{h(\mathbf{r})}$$

So, to calculate structure functions, it is enough to consider only a set of scaling solutions that can be numbered by h. This property is called local scale invariance.

The Large Fluctuations Theorem states that the probability of measured velocity difference $\Delta v(l)$ to have the scaling h is a power-law function of l:

$$\mathsf{P} \propto l^{3-D(h)}$$

Knowing $D(h)$, one could in principle calculate all structure functions:

$$\langle \Delta v^n \rangle = \int \delta v^n \mathsf{P} dh = \int l^{nh} l^{3-D(h)} d\mu(h)$$

Here $d\mu(h)$ is the measure that is responsible for relative weights of different values h. In the limit $l \to 0$, only the smallest exponent contributes to the integral; it then follows

$$\zeta_n = \min_h \left(nh + 3 - D(h) \right) \tag{1}$$

Without loss of generality, one can treat $D(h)$ as a concave function.

As follows from (1), the point h_0 where $D(h)$ reaches its maximum corresponds to $n=0$: the requirement $\zeta_0 = 0$ leads to $D(h_0) = 3$.

The definitional domain of D is restricted by some h_{min} and h_{max}. Since we are interested in positive values n, we will hereafter restrict ourselves by $h \leq h_0$. What about h_{min}, from the condition [11] $\zeta_n'' < 0$ and the finiteness of the Mach number (or from the absence of singularities of Δv as $l \to 0$) it follows $\zeta_n' \geq 0$ for any n. The minimum in (1) is reached at $h_n = \zeta_n'$. Since negative values of the derivative are forbidden, we get $h_{min} \geq 0$. The behavior of $D(h)$ near h_{min} determines the behavior of ζ_n at

$$n \simeq n_* = D'(h_{min}) .$$

For larger n, the minimum in (1) is reached at the boundary $h = h_{min}$. Thus, the behavior of ζ_n at very large n depends only on the value of h_{min}: the asymptotic behavior of ζ_n for all non-zero h_{min} is

$$\zeta_n|_{n>n_*} = nh_{min} + 3 - D(h_{min}) \sim_{n \to \infty} nh_{min} \tag{2}$$

If $h_{min} = 0$, the possible growth of ζ_n at $n > n_*$ is steeper than linear; if $D(h_{min})$ is finite, we get $\zeta_n|_{n>n_*} = const.$

We now proceed to the Vortex filament model. In [3] we derived the equation

$$\ddot{\omega}_i = \rho_{ij}\omega_j , \quad \rho_{ij} = -\nabla_i \nabla_j p \tag{3}$$

Here $\boldsymbol{\omega}$ is the vorticity of the flow, $\boldsymbol{\omega} = \nabla \times \mathbf{v}$; . ρ_{ij} is the pressure hessian. This equation is the direct consequence of the inviscid limit of the NSE and describes the evolution of vorticity along the trajectory of a liquid particle. It was for the first time obtained in [22]. The main assumption of the theory is that inside vortex filaments the 'longitudinal' (in relation to $\boldsymbol{\omega}$) part of the hessian doesn't depend on the local vorticity, and is determined by the large-scale component of a flow.[1] This makes (3) a linear stochastic equation in relation to $\boldsymbol{\omega}$. In [3] we wrote the corresponding equation for probability density function (PDF), $f(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}, t)$. Solving the equation and integrating over all angles and over $\dot{\omega}$, we found the intermediate asymptotic solution for the PDF at $\omega \to \infty$:

$$P(\omega) \propto 1/\omega^4$$

In [10] we used it to derive the condition

$$\min_h (3h + 2 - D(h)) = 0 \tag{4}$$

for the function $D(h)$. As follows from (1), this condition is just equivalent to

$$\zeta_3 = 1 \tag{5}$$

which is the well-known Kolmogorov's '4/5 law'.

The MF model is a dimensional theory, so it does not distinguish longitudinal and transverse scaling exponents. Also the condition (4) is valid for both of them. To describe the difference, it is natural to use two different functions $D(h)$ for $\zeta_n^\perp$ and $\zeta_n^\parallel$ [17]. But what is the relation between them, why does the difference appear?

In [4, 5] we introduced a simple model of an axially symmetric vortex filament:

$$v_r = a(t)r , \quad v_z = b(t)z , \quad v_\phi = \omega(t)r \tag{6}$$

From the incompressibility condition and the Euler equation it then follows

$$\begin{aligned} 2a + b &= 0 , \quad \dot{a} + a^2 - \omega^2 = -P_1(t) , \\ \dot{\omega} + 2a\omega &= 0 , \quad \dot{b} + b^2 = -P_2(t) , \\ p &= \tfrac{1}{2}P_1(t)r^2 + \tfrac{1}{2}P_2(t)z^2 \end{aligned} \tag{7}$$

Taking the second derivative of ω, we get

$$\ddot{\omega} = -P_2(t)\omega ,$$

[1] An indirect confirmation of this statement is presented in [23], where the maximal eigenvalue of the pressure hessian is shown to be orthogonal to vorticity.

which corresponds to (3). In accordance with the assumption discussed above, let $P_2(t)$ be a random function independent of ω. Then all moments of $\omega(t)$ increase exponentially as a function of time, $a \sim \dot\omega/\omega \ll \omega$ for large t.

It is easy to calculate that in this model $S_n^\perp \propto \langle \omega^n \rangle l^n$, $S_n^\parallel \propto \langle a^n \rangle l^n$. In [8] we discussed the connection between t and l and showed that $t \propto -\ln l$. Hence, in this model $S_n^\perp \gg S_n^\parallel$, $\zeta_n^\perp < \zeta_n^\parallel$.

The equality of scaling exponents can be restored if we take the other branch of the same filament into account, assuming that the filament is closed and 'turns back' at very large $|z|$. Then there are two filaments with equal and opposite vorticities. The second branch produces a perturbation of velocity $\delta v \sim \alpha\omega$, where the coefficient α is inversely proportional to the distance between the two branches. (This is just analogous to an axial electric current producing magnetic field in vacuum; one can write the corresponding solution to the Euler equation by developing the perturbation as series in α.) The perturbation violates the axial symmetry, so the scaling exponents become equal. However, the pre-exponent of $S_n^\parallel$ is much smaller than that of $S_n^\perp$ if α is small. The value of α increases as we approach the turnover point, and the two branches become nearer. This gives us a hint that the regions where a filament is strongly curved may make a small contribution to $S_n^\perp$ but contribute much to $S_n^\parallel$. This effect is stronger for large n, since $S_n^\parallel \sim \alpha^n S_n^\perp$. So, for infinitely large n we can expect that transverse structure functions are dominated by extremely stretched-out and very thin, roughly axially symmetric filaments with very high vorticities, while for longitudinal structure functions one needs extremely curved parts of these filaments.

The discussed example is one of numerous solutions to the Euler equation that are presented in a turbulent flow. The observed exponents $\zeta_n^\perp$ and $\zeta_n^\parallel$ are produced by contributions of many filaments. Different filaments (and different parts of them) make contributions to different n (or, in terms of MF theory, to different h). This picture is very complicated, but, thanks to the MF model, we don't need to know it in details. Approximating $D(h)$ by a second-order polynomial [10] and making use of (4), we only have to fix one more point of the curve.

For this purpose, we consider the minimal possible $h = h_{\min}$. This corresponds to $n \to \infty$. In [10] we have shown that $h_{\min} = 0$: values $h < 0$ are forbidden by the condition $d\zeta_n/dn \geq 0$ (see p.2).

The existence of $h = 0$ is proved by the possibility of a cylindric filament with rotating velocity independent of r:

$$\mathbf{v} = [\mathbf{e}_z, \mathbf{r}/r] \tag{8}$$

One can check that in this extreme case, indeed, $\delta v(l) \sim l^0$. The velocity (8) satisfies the Euler equation with

pressure logarithmically divergent; this means that for any positive h, there exists a corresponding velocity distribution with converging pressure.

Calculation of the correlators directly from (8) gives under the limit $n \to \infty$:

$$\langle |\Delta v \times \mathbf{1}/l|^n \rangle \propto \frac{2^n}{n} l^2 , \quad \langle |\Delta v \cdot \mathbf{1}/l|^n \rangle \propto n^{-5/2} l^2 \tag{9}$$

The proportionality to l^2 is caused by the axial geometry of the filament (integrating $r\,dr$). This corresponds to the definition of D given in, e.g., [11]: the probability that at least one of a pair of points would get inside the filament is proportional to l^2. Thus, we get

$$D^\perp(0) = 1 \tag{10}$$

To find the difference between $S_n^\perp$ and $S_n^\parallel$, we now remind that, at infinitely large n, $S_n^\perp$ are contributed mostly by 'cylindric' parts of filaments, while $S_n^\parallel$ are dominated by point-like regions where filaments are bent very strongly. Actually, the behavior of the pre-exponents in (9) shows that the contribution of the 'extreme' filament to $S_n^\perp$ increases as $n \to \infty$. To the opposite, its contribution to $S_n^\parallel$ for large n (i.e., small h) is very small. There must be other solutions to the Euler equation to determine the behavior of $D^\parallel(h)$ for small h and, equivalently, to make the most contribution to $S_n^\parallel$ for large n. These solutions correspond to 'strongly curved' extreme filament. To satisfy $h = 0$, velocity must be independent of r. It may, for example, take the form $\mathbf{v} = (v_r(\theta), v_\theta(\theta), 0)$. Such a solution exists but it cannot be written analytically. At $\theta = 0$, the radial velocity diverges weakly: $v_\theta \propto -\theta\sqrt{\ln(1/\theta)}, v_r \propto \sqrt{\ln(1/\theta)}$; the pressure divergence is $p \propto \ln r$, just as in (8). Since $\delta v \sim l^0$ in the case, and averaging includes $r^2 dr$, the correlator is proportional to l^3. This corresponds to

$$D^\parallel(0) = 0 \tag{11}$$

The difference between the boundary values (10) and (11) determines the difference between the functions $D_\perp$ and $D^\parallel$.

We seek the solution $D(h)$ in the simplest form

$$D(h) = 3 - b(h - h_0)^2$$

and use (4) and (10),(11) to find the two unknown parameters. We obtain two equations:

$$h_0^\perp - \tfrac{3}{8}(h_0^\perp)^2 = \tfrac{1}{3} , \quad b^\perp = 2/(h_0^\perp)^2 \quad \text{for } D^\perp$$
$$h_0^\parallel - \tfrac{1}{4}(h_0^\parallel)^2 = \tfrac{1}{3} , \quad b^\parallel = 3/(h_0^\parallel)^2 \quad \text{for } D^\parallel$$

Each of the equations has two roots. The bigger roots are non-physical, because in this case the curve ζ_n would be constant already at $n \geq n_* = D_h'(0) = 2bh_0 \sim 2$, and (4) would not hold. For the second roots, in accordance with (1), we have

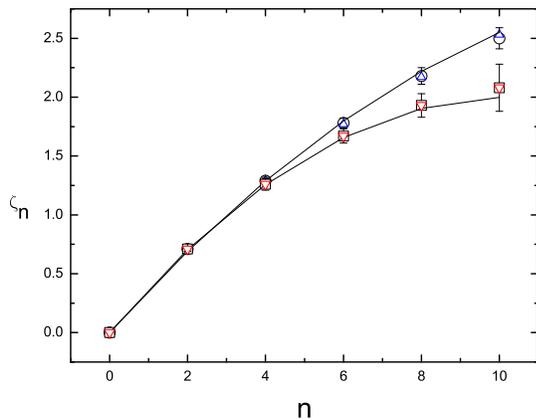$$\zeta_n = nh_0 - n^2/4b , \qquad n \leq n_* = 2bh_0 \tag{12}$$

FIG. 1: Euler longitudinal (upper branch) and transverse (lower branch) scaling exponents: the results of DNS from [16] ($\bigcirc$, $\square$) and [15] ($\triangle$. $\bigtriangledown$), and the prediction of the theory (13) (lines).

Substituting the values of h_0, b, we get the scaling exponents:

$$\begin{aligned}
\zeta_n^{\parallel} &= \begin{cases} 0.367n - 1.12 \cdot 10^{-2} n^2 \,, & n \leq 16.3 \,; \\ 3 \,, & n > 16.3 \,. \end{cases} \\
\zeta_n^{\perp} &= \begin{cases} 0.391n - 1.91 \cdot 10^{-2} n^2 \,, & n \leq 10.2 \,; \\ 2 \,, & n > 10.2 \,. \end{cases}
\end{aligned} \quad (13)$$

In Fig. 1 we compare this theoretical prediction with experimental data by [16] and [15]. We see that the result of our simple model is very close to the experimental results and lies inside the error bars of the experiments.

Our simple model has one difficulty: the two parabolas plotted in Fig. 1 coincide at $n = 0$ and $n = 3$, hence they do not coincide at $n = 2$. This contradicts to the exact theoretical statement that $\zeta_2 = \chi_2$. However, this difficulty is caused by postulating the simplest parabolic shape for $D(h)$. It can easily be solved by adding one more parameter and assuming $D(h)$ to be a cubic polynomial. Because of very small divergence ($1.6 \cdot 10^{-2}$) between ζ_2 and χ_2 in Fig. 1, the coefficient by the eldest order would be very small ($\sim 10^{-4}$). It would change very slightly (unnoticeable for an eye) the lines presented in Fig. 1. The only thing they may change significantly is the rate of approaching the constant at large (but still intermediate) n (from 10 to 15, approximately). But this range of n is, anyway, badly described by the lowest-order polynomials: adding more degrees of freedom with very small coefficients, though unimportant for smaller n, would change the solutions for these n. However, the changes cannot be very big, since the exponents are still restricted by the values 2 and 3, respectively.

One more comment is that, knowind $D(h)$, one can use the MF model to calculate, e.g., the PDF of velocity gradients or accelerations. In [1] it is shown that once $D(h)$ fits ζ_n well, it would also fit well the other quantities.

Thus, we propose an explanation of the difference between $\zeta_n^{\parallel}$ and $\zeta_n^{\perp}$ based on the difference between the filaments that contribute to the two structure functions: roughly speaking, this is the difference between axially symmetric and strongly curved ones. This allows to find the values of ζ_n for very large n, and merging of the Vortex filament and Multifractal theories gives the whole functions. The obtained solutions (12)-(13) fit very well the observed scaling exponents $\zeta_n^{\parallel}$ and $\zeta_n^{\perp}$, and we hope that they reveal the nature of the difference between longitudinal and transverse structure functions.

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