Minimality in CR geometry and the CR Yamabe problem on CR manifolds with boundary

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Abstract. We study the minimality of an isometric immersion of a Riemannian manifold into a strictly pseudoconvex CR manifold M endowed with the Webster metric (associated to a fixed contact form on M), hence formulate a version of the CR Yamabe problem for CR manifolds-with-boundary. This is shown to be a nonlinear subelliptic problem of variational origin.

1. Introduction

Minimal surfaces N^2 in the lowest dimensional Heisenberg group $\mathbb{H}_1$, or more generally in a 3-dimensional nondegenerate CR manifold, have been recently considered by a number of people (cf. N. Arcozzi & F. Ferrari, [1], I. Birindelli & E. Lanconelli, [6], J-H. Cheng et alt., [7], N. Garofalo & S.D. Pauls, [14], and S.D. Pauls, [23]) motivated by the interest in a Heisenberg version of the Bernstein problem, or by anticipating an appropriate formulation of the CR Yamabe problem on a CR manifold-with-boundary and a CR analog to the positive mass theorem. All the notions of minimality dealt with are but ordinary minimality of N^2 with respect to the ambient Webster metric. This is demonstrated by our Theorem 5 (though confined to the case where the characteristic direction $T = \partial/\partial t$ of $\mathbb{H}_1$ is tangent to N^2). We also study minimality of a given isometric immersion $\Psi : N^m \to \mathbb{H}_n$ of a m-dimensional Riemannian manifold (N^m, g) into $(\mathbb{H}_n, g_{\theta_0})$ (the Heisenberg group carrying the Webster metric g_{θ_0} associated with the contact form $\theta_0 = dt + i\sum_{j=1}^n (z_j d\overline{z}^j - \overline{z}_j dz^j))$, cf. our Theorem 4. A first step towards a Weierstrass type representation of minimal surfaces in $\mathbb{H}_n$ is taken in Theorem 7.

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The Yamabe problem on a compact n-dimensional ($n \geq 3$) Riemannian manifold (M, g) with boundary ∂M is to deform conformally the given metric $\hat{g} = u^{4/(n-2)} g$ ($u > 0$) such that $(M, \hat{g})$ has constant scalar curvature and ∂M is minimal in $(M, \hat{g})$. This is equivalent to solving the boundary value problem

$$\Delta u - \frac{n-2}{4(n-1)} \rho_g u + C u^{(n+2)/(n-2)} = 0 \ \text{ in } \ M, \tag{1}$$

$$\frac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u = 0 \ \text{ on } \ \partial M, \tag{2}$$

where Δ and ρ_g are respectively the Laplace-Beltrami operator and the scalar curvature of (M, g), h_g is the mean curvature of $\partial M \hookrightarrow (M, g)$, and η is a unit outward normal on ∂M with respect to g. When M is closed ($\partial M = \emptyset$) the full solution to (1) is described in [19]. When $\partial M \neq \emptyset$ the problem (1)-(2) was solved by J.F. Escobar, [10], under the assumptions that 1) $n \in \{3, 4, 5\}$, or 2) $n \geq 3$ and ∂M has some nonumbilic point, or 3) $n \geq 6$, ∂M is totally umbilical, and either M is locally conformally flat or the Weyl tensor doesn't vanish identically on ∂M. A CR analog of the Yamabe problem was formulated by D. Jerison & J.M. Lee, [15], though only on closed CR manifolds. Precisely, if M is a $(2n+1)$-dimensional closed strictly pseudoconvex CR manifold on which a contact form θ has been fixed then the CR Yamabe problem is to look for a contact form $\hat{\theta} = u^{p-2}\theta$ ($p = 2 + 2/n$) such that the Tanaka-Webster connection of $(M, \hat{\theta})$ has constant pseudohermitian scalar curvature $\hat{\rho} = \lambda$. This is equivalent to solving

$$-(2 + 2/n)\, \Delta_b u + \rho\, u = \lambda\, u^{p-1} \tag{3}$$

(the *CR Yamabe equation*) where Δ_b and ρ are respectively the sublaplacian[1] and the pseudohermitian scalar curvature of (M, θ). D. Jerison & J.M. Lee solved (cf. [16]-[17]) the problem (3) under the assumption that[2] $\lambda(M) < \lambda(S^{2n+1})$, where $\lambda(M)$ is the CR invariant

$$\inf\{\int_M (b_n \|\pi_H \nabla u\|^2 + \rho u^2) \theta \wedge (d\theta)^n : \int_M |u|^p \theta \wedge (d\theta)^n = 1\}.$$

Moreover, the inequality $\lambda(M) \leq \lambda(S^{2n+1})$ holds true. The remaining case $\lambda(M) = \lambda(S^{2n+1})$ was settled by N. Gamara & R. Yacoub, [12]. It is noteworthy that the proof in [12] doesn't rely on a CR analog to the positive mass theorem, but rather on techniques within the theory of

[1] As to the sign convention the sublaplacian in [16] is $-\Delta_b$.

[2] If $n \geq 2$ and M is not locally CR equivalent to S^{2n+1} then $\lambda(M) < \lambda(S^{2n+1})$, cf. [16].

critical points at infinity (by analogy with A. Bahri & H. Brezis, [2]). When $\partial M \neq \emptyset$ no formulation of the CR Yamabe problem is available as yet, perhaps due to the previous lack of a natural CR analog to minimality.

Our approach (as well as in [16]) is to formulate the CR Yamabe problem as the Yamabe problem for the Fefferman metric F_θ, a Lorentz metric on the total space $C(M)$ of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$ (cf. [18]). That is, to look for a positive function $u \in C^\infty(M)$ such that the Fefferman metric $F_{\hat{\theta}}$ corresponding to the contact form $\hat{\theta} = u^{p-2}\theta$ has constant scalar curvature. What is the appropriate boundary condition?

When ∂M is nonempty $C(M)$ is a manifold-with-boundary as well, and (by Theorem 1) the tangent space $T_z(\partial C(M))$ is nondegenerate in $(T_z(C(M)), F_{\theta,z})$ at all points z, except for those projecting on $\mathrm{Sing}(T^T)$, the singular points of the tangential component (with respect to ∂M) of the characteristic direction T of $d\theta$. It also turns out that $\partial C(M) \setminus \pi^{-1}(\mathrm{Sing}(T^T))$ is a Lorentz manifold (with the metric induced by F_θ). Therefore, when $\mathrm{Sing}(T^T) = \emptyset$ we may request that $\partial C(M)$ be minimal in $(C(M), F_{\hat{\theta}})$. By Theorem 2 this projects to the natural boundary condition (46) on ∂M, thus leading to the *CR Yamabe problem* (45)-(46) on a CR manifold-with-boundary. This is shown (cf. Theorem 6) to be a nonlinear subelliptic problem of variational origin.

Acknowledgements. The Author is grateful to E. Lanconelli for stimulating conversations on the arguments in this paper and for introducing him to the results in the preprint [6]. Also, the Author wishes to express his gratitude for the hospitality and excellent working atmosphere in the Department of Mathematics of the University of Bologna and for discussions with N. Arcozzi and F. Ferrari (who kindly provided the preprint [1]).

2. CR manifolds with boundary

Let M be an oriented m-dimensional C^∞ manifold-with-boundary ∂M. A *CR structure* is a complex subbundle $T_{1,0}(M)$ of the complexified tangent bundle $T(M) \otimes \mathbb{C}$, of complex rank n $(0 < n \leq [m/2])$, such that

$$T_{1,0}(M) \cap T_{0,1}(M) = (0),$$

$$Z, W \in \Gamma^\infty(T_{1,0}(M)) \Longrightarrow [Z, W] \in \Gamma^\infty(T_{1,0}(M)).$$

Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (complex conjugation). The pair $(M, T_{1,0}(M))$ is a *CR manifold* (with boundary) and the integer n is its *CR dimension*. Also $k = m - 2n$ is its *CR codimension* and the pair (n, k) is its *type*.

There is a natural first order differential operator $\overline{\partial}_b$ (the *tangential Cauchy-Riemann operator*) given by $(\overline{\partial}_b u)\overline{Z} = \overline{Z}(u)$, for any C^1 function $u : M \to \mathbb{C}$ and any $Z \in T_{1,0}(M)$. Then $\overline{\partial}_b u = 0$ are the *tangential Cauchy-Riemann equations*. A solution to the tangential Cauchy-Riemann equations is a *CR function* on M. Let $\mathrm{CR}^r(M)$ denote the space of all CR functions on M of class C^r.

The boundary ∂M is *noncharacteristic* for $T_{1,0}(M)$ if for any local frame $\{T_\alpha : 1 \le \alpha \le n\}$ of $T_{1,0}(M)$ defined on the open subset $U \subseteq M$ one has $T_\alpha \notin T(\partial M) \otimes \mathbb{C}$ (i.e. $T_{\alpha,x} \notin T_x(\partial M) \otimes \mathbb{C}$, for some $x \in U \cap \partial M$) for some $1 \le \alpha \le n$.

The *Levi distribution* of the CR manifold $(M, T_{1,0}(M))$ is

$$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

It carries the complex structure

$$J : H(M) \to H(M), \quad J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(M).$$

Assume from now on that M is a CR manifold of type $(n, 1)$ (of *hypersurface type*). $H(M)$ is oriented by J, hence the conormal bundle

$$H(M)^\perp_x = \{\omega \in T^*_x(M) : \mathrm{Ker}(\omega) \supseteq H(M)_x\}, \quad x \in M,$$

is an oriented real line bundle, hence trivial. Let then θ be a global nowhere vanishing section in $H(M)^\perp$ (a *pseudohermitian structure* on M). The *Levi form* is

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M),$$

and M is *nondegenerate* (respectively *strictly pseudoconvex*) if L_θ is nondegenerate (respectively positive definite) for some θ. Also M is *Levi flat* if $L_\theta = 0$ (equivalently, if $H(M)$ is integrable). An alternative definition of the Levi form is

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M).$$

Note that L_θ and the $\mathbb{C}$-linear extension of G_θ coincide on $T_{1,0}(M) \otimes T_{0,1}(M)$. If M is nondegenerate then any pseudohermitian structure θ is a *contact form*, i.e. $\theta \wedge (d\theta)^n$ is a volume form on M. Let M be a nondegenerate CR manifold and θ a fixed contact form (the pair (M, θ) is commonly referred to as a *pseudohermitian manifold*). There is a unique vector field T on M such that $\theta(T) = 1$ and $(d\theta)(T, X) = 0$, for any $X \in T(M)$ (T is the *characteristic direction* of $d\theta$). The *Webster metric* of (M, θ) is given by

$$g_\theta(X,Y) = G_\theta(X, JY), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1,$$

for any $X, Y \in H(M)$. g_θ is a semi-Riemannian (Riemannian, if M is strictly pseudoconvex and L_θ is positive definite) metric on M.

Proposition 1. *Let M be a nondegenerate CR manifold-with-boundary. Then the boundary ∂M is noncharacteristic for $T_{1,0}(M)$.*

The proof is by contradiction. Assume that there is a local frame $\{T_\alpha\}$ of $T_{1,0}(M)$ on $U \subseteq M$ such that $T_\alpha \in T(\partial M) \otimes \mathbb{C}$, for all $1 \leq \alpha \leq n$. Then $T_{1,0}(M)_x \subset T_x(\partial M) \otimes \mathbb{C}$ for any $x \in U \cap \partial M$. Then, by taking complex conjugates, $T_{0,1}(M)_x \subset T_x(\partial M) \otimes \mathbb{C}$ hence, by looking at dimensions, $H(M)_x = T_x(\partial M)$, i.e. $L_{\theta,x} = 0$, a contradiction. □

From now on we assume that M is nondegenerate. For each boundary point $x \in \partial M$ we set

$$T_{1,0}(\partial M)_x = T_{1,0}(M)_x \cap [T_x(\partial M) \otimes \mathbb{C}].$$

Let $\{T_\alpha : 1 \leq \alpha \leq n\}$ be a local frame of $T_{1,0}(M)$, defined on the local coordinate neighborhood $(U, \varphi = (x^1, \cdots, x^{2n+1}))$. $U \cap \partial M$ consists of the points $x \in U$ such that $\varphi(x) \in \partial \mathbb{R}^{2n+1}_+ = \mathbb{R}^{2n} \times \{0\}$. We may write $T_\alpha = f^A_\alpha \, \partial/\partial x^A$, for some C^∞ functions $f^A_\alpha : U \to \mathbb{C}$. By Proposition 1 there is α, say $\alpha = n$, such that $T_\alpha \notin T(\partial M) \otimes \mathbb{C}$. Then $f^{2n+1}_n(x_0) \neq 0$ for some $x_0 \in U \cap \partial M$, and then $f^{2n+1}_n \neq 0$ on a whole neighborhood of x_0, which we may denote again by U. Then

$$\{T_j - \left(\lambda^{2n+1}_j / \lambda^{2n+1}_n\right) \; T_n : 1 \leq j \leq n-1\}$$

is a local frame of $T_{1,0}(\partial M)$ on $U \cap \partial M$, hence $T_{1,0}(\partial M)$ has rank $n-1$. We got

Proposition 2. *Let M be a nondegenerate CR manifold-with-boundary, of CR dimension n. Then its boundary ∂M is a CR manifold of type $(n-1, 2)$, i.e. $T_{1,0}(\partial M) = T_{1,0}(M) \cap [T(\partial M) \otimes \mathbb{C}]$ is a CR structure of CR codimension 2.*

Let us look at a few examples. For instance, let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group, with the CR structure spanned by

$$Z_j = \frac{\partial}{\partial z_j} + i \, \overline{z}_j \, \frac{\partial}{\partial t}, \quad 1 \leq j \leq n,$$

(if $n = 1$ then $\overline{Z}_1$ is the *Lewy operator*, cf. [20]). $\mathbb{H}_n$ is a Lie group with the group law

$$(z,t) \cdot (w,s) = (z + w, t + s + 2 \, \mathrm{Im}(z \cdot \overline{w})),$$

for (z,t), $(w,t) \in \mathbb{H}_n$, where $z \cdot \overline{w} = \delta_{jk} z^j \overline{w}^k$ (with the convention $z^j = z_j$), and Z_j are left invariant.

Example 1. $\mathbb{H}_n^+ = \{(z,t) \in \mathbb{H}_n : t \geq 0\}$ is a CR manifold-with-boundary $\partial \mathbb{H}_n^+ = \mathbb{C}^n \times \{0\}$. Let $U = \{(z,t) \in \mathbb{H}_n^+ : z_n \neq 0\}$. Then

$$\{\frac{\partial}{\partial z^a} - \frac{\overline{z}_a}{\overline{z}_n} \frac{\partial}{\partial z_n} : 1 \leq a \leq n-1\} \tag{4}$$

is a local frame of $T_{1,0}(\partial \mathbb{H}_n^+)$ on $U \cap \partial \mathbb{H}_n^+$. In particular, the tangential Cauchy-Riemann equations on $\partial \mathbb{H}_n^+$ are

$$z_n \frac{\partial u}{\partial \overline{z}_a} - z_a \frac{\partial u}{\partial \overline{z}_n} = 0, \quad 1 \leq a \leq n-1.$$

□

The *Heisenberg norm* is $|x| = (|z|^4 + t^2)^{1/4}$, for any $x = (z,t) \in \mathbb{H}_n$, where $|z|^2 = z \cdot \overline{z}$.

Example 2. $\Omega_r = \{x \in \mathbb{H}_n : |x| \leq r\}$ $(r > 0)$ is a CR manifold-with-boundary $\partial \Omega_r = \Sigma_r = \{x \in \mathbb{H}_n : |x| = r\}$ (the *Heisenberg sphere*, cf. [13]). Let us set $\phi(z,t) = |z|^2 - i\ t$. Note that $\overline{\partial}_b \phi = 0$, i.e. $\phi \in \mathrm{CR}^\infty(\mathbb{H}_n)$. Taking into account that

$$Z_j(|x|) = \frac{\overline{\phi}}{2\,|x|^3}\, \overline{z}_j\,, \quad 1 \leq j \leq n,$$

it follows that (4) is a local frame of $T_{1,0}(\Sigma_r)$ on $\Sigma_r \cap \{z \in \mathbb{H}_n : z_n \neq 0\}$. The *Folland-Stein operators* are

$$\mathcal{L}_\alpha = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \overline{Z}_j + \overline{Z}_j Z_j) + i\ \alpha\ T, \quad \alpha \in \mathbb{C}, \tag{5}$$

where $T = \partial / \partial t$. Let us consider the function

$$\varphi_\alpha(z,t) = \phi(z,t)^{-(n+\alpha)/2}\ \overline{\phi(z,t)}^{\ -(n-\alpha)/2},$$

and the constant $c_\alpha = 2^{2-2n} \pi^{n+1} / \left(\Gamma(\frac{n+\alpha}{2})\ \Gamma(\frac{n-\alpha}{2})\right)$. $\alpha \in \mathbb{C}$ is *admissible* if $c_\alpha \neq 0$ (equivalently if $\pm\alpha \in \{n,\, n+2,\, n+4, \cdots\}$). The Folland-Stein operators (5) form a family of operators of the form $A + \alpha B$ (where A is a second order hypoelliptic operator and B is a first order operator) which are hypoelliptic for any admissible α (cf. [11], p. 444). This is by now classical, and as well known the key ingredient in the proof is to build a fundamental solution to (5) i.e. to show that $\mathcal{L}_\alpha(\varphi_\alpha / c_\alpha) = \delta$, for any admissible α. It is noteworthy that the

Heisenberg spheres Σ_r are the level sets of

$$\varphi_0(z,t) = |\phi(z,t)|^{-n} = \left(|z|^4 + t^2\right)^{-n/2} .$$

Let θ_0 be the canonical pseudohermitian structure on $\mathbb{H}_n$ i.e.

$$\theta_0 = dt + i \sum_{j=1}^{n} (z_j d\overline{z}^j - \overline{z}_j dz^j).$$

$\mathbb{H}_n$ is strictly pseudoconvex and L_{θ_0} is positive definite. Moreover, the Webster metric of $(\mathbb{H}_n, \theta_0)$ is expressed by

$$g_{\theta_0}(X_j, X_k) = g_{\theta_0}(Y_j, Y_k) = \delta_{jk}\,, \quad g_{\theta_0}(X_j, Y_k) = 0,$$

$$g_{\theta_0}(X_j, T) = g_{\theta_0}(Y_j, T) = 0, \quad g_{\theta_0}(T, T) = 1,$$

where

$$X_j = \frac{1}{\sqrt{2}}\,(Z_j + \overline{Z}_j), \quad Y_j = \frac{i}{\sqrt{2}}\,(Z_j - \overline{Z}_j).$$

Proposition 3. *The Heisenberg spheres form a foliation of* $(\mathbb{H}_n, g_{\theta_0})$ *whose normal bundle is the span of*

$$V = T + (\phi/t) z^j Z_j + (\overline{\phi}/t) \overline{z}^j \overline{Z}_j\,. \tag{6}$$

Then perhaps (6) is the Heisenberg analog to the radial vector field in $\mathbb{R}^{2n+1}$ (see [13], p. 331-332).

Proof of Proposition 3. Let us set

$$E_j = Z_j + \overline{Z}_j - \frac{1}{t}(\phi z_j + \overline{\phi}\overline{z}_j)T, \;\; F_j = i(Z_j - \overline{Z}_j) + \frac{i}{t}(\phi z_j - \overline{\phi}\overline{z}_j)T.$$

Then $\{E_j, F_j\}$ is a local frame of the tangent bundle of the foliation and a calculation shows that (6) satisfies $g_{\theta_0}(E_j, V) = g_{\theta_0}(F_j, V) = 0$. $\square$

Let M and N be two CR manifolds with boundary. A *CR map* is a C^∞ map $f : M \to N$ such that $(d_x f) T_{1,0}(M)_x \subseteq T_{1,0}(N)_{f(x)}$, for any $x \in M$. A *CR immersion* is an immersion and a CR map. A CR immersion $f : M \to N$ is *neat* if i) $f(M) \cap \partial N = f(\partial M)$ and ii) for each point $x \in \partial M$ there is a local chart $\psi : V \to \mathbb{R}^{m+p}_+$ of N such that $f(x) \in V$ and $\psi^{-1}(\mathbb{R}^m_+) = V \cap f(M)$ $(m = \dim(M))$.

Example 3. $\Sigma_r^+ = \Sigma_r \cap \mathbb{H}_n^+$ is a CR manifold-with-boundary $\partial \Sigma_r^+ = S^{2n-1}(r) \times \{0\}$ and the inclusion $\Sigma_r^+ \to \mathbb{H}_n^+$ is a neat CR immersion. $\square$

Example 4. $S^{2n+1}_+ = S^{2n+1} \cap \mathbb{R}^{2n+2}_+$ is a CR manifold-with-boundary $\partial S^{2n+1}_+ = S^{2n} \times \{0\}$. Let $\mathcal{C}$ be the Cayely transform

$$\mathcal{C}(\zeta) = (\frac{\zeta'}{1+\zeta^{n+1}}\,,\, i\,\frac{1-\zeta^{n+1}}{1+\zeta^{n+1}}), \;\; \zeta = (\zeta', \zeta^{n+1}), \;\; 1+\zeta^{n+1} \neq 0,$$

and $f : \mathbb{H}_n \to \partial\Omega_{n+1}$ the CR isomorphism $f(z,t) = (z, t + i|z|^2)$ with the obvious inverse $f^{-1}(z,w) = (z, \mathrm{Re}(w))$. Here Ω_{n+1} is the Siegel domain

$$\Omega_{n+1} = \{(z,w) \in \mathbb{C}^{n+1} : \mathrm{Im}(w) > |z|^2\}.$$

Then $F = f^{-1} \circ \mathcal{C}$ is a neat CR diffeomorphism

$$F : S^{2n+1}_+ \setminus \{(0, \cdots, 0, -1)\} \to \mathbb{H}^+_n.$$

Indeed if $\zeta \in S^{2n+1}_+$ and $\zeta^{n+1} = u + iv$ $(v \geq 0)$ and $(z,t) = F(\zeta)$ then $t = 2v/[(1+u)^2 + v^2] \geq 0$. In particular F descends to a CR diffeomorphism $(S^{2n} \times \{0\}) \setminus \{(0, \cdots, 0, -1)\} \approx \mathbb{C}^n \times \{0\}$. $\square$

Let M be a nondegenerate CR manifold-with-boundary. A complex p-form η on M is a $(p,0)$-*form* if $T_{0,1}(M) \rfloor \eta = 0$. Let $\Lambda^{p,0}(M) \to M$ be the bundle of all $(p,0)$-forms. If M has CR dimension n then the top degree $(p,0)$-forms are the $(n+1,0)$-forms. $K(M) = \Lambda^{n+1,0}(M)$ is the *canonical bundle* over M. There is a natural action of $\mathbb{R}_+ = (0, +\infty)$ on $K(M) \setminus \{0\}$. Let $C(M)$ be the quotient space and $\pi : C(M) \to M$ the projection. This construction leads to a principal bundle $S^1 \to C(M) \to M$ (the *canonical circle bundle* over M). Let θ be a pseudohermitian structure on M and T the characteristic direction of $d\theta$. Given a local frame $\{T_\alpha\}$ of $T_{1,0}(M)$ on a local coordinate neighborhood (U, x^A), let θ^α be the locally defined complex 1-forms determined by

$$\theta^\alpha(T_\beta) = \delta^\alpha_\beta\,, \;\; \theta^\alpha(T_{\overline{\beta}}) = 0, \;\; \theta^\alpha(T) = 0.$$

Here $T_{\overline{\alpha}} = \overline{T}_\alpha$. Then

$$\pi^{-1}(U) \to U \times S^1\,, \;\; [z] \mapsto (x\,,\, \lambda/|\lambda|),$$

$$z = \lambda\,(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n)_x\,, \;\; \lambda \in \mathbb{C} \setminus \{0\}, \;\; x \in M,$$

is a local trivialization chart of the canonical circle bundle. Let us set $\gamma : \pi^{-1}(U) \to \mathbb{R}$, $\gamma([z]) = \arg(\lambda)$ (where $\arg : \mathbb{C} \to [0, 2\pi)$). Then $(\pi^{-1}(U),\, \tilde{x}^A = x^A \circ \pi,\, \gamma)$ are naturally induced local coordinates on $C(M)$ and $\pi^{-1}(U \cap \partial M)$ consists of all $c \in \pi^{-1}(U)$ with $\tilde{x}^{2n+1}(c) = 0$, i.e. $C(M)$ is a manifold-with-boundary modelled on $\mathbb{R}^{2n+1}_+ \times \mathbb{R}$. We obtained

Lemma 1. *Let M be a nondegenerate CR manifold-with-boundary. Then the total space $C(M)$ of the canonical circle bundle is a manifold-with-boundary $\partial C(M) = \pi^{-1}(\partial M)$. In particular $\partial C(M)$ is a principal S^1-bundle over ∂M.*

Let ∇ be the unique linear connection on M (the *Tanaka-Webster connection*) satisfying the axioms 1) $H(M)$ is parallel with respect to ∇, 2) $\nabla J = 0$, $\nabla g_\theta = 0$, and 3) the torsion T_∇ of ∇ is *pure*, i.e. $T_\nabla(Z,W) = 0$, $T_\nabla(Z,\overline{W}) = 2iG_\theta(Z,\overline{W})T$, and $\tau \circ J + J \circ \tau = 0$. Here $\tau(X) = T_\nabla(T,X)$ is the *pseudohermitian torsion*. We set $A(X,Y) = g_\theta(\tau X, Y)$, for any $X, Y \in T(M)$. By a result of S. Webster, [24], A is symmetric.

With respect to a local frame $\{T_\alpha : 1 \leq \alpha \leq n\}$ of $T_{1,0}(M)$, defined on an open set $U \subseteq M$, it is customary to set $g_{\alpha\overline{\beta}} = L_\theta(T_\alpha, T_{\overline{\beta}})$ (the local coefficients of the Levi form), $\nabla T_\beta = \omega_\beta^\alpha \otimes T_\alpha$ (the connection 1-forms) and $R^\nabla(T_A,T_B)T_C = R_C{}^D{}_{AB}T_D$ (the curvature components). The range of the indices $A, B, C, \cdots$ is $\{0, 1, \cdots, n, \overline{1}, \cdots, \overline{n}\}$ (with the convention $T_0 = T$). Next, the *pseudohermitian Ricci tensor* is $R_{\lambda\overline{\mu}} = R_\lambda{}^\alpha{}_{\alpha\overline{\mu}}$ and the *pseudohermitian scalar curvature* is $\rho = g^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}}$. When M is strictly pseudoconvex and θ is a pseudohermitian structure such that L_θ is positive definite $C(M)$ carries a Lorentz metric F_θ such that $F_{\hat{\theta}} = e^{u\circ\pi}F_\theta$, where $\hat{\theta} = e^u\theta$, $u \in C^\infty(M)$ (in particular the *restricted conformal class* $[F_\theta] = \{e^{u\circ\pi}F_\theta : u \in C^\infty(M)\}$ is a CR invariant). Cf. J.M. Lee, [18], F_θ is given by

$$F_\theta = \pi^*\tilde{G}_\theta + 2(\pi^*\theta) \odot \sigma, \tag{7}$$

$$\sigma = \frac{1}{n+2}\{d\gamma + \pi^*(i\,\omega_\alpha^\alpha - \frac{i}{2}\,g^{\alpha\overline{\beta}}dg_{\alpha\overline{\beta}} - \frac{\rho}{4(n+1)}\,\theta)\}. \tag{8}$$

F_θ is the *Fefferman metric* of (M,θ). Here $\tilde{G}_\theta$ is the degenerate $(0,2)$-tensor field on M given by

$$\tilde{G}_\theta(X,Y) = (d\theta)(X,JY), \quad \tilde{G}_\theta(T,Z) = 0,$$

for any $X, Y \in H(M)$ and any $Z \in T(M)$. Also $\odot$ denotes the symmetric tensor product.

Let $S = \partial/\partial\gamma$ be the tangent to the S^1-action. σ is a connection 1-form in $S^1 \to C(M) \to M$. If $X \in T(M)$ is a tangent vector field on M then $X^\uparrow \in T(C(M))$ will denote the horizontal lift of X with respect to the connection $\mathcal{H} = \mathrm{Ker}(\sigma)$. Although the submersion $\pi : C(M) \to M$ is not semi-Riemannian (its fibres are degenerate) a technique similar to that in [21] leads to

Lemma 2. *For any* $X, Y \in H(M)$

$$\nabla^{C(M)}_{X^\uparrow} Y^\uparrow = (\nabla_X Y)^\uparrow - (d\theta)(X,Y)T^\uparrow - (A(X,Y) + (d\sigma)(X^\uparrow, Y^\uparrow))\hat{S},$$

$$\nabla^{C(M)}_{X^\uparrow} T^\uparrow = (\tau X + \phi X)^\uparrow,$$

$$\nabla^{C(M)}_{T^\uparrow} X^\uparrow = (\nabla_T X + \phi X)^\uparrow + 2(d\sigma)(X^\uparrow, T^\uparrow)\hat{S},$$

$$\nabla^{C(M)}_{X^\uparrow} \hat{S} = \nabla^{C(M)}_{\hat{S}} X^\uparrow = (JX)^\uparrow,$$

$$\nabla^{C(M)}_{T^\uparrow} T^\uparrow = V^\uparrow, \;\; \nabla^{C(M)}_{\hat{S}} \hat{S} = 0,$$

$$\nabla^{C(M)}_{\hat{S}} T^\uparrow = \nabla^{C(M)}_{T^\uparrow} \hat{S} = 0,$$

where $\phi : H(M) \to H(M)$ *is given by* $G_\theta(\phi X, Y) = (d\sigma)(X^\uparrow, Y^\uparrow)$, *and* $V \in H(M)$ *is given by* $G_\theta(V, Y) = 2(d\sigma)(T^\uparrow, Y^\uparrow)$. *Also* $\hat{S} = ((n+2)/2)S$.

Lemma 2 relates the Levi-Civita connection $\nabla^{C(M)}$ of $(C(M), F_\theta)$ to the Tanaka-Webster connection of (M, θ). Cf. [4] for the proof of Lemma 2.

3. The geometry of the first fundamental form of the boundaries

Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that G_θ is positive definite. Let $T(\partial M)^\perp \to \partial M$ be the normal bundle of $\partial M \hookrightarrow (M, g_\theta)$. Let $\tan_x : T_x(M) \to T_x(\partial M)$ and $\mathrm{nor}_x : T_x(M) \to T(\partial M)^\perp_x$ be the projections associated with the direct sum decomposition

$$T_x(M) = T_x(\partial M) \oplus T(\partial M)^\perp_x \, , \quad x \in \partial M.$$

If T is the characteristic direction of $d\theta$ then we set $T^\perp = \mathrm{nor}(T)$ and $T^T = \tan(T)$.

Theorem 1. *Let* $\mathrm{Null}(j^* F_\theta)$ *consist of all* $V \in T(\partial C(M))$ *such that* $F_\theta(V, W) = 0$, *for any* $W \in T(\partial C(M))$. *Let us consider the closed set* $\mathrm{Sing}(T^T) = \{x \in \partial M : T^T_x = 0\}$ *and set* $\Omega = \partial M \setminus \mathrm{Sing}(T^T)$. *Then*

$$\mathrm{Null}(j^* F_\theta)_z = \begin{cases} 0, & z \in \pi^{-1}(\Omega), \\ \mathrm{Ker}(d_z \pi), & z \in \pi^{-1}(\mathrm{Sing}(T^T)), \end{cases}$$

for any $z \in \partial C(M)$. *Moreover* $(\pi^{-1}(\Omega)\, ,\, j^* F_\theta)$ *is a Lorentz manifold.*

Here $j : \partial C(M) \hookrightarrow C(M)$ is the inclusion. Hence $\partial C(M)$ is degenerate at each point $z \in \pi^{-1}(\mathrm{Sing}(T^T))$. In particular, if ∂M is tangent to T then the boundary $(\partial C(M)\, ,\, j^* F_\theta)$ is a Lorentz manifold.

Example 1. (*continued*)
$T(\partial \mathbb{H}_n^+)$ is the span of $\{X_j - \sqrt{2}\, y_j T,\ Y_j + \sqrt{2}\, x_j T : 1 \leq j \leq n\}$ hence $\xi = T + \sqrt{2}\, y^j X_j - \sqrt{2}\, x^j Y_j$ is normal to $\partial \mathbb{H}_n^+$ (with $z^j = x^j + i y^j$). Then T decomposes as

$$T = a^j (X_j - \sqrt{2}\, y_j T) + b^j (Y_j + \sqrt{2}\, x_j T) + c\xi,$$

$$a^j = -\frac{\sqrt{2}\, y^j}{1 + 2|z|^2}\, , \quad b^j = \frac{\sqrt{2}\, x^j}{1 + 2|z|^2}\, , \quad c = \frac{1}{1 + 2|z|^2}\, .$$

Then $T^\perp = c\xi$ and (with the conventions in Theorem 1) $\mathrm{Sing}(T^T) = \{0\}$ hence $(\partial C(\mathbb{H}_n^+) \setminus \pi^{-1}(0)\, ,\, j^* F_{\theta_0})$ is a Lorentz manifold. □

Proof of Theorem 1. Let $V \in T(\partial C(M))$ such that $F_\theta(V, W) = 0$ for any $W \in T(\partial C(M))$ i.e.

$$(\pi^* \tilde{G}_\theta)(V, W) + (\pi^* \theta)(V) \sigma(W) + (\pi^* \theta)(W) \sigma(V) = 0.$$

By taking into account

$$T(C(M)) = \mathrm{Ker}(\sigma) \oplus \mathrm{Ker}(d\pi) \tag{9}$$

we may decompose $V = V_H + V_V$, with $V_H \in \mathrm{Ker}(\sigma)$. Then

$$\tilde{G}((d\pi)V_H, (d\pi)W_H) + \theta((d\pi)V_H)\sigma(W_V) + \theta((d\pi)W_H)\sigma(V_V) = 0. \tag{10}$$

As $\partial C(M)$ is a saturated set, it is tangent to the S^1-action. Hence we may apply (10) for $W = S \in \mathrm{Ker}(d\pi) \subset T(\partial C(M))$. As $\sigma(S) = 1/(n+2)$ we obtain

$$\theta((d\pi)V_H) = 0,$$

i.e. $(d\pi)V_H \in H(M)$, hence (10) becomes

$$\tilde{G}_\theta((d\pi)V_H, (d\pi)W_H) + \theta((d\pi)W_H)\sigma(V_V) = 0. \tag{11}$$

Applying (11) for $W = V$ gives

$$G_\theta((d\pi)V_H, (d\pi)V_H) = 0$$

hence $(d\pi)V_H = 0$, and then $V_H = 0$ (due to $\mathrm{Ker}(\sigma) \cap \mathrm{Ker}(d\pi) = (0)$). Therefore, on one hand

$$\mathrm{Null}(j^* F_\theta) \subseteq \mathrm{Ker}(d\pi) \tag{12}$$

and on the other (11) becomes

$$\theta((d\pi)W_H)\sigma(V_V) = 0. \tag{13}$$

Let $x_0 \in \Omega$ (so that $T^T_{x_0} \neq 0$) and $z_0 \in \pi^{-1}(x_0)$. We may apply (13) for $W = (T^T)^\uparrow$, at the point z_0. Yet

$$(\pi^*\theta)(W_H)_{z_0} = \theta(T^T)_{x_0} = \|T^T\|^2_{x_0} \neq 0$$

hence (by (13)) $\sigma(V_V)_{z_0} = 0$, or $(V_V)_{z_0} = 0$ and we may conclude that $\mathrm{Null}(j^*F_\theta)_{z_0} = (0)$. To complete the proof of Theorem 1 it suffices to show that $\mathrm{Null}(j^*F_\theta)_z$ is 1-dimensional, for any $z \in \pi^{-1}(C)$. Let us set $x = \pi(z)$. Then, for any $W \in T(\partial C(M))$

$$F_\theta(S, W)_z = (\pi^*\theta)(W)_z \sigma(S)_z = \frac{1}{n+2}\,\theta_x((d_z\pi)W_z) =$$
$$= g_{\theta,x}(T^\perp_x, (d_z\pi)W_z) = 0$$

as $(d_z\pi)W_z$ is tangent to ∂M. Hence $S_z \in \mathrm{Null}(j^*F_\theta)_z$ (and we may apply (12)).

Since $F_\theta(S,S) = 0$ and S is tangent to $\partial C(M)$, F_θ is indefinite on $T(\partial C(M))$. However (by the first part of Theorem 1) F_θ is non-degenerate on $T(\pi^{-1}(\Omega))$ hence $(j^*F_\theta)_z$ has signature $(2n, 1)$ at each $z \in \pi^{-1}(\Omega)$. □

Proposition 4. *Let M be a strictly pseudoconvex CR manifold-with-boundary and θ a contact form with G_θ positive definite. Let T be the characteristic direction of $d\theta$. The property that $T \in T(\partial M)$ is not CR invariant. If $T \in T(\partial M)$ and $\hat{T}$ is the characteristic direction of $d\hat{\theta}$, where $\hat{\theta} = e^{2u}\theta$ ($u \in C^\infty(M)$), then* $\mathrm{Sing}(\hat{T}^T) = \emptyset$.

Proof. Let us consider a local orthonormal (with respect to g_θ) frame of $T(\partial M)$ of the form $\{E_1, \cdots, E_{2n-1}, T\}$, so that $E_a \in H(M)$, $1 \leq a \leq 2n-1$. Next, let us complete $\{E_a\}$ to a local orthonormal frame $\{E_1, \cdots, E_{2n}\}$ of $H(M)$ and set $T_\alpha = (1/\sqrt{2})(E_\alpha + iE_{\alpha+n})$, $1 \leq \alpha \leq n$. Given another contact form $\hat{\theta} = e^{2u}\theta$ ($u \in C^\infty(M)$) the characteristic direction of $d\hat{\theta}$ is expressed by

$$\hat{T} = e^{-2u}(T + iu^{\overline{\alpha}}T_{\overline{\alpha}} - iu^\alpha T_\alpha) =$$
$$= e^{-2u}\{T + \frac{i}{\sqrt{2}}(u^{\overline{\alpha}} - u^\alpha)E_\alpha + \frac{1}{\sqrt{2}}(u^{\overline{\alpha}} + u^\alpha)E_{\alpha+n}\}$$

dove $u^\alpha = u_\alpha = T_\alpha(u)$ (as $L_\theta(T_\alpha, T_{\overline{\beta}}) = \delta_{\alpha\beta}$). Let ξ be a unit normal on ∂M. Then $\xi \in \{\pm E_{2n}\}$ hence $\mathrm{Sing}(\hat{T}^T) = \mathrm{Sing}(T) = \emptyset$. □

If $z \in C(M)$ we denote by $\beta_z : T_{\pi(z)}(M) \to \mathrm{Ker}(\sigma_z)$ the inverse of the $\mathbb{R}$-linear isomorphism $d_z\pi : \mathrm{Ker}(\sigma_z) \to T_{\pi(z)}(M)$. It is an elementary matter that

Lemma 3. *Given $v \in T_x(\partial M)$ its horizontal lift $\beta_z v$, $z \in \pi^{-1}(x)$, is tangent to $\partial C(M)$.*

Indeed, let $a : (-\epsilon, \epsilon) \to \partial M$ be a smooth curve such that $a(0) = x$ and $\dot{a}(0) = v$. Let $X \in T(\partial M)$ be a tangent vector field such that $X_x = v$. Let $a^\uparrow : (-\epsilon, \epsilon) \to C(M)$ be the unique horizontal lift of a, issuing at z. As $\pi(a^\uparrow(t)) = a(t)$ one has $a^\uparrow(t) \in \partial C(M)$, $|t| < \epsilon$. On the other hand $\dot{a}^\uparrow(0) \in \mathrm{Ker}(\sigma_z)$ and it projects on v hence

$$T_z(\partial C(M)) \ni \dot{a}^\uparrow(0) = X_z^\uparrow = \beta_z v.$$

□

We set $T(\partial M)^\uparrow = \{\beta X : X \in T(\partial M)\}$ and $\mathcal{V}_z = \mathrm{Ker}(d_z \pi)$, for $z \in \partial C(M)$. As observed above, $\partial C(M)$ is tangent to the S^1-action hence $\mathcal{V}$ is a smooth distribution on $\partial C(M)$.

Lemma 4. *Let M be a strictly pseudoconvex CR manifold-with-boundary. One has the decomposition*

$$T(\partial C(M)) = T(\partial M)^\uparrow \oplus \mathcal{V} \tag{14}$$

Moreover, if ∂M is tangent to the characteristic direction T of $d\theta$ then

$$T(\partial C(M))^\perp \subseteq \mathrm{Ker}(\sigma), \quad (d\pi) T(\partial C(M))^\perp \subseteq H(M), \tag{15}$$

$$\mathrm{Ker}(\sigma) = T(\partial M)^\uparrow \oplus T(\partial C(M))^\perp. \tag{16}$$

Here $T(\partial C(M))^\perp \to \partial C(M)$ is the normal bundle of $j : \partial C(M) \hookrightarrow (C(M), F_\theta)$.

Proof of Lemma 4. Note that

$$T(\partial M)^\uparrow \cap \mathcal{V} \subseteq \mathrm{Ker}(\sigma) \cap \mathrm{Ker}(d\pi) = (0),$$

hence the sum $T(\partial M)^\uparrow + \mathcal{V}$ is direct. The arguments preceding Lemma 4 show that $T(\partial M)^\uparrow \oplus \mathcal{V} \subseteq T(\partial C(M))$. Viceversa, let $V \in T(\partial C(M)) \subset T(C(M))$. Then (by the decomposition (9))

$$V = X^\uparrow + f\, S, \tag{17}$$

for some $X \in T(M)$ and $f \in C^\infty(C(M))$. Then

$$X_{\pi(z)} = (d_z \pi) V_z \in T_{\pi(z)}(\partial M), \quad z \in \partial C(M),$$

i.e. $X \in T(\partial M)$ and then $T(\partial C(M)) \subseteq T(\partial M)^\uparrow \oplus \mathcal{V}$. To check (15) let $V \in T(\partial C(M))^\perp \subset T(C(M))$ and use (9) to decompose as in (17). By assumption $T \in T(\partial M)$ hence $T^\uparrow \in T(\partial C(M))$ and then

$$0 = F_\theta(V, T^\uparrow) = \tilde{G}((d\pi)V, (d\pi)T^\uparrow) + \theta((d\pi)T^\uparrow)\sigma(V) =$$

$$= \tilde{G}_\theta(X, T) + \frac{f}{n+2} = \frac{f}{n+2}$$

i.e. $f = 0$, or $V = X^\uparrow \in \mathrm{Ker}(\sigma)$. To check the second statement in (15) let

$$V \in T(\partial C(M))^\perp \subseteq \mathrm{Ker}(\sigma) = T(M)^\uparrow = H(M)^\uparrow \oplus (\mathbb{R}T)^\uparrow$$

i.e. $V = Y^\uparrow + fT^\uparrow$, for some $Y \in H(M)$. Moreover $S \in \mathrm{Ker}(d\pi) \subset T(\partial C(M))$, hence S and V are orthogonal

$$0 = F_\theta(S, V) = \theta((d\pi)V)\sigma(S) = \frac{f}{n+2}$$

i.e. $f = 0$, or $V \in H(M)^\uparrow$. (15) is proved and may be equivalently written

$$T(\partial C(M))^\perp \subseteq H(M)^\uparrow.$$

When $T^\perp = 0$ the space $T(\partial C(M))$ is nondegenerate in $(T(C(M)), F_\theta)$ hence so does the perp space $T(\partial C(M))^\perp$. Also

$$T(C(M)) = T(\partial C(M)) \oplus T(\partial C(M))^\perp.$$

Let us prove (16). First

$$T(\partial M)^\uparrow \cap T(\partial C(M))^\perp \subseteq T(\partial C(M)) \cap T(\partial C(M))^\perp = (0)$$

hence the sum $T(\partial M)^\uparrow + T(\partial C(M))^\perp$ is direct and (by (15))

$$T(\partial M)^\uparrow \oplus T(\partial C(M))^\perp \subseteq \mathrm{Ker}(\sigma). \tag{18}$$

Finally (by (14))

$$\mathrm{Ker}(\sigma) \oplus \mathrm{Ker}(d\pi) = T(C(M)) = T(\partial C(M)) \oplus T(\partial C(M))^\perp =$$
$$= T(\partial M)^\uparrow \oplus \mathrm{Ker}(d\pi) \oplus T(\partial C(M))^\perp$$

and (18) yields (16). □

From now on we assume that ∂M is tangent to T. Then let us consider a local orthonormal frame $\{E_1, \cdots, E_{2n-1}, T\}$ of $T(\partial M)$, with respect to $i^* g_\theta$ (the first fundamental form of $i : \partial M \hookrightarrow M$), defined on some open set $U \subseteq \partial M$. In particular $E_a \in H(M)$, $1 \le a \le 2n-1$.

Lemma 5. *Let M be a strictly pseudoconvex CR manifold-with-boundary. Let θ be a contact form on M such that G_θ is positive definite and let T be the characteristic direction of $d\theta$. Assume that ∂M is tangent to T. Then*

$$\{E_1^\uparrow, \cdots, E_{2n-1}^\uparrow, T^\uparrow \pm \frac{n+2}{2} S\}$$

is a local orthonormal frame of $T(\partial C(M))$, with respect to $j^* F_\theta$, defined on the open set $\pi^{-1}(U) \subseteq \partial C(M)$. In particular $T^\uparrow - ((n+2)/2)S$ is a global timelike vector field on $\partial C(M)$, i.e. $(\partial C(M), j^* F_\theta)$ is a spacetime.

See also [5]. The proof is straightforward.

4. The geometry of the second fundamental form of the boundaries

As $(\partial C(M), j^* F_\theta)$ is a Lorentz submanifold of $(C(M), F_\theta)$ we may write the Gauss equation

$$\nabla^{C(M)}_X Y = \nabla^{\partial C(M)}_X Y + \mathbb{B}(X,Y),$$

for any $X, Y \in T(\partial C(M))$. Here $\nabla^{\partial C(M)}$ is the induced connection and $\mathbb{B}$ is the second fundamental form of $j : \partial C(M) \hookrightarrow C(M)$. Cf. e.g. [22], p. 100. At this point, we wish to compute the mean curvature vector of j

$$\mathbb{H} = \frac{1}{2n+1}\, \mathrm{trace}_{j^* F_\theta}(\mathbb{B}).$$

To this end it is convenient to use the local frame in Proposition 5.

Theorem 2. *Let M be a strictly pseudoconvex CR manifold-with-boundary, of CR dimension n, and θ a contact form on M such that G_θ is positive definite. Assume that ∂M is tangent to the characteristic direction T of $d\theta$. Let $\{E_1, \cdots, E_{2n-1}, T\}$ be a local g_θ-orthonormal frame of $T(\partial M)$ and ξ a unit normal vector field on ∂M, both defined on the open set $U \subseteq \partial M$. Then the mean curvature vector $\mathbb{H}$ of the immersion $j : \partial C(M) \hookrightarrow C(M)$ is given by*

$$\mathbb{H}_z = \frac{1}{2n+1} \sum_{a=1}^{2n-1} g_\theta(\nabla_{E_a} E_a\, ,\, \xi)_{\pi(z)}\, \xi^{\uparrow}_z \tag{19}$$

for any $z \in \pi^{-1}(U)$. Here ∇ is the Tanaka-Webster connection of (M,θ). In particular $\mathbb{H} = (2n/(2n+1))\, H^{\uparrow}$, where H is the mean curvature vector of the immersion $i : \partial M \hookrightarrow M$. Therefore, $\partial C(M)$ is minimal in $(C(M), F_\theta)$ if and only if ∂M is minimal in (M, g_θ).

Example 5. $\mathbb{R}^{2n}_+ \times \mathbb{R}$ is a strictly pseudoconvex CR manifold (with the CR structure induced from $\mathbb{H}_n$) whose boundary $N = \partial(\mathbb{R}^{2n}_+ \times \mathbb{R})$ is tangent to $T = \partial/\partial t$. The normal bundle of the boundary is the span of $\xi = \partial/\partial y^n - 2x_n T$. By the Gauss formula, the second fundamental form of the boundary is given by

$$B(\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^n}) = -4y_n\xi, \quad B(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^n}) = -2y_\alpha\xi, \quad B(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}) = 0,$$

$$B(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\beta}) = 0, \quad B(\frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^\beta}) = 2x_\beta\xi, \quad B(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}) = 0,$$

$$B(\frac{\partial}{\partial x^\alpha}, T) = 0, \quad B(\frac{\partial}{\partial x^n}, T) = \xi, \quad B(\frac{\partial}{\partial y^\alpha}, T) = 0, \quad B(T,T) = 0.$$

Here $1 \leq \alpha, \beta \leq 2n-1$. On the other hand, the induced metric on N is given by

$$g : \begin{pmatrix} 2(\delta_{ij} + 2y_i y_j) & -4y_i x_\beta & -2y_i \\ -4x_\alpha y_j & 2(\delta_{\alpha\beta} + 2x_\alpha x_\beta) & 2x_\alpha \\ -2y_j & 2x_\beta & 1 \end{pmatrix}$$

hence (by an argument similar to the proof of Lemma 7) the corresponding cometric on $T^*(N)$ is given by

$$g^{-1} : \begin{pmatrix} \frac{1}{2}\delta^{ij} & 0 & y^i \\ 0 & \frac{1}{2}\delta^{\alpha\beta} & -x^\alpha \\ y^j & -x^\beta & 1 + 2|x'|^2 + 2|y|^2 \end{pmatrix} \tag{20}$$

where $x' = (x_1, \cdots, x_{2n-1})$, $|x'|^2 = x_\alpha x^\alpha$ and $|y|^2 = y_j y^j$. Finally a calculation (based on (20)) shows that $2nH = g^{ab} B(\partial_a, \partial_b) = 0$, i.e. N is minimal in $(\mathbb{R}^{2n}_+ \times \mathbb{R}, g_{\theta_0})$. In particular (by Theorem 2) $\partial C(\mathbb{R}^{2n}_+ \times \mathbb{R})$ is minimal in $(C(\mathbb{R}^{2n}_+ \times \mathbb{R}), F_{\theta_0})$. $\square$

Let $\{X_A : 1 \leq A \leq 2n+1\}$ be a local F_θ-orthonormal frame of $T(\partial C(M))$, i.e. $F_\theta(X_A, X_B) = \epsilon_A \delta_{AB}$, with $\epsilon_1 = \cdots = \epsilon_{2n} = 1 = -\epsilon_{2n+1}$. Then $\mathbb{H}$ is locally given by

$$\mathbb{H} = \frac{1}{2n+1} \sum_A \epsilon_A \mathbb{B}(X_A, X_A).$$

Proof of Theorem 2. Using the local frame furnished by Lemma 5 we obtain

$$(2n+1)\mathbb{H} = \sum_{a=1}^{2n-1} \mathbb{B}(E_a^\uparrow, E_a^\uparrow) + 2(n+2)\mathbb{B}(T^\uparrow, S). \tag{21}$$

As a consequence of Lemma 2 we have

$$\nabla^{C(M)}_{E_a^\uparrow} E_a^\uparrow = (\nabla_{E_a} E_a)^\uparrow - \frac{n+2}{2} A(E_a, E_a) S, \tag{22}$$

$$\nabla^{C(M)}_{T^\uparrow} S = 0. \tag{23}$$

The equation (23) implies $\mathbb{B}(T^\uparrow, S) = 0$ (with the corresponding simplification of (21)). As $T \in T(\partial M)$ we have

$$T(\partial M)^\perp \subseteq H(M).$$

We need the following

Lemma 6.

Assume that ∂M is tangent to T. Let $T(\partial M)^\perp \to \partial M$ be the normal bundle of the immersion $i : \partial M \hookrightarrow M$. Then

$$[T(\partial M)^\perp]^\uparrow = T(\partial C(M))^\perp. \tag{24}$$

Proof of Lemma 6. Let $\xi \in T(\partial M)^\perp$ and $V \in T(\partial C(M)) = T(\partial M)^\uparrow \oplus \text{Ker}(d\pi)$, i.e. $V = X^\uparrow + f\, S$. Let us set $X_H := X - \theta(X)T \in H(M)$. Then

$$F_\theta(V, \xi^\uparrow) = \tilde{G}_\theta(X, \xi) + f\ F_\theta(S, \xi^\uparrow) =$$

$$= G_\theta(X_H, \xi) + f\ \theta(\xi)\sigma(S) = g_\theta(X_H, \xi) = 0$$

because $X, T \in T(\partial M)$ implies $X_H \in T(\partial M)$. It follows that

$$[T(\partial M)^\perp]^\uparrow \subseteq T(\partial C(M))^\perp.$$

The desired equality follows by inspecting dimensions. $\square$

Let ξ be a unit normal vector field on ∂M, defined on the open set $U \subseteq N$. Then (by Lemma 6) $\xi^\uparrow$ is a unit normal vector field on $\partial C(M)$. Then (by the Gauss equation and by (22))

$$F_\theta(\mathbb{B}(E_a^\uparrow, E_a^\uparrow)\,,\, \xi^\uparrow) = F_\theta(\nabla^{C(M)}_{E_a^\uparrow} E_a^\uparrow\,,\, \xi^\uparrow) = F_\theta((\nabla_{E_a} E_a)^\uparrow\,,\, \xi^\uparrow) =$$

$$= \tilde{G}_\theta(\nabla_{E_a} E_a\,,\, \xi) = g_\theta(\nabla_{E_a} E_a\,,\, \xi)$$

which yields (19). $\square$

The Levi-Civita connection ∇^{g_θ} of (M, g_θ) is related to the Tanaka-Webster connection ∇ of (M, θ) by

$$\nabla^{g_\theta}_X Y = \nabla_X Y + (\Omega(X,Y) - A(X,Y))T + \tag{25}$$

$$+ \tau(X)\theta(Y) + \theta(X)JY + \theta(Y)JX,$$

for any $X, Y \in T(M)$. Here $\Omega = -d\theta$. Cf. e.g. [3], p. 238. Thus, for any $X, Y \in H(M)$

$$\nabla^{g_\theta}_X Y = \nabla_X Y + (\Omega(X,Y) - A(X,Y))T$$

and then

$$\nabla^{g_\theta}_{E_a} E_a = \nabla_{E_a} E_a - A(E_a, E_a)T$$

implies (as $g_\theta(T, \xi) = 0$)

$$(2n+1)\mathbb{H} = \sum_a g_\theta(\nabla^{g_\theta}_{E_a} E_a\,,\, \xi)\xi^\uparrow =$$

$$= \sum_a g_\theta(B(E_a, E_a)\,,\, \xi)\xi^\uparrow = 2n g_\theta(H\,,\, \xi)\xi^\uparrow$$

because $\nabla^{g_\theta}_T T = 0$ implies $B(T,T) = 0$. Here B is the second fundamental form of $i : \partial M \hookrightarrow M$ and $H = (1/(2n))\,\mathrm{trace}_{g_\theta}(B)$ is its mean curvature vector. Then $\mathbb{H} = (2n/(2n+1))\,H^{\uparrow}$. □

Theorem 3. *Let M be a strictly pseudoconvex CR manifold-with-boundary and θ such that $T \in T(\partial M)$. Then $\partial C(M)$ has nonumbilic points in $(C(M), F_\theta)$. Moreover ∂M is totally umbilical in (M, g_θ) if and only if*

$$\mathbb{B}(X^{\uparrow}, Y^{\uparrow}) = \frac{2n+1}{2n}\, F_\theta(X^{\uparrow}, Y^{\uparrow})\mathbb{H},$$
$$\mathbb{B}(X^{\uparrow}, T^{\uparrow}) = \{(d\sigma)(X^{\uparrow}, \xi^{\uparrow}) + g_\theta(X, J\xi)\}\xi^{\uparrow},$$

for any $X, Y \in T(\partial M) \cap H(M)$.

Proof. By (25) and the Gauss formula for the immersion $\partial M \hookrightarrow (M, g_\theta)$

$$B(X,Y) = g_\theta(\nabla_X Y, \xi)\xi, \quad B(X,T) = g_\theta(\tau X + JX, \xi)\xi,$$

for any $X, Y \in T(\partial M) \cap H(M)$. Next, by Lemma 2 and the Gauss formula for the immersion $\partial C(M) \hookrightarrow (C(M), F_\theta)$

$$\mathbb{B}(X^{\uparrow}, Y^{\uparrow}) = B(X,Y)^{\uparrow}, \tag{26}$$

$$\mathbb{B}(X^{\uparrow}, T^{\uparrow}) = B(X,T)^{\uparrow} + \{(d\sigma)(X^{\uparrow}, \xi^{\uparrow}) + g_\theta(X, J\xi)\}\xi^{\uparrow}, \tag{27}$$

$$\mathbb{B}(X^{\uparrow}, \hat{S}) = -g_\theta(X, J\xi)\xi^{\uparrow}, \quad \mathbb{B}(T^{\uparrow}, \hat{S}) = 0. \tag{28}$$

Note that $J\xi$ is tangent to ∂M. Assume that $\mathbb{B} = F_\theta \otimes \mathbb{H}$. Then (by (28)) $J\xi$ is orthogonal to ∂M, hence $\xi = 0$, a contradiction. The last statement in Theorem 3 follows from $B = g_\theta \otimes H$ and (26)-(27). □

5. Minimal submanifolds

The purpose of this section to investigate minimal submanifolds in the Heisenberg group $\mathbb{H}_n$. First, we establish the relationship between the notion of X-minimality of N. Arcozzi & F. Ferrari, cf. (3) in [1], I. Birindelli & E. Lanconelli, cf. (3.23) in [6], and N. Garofalo & S.D. Pauls, cf. (2.5) in [14] (see also [23]) and minimality of an isometric immersion (between Riemannian manifolds). Second, we prove the following

Theorem 4. *Let $\Psi : N \to \mathbb{H}_n$ be an isometric immersion of a m-dimensional Riemannian manifold (N, g) into $(\mathbb{H}_n, g_{\theta_0})$. Then Ψ is minimal if and only if*

$$\Delta\Psi = 2JT^{\perp} \tag{29}$$

where Δ is the Laplace-Beltrami operator of (N, g). In particular, there are no minimal isometric immersions Ψ of a compact Riemannian manifold N into the Heisenberg group such that T is tangent to $\Psi(N)$.

Compare to Theorem 6.2 and Corollaries 6.1 and 6.2 in [8], p. 45-48. Let $M = \mathbb{H}_1$ be the lowest dimensional Heisenberg group and $\varphi : \mathbb{H}_1 \to \mathbb{R}$ a C^2 function. Let us set

$$N = \{x \in \mathbb{H}_1 : \varphi(x) = 0\}$$

and assume there is an open neighborhood $O \supset N$ such that

$$|\nabla \varphi(x)| \geq \alpha > 0, \quad x \in O. \tag{30}$$

Here $\nabla\varphi$ is the Euclidean gradient of φ. Let (z, t) be the natural coordinates on $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$ and set $Z = Z_1 = \partial/\partial z + i\,\overline{z}\,\partial/\partial t$ (the generator of $T_{1,0}(\mathbb{H}_1)$). Let $\theta_0 = dt + i(z\,d\overline{z} - \overline{z}\,dz)$ be the canonical contact form on $\mathbb{H}_1$. Note that $L_{\theta_0}(Z, \overline{Z}) = 1$. The Tanaka-Webster connection of $(\mathbb{H}_1, \theta_0)$ is given by

$$\Gamma^A_{BC} = 0, \quad A, B, C \in \{1, \overline{1}, 0\}.$$

Let us set $X_1 = \frac{1}{\sqrt{2}}(Z + \overline{Z})$ and $X_2 = Y_1 = \frac{i}{\sqrt{2}}(Z - \overline{Z})$. We shall prove the following

Theorem 5. *Let $N = \{x \in \mathbb{H}_1 : \varphi(x) = 0\}$ be a surface in $\mathbb{H}_1$ such that (30) holds. Assume that N is tangent to the characteristic direction $T = \partial/\partial t$ of $(\mathbb{H}_1, \theta_0)$. Let ξ be a unit normal vector field on N. Then the mean curvature vector of N in $(\mathbb{H}_1, g_{\theta_0})$ is given by*

$$H = -\frac{1}{2}\sum_{j=1}^{2} X_j\left(\frac{X_j\varphi}{|X\varphi|}\right)\,\xi. \tag{31}$$

Here $|X\varphi|^2 = (X_1\varphi)^2 + (X_2\varphi)^2$ is the X-gradient of φ.

Proof of Theorem 5. $T(N)$ is the span of $\{E\,,\,T\}$ while $T(N)^\perp$ is the span of ξ, where

$$E = \frac{1}{|X\varphi|}\{(X_2\varphi)X_1 - (X_1\varphi)X_2\}, \;\; \xi = \frac{1}{|X\varphi|}\{(X_1\varphi)X_1 + (X_2\varphi)X_2\},$$

so that $g_{\theta_0}(E, E) = 1$ and $g_{\theta_0}(\xi, \xi) = 1$. A calculation (based on $\nabla_{X_j}X_k = 0$) leads to

$$\nabla_{X_1}E = X_1\left(\frac{X_2\varphi}{|X\varphi|}\right)X_1 - X_1\left(\frac{X_1\varphi}{|X\varphi|}\right)X_2\,,$$

$$\nabla_{X_2}E = X_2\left(\frac{X_2\varphi}{|X\varphi|}\right)X_1 - X_2\left(\frac{X_1\varphi}{|X\varphi|}\right)X_2\,,$$

hence

$$\nabla_E E = \frac{1}{|X\varphi|}\left\{\left[(X_2\varphi)X_1\left(\frac{X_2\varphi}{|X\varphi|}\right) - (X_1\varphi)X_2\left(\frac{X_2\varphi}{|X\varphi|}\right)\right]X_1 + \right. \tag{32}$$
$$\left. + \left[(X_1\varphi)X_2\left(\frac{X_1\varphi}{|X\varphi|}\right) - (X_2\varphi)X_1\left(\frac{X_1\varphi}{|X\varphi|}\right)\right]X_2\right\}.$$

Then (by (32))

$$g_{\theta_0}(\nabla_E E, \xi) = -\sum_{j=1}^{2} X_j\left(\frac{X_j\varphi}{|X\varphi|}\right) + \tag{33}$$
$$+\frac{1}{|X\varphi|^2}\left\{(X_1\varphi)^2\ X_2\left(\frac{X_2\varphi}{|X\varphi|}\right) + (X_1\varphi)^2\ X_1\left(\frac{X_1\varphi}{|X\varphi|}\right) + \right.$$
$$\left. + (X_1\varphi)(X_2\varphi)\ X_1\left(\frac{X_2\varphi}{|X\varphi|}\right) + (X_1\varphi)(X_2\varphi)\ X_2\left(\frac{X_1\varphi}{|X\varphi|}\right)\right\}.$$

Using the identity

$$|X\varphi| X_j(|X\varphi|) = (X_1\varphi)\ X_j X_1\varphi + (X_2\varphi)\ X_j X_2\varphi$$

one may show that the second term in the right hand member of (33) is $|X\varphi|^{-4}$ times

$$(X_2\varphi)^2\{(X_2X_2\varphi)|X\varphi| - (X_2\varphi)X_2(|X\varphi|)\} +$$
$$+(X_1\varphi)^2\{(X_1X_1\varphi)|X\varphi| - (X_1\varphi)X_1(|X\varphi|)\} +$$
$$+(X_1\varphi)(X_2\varphi)\{(X_1X_2\varphi)|X\varphi| - (X_2\varphi)X_1(|X\varphi|)\} +$$
$$+(X_1\varphi)(X_2\varphi)\{(X_2X_1\varphi)|X\varphi| - (X_1\varphi)X_2(|X\varphi|)\} =$$
$$= -\{(X_1\varphi)X_1(|X\varphi|) + (X_2\varphi)X_2(|X\varphi|)\}\{(X_1\varphi)^2 + (X_2\varphi)^2\} +$$
$$+|X\varphi|\{(X_1\varphi)^2 X_1X_1\varphi + 2(X_1\varphi)(X_2\varphi)X_1X_2\varphi + (X_2\varphi)^2 X_2X_2\varphi\}$$

(as $[X_1, X_2] = -2\,T$ and $T(\varphi) = 0$) or

$$-(X_2\varphi)|X\varphi|\{(X_1\varphi)X_2X_1\varphi + (X_2\varphi)X_2X_2\varphi\} -$$
$$-(X_1\varphi)|X\varphi|\{(X_1\varphi)X_1X_1\varphi + (X_2\varphi)X_1X_2\varphi\} +$$
$$+|X\varphi|\{(X_1\varphi)^2 X_1X_1\varphi + 2(X_1\varphi)(X_2\varphi)X_1X_2\varphi + (X_2\varphi)^2 X_2X_2\varphi\} = 0$$

hence (33) leads to (31). □

Let us prove Theorem 4. Let $(x^1, \cdots, x^{2n}, x^0)$ be the Cartesian coordinates on $\mathbb{R}^{2n+1}$ and $(U, u^1, \cdots, u^m)$ a local coordinate system on N. Let $H(\Psi)$ be the mean curvature vector of $\Psi : N \to \mathbb{H}_n$. Then $H(\Psi) = H^A \partial_A$, where ∂_A is short for $\partial/\partial x^A$. Let $g_0 = g_{\theta_0}$ be the Webster metric of $(\mathbb{H}_n, \theta_0)$ and D^0 the Levi-Civita connection of $(\mathbb{H}_n, g_0)$. We set $B^A_\alpha = \partial\Psi^A/\partial u^\alpha$, so that $\Psi_*(\partial/\partial u^\alpha) = B^A_\alpha \partial_A$. Let $\{E_1, \cdots, E_m\}$ be a local orthonormal (with respect to g) frame of $T(N)$, defined on

U. Then $E_\alpha = E_\alpha^\beta \partial/\partial u^\beta$. Taking into account that $E_\alpha^\beta B_\beta^A = E_\alpha(\Psi^A)$, the Gauss formula of Ψ

$$D^0_{E_\alpha} E_\beta = \Psi_* D_{E_\alpha} E_\beta + B(E_\alpha, E_\beta)$$

may be written

$$\{E_\alpha(E_\beta \Psi^A) - (D_{E_\alpha} E_\beta)(\Psi^A)\}\partial_A =$$
$$= B(E_\alpha, E_\beta) - E_\alpha(\Psi^A) E_\beta(\Psi^B) D^0_{\partial_A} \partial_B \,.$$

Here D is the Levi-Civita connection of (N, g) and B is the second fundamental form of Ψ. Contraction of α and β gives

$$(\Delta \Psi^A)\partial_A = mH(\Psi) - \sum_{\alpha=1}^{m} E_\alpha(\Psi^A) E_\alpha(\Psi^B) D^0_{\partial_A} \partial_B \tag{34}$$

Since

$$\partial_j = \frac{\partial}{\partial x^j} = Z_j + \overline{Z}_j - 2y^j T, \;\; \partial_{j+n} = \frac{\partial}{\partial y^j} = i(Z_j - \overline{Z}_j) + 2x^j T, \tag{35}$$

it follows that the Tanaka-Webster connection of $(\mathbb{H}_n, \theta_0)$ satisfies

$$\nabla_{\partial_j} \partial_k = \nabla_{\partial_{j+n}} \partial_{k+n} = 0,$$

$$\nabla_{\partial_j} \partial_{k+n} = -\nabla_{\partial_{j+n}} \partial_k = 2\delta_{jk} T, \tag{36}$$
$$\nabla_{\partial_A} T = \nabla_T \partial_B = 0.$$

Let J be the complex structure in $H(\mathbb{H}_n)$, extended to a $(1,1)$-tensor field on $\mathbb{H}_n$ by requesting that $JT = 0$. Using $D^0 = \nabla - (d\theta_0) \otimes T + 2(\theta_0 \odot J)$ it follows that

$$E_\alpha(\Psi^A) E_\beta(\Psi^B) D^0_{\partial_A} \partial_B = E_\alpha(\Psi^A) E_\beta(\Psi^B) \nabla_{\partial_A} \partial_B -$$
$$-(d\theta)(E_\alpha, E_\beta) T + \theta(E_\alpha) J \Psi_* E_\beta + \theta(E_\beta) J \Psi_* E_\alpha$$

where $\theta = \Psi^* \theta_0$. On the other hand, by (36)

$$E_\alpha(\Psi^A) E_\beta(\Psi^B) \nabla_{\partial_A} \partial_B =$$
$$= 2 \sum_{j=1}^{n} \{E_\alpha(\Psi^j) E_\beta(\Psi^{j+n}) - E_\alpha(\Psi^{j+n}) E_\beta(\Psi^j)\} T.$$

Also $\sum_\alpha \theta(E_\alpha) = \sum_\alpha g_0(T, \Psi_* E_\alpha) = \sum_\alpha g(T^T, E_\alpha)$ hence

$$\sum_\alpha \theta(E_\alpha) J \Psi_* E_\alpha = J \Psi_* T^T = -J T^\perp,$$

so that (34) becomes $mH(\Psi) = \Delta \Psi - 2JT^\perp$ (yielding (29)). □

Our Theorem 5 demonstrates that the Webster metric is the "correct" choice of ambient metric. Nevertheless, even the geometry of a hyperplane in $(\mathbb{H}_n\,,\,g_0)$ turns out to be rather involved. In the sequel, we work out explicitly the case of $\{z \in \mathbb{H}_n : t = 0\}$.

Example 1. (*continued*) Let $\Psi : \partial\mathbb{H}_n^+ \to \mathbb{H}_n^+$ be the inclusion and $g = \Psi^* g_0$ (the first fundamental form of Ψ). Let Δ be the Laplace-Beltrami operator of $(\partial\mathbb{H}_+^n, g)$. We may state

Proposition 5. *The coordinate functions z^j on $\partial\mathbb{H}_n^+ \approx \mathbb{C}^n$ satisfy $\Delta z^j = 2\,z^j/(1+2|z|^2)$. Consequently the boundary of $(\mathbb{H}_n^+, g_0)$ is minimal.*

Note that

$$\theta_0(\partial_i) = -2y_i\,, \quad \theta_0(\partial_{i+n}) = 2x_i\,,$$
$$(d\theta_0)(\partial_i, \partial_j) = (d\theta_0)(\partial_{i+n}, \partial_{j+n}) = 0, \;\; (d\theta_0)(\partial_i, \partial_{j+n}) = 2\delta_{ij}\,,$$
$$J\partial_j = \partial_{j+n} - 2x_j T, \quad J\partial_{j+n} = -\partial_j - 2y_j T.$$

Then by (25) (with $\tau = 0$) and by (36) it follows that

$$D^0_{\partial_i}\partial_j = -2(y_i\delta^k_j + y_j\delta^k_i)\frac{\partial}{\partial y^k} + 4(y_i x_j + y_j x_i)T,$$
$$D^0_{\partial_i}\partial_{j+n} = 2(y_i\frac{\partial}{\partial x^j} + x_j\frac{\partial}{\partial y^i}) + 4(y_i y_j - x_i x_j)T,$$
$$D^0_{\partial_{i+n}}\partial_{j+n} = -2(x_i\delta^k_j + x_j\delta^k_i)\frac{\partial}{\partial x^k} - 4(x_i y_j + x_j y_i)T.$$

Next, we shall need the Gauss formula

$$D^0_{\partial_a}\partial_b = D_{\partial_a}\partial_b + B(\partial_a, \partial_b),$$

where D is the Levi-Civita connection of $(\partial\mathbb{H}_n^+, g)$. We obtain

$$B(\partial_i, \partial_j) = 4c(y_i x_j + y_j x_i)\xi,$$
$$B(\partial_i, \partial_{j+n}) = 4c(y_i y_j - x_i x_j)\xi, \tag{37}$$
$$B(\partial_{i+n}, \partial_{j+n}) = -4c(x_i y_j + x_j y_i)\xi,$$

hence Ψ is not totally geodesic, and if $D_{\partial_a}\partial_b = \Gamma^c_{ab}\partial_c$ then

$$\Gamma^k_{ij} = -4c(y_i x_j + y_j x_i)y^k\,, \;\; \Gamma^{k+n}_{i+n\;j+n} = -4c(x_i y_j + x_j y_i)x^k\,,$$
$$\Gamma^{k+n}_{ij} = 4c(y_i x_j + y_j x_i)x^k - 2(y_i\delta^k_j + y_j\delta^k_i),$$
$$\Gamma^k_{i\;j+n} = 2y_i\delta^k_j - 4c(y_i y_j - x_i x_j)y^k\,, \tag{38}$$
$$\Gamma^{k+n}_{i\;j+n} = 2x_j\delta^k_i + 4c(y_i y_j - x_i x_j)x^k\,,$$
$$\Gamma^k_{i+n\;j+n} = -2(x_i\delta^k_j + x_j\delta^k_i) + 4c(x_i y_j + x_j y_i)y^k\,.$$

We need the following

Lemma 7. *The local coefficients of the cometric g^{-1} on $T^*(\partial\mathbb{H}_n^+)$ are given by*

$$g^{-1}:\begin{pmatrix} \frac{1}{2}\delta^{ij}-cy^iy^j & cy^ix^j \\ cx^iy^j & \frac{1}{2}\delta^{ij}-cx^ix^j \end{pmatrix}. \tag{39}$$

Consequently

$$\Delta u=\frac{1}{2}\,\Delta_0 u+2c\frac{\partial u}{\partial r}-c\{y^iy^j\frac{\partial^2 u}{\partial x^i\partial x^j}-2y^ix^j\frac{\partial^2 u}{\partial x^i\partial y^j}+x^ix^j\frac{\partial^2 u}{\partial y^i\partial y^j}\}$$

for any $u\in C^2(\partial\mathbb{H}_n^+)$, where Δ_0 is the ordinary Laplacian on $\mathbb{R}^{2n}$ and $\partial/\partial r$ is the radial vector field $x^j(\partial/\partial x^j)+y^j(\partial/\partial y^j)$.

By Lemma 7 it follows that $\Delta x^j=2cx^j$ and $\Delta y^j=2cy^j$, hence the first statement in Proposition 5. On the other hand $T^\perp=c\xi$ implies $JT^\perp=c\,\partial/\partial r$ hence (by Theorem 4) $H(\Psi)=0$. Note that the mean curvature vector may be also computed from $2nH(\Psi)=g^{ab}B(\partial_a,\partial_b)$ by (37) and (39).

It remains that we prove Lemma 7. The first statement is elementary yet rather involved. The identities $g_{ac}g^{cb}=\delta_a^b$ may be written

$$\begin{cases} 2(\delta_{ij}+2y_iy_j)g^{jk}-4y_ix_jg^{j+n,k}=\delta_i^k\,, \\ 2(\delta_{ij}+2y_iy_j)g^{j,k+n}-4y_ix_jg^{j+n,k+n}=0, \\ -4x_iy_jg^{jk}+2(\delta_{ij}+2x_ix_j)g^{j+n,k}=0, \\ -4x_iy_jg^{j,k+n}+2(\delta_{ij}+2x_ix_j)g^{j+n,k+n}=\delta_i^k\,. \end{cases} \tag{40}$$

Contraction of the first two equations (respectively of the last two equations) by y^i (respectively by x^i) gives

$$(1+2|y|^2)y_jg^{jk}-2|y|^2x_jg^{j+n,k}=y^k\,,$$

$$(1+2|y|^2)y_jg^{j,k+n}-2|y|^2x_jg^{j+n,k+n}=0,$$

$$2|x|^2y_jg^{jk}-(1+2|x|^2)x_jg^{j+n,k}=0,$$

$$-2|x|^2y_jg^{j,k+n}+(1+2|x|^2)x_jg^{j+n,k+n}=x^k\,,$$

where from

$$y_jg^{jk}=\frac{c}{2}(1+2|x|^2)y^k\,,\quad x_jg^{j+n,k}=c|x|^2y^k\,,$$

$$y_jg^{j,k+n}=c|y|^2x^k\,,\quad x_jg^{j+n,k+n}=\frac{c}{2}(1+2|y|^2)x^k\,,$$

and substitution back into (40) yields (39). To compute the Laplacian

$$\Delta u=\frac{\partial}{\partial x^a}\left(g^{ab}\frac{\partial u}{\partial x^b}\right)+g^{ab}\frac{\partial}{\partial x^a}\left(\log\sqrt{G}\right)\frac{\partial u}{\partial x^b}$$

(with $G = \det[g_{ab}]$) we recall that $\partial(\log\sqrt{G})/\partial x^a = \Gamma^b_{ba}$ hence (by (38))

$$\frac{\partial}{\partial x^a}\left(\log\sqrt{G}\right) = 2cx_a\,, \quad 1 \leq a \leq 2n.$$

Then (39) yields the result. □

6. The CR Yamabe problem

Let M be a compact strictly pseudoconvex CR manifold-with-boundary, of CR dimension n, and θ a contact form on M with G_θ positive definite. Let us assume that ∂M is tangent to the characteristic direction T of $d\theta$.

Lemma 8. *Let us set $p = 2 + 2/n$ and $f = (p-2)\log u$, with $u \in C^\infty(M)$, $u > 0$. If $\hat{\theta} = e^f\theta$ then $\partial C(M)$ is minimal in $(C(M), F_{\hat{\theta}})$ if and only if*

$$\frac{\partial(u\circ\pi)}{\partial\eta} - n\,F_\theta(\mathbb{H},\eta)\,u\circ\pi = 0 \quad \text{on} \quad \partial C(M), \tag{41}$$

where η and $\mathbb{H}$ are respectively an outward unit normal and the mean curvature vector of the immersion $\partial C(M) \hookrightarrow (C(M), F_\theta)$. In particular, if ξ and H are an outward unit normal and the mean curvature vector of the immersion $\partial M \hookrightarrow (M, g_\theta)$ then (41) projects to

$$\frac{\partial u}{\partial\xi} - \frac{2n^2}{2n+1}\,g_\theta(H,\xi)\,u = 0 \quad \text{on} \quad \partial M. \tag{42}$$

The first statement in Lemma 8 is of course well known in conformal geometry. We give a brief proof for the convenience of the reader. If $\hat{\theta} = e^f\theta$ the corresponding Fefferman metric is $F_{\hat{\theta}} = e^{f\circ\pi}F_\theta$ hence the Levi-Civita connections $\hat{D}$ and D (of $F_{\hat{\theta}}$ and F_θ, respectively) are related by

$$\hat{D}_V W = D_V W + \frac{1}{2}\{V(f)W + W(f)V - F_\theta(V,W)D(f\circ\pi)\}, \tag{43}$$

for any $V, W \in T(C(M))$, where $D(f\circ\pi)$ is the gradient of $f\circ\pi$ with respect to F_θ. Our assumption $T \in T(\partial M)$ and Proposition 1 imply that $T(\partial C(M))$ is nondegenerate in $T(C(M))$ with respect to F_θ, hence with respect to $F_{\hat{\theta}}$ as well. Let $\mathbb{B}$ and $\hat{\mathbb{B}}$ be the second fundamental forms of the immersions $\partial C(M) \hookrightarrow (C(M), F_\theta)$ and $\partial C(M) \hookrightarrow (C(M), F_{\hat{\theta}})$. Then (by (43) and the Gauss formula)

$$\hat{\mathbb{B}} = \mathbb{B} - \frac{1}{2}\,F_\theta \otimes (D(f\circ\pi))^\perp\,. \tag{44}$$

Taking traces in (44) shows that the mean curvature vectors of the two immersions are related by $\hat{\mathbb{H}} = e^{-f}\{\mathbb{H} - \frac{1}{2}(D(f\circ\pi))^{\perp}\}$ hence $\partial C(M)$ is minimal in $(C(M), F_{\hat{\theta}})$ if and only if $\mathbb{H} = (1/2)(D(f\circ\pi))^{\perp}$ and (41) is proved. Let ξ be an outward unit normal on ∂M in (M, g_θ). Then $\eta = \xi^{\uparrow}$ is an outward unit normal on $\partial C(M)$ in $(C(M), F_\theta)$. Then (by Theorem 2) the mean curvatures of $\partial M \hookrightarrow (M, g_\theta)$ and $\partial C(M) \hookrightarrow (C(M), F_\theta)$ are related by

$$F_\theta(\mathbb{H}, \eta) = \frac{2n}{2n+1}\, g_\theta(H, \xi)\circ\pi$$

hence (41) projects on M to give (42). $\square$

We may consider the problem

$$-b_n\,\Delta_b u + \rho\, u = \lambda\, u^{p-1} \quad \text{in } M, \tag{45}$$

$$\frac{\partial u}{\partial \xi} - \frac{2n^2}{2n+1}\,\mu_\theta\, u = 0 \quad \text{on } \partial M, \tag{46}$$

(the *CR Yamabe problem* on a CR manifold-with-boundary) where

$$\Delta_b u = \operatorname{div}(\nabla^H u), \quad u \in C^2(M),$$

is the *sublaplacian* of (M, θ), $b_n = 2 + 2/n$, λ is a constant, and $\mu_\theta = g_\theta(H, \xi) \in \{\pm\|H\|\}$. Also ∇u is the gradient of u with respect to g_θ and $\nabla^H u = \pi_H \nabla u$ (the *horizontal gradient*) where $\pi_H : T(M) \to H(M)$ is the projection associated with the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$. The divergence operator is meant with respect to the volume form $\omega = \theta\wedge(d\theta)^n$. The problem (45)-(46) is a nonlinear subelliptic problem of variational origin. Indeed, we may state

Theorem 6. *Let us set*

$$A_\theta(u) = \int_M \{b_n\|\nabla^H u\|^2 + \rho\, u^2\}\omega - a_n \int_{\partial M} \mu_\theta\, u^2\, d\sigma\,,$$

$$B_\theta(u) = \int_M |u|^p \omega\,,$$

where $\sigma = \operatorname{vol}(i^* g_\theta)$*, the canonical volume form associated with the induced metric* $i^* g_\theta$ *on* ∂M*, and* $a_n = 2^{n+2}\,(n+1)!\,n/(2n+1)$*. Moreover, let*

$$Q_\theta(u) = \frac{A_\theta(u)}{B_\theta(u)}\,, \quad Q(M) = \inf\{Q_\theta(u) : u \in C^\infty(M),\ \ u > 0\}.$$

If $u \in C^\infty(M)$ *is a positive function such that* $Q_\theta(u) = Q(M)$ *then* u *is a solution to* (45)-(46) *with* $\lambda = (p/2)Q(M)$*, a CR invariant of* M*.*

Proof. If $\{T_\alpha\}$ is a local frame of $T_{1,0}(M)$ then the horizontal gradient is expressed by $\nabla^H u = u^\alpha T_\alpha + u^{\overline{\alpha}} T_{\overline{\alpha}}$, where $u^\alpha = g^{\alpha\overline{\beta}} u_{\overline{\beta}}$ and $u_{\overline{\beta}} = T_{\overline{\beta}}(u)$, hence $\|\nabla^H u\|^2 = 2 u_\alpha u^\alpha$. Then

$$\frac{d}{dt}\{A_\theta(u+th)\}_{t=0} = 2\int_M \{b_n(u^\alpha h_\alpha + u_\alpha h^\alpha) + \rho u h\}\omega - 2a_n \int_{\partial M} \mu_\theta\, u\, h\, d\sigma,$$

for any $h \in C^2(\mathrm{Int}(M)) \cap C^1(M)$ (where $\mathrm{Int}(M) = M \setminus \partial M$). On the other hand

$$\int_M u^\alpha h_\alpha\, \omega = \int_M \{T_\alpha(u^\alpha h) - h T_\alpha(u^\alpha)\}\omega =$$

$$= \int_M \mathrm{div}(h u^\alpha T_\alpha)\omega - \int_M \{T_\alpha(u^\alpha) + u^\alpha \mathrm{div}(T_\alpha)\} h\, \omega.$$

Note that $\mathrm{div}(T_\alpha) = \Gamma^\beta_{\beta\alpha}$ hence $T_\alpha(u^\alpha) + u^\alpha \mathrm{div}(T_\alpha) = u^\alpha{}_\alpha$, where $u^\alpha{}_\beta = g^{\alpha\overline{\gamma}} u_{\overline{\gamma}\beta}$ and $u_{\alpha\overline{\beta}} = (\nabla^2 u)(T_\alpha, T_{\overline{\beta}})$. The complex Hessian is meant with respect to the Tanaka-Webster connection i.e.

$$(\nabla^2 u)(X,Y) = (\nabla_X du)Y = X(Y(u)) - (\nabla_X Y)(u),$$

for any $X, Y \in \mathcal{X}(M)$. Note that $\omega = c_n\, d\,\mathrm{vol}(g_\theta)$ (with $c_n = 2^n n!$). Then (by Green's lemma)

$$\int_M u^\alpha h_\alpha\, \omega = c_n \int_{\partial M} h\, u^\alpha g_\theta(T_\alpha, \xi) d\sigma - \int_M u^\alpha{}_\alpha\, h\, \omega.$$

As the sublaplacian is locally given by

$$\Delta_b u = u^\alpha{}_\alpha + u^{\overline{\alpha}}{}_{\overline{\alpha}}$$

we may conclude that

$$\frac{d}{dt}\{A_\theta(u+th)\}_{t=0} = 2\int_M (-b_n\, \Delta_b u + \rho\, u) h\, \omega + \tag{47}$$

$$+2\int_{\partial M} [b_n c_n g_\theta(\nabla^H u, \xi) - a_n \mu_\theta u] h\, d\sigma.$$

Also

$$\frac{d}{dt}\{B_\theta(u+th)\}_{t=0} = p\int_M u^{1+2/n} h\, \omega. \tag{48}$$

As $T \in T(\partial M)$ one has $\xi \in H(M)$ hence $g_\theta(\nabla^H u, \xi) = \xi(u)$ (also denoted by $\partial u/\partial\xi$). If u achieves $Q(M)$

$$\frac{d}{dt}\{Q_\theta(u+th)\}_{t=0} = 0$$

hence

$$2\int_M (-b_n\, \Delta_b u + \rho\, u) h\, \omega + 2\int_{\partial M} [b_n c_n\, \xi(u) - a_n \mu_\theta u] h\, d\sigma -$$

$$-p\, Q_\theta(u) \int_M u^{1+2/n} h\, \omega = 0.$$

In particular this holds for $h|_{\partial M} = 0$ hence

$$-b_n\, \Delta_b u + \rho u = (p/2) Q(M) u^{1+2/n}$$

and going back to arbitrary h

$$\frac{\partial u}{\partial \xi} - \frac{a_n}{b_n c_n}\, \mu_\theta\, u = 0 \quad \text{on} \quad \partial M$$

which is (46) because $a_n/(b_n c_n) = 2n^2/(2n+1)$. The proof that $Q(M)$ is a CR invariant is similar to the arguments in [16], p. 174-175. Let $E^+ \to M$ be the $\mathbb{R}_+$-bundle spanned by θ and let us set

$$E^\alpha_x = \{\nu : E^+_x \to \mathbb{R} : \nu(t\theta_x) = t^{-\alpha} \nu(\theta_x), \text{ for all } t > 0\}, \quad (\alpha > 0)$$

for any $x \in M$. Then $(\nu_\theta)_x(t\theta_x) = 1/t$ defines a global frame $\{\nu_\theta\}$ of $E^1 \to M$ (and of course $\{\nu^\alpha_\theta\}$ is a global frame of $E^\alpha \to M$). We need the *CR invariant sublaplacian*

$$L : \Gamma^\infty(E^{n/2}) \to \Gamma^\infty(E^{1+n/2}), \quad L(u\, \nu_\theta^{n/2}) = (-b_n\, \Delta_b u + \rho\, u) \nu_\theta^{1+n/2}.$$

By definition $\int_M u\, \nu_\theta^{n+1} = \int_M u\, \omega$. A section $s = u\nu^\alpha_\theta$ in E^α is *positive* if $u > 0$. Finally, the fact that $Q(M)$ is a CR invariant follows from

$$Q(M) = \inf\{\int_M (Ls) \otimes s \;:\; s \in \Gamma^\infty(E^{n/2}) \tag{49}$$

$$\text{a positive section such that } \int_M s^p = 1\}.$$

The identity (49) follows from the fact that the sets $\{A_\theta(u) : B_\theta(u) = 1, u > 0\}$ and $\{A_\theta(u)/B_\theta(u) : u > 0\}$ coincide and from the calculation

$$\int_M (Ls) \otimes s = \int_M (-b_n\, u \Delta_b u + \rho u^2) \omega,$$

$$\int_M u(\Delta_b u)\, \omega = \int_M \{\text{div}(u \nabla^H u) - \|\nabla^H u\|^2\} \omega =$$

$$= c_n \int_{\partial M} u \frac{\partial u}{\partial \xi}\, d\sigma - \int_M \|\nabla^H u\|^2 \omega,$$

hence (by (46)) $\int_M (Ls) \otimes s = A_\theta(u)$, for any $s = u\nu_\theta^{n/2} \in \Gamma^\infty(E^{n/2})$.

7. Minimal surfaces in $\mathbb{H}_n$

Let (N, g) be a 2-dimensional Riemannian manifold and $\Psi : N \to \mathbb{H}_n$ a minimal isometric immersion of (N, g) into $(\mathbb{H}_n, g_0)$. Let $(U, z = x + iy)$ be isothermal local coordinates on N, i.e. locally

$$g = 2E(dx^2 + dy^2),$$

for some $E \in C^\infty(U)$, $E > 0$. As well known the Laplace-Beltrami operator of (N, g) is locally given by

$$\Delta u = \frac{2}{E} \frac{\partial^2 u}{\partial z \partial \overline{z}}, \quad u \in C^2(N).$$

Let us set $F^j = \Psi^j + i\Psi^{j+n}$, $1 \leq j \leq n$, and $f = \Psi^0$. Also, we consider $K : U \to \mathbb{C}$ given by

$$K = \frac{\partial f}{\partial z} + i \sum_{j=1}^{n} (F^j \frac{\partial \overline{F}^j}{\partial z} - \overline{F}^j \frac{\partial F^j}{\partial z}).$$

Lemma 9. *The normal component of the characteristic vector field $T = \partial/\partial t$ of $d\theta_0$ is locally given by*

$$T^\perp = (1 - \frac{2}{E}|K|^2)T - \tag{50}$$

$$-\frac{1}{E}\{(\overline{K}\frac{\partial F^j}{\partial z} + K\frac{\partial F^j}{\partial \overline{z}})Z_j + (\overline{K}\frac{\partial \overline{F}^j}{\partial z} + K\frac{\partial \overline{F}^j}{\partial \overline{z}})\overline{Z}_j\}.$$

Proof. The characteristic direction decomposes as $T = \Psi_* T^T + T^\perp$, where $T^T = \lambda \partial/\partial z + \overline{\lambda}\partial/\partial \overline{z}$, for some $\lambda \in C^\infty(U)$. Taking the inner product with $\Psi_* \partial/\partial \overline{z}$ yields $\lambda = \overline{K}/E$ hence (35) yields (50). □

Lemma 10. *Let $\Psi : N \to \mathbb{H}_n$ be an isometric immersion of (N, g) into $(\mathbb{H}_n, g_0)$. Then*

$$2\sum_{j=1}^{n} \frac{\partial F^j}{\partial z}\frac{\partial F^j}{\partial \overline{z}} + K^2 = 0, \tag{51}$$

$$\sum_{j=1}^{n} \left(\left| \frac{\partial F^j}{\partial z} \right|^2 + \left| \frac{\partial F^j}{\partial \overline{z}} \right|^2 \right) + |K|^2 \neq 0. \tag{52}$$

Proof. A calculation based on (35) shows that the Webster metric of $(\mathbb{H}_n, \theta_0)$ is given (with respect to the frame $\{\partial/\partial x^j,\ \partial/\partial y^j,\ \partial/\partial t\}$) by

$$g_0 : \begin{pmatrix} 2(\delta_{jk} + 2y_j y_k) & -4y_j x_k & -2y_j \\ -4x_j y_k & 2(\delta_{jk} + 2x_j x_k) & 2x_j \\ -2y_k & 2x_k & 1 \end{pmatrix}$$

hence

$$g_\theta(\Psi_* \frac{\partial}{\partial z}\,,\ \Psi_* \frac{\partial}{\partial \overline{z}}) = \Psi_z^A \Psi_{\overline{z}}^B g_{AB} = |K|^2 + \sum_j (|F_z^j|^2 + |F_{\overline{z}}^j|^2),$$

$$g_\theta(\Psi_* \frac{\partial}{\partial z}\,,\ \Psi_* \frac{\partial}{\partial z}) = \Psi_z^A \Psi_z^B g_{AB} = K^2 + \sum_j F_z^j F_{\overline{z}}^j,$$

(where $g_{AB} = g_0(\partial_A, \partial_B)$). Since Ψ is an isometric immersion

(53) $$g_0(\Psi_* \frac{\partial}{\partial x}\ ,\ \Psi_* \frac{\partial}{\partial y}) = 0,$$

(54) $$g_0(\Psi_* \frac{\partial}{\partial x}\ ,\ \Psi_* \frac{\partial}{\partial x}) = g_0(\Psi_* \frac{\partial}{\partial y}\ ,\ \Psi_* \frac{\partial}{\partial y}),$$

and then (53)-(54) yield (51)-(52), respectively. $\square$

Note that (again by (35))

$$\Delta\Psi = (\Delta\psi^A)\partial_A = (\Delta F^j) Z_j + (\Delta \overline{F}^j) \overline{Z}_j +$$

$$+ \{\Delta f + 2 \sum_{j=1}^{n} (\Psi^j \Delta \Psi^{j+n} - \Psi^{j+n} \Delta \Psi^j)\} T$$

and (by Lemma 9)

$$iE\, JT^\perp = (\overline{K} F_z^j + K F_{\overline{z}}^j) Z_j - (\overline{K}\ \overline{F}_z^j + K \overline{F}_{\overline{z}}^j) \overline{Z}_j$$

hence the minimality condition (29) becomes

(55) $$\Delta F^j = -\frac{2i}{E} (\overline{K} F_z^j + K F_{\overline{z}}^j), \quad 1 \le j \le n,$$

and $\Delta f = \frac{i}{2} \sum_j (\overline{F}^j \Delta F^j - F^j \Delta \overline{F}^j)$ or (by (55))

(56) $$\Delta f = \frac{1}{E} \{\overline{K} (|F|^2)_z + K (|F|^2)_{\overline{z}}\}.$$

Let N be a Riemann surface. An immersion $\Psi : N \to \mathbb{H}_n$ is *conformal* if (53)-(54) hold, for any local complex coordinate system $(U, z = x + iy)$

on N. Moreover (55)-(56) lead to the following definition. A *minimal surface* in $\mathbb{H}_n$ is a Riemann surface N together with a conformal immersion $\Psi : N \to \mathbb{H}_n$ such that

$$F^j_{z\overline{z}} + i(\overline{K}F^j_z + KF^j_{\overline{z}}) = 0, \quad 1 \le j \le n, \tag{57}$$

$$f_{z\overline{z}} - \frac{1}{2}\{\overline{K}(|F|^2)_z + K(|F|^2)_{\overline{z}}\} = 0. \tag{58}$$

Here $|F|^2 = \sum_j F^j \overline{F}^j$. We may state the following

Theorem 7. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $\Psi : \Omega \to \mathbb{H}_n$ a minimal surface such that $JT^\perp = 0$ (e.g. $\Psi(\Omega)$ is tangent to the characteristic direction of $d\theta_0$). Let us set $\Phi = \partial\Psi/\partial z$. Then Φ is holomorphic and* (51)-(52) *hold in Ω. Viceversa, let $\Phi : \Omega \to \mathbb{C}^{2n+1}$ be a holomorphic map and let us set*

$$\Psi^A(z) = \mathrm{Re}\int_o^z \Phi^j(\zeta)d\zeta, \quad A \in \{0, 1, \cdots, 2n\}, \tag{59}$$

for any $z \in \Omega$, where $o \in \Omega$ is a fixed base point. Let $K : \Omega \to \mathbb{C}$ be given by

$$K = \Phi^0 - 2\sum_{j=1}^n \{\Phi^j \,\mathrm{Re}\int_o^z \Phi^{j+n}(\zeta)d\zeta + \Phi^{j+n}\,\mathrm{Re}\int_o^z \Phi^j(\zeta)d\zeta\}.$$

If the following identities hold in Ω

$$2\sum_{j=1}^n \{|\Phi^j|^2 - |\Phi^{j+n}|^2 + i(\Phi^{j+n}\overline{\Phi}^j + \Phi^j\overline{\Phi}^{j+n})\} + K^2 = 0, \tag{60}$$

$$2\sum_{j=1}^n (|\Phi^j|^2 + |\Phi^{j+n}|^2) + |K|^2 \ne 0, \tag{61}$$

$$\overline{K}(\Phi^j + i\Phi^{j+n}) + K(\overline{\Phi}^j + i\overline{\Phi}^{j+n}) = 0, \quad 1 \le j \le n, \tag{62}$$

then $\Psi : \Omega \to \mathbb{H}_n$ is a minimal immersion such that $JT^\perp = 0$.

Compare to Theorem 8.1 in [8], p. 58. *Proof of Theorem* 7. (51)-(52) follow from Lemma 10. Next $JT^\perp = 0$ and (55)-(56) yield $\partial\Phi/\partial\overline{z} = 0$ in Ω.

Viceversa, given a holomorphic map $\Phi : \Omega \to \mathbb{C}^{2n+1}$ the function Ψ^A given by (59) is well defined (by the classical theorem of Cauchy the integral doesn't depend upon the choice of path from o to z) and

$\partial\Psi/\partial z = \Phi$ hence (60)-(61) yield (51)-(52) so that (53)-(54) are satisfied and $g_0(\Psi_*\partial/\partial x\, , \Psi_*\partial/\partial x) \neq 0$, i.e. Ψ is a conformal immersion. Finally (62) may be written

$$\overline{K} F^j_z + K F^j_{\overline{z}} = 0, \quad 1 \leq j \leq n,$$

which is equivalent (by Lemma 9) to $JT^\perp = 0$ and (57)-(58) imply minimality. □

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