

Quantum Geometry and Gravity: Recent Advances

Abhay Ashtekar

*Center for Gravitational Physics and Geometry,
Physics Department, Penn State, University Park, PA 16802, USA*

Abstract

Over the last three years, a number of fundamental physical issues were addressed in loop quantum gravity. These include: A statistical mechanical derivation of the horizon entropy, encompassing astrophysically interesting black holes as well as cosmological horizons; a natural resolution of the big-bang singularity; the development of spin-foam models which provide background independent path integral formulations of quantum gravity and ‘finiteness proofs’ of some of these models; and, the introduction of semi-classical techniques to make contact between the background independent, non-perturbative theory and the perturbative, low energy physics in Minkowski space. These developments spring from a detailed quantum theory of geometry that was systematically developed in the mid-nineties and have added a great deal of optimism and intellectual excitement to the field.

The goal of this article is to communicate these advances in general physical terms, accessible to researchers in all areas of gravitational physics represented in this conference.

I. INTRODUCTION

Let us begin by recalling some of the central conceptual and physical questions of quantum gravity.

- *Big-Bang and other singularities:* It is widely believed that the prediction of a singularity, such as the big-bang of classical general relativity, is primarily a signal that the theory has been pushed beyond the domain of its validity. A key question to any quantum gravity theory, then, is: What replaces the big-bang? Qualitatively, classical geometry may be a mean field like ‘magnetization’, which provides an excellent macroscopic description of a ferromagnet. However, at the Curie temperature, magnetization goes to zero and susceptibility diverges. But there is no physical infinity; we simply have to turn to the correct *microscopic* description in terms of spin-systems to describe physics. Does something similar happen at the big-bang and other singularities? Is there a mathematically consistent description of the quantum state of the universe which replaces the classical big-bang? What is the analog of the microscopic spin-system that underlies magnetism? What can we say about the ‘initial conditions’, i.e., the quantum state of geometry and matter that correctly describes the big-bang? If they have to be imposed externally, is there a *physical* guiding principle?

- *Black holes:* In the early seventies, using imaginative thought experiments, Bekenstein argued that black holes must carry an entropy proportional to their area. About the same time, Bardeen, Carter and Hawking (BCH) showed that black holes in equilibrium obey two basic laws, which have the same form as the zeroth and the first laws of thermodynamics, provided one equates the black hole surface gravity κ to some multiple of the temperature T in thermodynamics and the horizon area a_{hor} to a corresponding multiple of the entropy S . However, at first this similarity was thought to be only a formal analogy because the BCH analysis was based on *classical* general relativity and simple dimensional considerations show that the proportionality factors must involve Planck's constant \hbar . Two years later, using quantum field theory on a black hole background space-time, Hawking showed that black holes in fact radiate quantum mechanically as though they are black bodies at temperature $T = \hbar\kappa/2\pi$. Using the analogy with the first law, one can then conclude that the black hole entropy should be given by $S_{\text{BH}} = a_{\text{hor}}/4G\hbar$. This conclusion is striking and deep because it brings together the three pillars of fundamental physics —general relativity, quantum theory and statistical mechanics. However, the argument itself is a rather hodge-podge mixture of classical and semi-classical ideas, reminiscent of the Bohr theory of atom. A natural question then is: what is the analog of the more fundamental, Pauli-Schrödinger theory of the Hydrogen atom? More precisely, what is the statistical mechanical origin of black hole entropy? What is the nature of a quantum black hole and what is the interplay between the quantum degrees of freedom responsible for entropy and the exterior curved geometry? Can one derive the Hawking effect from first principles of quantum gravity? Is there an imprint of the classical singularity on the final quantum description, e.g., through ‘information loss’ ?

- *Planck scale physics and the low energy world:* Perhaps the central lesson of general relativity is that *gravity is geometry*. There is no longer a background metric, no inert stage on which dynamics unfolds. Geometry itself is dynamical. Therefore, one expects that a fully satisfactory quantum gravity theory would also be free of a background space-time geometry. However, of necessity, a background independent description must use physical concepts and mathematical tools that are quite different from those of the familiar, low energy physics. A major challenge then is to show that this low energy description does arise from the pristine, Planckian world in an appropriate sense. In this ‘top-down’ approach, does the fundamental theory admit a “sufficient number” of semi-classical states? Do these semi-classical sectors provide enough of a background geometry to anchor low energy physics? can one recover the familiar description? Furthermore, can one pin point why the standard ‘bottom-up’ perturbative approach fail? That is, what is the essential feature which makes the fundamental description mathematically coherent but is absent in the standard perturbative quantum gravity?

Of course, this is by no means a complete list of challenges. There are many others: the issue of time, of measurement theory and the associated questions of interpretation of the quantum framework, the issue of diffeomorphism invariant observables and practical methods of computing their properties, practical methods of computing time evolution and S-matrices, exploration of the role of topology and topology change, The purpose of this report is to summarize recent advances in the non-perturbative approach based on quantum geometry which has led to illuminating answers to many of these questions and opened-up avenues to address others. The plenary session of this conference also covered simplicial

quantum gravity and string theory which, in a certain sense, complement our approach. And there are other approaches as well, ranging from twistors and causal sets to Euclidean path integrals. Unfortunately, due to space limitation, I will not be able to discuss these; indeed, even within the approach I focus on, I can discuss only a few illustrative examples. I apologize in advance to the authors whose very interesting contributions could not be referred to in this brief report.

II. A BIRD'S EYE VIEW OF LOOP QUANTUM GRAVITY

In this section, I will briefly summarize the salient features and current status of loop quantum gravity. The emphasis is on structural and conceptual issues; detailed treatments can be found in references [1-9] and papers they refer to.

A. Viewpoint

In this approach, one takes the central lesson of general relativity seriously: gravity *is* geometry whence, in a fundamental theory, there should be no background metric. In quantum gravity, geometry and matter should *both* be ‘born quantum mechanically’. Thus, in contrast to approaches developed by particle physicists, one does not begin with quantum matter on a background geometry and use perturbation theory to incorporate quantum effects of gravity. There *is* a manifold but no metric, or indeed any other physical fields, in the background.¹

In the classical gravity, Riemannian geometry provides the appropriate mathematical language to formulate the physical, kinematical notions as well as the final dynamical equations. This role is now taken by *quantum* Riemannian geometry, discussed below. In the classical domain, general relativity stands out as the best available theory of gravity, some of whose predictions have been tested to an amazing accuracy, surpassing even the legendary tests of quantum electrodynamics. Therefore, it is natural to ask: *Does quantum general relativity, coupled to suitable matter* (or supergravity, its supersymmetric generalization) *exist as consistent theories non-perturbatively?* There is no a priori implication that such a theory would be the final, complete description of Nature. Nonetheless, this is a fascinating open question, at least at the level of mathematical physics.

In the particle physics circles, the answer is often assumed to be in the negative, not because there is concrete evidence against non-perturbative quantum gravity, but because of an analogy to the theory of weak interactions. There, one first had a 4-point interaction model due to Fermi which works quite well at low energies but which fails to be renormalizable. Progress occurred not by looking for non-perturbative formulations of the Fermi

¹In 2+1 dimensions, although one begins in a completely analogous fashion, in the final picture one can get rid of the background manifold as well. Thus, the fundamental theory can be formulated combinatorially [3,1]. To achieve this goal in 3+1 dimensions, one needs a much better understanding of the theory of (intersecting) knots in 3 dimensions.

model but by replacing the model by the Glashow-Salam-Weinberg renormalizable theory of electro-weak interactions, in which the 4-point interaction is replaced by W^\pm and Z propagators. Therefore, it is often assumed that perturbative non-renormalizability of quantum general relativity points in a similar direction. However this argument overlooks the crucial fact that, in the case of general relativity, there is a qualitatively new element. Perturbative treatments pre-suppose that the space-time can be assumed to be a continuum *at all scales* of interest to physics under consideration. In the gravitational case, the scale of interest is given by the Planck length ℓ_{Pl} and there is no physical basis to pre-suppose that the continuum picture should be valid down to that scale. The failure of the standard perturbative treatments may simply be due to this grossly incorrect assumption and a non-perturbative treatment which correctly incorporates the physical micro-structure of geometry may well be free of these inconsistencies.

As indicated above, even if quantum general relativity did exist as a mathematically consistent theory, there is no a priori reason to assume that it would be the ‘final’ theory of all known physics. In particular, as is the case with classical general relativity, while requirements of background independence and general covariance do restrict the form of interactions between gravity and matter fields and among matter fields themselves, the theory would not have a built-in principle which *determines* these interactions. Put differently, such a theory would not be a satisfactory candidate for unification of all known forces. However, just as general relativity has had powerful implications in spite of this limitation in the classical domain, quantum general relativity should have qualitatively new predictions, pushing further the existing frontiers of physics. Indeed, unification does not appear to be an essential criterion for usefulness of a theory even in other interactions. QCD, for example, is a powerful theory even though it does not unify strong interactions with electro-weak ones. Furthermore, the fact that we do not yet have a viable candidate for the grand unified theory does not make QCD any less useful.

B. Quantum Geometry

Although there is no natural unification of dynamics of all interactions in loop quantum gravity, it does provide a kinematical unification. More precisely, in this approach one begins by formulating general relativity in the mathematical language of connections, the basic variables of gauge theories of electro-weak and strong interactions. Thus, now the configuration variables are not metrics as in Wheeler’s geometrodynamics, but certain spin connections; the emphasis is shifted from distances and geodesics to holonomies and Wilson loops [1]. Consequently, the basic kinematical structures are the same as those used in gauge theories. A key difference, however, is that while a background space-time metric is available and crucially used in gauge theories, there are no background fields whatsoever now. This absence is forced on us by the requirement of diffeomorphism invariance (or ‘general covariance’).

This is a key difference and it causes a host of conceptual as well as technical difficulties in the passage to quantum theory. For, most of the techniques used in the familiar, Minkowskian quantum theories are deeply rooted in the availability of a flat back-ground metric. It is this structure that enables one to single out the vacuum state, perform Fourier

transforms to decompose fields canonically in to creation and annihilation parts, define masses and spins of particles and carry out regularizations of products of operators. Already when one passes to quantum field theory in *curved* space-times, extra work is needed to construct mathematical structures that can adequately capture underlying physics. In our case, the situation is much more drastic: there is no background metric what so ever! Therefore new physical ideas and mathematical tools are now necessary. Fortunately, they were constructed by a number of researchers in the mid-nineties and have given rise to a detailed quantum theory of geometry [4–7].

Because the situation is conceptually so novel and because there are no direct experiments to guide us, reliable results require a high degree of mathematical precision to ensure that there are no hidden infinities. Achieving this precision has been a priority in the program. Thus, while one is inevitably motivated by heuristic, physical ideas and formal manipulations, the final results are mathematically rigorous. In particular, due care is taken in constructing function spaces, defining measures and functional integrals, regularizing products of field operator, and calculating eigenvectors and eigenvalues of geometric operators. Consequently, the final results are all free of divergences, well-defined, and respect the background independence and diffeomorphism invariance.

Let me now turn to specifics. Our basic configuration variable is an $SU(2)$ -connection, A_a^i on a 3-manifold Σ representing ‘space’ and, as in gauge theories, the momenta are the ‘electric fields’ E_i^a . However, in the present gravitational context, they acquire an additional meaning: they can be naturally interpreted as orthonormal triads (with density weight 1) and determine the dynamical, Riemannian geometry of Σ . Thus, in contrast to Wheeler’s geometrodynamics, the Riemannian structures, including the positive-definite metric on Σ , is now built from *momentum* variables. The basic kinematic objects are holonomies of A_a^i , which dictate how spinors are parallel transported along curves, and the triads E_i^a , which determine the geometry of Σ . (Matter couplings to gravity have also been studied extensively [2,1].)

In the quantum theory, the fundamental excitations of geometry are most conveniently expressed in terms of holonomies [3,4]. They are thus *one-dimensional, polymer-like* and, in analogy with gauge theories, can be thought of as ‘flux lines of the electric field’. More precisely, they turn out to be flux lines of areas, the simplest gauge invariant quantities constructed from E_i^a : an elementary flux line deposits a quantum of area on any 2-surface S it intersects. Thus, if quantum geometry were to be excited along just a few flux lines, most surfaces would have zero area and the quantum state would not at all resemble a classical geometry. This state would be analogous, in Maxwell theory, to a ‘genuinely quantum mechanical state’ with just a few photons. In the Maxwell case, one must superpose photons coherently to obtain a semi-classical state that can be approximated by a classical electromagnetic field. Similarly, here, semi-classical geometries can result only if a huge number of these elementary excitations are superposed in suitable dense configurations [13,14]. The state of quantum geometry around you, for example, must have so many elementary excitations that $\sim 10^{68}$ of them intersect the sheet of paper you are reading. Even in such states, the geometry is still distributional, concentrated on the underlying elementary flux lines; but if suitably coarse-grained, it can be approximated by a smooth metric. Thus, the continuum picture is only an approximation that arises from coarse graining of semi-classical states.

These quantum states span a specific Hilbert space \mathcal{H} consisting of wave functions of

connections which are square integrable with respect to a natural, diffeomorphism invariant measure [4]. This space is very large. However, it can be conveniently decomposed into a family of orthonormal, *finite* dimensional sub-spaces $\mathcal{H} = \bigoplus_{\gamma, \vec{j}} \mathcal{H}_{\gamma, \vec{j}}$, labelled by graphs γ each edge of which itself is labelled by a spin (i.e., half-integer) j [5]. (The vector \vec{j} stands for the collection of half-integers associated with all edges of γ .) One can think of γ as a ‘floating lattice’ in Σ —‘floating’ because its edges are arbitrary, rather than ‘rectangular’. (Indeed, since there is no background metric on Σ , a rectangular lattice has no invariant meaning.) Mathematically, $\mathcal{H}_{\gamma, \vec{j}}$ can be regarded as the Hilbert space of a spin-system. These spaces are extremely simple to work with; this is why very explicit calculations are feasible. Elements of $\mathcal{H}_{\gamma, \vec{j}}$ are referred to as *spin-network states* [5].

As one would expect from the structure of the classical theory, the basic quantum operators are the holonomies \hat{h}_p along paths p in Σ and the triads \hat{E}_i^a [6]. (Both sets are densely defined and self-adjoint on \mathcal{H} .) Furthermore, a striking result is that *all eigenvalues of the triad operators are discrete*. This key property is, in essence, the origin of the fundamental discreteness of quantum geometry. For, just as the classical Riemannian geometry of Σ is determined by the triads E_i^a , all Riemannian geometry operators —such as the area operator \hat{A}_S associated with a 2-surface S or the volume operator \hat{V}_R associated with a region R — are constructed from \hat{E}_i^a . However, since even the classical quantities A_S and V_R are non-polynomial functionals of the triads, the construction of the corresponding \hat{A}_S and \hat{V}_R is quite subtle and requires a great deal of care. But their final expressions are rather simple [6].

In this regularization, the underlying background independence turns out to be a blessing. For, diffeomorphism invariance constrains the possible forms of the final expressions *severely* and the detailed calculations then serve essentially to fix numerical coefficients and other details. Let us illustrate this point with the example of the area operators \hat{A}_S . Since they are associated with 2-surfaces S while the states are 1-dimensional excitations, the diffeomorphism covariance requires that the action of \hat{A}_S on a state $\Psi_{\gamma, \vec{j}}$ must be concentrated at the intersections of S with γ . The detailed expression bears out this expectation: the action of \hat{A}_S on $\Psi_{\gamma, \vec{j}}$ is dictated simply by the spin labels j_I attached to those edges of γ which intersect S . For all surfaces S and 3-dimensional regions R in Σ , \hat{A}_S and \hat{V}_R are densely defined, self-adjoint operators. *All their eigenvalues are discrete* [6]. Naively, one might expect that the eigenvalues would be uniformly spaced, given by, e.g., integral multiples of the Planck area or volume. This turns out *not* to be the case; the distribution of eigenvalues is quite subtle. In particular, the eigenvalues crowd rapidly as areas and volumes increase. In the case of area operators, the complete spectrum is known in a *closed form*, and the first several hundred eigenvalues have been explicitly computed numerically. For a large eigenvalue a_n , the separation $\Delta a_n = a_{n+1} - a_n$ between consecutive eigenvalues decreases exponentially: $\Delta a_n \leq \ell_{\text{Pl}}^2 \exp -(\sqrt{a_n}/\ell_{\text{Pl}})!$ Because of such strong crowding, the continuum approximation becomes excellent quite rapidly just a few orders of magnitude above the Planck scale. At the Planck scale, however, there is a precise and very specific replacement. This is the arena of quantum geometry. The premise is that the standard perturbation theory fails because it ignores this fundamental discreteness.

There is however a further subtlety [2,7]. This non-perturbative quantization has a one parameter family of ambiguities labelled by $\gamma > 0$. This γ is called the Barbero-

Immirzi parameter and is rather similar to the well-known θ -parameter of QCD. In QCD, a single classical theory gives rise to inequivalent sectors of quantum theory, labelled by θ . Similarly, γ is classically irrelevant but different values of γ correspond to unitarily inequivalent representations of the algebra of geometric operators. The overall mathematical structure of all these sectors is very similar; the only difference is that the eigenvalues of all geometric operators scale with γ . For example, the simplest eigenvalues of the area operator \hat{A}_S in the γ quantum sector is given by ²

$$a_{\{j\}} = 8\pi\gamma\ell_{\text{Pl}}^2 \sum_I \sqrt{j_I(j_I + 1)} \quad (1)$$

where $\{j\}$ is a collection of 1/2-integers j_I , with $I = 1, \dots, N$ for some N . Since the representations are unitarily inequivalent, as usual, one must rely on Nature to resolve this ambiguity: Just as Nature must select a specific value of θ in QCD, it must select a specific value of γ in loop quantum gravity. With one judicious experiment —e.g., measurement of the lowest eigenvalue of the area operator \hat{A}_S for a 2-surface S of any given topology— we could determine the value of γ and fix the theory. Unfortunately, such experiments are hard to perform! However, we will see in Section III B that the Bekenstein-Hawking formula of black hole entropy provides an indirect measurement of this lowest eigenvalue of area for the 2-sphere topology and can therefore be used to fix the value of γ .

C. Quantum dynamics

Quantum geometry provides a mathematical arena to formulate non-perturbative dynamics of candidate quantum theories of gravity, without any reference to a background classical geometry. In the case of general relativity, it provides the tools to write down quantum Einstein's equations in the Hamiltonian approach and calculate transition amplitudes in the path integral approach. Until recently, effort was focussed primarily on the Hamiltonian approach. However, over the last three years, path integrals —called *spin foams*— have drawn a great deal of attention. This work has led to fascinating results suggesting that, thanks to the fundamental discreteness of quantum geometry, path integrals defining quantum general relativity may be finite. These developments will be discussed in some detail in Section IV A. In this Section, I will summarize the status of the Hamiltonian approach. For brevity, I will focus on source-free general relativity, although there has been considerable work also on matter couplings.

²In particular, the lowest non-zero eigenvalue of area operators is proportional to γ . This fact has led to a misunderstanding: in circles outside loop quantum gravity, γ is sometimes thought of as a regulator responsible for discreteness of quantum geometry. As explained above, this is *not* the case; γ is analogous to the QCD θ and quantum geometry is discrete in *every* permissible γ -sector. Note also that, at the classical level, the theory is equivalent to general relativity only if γ is *positive*; if one sets $\gamma = 0$ by hand, one can not recover even the kinematics of general relativity. Similarly, at the quantum level, setting $\gamma = 0$ would lead to a meaningless theory in which *all* eigenvalues of geometric operators vanish identically.

For simplicity, let me suppose that the ‘spatial’ 3-manifold Σ is compact. Then, in any theory without background fields, Hamiltonian dynamics is governed by constraints. Roughly, this is because, in these theories, diffeomorphisms correspond to gauge in the sense of Dirac. Recall that, on the Maxwell phase space, gauge transformations are generated by the functional $\mathcal{D}_a E^a$ which is constrained to vanish on physical states due to Gauss law. Similarly, on phase spaces of background independent theories, diffeomorphisms are generated by Hamiltonians which are constrained to vanish on physical states. In the case of general relativity, there are three sets of constraints. The first set consists of the three Gauss equations

$$\mathcal{G}_i := \mathcal{D}_a E_i^a = 0,$$

which, as in Yang-Mills theories, generates internal $SU(2)$ rotations on the connection and the triad fields. The second set consists of a co-vector (or diffeomorphism) constraint

$$\mathcal{C}_b := E^a F_{ab} = 0,$$

which generates spatial diffeomorphism on Σ (modulo internal rotations generated by \mathcal{G}_i). Finally, there is the key scalar (or Hamiltonian) constraint

$$\mathcal{S} := \epsilon^{ijk} E_i^a E_j^b F_{ab} k + \dots = 0$$

which generates time-evolutions. (The \dots are extrinsic curvature terms, expressible as Poisson brackets of the connection, the total volume constructed from triads and the first term in the expression of \mathcal{S} given above. We will not need their explicit forms.) Our task in quantum theory is three-folds: i) Elevate these constraints (or their ‘exponentiated versions’) to well-defined operators on the kinematic Hilbert space \mathcal{H} ; ii) Select physical states by asking that they be annihilated by these constraints; iii) introduce an inner-product and interesting observables, and develop approximation schemes, truncations, etc to explore physical consequences.

Step i) has been completed. Since the action of the Gauss and the co-vector constraints have a simple geometrical meaning, completion of i) in these cases is fairly straightforward. For the scalar constraint, on the other hand, there are no such guiding principles whence the procedure is subtle. In particular, specific regularization choices have to be made. Consequently, the answer is not unique. At the present stage of the program, such ambiguities are inevitable; one has to consider all viable candidates and analyze if they lead to sensible theories. However, the availability of well-defined Hamiltonian constraint operators is by itself a notable technical success. For example, the analogous problem in quantum geometrodynamics —a satisfactory regularization of the Wheeler-DeWitt equation— is still open although the formal equation was written down some thirty five years ago. To be specific, I will focus on the procedure developed by Rovelli, Smolin, Lewandowski and others which culminated in a specific construction due to Thiemann [9].

Step ii) has been completed for the Gauss and the co-vector constraints [8]. The mathematical implementation required a very substantial extension [8,1] of the algebraic quantization program initiated by Dirac, and the use of the spin-network machinery [5] of quantum geometry. Again, the detailed implementation is a non-trivial technical success and the analogous task has not been completed in geometrodynamics because of difficulties associated with infinite dimensional spaces. Thiemann’s quantum scalar constraint is defined

on the space of solutions to the Gauss and co-vector constraints. The problem of finding a general solution to the scalar constraint has been systematically reduced to that of finding *elementary* solutions, a task that requires only analysis of linear operators on certain finite dimensional spaces. In this sense, step ii) has been completed for all constraints.

This is a striking result. However, it is still unclear whether this theory is physically satisfactory; at this stage, it is in principle possible that it captures only an ‘exotic’ sector of quantum gravity. *A key open problem in loop quantum gravity is to show that the scalar/Hamiltonian constraint*—either Thiemann’s or an alternative such as the one of Gambini and Pullin [9]—admits a ‘sufficient number’ of semi-classical states. Progress on this problem has been slow because, as explained in Section I, the general issue of semi-classical limits is itself difficult in background independent approaches. However, as discussed in Section IV B below, a systematic understanding has now begun to emerge and is providing the ‘infra-structure’ needed to analyze the key problem mentioned above. More generally, while there are promising ideas to complete step iii), substantial further work is necessary to fully solve this problem. Recent advance in quantum cosmology, described in Section III A, is an example of progress in this direction and it provides a strong support for the Thiemann scheme, but of course only within the limited context of mini-superspaces.

To summarize, the crux of dynamics in the Hamiltonian approach lies in quantum constraints. While the quantum Gauss and co-vector/diffeomorphism constraints have been solved, it is not clear if any of the proposed strategies to solve the scalar/Hamiltonian constraint incorporates the familiar low energy physics.

Remark: There has been another concern about this class of regularizations of the scalar constraint which, however, is less specific. It stems from the structure of the constraint algebra. To analyze this issue, the domain of definition of the scalar constraint had to be extended to *certain* states which are not diffeomorphism invariant, so that the commutators could be meaningfully calculated. It was then found that the commutator between any two Hamiltonian constraints vanishes identically [9], while in the classical theory, the corresponding Poisson brackets vanishes only on solutions to the diffeomorphism constraint. However, it was also shown that the operator representing the right side of the classical Poisson bracket *also vanishes* on all the quantum states considered, including the ones which are not diffeomorphism invariant. Therefore, while the vanishing of the commutator of the Hamiltonian constraint was unexpected, this analysis does not reveal a clear-cut problem with these regularizations.

Furthermore, one can follow this scheme step by step in 2+1 gravity where one knows what the result should be. One can obtain the ‘elementary solutions’ mentioned above and show that all the standard quantum states—including the semi-classical ones—can be recovered as linear combinations of these elementary ones. As is almost always the case with constrained systems, there are *many more solutions* and the ‘spurious ones’ have to be eliminated by the requirement that the physical norm be finite. In 2+1 gravity, the connection formulation used here naturally leads to a complete set of Dirac observables and the inner-product can be essentially fixed by the requirement that they be self-adjoint. In 3+1 gravity, by contrast, we do not have this luxury and the problem of constructing the physical inner-product is therefore much more difficult. However, the concern here is that of weeding out unwanted solutions rather than having a ‘sufficient number’ of semi-classical ones, a significantly less serious issue at the present stage.

III. APPLICATIONS OF QUANTUM GEOMETRY

In this section, I will summarize two recent developments that answer several of the questions raised under first two bullets in the Introduction.

A. Big bang

Over the last three years, quantum geometry has led to some striking results of direct physical interest. The first of these concerns the fate of the big-bang singularity.

Traditionally, in quantum cosmology one has proceeded by first imposing spatial symmetries —such as homogeneity and isotropy— to freeze out all but a finite number of degrees of freedom *already at the classical level* and then quantizing the reduced system. In the simplest case, the basic variables of the reduced classical system are the scale factor a and matter fields ϕ . The symmetries imply that space-time curvature goes as $\sim 1/a^2$ and Einstein's equations predict a big-bang, where the scale factor goes to zero and the curvature blows up. As indicated in Section I, this is reminiscent of what happens to ferro-magnets at the Curie temperature: magnetization goes to zero and the susceptibility diverges. By analogy, the key question is: Do these 'pathologies' disappear if we re-examine the situation in the context of an appropriate quantum theory? In traditional quantum cosmologies, without an additional input, they do not. That is, typically, to resolve the singularity one either has to introduce matter with unphysical properties or introduce boundary conditions, e.g., by invoking new principles.

In a series of seminal papers [10], Bojowald has shown that the situation in loop quantum cosmology is quite different: the underlying quantum geometry makes a *qualitative* difference very near the big-bang. In the standard procedure summarized above, the reduction is carried out at the classical level and this removes all traces of the fundamental discreteness. Therefore, the key idea in Bojowald's analysis is to retain the essential features of quantum geometry by first quantizing the kinematics of the *full theory* as in Section IIB and then restricting oneself to *quantum* states which are spatially homogeneous and isotropic. As a result, the scale factor operator \hat{a} has *discrete eigenvalues*. The continuum limit is reached rapidly. For example, the gap between an eigenvalue of \hat{a} of $\sim 1\text{cm}$ and the next one is less than $\sim 10^{-30} \ell_{\text{Pl}}$! Nonetheless, near $a \sim \ell_{\text{Pl}}$ there are surprises and predictions of loop quantum cosmology are very different from those of traditional quantum cosmology.

The first surprise occurs already at the kinematical level. Recall that, in the classical theory curvature is essentially given by $1/a^2$, and blows up at the big-bang. What is the situation in quantum theory? Denote the Hilbert space of spatially homogeneous, isotropic kinematical quantum states by \mathcal{H}_{HI} . A self-adjoint operator $\widehat{\text{curv}}$ corresponding to curvature can be constructed on \mathcal{H}_{HI} and turns out to be *bounded from above*. This is very surprising because \mathcal{H}_{HI} admits an eigenstate of the scale factor operator \hat{a} with a discrete, zero eigenvalue! At first, it may appear that this could happen only by an artificial trick in the construction of $\widehat{\text{curv}}$ and that this quantization can not possibly be right because it seems to represent a huge departure from the classical relation $(\text{curv}) a^2 = 1$. However, these concerns turn out to be misplaced. The procedure for constructing $\widehat{\text{curv}}$ is natural and, furthermore, descends from the full theory; Bojowald essentially repeats a key step

in Thiemann's procedure of defining the quantum scalar/Hamiltonian constraint in the full theory. Let us examine the properties of $\widehat{\text{curv}}$. Its upper bound u_{curv} is finite but absolutely huge:

$$u_{\text{curv}} \sim \frac{256}{81} \frac{1}{\ell_{\text{Pl}}^2} \equiv \frac{256}{81} \frac{1}{G\hbar} \quad (2)$$

or, about 10^{77} times the curvature at the horizon of a solar mass black hole. The functional form of the upper bound is also illuminating. Recall that, in the case of an hydrogen atom, energy is unbounded from below classically but, thanks to \hbar , we obtain a finite value, $E_0 = -(me^4/\hbar^2)$, in quantum theory. Similarly, u_{curv} is finite because \hbar is non-zero and tends to the classical answer as \hbar tends to zero. At curvatures as large as u_{curv} , it is natural to expect large departures from classical relations such as $(\text{curv})a^2 = 1$. But is this relation recovered in the semi-classical regime? The answer is in the affirmative. In fact it is somewhat surprising how quickly this happens. As one would expect, one can simultaneously diagonalize \hat{a} and $\sqrt{\widehat{\text{curv}}}$. If we denote their eigenvalues by a_n and b_n respectively, then $(a_n \cdot b_n - 1)$ is of the order 10^{-4} at $n = 100$ and decreases rapidly as n increases. These properties show that, in spite of the initial surprise, the quantization procedure is viable. Furthermore, one can apply it also to more familiar systems such as a particle moving on a circle and obtain results which at first seem surprising but are in complete agreement with the standard quantum theory of these systems.

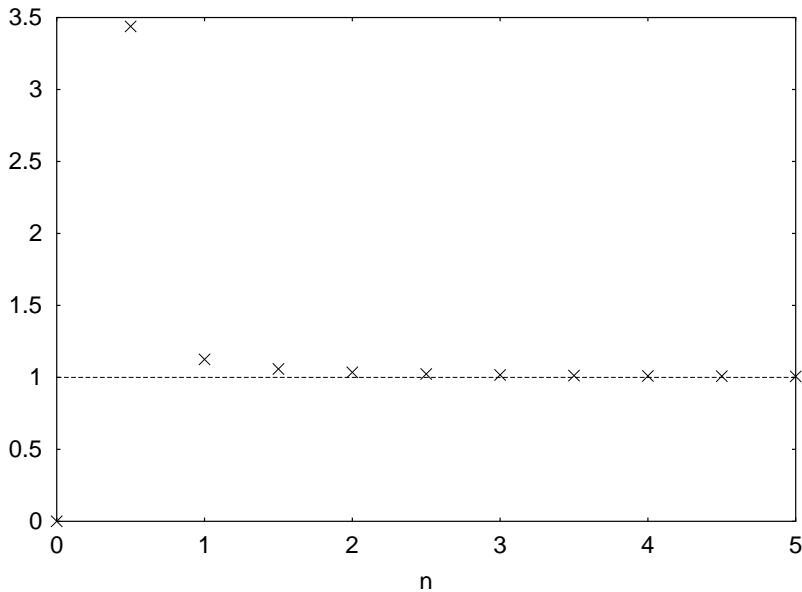


FIG. 1. The product $a_n \cdot b_n$ as a function of n . The dashed line is the classical value of $a \cdot \sqrt{\text{curv}}$.

Since the curvature is bounded above in the entire Hilbert space, one might hope that the quantum evolution may be well-defined right through the big-bang singularity. Is this in fact the case? The second surprise is that although the quantum evolution is close to that of the Wheeler-DeWitt equation of standard quantum cosmology for large a , there are dramatic

differences near the big-bang which makes it well defined even *at* the big-bang, without any additional input. As one might expect, the ‘evolution’ is dictated by the quantum scalar constraint operator. To obtain this operator, Bojowald again follows, step by step, the procedure used by Thiemann in the full theory. Let us expand out the full quantum state as $|\Psi\rangle = \sum_n \psi_n(\phi) |n\rangle$ where $|n\rangle$ are the eigenstates of the scale factor operator and ϕ denotes matter fields. Then, the scalar constraint takes the form:

$$c_n \psi_{n+8}(\phi) + d_n \psi_{n+4}(\phi) + e_n \psi_n(\phi) + f_n \psi_{n-4}(\phi) + g_n \psi_{n-8}(\phi) = \gamma \ell_{\text{Pl}}^2 \hat{H}_\phi \psi_n(\phi) \quad (3)$$

where c_n, \dots, g_n are fixed numerical coefficients, γ the Barbero-Immirzi parameter and \hat{H}_ϕ is the matter Hamiltonian. (Again, using the Thiemann regularization, one can show that the matter Hamiltonian is a well-defined operator.) Primarily, being a constraint equation, (3) constrains the physically permissible $\psi_n(\phi)$. However, *if* we choose to interpret the scale factor (more precisely, the square of the scale factor times the determinant of the triad) as a time variable, (3) can be interpreted as an ‘evolution equation’ which evolves the state through discrete time steps. In a (large) neighborhood of the big-bang singularity, this ‘deparametrization’ is viable. For the choice of factor ordering used in the Thiemann regularization, one can evolve in the past through $n = 0$, i.e. right through the classical singularity. Thus, the infinities predicted by the classical theory at the big-bang are artifacts of assuming that the classical, continuum space-time approximation is valid right up to the big-bang. In the quantum theory, the state can be evolved through the big-bang without any difficulty. However, the classical space-time description fails near the big-bang; quantum evolution is well-defined but the classical space-time ‘dissolves’.

The ‘evolution’ equation (3) has other interesting features. To begin with, the space of solutions is 16 dimensional. Can we single out a preferred solution by imposing a *physical* condition? One possibility is to impose *pre-classicality*, i.e., to require that the quantum state not oscillate rapidly from one step to the next at *late* times when we know our universe behaves classically. Although this is an extra input, it is not a theoretical prejudice about what should happen at (or near) the big-bang but an observationally motivated condition that is clearly satisfied by our universe. The coefficients c_n, \dots, g_n of (3) are such that this condition singles out a solution uniquely. One can ask what this state does at negative times, i.e., before the big-bang. (Time becomes negative because triads flip orientation on the ‘other side’.) Preliminary indications are that the state does not become pre-classical there. If this is borne out by detailed calculations, then the qualitative analogy with ferro-magnets would become closer, our side of the big-bang being analogous to the ferro-magnetic phase in which classical geometry (the analog of magnetization) is both meaningful and useful and the ‘other’ side being analogous to the para-magnetic phase where it is not. Another interesting feature is that the standard Wheeler-DeWitt equation is recovered if we take the limit $\gamma \rightarrow 0$ and $n \rightarrow \infty$ such that the eigenvalues of \hat{a} take on continuous values. This is completely parallel to the limit we often take to coarse grain the quantum description of a rotor to ‘wash out’ discreteness in angular momentum eigenvalues and arrive at the classical description. From this perspective, then, one is led to say that the most striking of the consequences of loop quantum gravity are not seen in standard quantum cosmology because it ‘washes out’ the fundamental discreteness of quantum geometry.

Finally, the detailed calculations have revealed another surprising feature. The fact that the quantum effects become prominent near the big bang, completely invalidating the

classical predictions, is pleasing but not unexpected. However, prior to these calculations, it was not clear how soon after the big-bang one can start trusting semi-classical notions and calculations. It would not have been surprising if we had to wait till the radius of the universe became, say, a few million times the Planck length. These calculations strongly suggest that a few tens of Planck lengths should suffice. This is fortunate because it is now feasible to develop quantum numerical relativity; with computational resources commonly available, grids with $(10^6)^3$ points are hopelessly large but one with $(100)^3$ points could be manageable.

B. Black-holes

Loop quantum cosmology illuminates dynamical ramifications of quantum geometry but within the context of mini-superspaces where all but a finite number of degrees of freedom are frozen. In this sub-section, I will discuss a complementary application where one considers the full theory but probes consequences of quantum geometry which are not sensitive to full quantum dynamics —the application of the framework to the problem of black hole entropy. This discussion is based on joint work with Baez, Corichi and Krasnov [11] which itself was motivated by earlier work of Krasnov, Rovelli and others.

As explained in the Introduction, since mid-seventies, a key question in the subject has been: What is the statistical mechanical origin of the black hole entropy $S_{\text{BH}} = (a_{\text{hor}}/4\ell_{\text{Pl}}^2)$? What are the microscopic degrees of freedom that account for this entropy? This relation implies that a solar mass black hole must have $(\exp 10^{77})$ quantum states, a number that is *huge* even by the standards of statistical mechanics. Where do all these states reside? To answer these questions, in the early nineties Wheeler had suggested the following heuristic picture, which he christened ‘It from Bit’. Divide the black hole horizon into elementary cells, each with one Planck unit of area, ℓ_{Pl}^2 and assign to each cell two microstates, or one ‘bit’. Then the total number of states \mathcal{N} is given by $\mathcal{N} = 2^n$ where $n = (a_{\text{hor}}/\ell_{\text{Pl}}^2)$ is the number of elementary cells, whence entropy is given by $S = \ln \mathcal{N} \sim a_{\text{hor}}$. Thus, apart from a numerical coefficient, the entropy (‘It’) is accounted for by assigning two states (‘Bit’) to each elementary cell. This qualitative picture is simple and attractive. But the key open issue was: can these heuristic ideas be supported by a systematic analysis from first principles? Quantum geometry has supplied the required analysis.³

A systematic approach requires that we first specify the class of black holes of interest. Since the entropy formula is expected to hold unambiguously for black holes in equilibrium, most analyses were confined to *stationary*, eternal black holes (i.e., in 4-dimensional general relativity, to the Kerr-Newman family). From a physical viewpoint however, this assumption seems overly restrictive. After all, in statistical mechanical calculations of entropy of ordinary systems, one only has to assume that the given system is in equilibrium, not the whole world. Therefore, it should suffice for us to assume that the black hole itself is in equilibrium; the

³However, I should add that this account does not follow chronology. Black hole entropy was computed in quantum geometry quite independently and the fact that the ‘It from Bit’ picture works so well in the final picture came as a surprise.

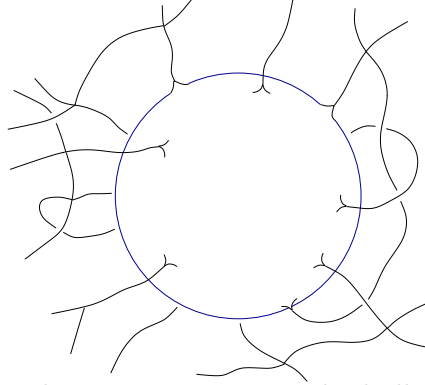


FIG. 2. Quantum Horizon. Polymer excitations in the bulk puncture the horizon, endowing it with quantized area. Intrinsically, the horizon is flat except at punctures where it acquires a quantized deficit angle. These angles add up to endow the horizon with a 2-sphere topology.

exterior geometry should not be forced to be time-independent. Furthermore, the analysis should also account for entropy of black holes which may be distorted or carry (Yang-Mills and other) hair. Finally, it has been known since the mid-seventies that the thermodynamical considerations apply not only to black holes but also to cosmological horizons. A natural question is: Can these diverse situations be treated in a single stroke? Within the quantum geometry approach, the answer is in the affirmative. The entropy calculations have been carried out in the recently developed framework of ‘isolated horizons’ which encompasses all these situations. Isolated horizons serve as ‘internal boundaries’ whose intrinsic geometries (and matter fields) are time-independent, although space-time geometry as well as matter fields in the external space-time region can be fully dynamical. The zeroth and first laws of black hole mechanics have been extended to isolated horizons. Entropy associated with an isolated horizon refers to the family of observers in the exterior for whom the isolated horizon is a physical boundary that separates the region which is accessible to them from the one which is not. This point is especially important for cosmological horizons where, without reference to observers, one can not even define horizons. States which contribute to this entropy are the ones which can interact with the states in the exterior; in this sense, they ‘reside’ on the horizon.

In the detailed analysis, one considers space-times admitting an isolated horizon as inner boundary and carries out a systematic quantization. The quantum geometry framework can be naturally extended to this case. The isolated horizon boundary conditions imply that the intrinsic geometry of the quantum horizon is described by the so called $U(1)$ Chern-Simons theory on the horizon. This is a well-developed, topological field theory. A deeply satisfying feature of the analysis is that there is a seamless matching of three otherwise independent structures: the isolated horizon boundary conditions, the quantum geometry in the bulk, and the Chern-Simons theory on the horizon. In particular, one can calculate eigenvalues of certain physically interesting operators using purely bulk quantum geometry without any knowledge of the Chern-Simons theory, or using the Chern-Simons theory without any knowledge of the bulk quantum geometry. The two theories have never heard of each other. Yet, thanks to the isolated horizon boundary conditions, the two infinite sets of numbers match exactly, providing a coherent description of the quantum horizon.

In this description, the polymer excitations of the bulk geometry, each labelled by a spin j_I , pierce the horizon, endowing it an elementary area a_{j_I} given by (1). The sum $\sum_I a_{j_I}$ adds up to the total horizon area a_{hor} . The intrinsic geometry of the horizon is flat except at these puncture, but at each puncture there is a *quantized* deficit angle. These add up to endow the horizon with a 2-sphere topology. For a solar mass black hole, a typical horizon state would have 10^{77} punctures, each contributing a tiny deficit angle. So, although the quantum geometry *is* distributional, it can be well approximated by a smooth metric.

The counting of states can be carried out as follows. First one constructs a micro-canonical ensemble by restricting oneself only to those states for which the total area, angular momentum, and charges lie in small intervals around fixed values $a_{\text{hor}}, J_{\text{hor}}, Q_{\text{hor}}^i$. (As is usual in statistical mechanics, the leading contribution to the entropy is independent of the precise choice of these small intervals.) For each set of punctures, one can compute the dimension of the surface Hilbert space, consisting of Chern-Simons states compatible with that set. One allows all possible sets of punctures (by varying both the spin labels and the number of punctures), subject to the constraint that the total area a_{hor} be fixed, and adds up the dimensions of the corresponding surface Hilbert spaces to obtain the number \mathcal{N} of permissible surface states. One finds that the horizon entropy S_{hor} is given by

$$S_{\text{hor}} := \ln \mathcal{N} = \frac{\gamma_o}{\gamma} \frac{a_{\text{hor}}}{\ell_{\text{Pl}}^2} + \mathcal{O}\left(\frac{\ell_{\text{Pl}}^2}{a_{\text{hor}}}\right), \quad \text{where} \quad \gamma_o = \frac{\ln 2}{\sqrt{3}\pi} \quad (4)$$

Thus, for large black holes, entropy is indeed proportional to the horizon area. This is a non-trivial result; for examples, early calculations often led to proportionality to the square-root of the area. However, even for large black holes, one obtains agreement with the Hawking-Bekenstein formula only in the sector of quantum geometry in which the Barbero-Immirzi parameter γ takes the value $\gamma = \gamma_o$. Thus, while all γ sectors are equivalent classically, the standard quantum field theory in curved space-times is recovered in the semi-classical theory only in the γ_o sector of quantum geometry. It is quite remarkable that thermodynamic considerations involving *large* black holes can be used to fix the quantization ambiguity which dictates such Planck scale properties as eigenvalues of geometric operators. Note however that the value of γ can be fixed by demanding agreement with the semi-classical result just in one case —e.g., a spherical horizon with zero charge, or a cosmological horizon in the de Sitter space-time, or, Once the value of γ is fixed, the theory is completely fixed and we can ask: Does this theory yield the Hawking-Bekenstein value of entropy of *all* isolated horizons, irrespective of the values of charges, angular momentum, and cosmological constant, the amount of distortion, or hair. The answer is in the affirmative. Thus, the agreement with quantum field theory in curved space-times holds in *all* these diverse cases.

Why does γ_o not depend on other quantities such as charges? This important property can be traced back to a key consequence of the isolated horizon boundary conditions: detailed calculations show that only the gravitational part of the symplectic structure has a surface term at the horizon; the matter symplectic structures have only volume terms. (Furthermore, the gravitational surface term is insensitive to the value of the cosmological constant.) Consequently, there are no independent surface quantum states associated with matter. This provides a natural explanation of the fact that the Hawking-Bekenstein entropy depends only on the horizon geometry and is independent of electro-magnetic (or other) charges.

Finally, let us return to Wheeler’s ‘It from Bit’. One can ask: what are the states that dominate the counting? Perhaps not surprisingly, they turn out to be the ones which assign to each puncture the smallest quantum of area (i.e., spin value $j = \frac{1}{2}$), thereby maximizing the number of punctures. In these states, each puncture defines Wheeler’s ‘elementary cell’ and his two states correspond to whether the deficit angle is positive or negative.

To summarize, quantum geometry naturally provides the micro-states responsible for the huge entropy associated with horizons. In this analysis, all black holes and cosmological horizons are treated in an unified fashion; there is no restriction, e.g., to near-extremal black holes. The sub-leading term has also been calculated and shown to be proportional to $\ln a_{\text{hor}}$. Finally, in this analysis quantum Einstein’s equations *are* used. In particular, had we not imposed the quantum Gauss and co-vector/diffeomorphism constraints on surface states, the spurious gauge degrees of freedom would have given an infinite entropy. However, because of the isolated horizon boundary conditions, the scalar/Hamiltonian constraint has to be imposed just in the bulk. Since in the entropy calculation one traces over bulk states, the final result is insensitive to the details of how this (or any other bulk) equation is imposed. Thus, as in other approaches to black hole entropy, the calculation does not require a complete knowledge of quantum dynamics.

IV. CURRENT WORK

Work is now in progress along many directions, ranging from the fate of the ‘final’ black hole singularity in quantum geometry and the associated issue of ‘information loss’, to predictions of quantum cosmology for structure formation, to practical methods of constructing Dirac observables, to possible experimental tests of quantum geometry. To illustrate these developments, in this section I will discuss two main thrusts.

A. Spin foams

Spin foams can be thought of as histories traced out by ‘time evolution’ of spin networks and provide a path integral approach to quantum dynamics. I will focus on the fascinating ‘finiteness results’ [12] obtained by Crane, Rovelli and especially Perez based on earlier work also by Baez, Barrett, De Pietri, Freidel, Krasnov, Mikovič, Reisenberger and others.

In the gravitational context, the path integral can play two roles. First, as in standard quantum field theories, it can be used to compute ‘transitions amplitudes’. However outside, say, perturbation theory about a background space-time, there still remain unresolved conceptual questions about the physical meaning of such amplitudes. The second role is ‘cleaner’: as in the Euclidean approach of Hawking and others, it can be considered as a device to extract physical states, i.e. solutions to all the quantum constraint equations. In this role as an *extractor*, it can shed new light on the difficult issue of regularization of the Hamiltonian constraint discussed in Section II C.

The well-defined quantum kinematics of Section II B has motivated specific proposals for the definition of path integrals, often called ‘state sum models’. Perhaps the most successful of these is the Barrett-Crane model (modified slightly to handle a technical issue). At the classical level, one regards general relativity as a topological field theory, *called the BF*

theory, supplemented with an algebraic constraint. The BF theory is itself a generalization of the Chern-Simons theory mentioned in Section IIIB and has been investigated in detail in the mathematical physics literature. However, the role of the additional constraint is very important. Indeed, the BF theory has no local degrees of freedom; it is the extra constraint that reduces the huge gauge freedom of the BF theory, thereby recovering the local degrees of freedom of general relativity. The crux of the problem in quantum gravity is the appropriate incorporation of this constraint. At the classical level, the constrained B-F theory is equivalent to general relativity. To obtain Euclidean general relativity, one has to start with the BF theory associated with $SO(4)$ while the Lorentzian theory results if one uses $SO(3,1)$ instead. The Barrett-Crane model is a specific proposal to define path integrals for the constrained BF theory in either case.

Fix a 4-manifold M bounded by two 3-manifolds Σ_1 and Σ_2 . Spin-network states on the two boundaries can be regarded as ‘initial’ and ‘final’ quantum geometries. One can then consider histories, i.e., quantum 4-geometries, joining them. Each history is a spin-foam. Each vertex of the initial spin-network on Σ_1 ‘evolves’ to give a 1-dimensional edge in the spin-foam and each edge, to give a 2-dimensional face. Consequently, each face carries a spin label j . However, in the course of ‘evolution’ *new vertices* can appear, making the dynamics non-trivial and yielding a non-trivial amplitude for an ‘initial’ spin-network with n_1 vertices to evolve in to a ‘final’ spin-network with n_2 vertices. For mathematical clarity as well as physical intuition, it is convenient to group spin-foams associated with the same 4-dimensional graph but differing from one another in the labels, such as the spins j carried by faces. Each group is said to provide a *discretization* of M . Physically, a discretization has essentially just the topological information. The geometrical information—such as the area associated with each face—resides in the labels. This is a key distinction from lattice gauge theories with a background metric, where a discretization itself determines, e.g., the edge lengths and hence how refined the lattice is.

A key recent development is the discovery that the non-perturbative path integral, defined by the (modified) Barrett-Crane model is, in a certain precise sense, equivalent to a manageable *group field theory* (GFT) in the sense specified below. The GFT is a rather simple quantum field theory, defined on four copies of the underlying group— $SL(2, C)$ in the case of Lorentzian gravity and $Spin(4)$ in the case of Euclidean. (Note that these are just double covers of the Lorentz group and the rotation group in Euclidean 4-space.) Thus GFTs live in high dimensions. The action has a ‘free part’ and an interaction term with a coupling constant λ . But the free part is non-standard and does not have the familiar kinetic term, whence the usual non-renormalizability arguments for higher dimensional, interacting theories do not apply. In fact, the first key recent result is that *this GFT is finite order by order in the Feynman perturbation expansion*. The second key result is $A_{BC}(n) = A_{GFT}(n)$, where $A_{BC}(n)$ is the Barret-crane amplitude obtained by summing over all geometries (i.e., spin labels j) for a fixed discretization and $A_{GFT}(n)$ is the coefficient of λ^n in the Feynman expansion of the GFT. Together, the two results imply that, in this approach to quantum gravity, *sum over geometries for a fixed discrete topology is finite*. This is a highly non-trivial result because, on each face, the sum over js ranges from zero to infinity; there is no cut-off.

These results do not show that the full path integral is finite because the discrete topology is kept fixed in the sum. Work is in progress on removing this dependence. But even as they stand, the results have a striking, qualitative similarity with the finiteness results that have

been recently obtained in other approaches. The order by order finiteness is reminiscent of the order by order ultraviolet finiteness of string perturbation theory. In the present case, the situation is somewhat better: there is both ultra-violet *and* infra-red finiteness. Qualitatively, the first is ensured by the discreteness of underlying geometry while the second by the fact that the sum converges even though arbitrarily large values of j s are allowed. The second result —equivalence of the GFT with a certain definition of quantum gravity— is reminiscent of the conjectures discussed by Maldacena in this conference. In both cases, there is a mathematical equivalence between quantum gravity and certain field theories which have no knowledge of the physical space-time. On the one hand, results are stronger in the present case: both sides of the equality are separately defined and the thrust is on proving the equality, not just on constructing evidence in support of it. However, in contrast to Maldacena’s bold conjecture, here the equality in question is only order by order. Finally, recently Luscher and Reuter have found evidence for nonperturbative renormalizability of 4-dimensional Euclidean quantum general relativity (stemming from the existence of a non-trivial fixed point) [12]. It is natural to ask: Is there a relation to Perez’s finiteness results in the Euclidean signature?

B. Relation to low energy physics

The basic mathematical structures underlying loop quantum gravity are very different from those used in the text-book treatments of low energy physics. For example, in quantum geometry the fundamental excitations are one dimensional, polymer-like; a convenient basis of states is provided by spin-networks; and, eigenvalues of the basic geometric operators are discrete. By contrast, in the Fock framework of low energy physics the fundamental excitations are 3-dimensional, wavy; the convenient basis is labelled by the number of particles, their momenta and helicities; and, all geometric operators have continuous spectra. The challenge is to bridge the gap between these apparently disparate frameworks. These differences are inevitable, given that the standard quantum field theory is constructed in Minkowski space, while loop quantum gravity does not refer to any background geometry. Nonetheless, if loop quantum gravity is correct, the standard Fock framework should emerge in a suitable semi-classical approximation. Over the last decade, semi-classical states approximating classical geometries have been constructed with various degrees of precision [13]. However, the relation to Fock quantization has been explored only recently [14]. My summary of this development will be even briefer than that of other advances because this topic was discussed in some detail in Rovelli’s workshop D1i.

The recent developments are based on a key mathematical insight due to Varadarajan. Fock states do not belong to the Hilbert space \mathcal{H} of polymer excitations. However, neither do the physical states, i.e., the solutions to quantum constraints. This is not a peculiarity of general relativity. Even for quantum mechanical systems such as a free particle in Minkowski space, the physical states belong not to the kinematical Hilbert space, but to a natural extension of it, constructed from the refinement of the Dirac quantization program for constrained systems, mentioned in Section II C. Varadarajan showed that Fock states do belong to this extension \mathcal{E} of \mathcal{H} . Thus, the natural habitat \mathcal{E} for the physical states in loop quantum gravity also accommodates the standard Fock states of low energy physics.

At first, this result seems very surprising. For example, in the polymer description of a quantum Maxwell field, fluxes of electric field are quantized. On the Fock space, on the other hand, these flux operators are not even defined unless surfaces are thickened and, when they are, the quantization is lost. This phenomenon succinctly captures the tension between the polymer and Fock excitations. It turns out that the flux operators are indeed well-defined on \mathcal{E} but they do not map the Fock space to itself; flux quantization is lost because the Fock space is ‘too small’ and fails to accommodate any of the eigenvectors of these operators with discrete eigenvalues. The situation with quantum geometry is completely analogous: The discreteness of quantum geometry is lost if one insists that *all* quantum gravity states must reside in the Fock space of gravitons. To see the discreteness, one has to allow states which are ‘purely quantum mechanical’ and can not be regarded as excitations on *any* classical background geometry.

The interplay between the Fock and polymer excitations can now be studied in detail, thanks to a key property: each Fock state casts a ‘shadow’ in each finite dimensional space $\mathcal{H}_{\gamma, \vec{j}}$ of \mathcal{H} and can be fully recovered from the collection of these shadows. Using these shadows one can analyze a number of features of the Fock framework, such as construction of coherent states, expectations values and fluctuations of fields, needed in the analysis of semi-classical issues. Thus, one can now begin to analyze two key questions: Can the background-independent, non-perturbative theory reproduce the familiar low-energy physics on suitable coarse graining? and, Can one pin-point where and why the standard perturbation theory fails?

V. CONCLUSION

In the last two sections, I have summarized recent advances which have answered some of the long standing questions of quantum gravity raised in the Introduction. Personally, I find it very satisfying that a number of the key ideas came from younger researchers — from Bojowald in quantum cosmology, from Krasnov in the understanding of quantum black holes, Perez in spin foams and Varadarajan in the relation to low energy physics.

Throughout the development of loop quantum gravity, unforeseen simplifications have arisen regularly, leading to surprising solutions to seemingly impossible difficulties. Progress could occur because some of the obstinate problems which had slowed developments in background independent approaches, sometimes for decades, evaporated when ‘right’ perspectives were found. I will conclude with a few examples.

- Up until the early nineties, it was widely believed that spaces of connections do not admit non-trivial diffeomorphism invariant measures. This would have made it impossible to develop a background independent approach. Quite surprisingly, such a measure could be found by looking at connections from a slightly more general perspective. It is simple, natural, and has just the right structure to support quantum geometry. This geometry, in turn, supplied some missing links, e.g., by providing just the right expressions that Ponzano-Regge had to postulate without justification in their celebrated, early work.

- Fundamental discreteness first appeared in a startling fashion in the construction of the so-called weave states, which approximate a classical 3-geometry. In this construction, the polymer excitations were introduced as a starting point with the goal of taking the standard

continuum limit. It came as a major surprise that, if one wants to recover a given classical geometry *on large scales*, one can not take this limit, i.e., one can not pack the polymer excitations arbitrarily close together; there is an in-built discreteness.

- At a heuristic level, it was found that the Wilson loop functionals of a suitably defined connection around a smooth loop solve the notoriously difficult quantum scalar constraint automatically. No one had anticipated, even heuristically, such simple and natural solutions. This calculation suggested that the action of the constraint operator is concentrated at ‘nodes’, i.e., intersections, which in turn led to strategies for its regularization.

- As I indicated in some detail, unforeseen insights arose in the well-studied subject of quantum cosmology essentially by taking an adequate account of the quantum nature of geometry, i.e., by respecting the fundamental discreteness of the eigenvalues of the scale factor operator. Similarly, in the case of black holes, three quite distinct structures—the isolated horizon boundary conditions, the bulk quantum geometry and the surface Chern-Simons theory—blended together unexpectedly to provide a coherent theory of quantum horizons.

Repeated occurrence of such ‘unreasonable’ simplifications suggests that the ideas underlying loop quantum gravity may have captured an essential germ of truth.

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