Regularity of the extremal solution for some elliptic problems with singular nonlinearity and advection

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Abstract

In this note, we investigate the regularity of the extremal solution u^* for the semilinear elliptic equation $-\triangle u + c(x) \cdot \nabla u = \lambda f(u)$ on a bounded smooth domain of $\mathbb{R}^n$ with Dirichlet boundary condition. Here f is a positive nondecreasing convex function, exploding at a finite value $a \in (0, \infty)$. We show that the extremal solution is regular in the low dimensional case. In particular, we prove that for the radial case, all extremal solutions are regular in dimension two.

Keywords: singular nonlinearity, advection, extremal solution, regularity*.

1. Introduction

We consider the elliptic problem

$$\begin{cases} -\triangle u + c(x) \cdot \nabla u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda)$$

where $\lambda > 0$, Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, $c(x)$ is a smooth vector field over $\overline{\Omega}$ and $f : [0, a) \to \mathbb{R}_+$ with fixed $a \in (0, \infty)$ satisfies the following condition (H):

$$f \text{ is } C^2, \text{ positive, nondecreasing and convex in } [0, a) \text{ with } \lim_{t \to a^-} f(t) = \infty.$$

In the literature, f is refered as a *singular nonlinearity*. We say that u is a regular solution if $u \in C^2(\overline{\Omega})$, and we also deal with solutions in the following weak sense.

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Definition 1.1. *We say that u is a weak solution of (P_λ) if $0 \le u \le a$ a.e. in Ω such that $f(u)d(x,\partial\Omega) \in L^1(\Omega)$ and*

$$-\int_\Omega u\Delta\phi - \int_\Omega u\mathrm{div}(\phi c) = \lambda\int_\Omega f(u)\phi, \quad \forall\, \phi \in C^2(\overline{\Omega}) \cap H_0^1(\Omega).$$

Moreover, u is a weak super-solution of (P_λ) if " $=$ " is replaced by " $\ge$ " for all nonnegative functions $\phi \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$.

Clearly, a weak solution is regular if $\sup_\Omega u < a$. For regular solutions, we introduce a notion of stability.

Definition 1.2. *A regular solution u of (P_λ) is said to be stable if the principal eigenvalue of the linearized operator $L_{u,\lambda,c} := -\triangle + c\cdot\nabla - \lambda f'(u)$ is nonnegative in $H_0^1(\Omega)$.*

Exploiting some ideas in [11, 10], the solvability of (P_λ) is characterized by a parameter λ^*:

Proposition 1.1. *There exists $\lambda^* \in (0,\infty)$ such that*

- *For $0 < \lambda < \lambda^*$, the problem (P_λ) has a minimal solution u_λ, u_λ is regular and the map $\lambda \mapsto u_\lambda$ is increasing. Moreover, u_λ is the unique stable solution of (P_λ).*
- *For $\lambda = \lambda^*$, (P_{λ^*}) admits a unique weak solution $u^* := \lim_{\lambda\to\lambda^*} u_\lambda$, called the extremal solution.*
- *For $\lambda > \lambda^*$, (P_λ) admits no weak solution.*

Here the minimal solution means that $u_\lambda \le v$ for any solution v of (P_λ). We remark immediately a close similarity between (P_λ) and the Emden-Fowler equation with superlinear *regular nonlinearity*, that is

$$-\Delta u = \lambda g(u) \;\text{ in }\; \Omega\subset\mathbb{R}^n; \quad u = 0 \;\text{ on }\; \partial\Omega, \tag{1.1}$$

with $\lambda > 0$ and $g : [0,\infty) \to (0,\infty)$ satisfies

$$g \text{ is } C^2\text{, nondecreasing, convex and } \lim_{t\to\infty}\frac{g(t)}{t} = \infty. \tag{1.2}$$

In fact, there exists also a critical parameter $\overline{\lambda} \in (0,\infty)$ for (1.1) such that all conclusions in the above proposition hold true by replacing λ^* by $\overline{\lambda}$ (see [2, 11]). It is well known by classical examples as $g(u) = (1+u)^p$ with $p > 1$ or $g(u) = e^u$, the extremal solution u^* can be either a regular solution or a real weak solution in the distribution sense with $\sup_\Omega u = \infty$.

For general nonlinearity g satisfying (1.2), the regularity of the extremal solution u^* to (1.1) is obtained by Nedev [13] for any bounded smooth domain $\Omega \subset \mathbb{R}^n$ if $n = 2, 3$; by Cabré [4] for convex domains in $\mathbb{R}^4$; and for radial symmetry case in $\mathbb{R}^n$ with $n \le 9$ by Cabré & Capella [5]. In [17], it is proved that, under mild condition on g, the extremal solution u^* is regular for any smooth bounded domain $\Omega \subset \mathbb{R}^n$ if $n \le 9$.

We can ask the same question about the problem (P_λ): For f verifying (H), is it true that the extremal solution to (P_λ) is regular for general vector field c and general domain $\Omega \subset \mathbb{R}^n$ with low dimensions n? We will partly answer this question. It is worthy to mention that for studying the explosion phenomena in a flow, Berestycki *et al.* [1] have considered the problem (P_λ) with a general source f verifying (1.2).

Without loss of generality, fix $a = 1$ in the sequel. The problem (P_λ) can be linked to equation (1.1) up to the transformation $v = -\ln(1-u)$. In fact, let u solve (P_λ), v verifies then

$$\begin{cases} -\triangle v + |\nabla v|^2 + c(x)\cdot \nabla v = \lambda e^v f(1-e^{-v}) := \lambda g(v) & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases} \qquad (Q_\lambda)$$

Therefore g verifies (1.2) and $v^* = -\ln(1-u^*)$ is the extremal solution for the problem (Q_λ). Thus the regularity of u^* is equivalent to the boundedness of v^*, however the situation could be very different with the presence of advection terms (see [7, 16]). In last decade, a model describing the steady state of MEMS (Micro-Electro-Mechanical Systems) device given by Pelesko and Bernstein in [14], has drawn many attentions (see [9] and the references therein).

$$-\Delta u = \frac{\lambda}{(1-u)^2} \ \text{ in } \ \Omega \subset \mathbb{R}^n; \quad u = 0 \ \text{ on } \ \partial\Omega.$$

More generally, many precise studies have been done for the singular nonlinearities with negative exponent $f(u) = (1-u)^{-p}$ $(p > 0)$ in the advection-free situation, i.e. $c \equiv 0$. In that case, when Ω is moreover the unit ball in $\mathbb{R}^n$, it is known that u^* is regular if and only if (see [12, 10])

$$n < n_p := 2 + \frac{4p}{p+1} + 4\sqrt{\frac{p}{p+1}}. \tag{1.3}$$

Tending $p \to 0^+$ in (1.3), we see that $n_p \to 2$. Therefore we cannot expect in general better than dimension two to claim the regularity of u^*.

For the radial case of (P_λ), equally when Ω is a ball and $c(x)$ is the gradient of a smooth radial function, u_λ is radial by uniqueness of the minimal solution. We obtain the following optimal results which are new even for the advection-free case.

Theorem 1.1. *Assume that $n = 2$, $\Omega = B_1$. Let γ is a smooth radial function and $c = \nabla\gamma$, then the extremal solution u^* is regular for any f satisfying (H).*

Theorem 1.2. *For any f satisfying (H), $\Omega = B_1$ and smooth radial function γ, there exists $C > 0$ such that for all $\lambda \in (0, \lambda^*]$*

$$|u_\lambda'(r)| \le \begin{cases} Cr^{-1} & \text{if } n \ge 10;\\ Cr^{-\frac{n}{2}+1+\sqrt{n-1}} & \text{if } 3 \le n \le 9; \end{cases} \qquad \forall\, r = |x| \in (0,1]$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$.

Remark 1.1. *The above estimates are optimal. In fact, when* $f(u) = (1-u)^{-p}$, $p > 0$, $\Omega = B_1$ *and* $c \equiv 0$, *it is well known that* $u^*(x) = 1 - r^{\frac{2}{p+1}}$ *if* $n \geq n_p$ *with* n_p *given in* (1.3), *and we have*

$$n \geq n_p \quad \text{iff} \quad n \geq 10 \ \text{ or } \ 3 \leq n \leq 9, \ \frac{2}{p+1} \leq -\frac{n}{2} + 2 + \sqrt{n-1}.$$

But is the extremal solution u^* of (P_λ) regular with general singular nonlinearity f verifying (H), vector field c and smooth bounded domains in $\mathbb{R}^2$? The answer is affirmative under some additional mild condition on f.

Theorem 1.3. *Assume that* f *satisfies conditions* (H) *and the additional conditions,*

$$(H1) \qquad \limsup_{t\to 1^-} \frac{f(t)}{f'(t)(1-t)\ln^2(1-t)} < 1$$

and

$$(H2) \qquad \liminf_{t\to 1^-} \frac{f(t)f''(t)}{f'^2(t)} > 0.$$

Then u^* *is regular solution to* (P_{λ^*}) *if* $n = 2$, *i.e.* $\Omega \subset \mathbb{R}^2$.

Under more precise conditions on the growth of f, the extremal solution can be showed to be regular in some higher dimensions.

Theorem 1.4. *Let* f *verify (H) and* $g(v) = e^v f(1 - e^{-v})$. *Assume that* g *satisfies*

$$(H3) \qquad \liminf_{t\to\infty} \frac{g'(t)}{g(t)} = 1 + \delta > 1$$

and

$$(\widetilde{H}2) \qquad \liminf_{t\to\infty} \frac{g''(t)g(t)}{g'^2(t)} = \mu > \frac{1}{1+\delta}.$$

Then $v^* = -\ln(1-u^*)$ *is bounded (so* u^* *is regular) when*

$$n < 2 + \frac{4\delta}{1+\delta} + \frac{4\sqrt{\delta(\mu + \mu\delta - 1)}}{1+\delta}. \tag{1.4}$$

Consequently, if $\mu\delta > 1$, u^* is regular for all $n \leq 6$. Furthermore, if we can tend δ to ∞, which means $g = o(g')$ near ∞, then u^* is regular for $n < 6 + 4\sqrt{\mu}$ with any $\mu > 0$. However, we can never have $\mu > 1$, since otherwise g blows up at finite value and contradicts (1.2), so the best result we can expect is for $n \leq 9$. For example, if $f(u) = e^{\frac{1}{1-u}}$, then $g(v) = e^{v+e^v}$ verifies $\delta = \infty$ and $\mu = 1$.

Theorem 1.5. *Let* f *verify (H) and* $g(v) = e^v f(1-e^{-v})$. *Assume that* $g = o(g')$ *near* ∞. *Rewrite* $g(t) = g(0) + te^{h(t)}$ *in* $(0,\infty)$, *suppose there exists* $t_0 > 0$ *such that* $t^2 h'(t)$ *is nondecreasing for* $t \geq t_0$, *then for any bounded smooth domain* $\Omega \subset \mathbb{R}^n$ *with* $n \leq 9$, u^* *is a regular solution.*

Furthermore, when $g = o(g')$ near ∞, the condition $(\widetilde{H2})$ is just equivalent to $(H2)$, since

$$\frac{f''(t)f(t)}{f'^2(t)} = \frac{(g''-g')\,g}{(g'-g)^2}(s) = \left(\frac{g''g}{g'^2} - \frac{g}{g'}\right) \times \left(1 - \frac{g}{g'}\right)^{-2}(s), \qquad \forall\, t = 1 - e^{-s}.$$

It is also easy to see that $(H3)$ is equivalent to the condition

$$\liminf_{t\to 1^-} \frac{f'(t)(1-t)}{f(t)} = \delta > 0.$$

If the equality holds for the whole limit, we have the following optimal result. The case $f(u) = (1-u)^{-2}$ was obtained in [7] with a different argument.

Theorem 1.6. *Assume that*

$$\lim_{u\to 1^-} \frac{f'(u)(1-u)}{f(u)} = p > 0. \tag{1.5}$$

Then u^* is a regular solution if $n < n_p$ where n_p is defined in (1.3).

One of the main difficulties here is due to the vector field $c(x)$. When $c \neq 0$, the operator $-\Delta + c\cdot\nabla$ is not self-adjoint, we use ideas from [7] to get some energy estimates. However if c is a gradient, say $c = -\nabla\gamma$ in Ω, then $-\Delta + c\cdot\nabla$ can be rewritten as $e^{-\gamma}L_\gamma$ where $L_\gamma = -\mathrm{div}(e^\gamma\nabla)$ is a self-adjoint operator. In that case, (P_λ) admits a variational structure and we can expect more precise estimates of minimal solutions u_λ, as in the radial case.

The paper is organized as follows: In section 2, we prove quickly Proposition 1.1 and show some general consequences of the stability of u_λ. The section 3 is devoted to the proof of Theorems 1.3 to 1.6 for general domains. In section 4, we discuss the radial case. The norm $\|\cdot\|_q$ denotes always the standard L^q norm for any $q \in [1,\infty]$. The capital letter C denotes a generic positive constant independent of λ, it could be changed from one line to another.

2. Preliminaries

As mentioned above, $-\Delta + c\cdot\nabla$ is not a self-adjoint operator for general vector field c. However using Lemma 1 in [7], we have a kind of Hodge decomposition, which tells us that for any vector field $c \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$, there exist a smooth scalar function γ and a vector field $b \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ such that

$$c = -\nabla\gamma + b \ \text{ and } \ \mathrm{div}(e^\gamma b) = 0 \quad \text{in } \ \overline{\Omega}. \tag{2.1}$$

Therefore the problem (P_λ) can be rewritten as

$$-\,\mathrm{div}(e^\gamma\nabla u) + e^\gamma b\cdot\nabla u = \lambda e^\gamma f(u) \quad \text{in } \Omega. \tag{P'_λ}$$

On the other hand, we don't have a suitable variational characterization in general to use the stability assumption. Fortunately, we can adopt an energy inequality as in [7], which is derived from a generalized Hardy inequality of [6].

Proposition 2.2. *Let u_λ be minimal solution of (P_λ). For any $1 \leq \beta < 2$, we have*

$$\lambda \int_\Omega e^\gamma f'(u_\lambda)\psi^2 \leq \frac{2}{\beta}\int_\Omega e^\gamma |\nabla \psi|^2 + \frac{\|b\|_\infty^2}{2(2-\beta)}\int_\Omega e^\gamma \psi^2, \quad \forall\, \psi \in H_0^1(\Omega). \tag{2.2}$$

where b is the vector field in (2.1), $\|b\|_\infty = \max_{\overline{\Omega}} |b(x)|$.

Proof. We use a Hardy type inequality given by Theorem 2 in [7], which says that for a positive principal eigenfunction φ of $L_{u_\lambda,\lambda,c}$, for $\beta \in [1,2)$ and any $\psi \in H_0^1(\Omega)$,

$$\lambda \int_\Omega e^\gamma f'(u_\lambda)\psi^2 \leq \frac{2}{\beta}\int_\Omega e^\gamma |\nabla \psi|^2 + \int_\Omega \left[-\frac{2-\beta}{2}\frac{|\nabla\varphi|^2}{\varphi^2} + \frac{b\cdot\nabla\varphi}{\varphi}\right] e^\gamma \psi^2.$$

By Cauchy-Schwarz inequality, it is easy to see

$$-\frac{2-\beta}{2}\frac{|\nabla\varphi|^2}{\varphi^2} + \frac{b\cdot\nabla\varphi}{\varphi} \leq \frac{|b(x)|^2}{2(2-\beta)} \leq \frac{\|b\|_\infty^2}{2(2-\beta)},$$

so we are done. □

Another main ingredient of our approach is just the transformation $v = -\ln(1-u)$. Let ϕ and ξ be nonnegative C^1 functions satisfying $\phi(0) = \xi(0) = 0$ and $\xi' = \phi'^2$. Define $v_\lambda = -\ln(1-u_\lambda)$ and $g(v_\lambda) = e^{v_\lambda} f(1-e^{-v_\lambda})$. Using (Q_λ), we get $-\mathrm{div}(e^\gamma \nabla v_\lambda) + e^\gamma b \cdot \nabla v_\lambda \leq \lambda e^\gamma g(v_\lambda)$ in Ω. Let $\psi = \phi(v_\lambda)$ in (2.2), $\forall\, \lambda \in (0,\lambda^*)$,

$$\begin{aligned}
&\lambda \int_\Omega e^\gamma f'(u_\lambda)\phi^2(v_\lambda)\\
\leq\ & \frac{2}{\beta}\int_\Omega e^\gamma |\nabla \phi(v_\lambda)|^2 + \frac{\|b\|_\infty^2}{2(2-\beta)}\int_\Omega e^\gamma \phi^2(v_\lambda)\\
=\ & \frac{2}{\beta}\int_\Omega e^\gamma \nabla\xi(v_\lambda)\nabla v_\lambda + C_\beta \int_\Omega e^\gamma \phi^2(v_\lambda)\\
=\ & -\frac{2}{\beta}\int_\Omega \mathrm{div}(e^\gamma \nabla v_\lambda)\xi(v_\lambda) + C_\beta \int_\Omega e^\gamma \phi^2(v_\lambda)\\
\leq\ & \frac{2\lambda}{\beta}\int_\Omega e^\gamma g(v_\lambda)\xi(v_\lambda) - \frac{2}{\beta}\int_\Omega e^\gamma b\cdot\xi(v_\lambda)\nabla v_\lambda + C_\beta\int_\Omega e^\gamma\phi^2(v_\lambda)\\
=\ & \frac{2\lambda}{\beta}\int_\Omega e^\gamma g(v_\lambda)\xi(v_\lambda) + C_\beta \int_\Omega e^\gamma \phi^2(v_\lambda).
\end{aligned}$$

The last line is due to $\mathrm{div}(e^\gamma b) = 0$. We claim then

Proposition 2.3. *Let $1 \leq \beta < 2$. For any $\lambda \in (0,\lambda^*)$ and any nonnegative C^1 test functions ϕ, ξ verifying $\phi(0) = \xi(0) = 0$ and $\xi' = \phi'^2$, there hold*

$$\lambda \int_\Omega e^\gamma f'(u_\lambda)\phi^2(v_\lambda) \leq \frac{2\lambda}{\beta}\int_\Omega e^\gamma g(v_\lambda)\xi(v_\lambda) + C_\beta \int_\Omega e^\gamma \phi^2(v_\lambda) \tag{2.3}$$

and

$$\lambda \int_\Omega e^\gamma f'(u_\lambda)\phi^2(u_\lambda) \leq \frac{2\lambda}{\beta}\int_\Omega e^\gamma f(u_\lambda)\xi(u_\lambda) + C_\beta \int_\Omega e^\gamma \phi^2(u_\lambda). \tag{2.4}$$

The proof of (2.4) is completely similar to (2.3) but using (P'_λ) instead of (Q_λ).

We also make use the following behavior of f proved in [18].

Lemma 2.1. *For any f verifying (H), we have $\lim_{t\to 1} f(t)/f'(t) = 0$.*

Choose first $\phi(u) = e^u - 1$ in (2.4), then $\xi(u) = \frac{e^{2u}-1}{2}$ and

$$\lambda \int_\Omega e^\gamma f'(u_\lambda)\left(e^{u_\lambda} - 1\right)^2 \le \frac{\lambda}{\beta}\int_\Omega e^\gamma f(u_\lambda)\left(e^{2u_\lambda} - 1\right) + C_\beta \int_\Omega e^\gamma \left(e^{u_\lambda} - 1\right)^2.$$

Fix $\beta \in (1, 2)$. By Lemma 2.1,

$$\lambda \int_\Omega e^\gamma f'(u_\lambda) e^{2u_\lambda} \le C.$$

Consequently $\|f'(u_\lambda)\|_1$ is uniformly bounded, so is $\|f(u_\lambda)\|_1$. Multiplying (P_λ) by u_λ,

$$\int_\Omega |\nabla u_\lambda|^2 = \int_\Omega \frac{\operatorname{div}(c)}{2} u_\lambda^2 + \lambda \int_\Omega f(u_\lambda) u_\lambda \le C,$$

which gives

Proposition 2.4. *The family of minimal solutions $\{u_\lambda\}_{0<\lambda<\lambda^*}$ is uniformly bounded in $H_0^1(\Omega)$.*

Remark 2.1. *As far as we know, it is always an open question whether the similar H^1 energy estimation holds for minimal solutions of (1.1) with general regular nonlinearity satisfying (1.2) and general domain Ω when $n \ge 6$ (see [13] for $n \le 5$). For the advection-free case $c = 0$, it was proved in [18] that $u^* \in H^2 \cap H_0^1(\Omega)$ under the condition (H), it is also true for the gradient case $c = \nabla\gamma$ (see Lemma 4.1).*

Sketches of proof of Proposition 1.1. We follow the ideas coming from [1, 11, 10]. The main argument is the maximum principle for operators $-\Delta + c\cdot\nabla$ and L_γ under the Dirichlet boundary condition, we use also the super-sub solution method and monotone iteration.

Let $w \in H_0^1(\Omega)$ be the regular solution of $-\Delta w + c \cdot \nabla w = 1$ in Ω and fix $\alpha > 0$ such that $\alpha \max_\Omega w < 1$. It is easy to verify that αw is a supersolution of (P_λ) for $\lambda > 0$ small enough. As 0 is a subsolution and $\alpha w > 0$ in Ω, (P_λ) admits a regular solution for $\lambda > 0$ small enough. As any regular solution u of (P_λ) is also a supersolution for (P_μ) if $\mu \in (0, \lambda)$, the set of λ for which (P_λ) admits a regular solution is just an interval. Moreover, for these λ, using (H) and the monotone iteration $v_0 = 0$; $-\Delta v_{n+1} + c \cdot \nabla v_{n+1} = \lambda f(v_n)$ in Ω with $v_{n+1} = 0$ on $\partial\Omega$ for $n \in \mathbb{N}$, we get the minimal solution $u_\lambda = \lim_{n\to\infty} v_n$.

If we suppose that the principal eigenvalue of $L_{u_\lambda,\lambda,c}$ is negative, we can construct, as in [1] another solution $v \le u_\lambda$ using the associated first eigenfunction, this is just impossible by the definition of u_λ, hence u_λ is stable. The uniqueness of stable solution comes from Lemmas 2.16 and 2.17 in [8].

Take a positive first eigenfunction φ of L_γ with the Dirichlet boundary condition, by (P'_λ),

$$\lambda f(0)\int_\Omega e^\gamma\varphi \le \int_\Omega \lambda e^\gamma f(u)\varphi = \int_\Omega \lambda_1(L_\gamma)u\varphi - \int_\Omega \mathrm{div}(e^\gamma b\varphi)u \le C.$$

So λ is upper bounded. Define the critical threshold λ^* as the supermum of $\lambda > 0$ for which (P_λ) admits a regular solution, as u^* is the monotone limit of u_λ when $\lambda \to \lambda^*$, we deduce that $u^* \in H_0^1(\Omega)$ is a weak solution of (P_λ) by Proposition 2.4.

Suppose that u is a weak solution to (P_λ). By the monotonicity of f, it is easy to verify that for any $\delta > 1$, the function $v = \delta^{-1}u$ is a weak supersolution for $(P_{\lambda/\delta})$, then the monotone iteration will enable us a weak solution w of $(P_{\lambda/\delta})$ satisfying $0 \le w \le v \le \delta^{-1} < 1$. The regularity theory implies then w is a regular solution of $(P_{\lambda/\delta})$. This means that $\lambda/\delta \le \lambda^*$. Let δ tend to 1, we get $\lambda \le \lambda^*$. Therefore, no weak solution exists for $\lambda > \lambda^*$.

The uniqueness of the weak solution can be proved in the very similar way as in [11] using the monotonicity and convexity of f, with the strong maximum principle for the operator $-\Delta + c\cdot\nabla$ associated to Dirichlet boundary condition, so we omit the details.□

3. Regularity of u^* for general c and Ω

For proving our results, we will choose suitable functions ϕ to apply (2.3) or (2.4). We need also

Lemma 3.1. *For any $q > n/2$, there exists $C > 0$ such that the solution v of (Q_λ) satisfies $0 \le v \le C\|g(v)\|_q$ in Ω.*

Indeed, let w be the solution of $L(w) := -\Delta w + c\cdot\nabla w = \lambda g(v)$ in Ω with $w = 0$ on $\partial\Omega$. By regularity theory and Sobolev embedding, $\|w\|_\infty \le C\|w\|_{W^{2,q}(\Omega)} \le C'\lambda^*\|g(v)\|_q$ because $q > n/2 \ge 1$. Morover, as $L(w - v) \ge 0$, the maximum principle implies then $0 \le v \le w \le C\|g(v)\|_q$.

3.1. Proof of Theorem 1.3

For simplicity, we omit the index λ for u_λ or v_λ. Let $\phi(u) = v = -\ln(1-u)$ in (2.4), so $\xi(u) = (1-u)^{-1} - 1$. Fix $\beta \in (1, 2)$ but very close to 2. Repeating the proof of Theorem 2 in [18] with the assumption $(H1)$, there exists $C > 0$ such that

$$\lambda\int_\Omega e^\gamma\frac{f(u)}{1-u} < C + CC_\beta\int_\Omega e^\gamma\phi^2(u).$$

As $\phi^2(u) = o(\xi(u)) = o(f\xi)$ when $u \to 1^-$,

$$\lambda\int_\Omega e^\gamma\frac{f(u)}{1-u} \le C.$$

Using the equation (Q_λ) and $\partial_\nu v \le 0$ on $\partial\Omega$,

$$\begin{aligned}\int_\Omega |\nabla v|^2 = \lambda\int e^v f(1-e^{-v}) + \int_{\partial\Omega}\frac{\partial v}{\partial\nu}d\sigma - \int_\Omega c\cdot\nabla v &\le \lambda\int_\Omega\frac{f(u)}{1-u} + C\|\nabla v\|_2\\ &\le C + C\|\nabla v\|_2.\end{aligned}$$

Therefore $\|\nabla v\|_2 \le C$, the classical Moser-Trudinger inequality enables us, as $n = 2$

$$\int_\Omega e^{qv} \le C_q, \quad \forall\, q \ge 1. \tag{3.1}$$

Take now $\phi(u) = f(u) - f(0)$ in (2.4), we need to estimate

$$\begin{aligned}\zeta(u) := f'(u)\phi(u) - \frac{2}{\beta}\xi(u) &= f'(u)\phi(u) - \frac{2}{\beta}\int_0^u f'^2(s)ds\\ &= f'(u)f(u) - \frac{2}{\beta}\int_0^u f'^2(s)ds - Cf'(u)\\ &:= I(u) - \frac{2}{\beta}J(u) - Cf'(u).\end{aligned}$$

By $(H2)$, there exists $\delta > 0$ such that

$$I(u) - I(0) = \int_0^u \left[f'^2(s) + f''(s)f(s)\right] ds \ge (1+\delta)J(u) - Cf'(u), \ \ \forall\, u \in [0,1)$$

Let $\frac{4}{2+\delta} < \beta < 2$, we get $\zeta(u) \ge CI(u) - C$. Asserting this in (2.4),

$$\lambda \int_\Omega e^\gamma f'(u)f^2(u) \le C\int_\Omega e^\gamma f^2(u) + C.$$

Consequently, $\|f'(u)f^2(u)\|_1 \le C$. By Lemma 2.1, we deduce $\|f(u)\|_3 \le C$. Combining with (3.1), $\|g(v)\|_p \le C$ for any $p < 3$. The proof is completed by Lemma 3.1 as $n = 2$. $\square$

3.2. Proof of Theorem 1.4

Without loss of generality, we can assume that $g(0) = 1$. Let $\phi(t) = g^\alpha(t) - 1$ where $\alpha > 0$ is a constant to be determined later. Then

$$\begin{aligned}\xi(t) &= \int_0^t \phi'^2(s)ds\\ &= \alpha^2 \int_0^t g^{2\alpha-2}(s)g'^2(s)ds\\ &= \frac{\alpha^2}{2\alpha-1}g^{2\alpha-1}(t)g'(t) - \frac{\alpha^2}{2\alpha-1}\int_0^t g^{2\alpha-1}(s)g''(s)ds - C_\alpha.\end{aligned} \tag{3.2}$$

The condition $(\widetilde{H2})$ yields: Given any $\epsilon \in \left(0, \mu - \frac{1}{1+\delta}\right)$, there exists $C \ge 0$ such that $g(t)g''(t) \ge (\mu - \epsilon)g'^2(t) - C$ in $[0,\infty)$. Therefore

$$\begin{aligned}-\int_0^t g^{2\alpha-1}(s)g''(s)ds &\le -(\mu-\epsilon)\int_0^t g^{2\alpha-2}(s)g'^2(s)ds + C\\ &\le -\frac{\mu-\epsilon}{\alpha^2}\xi(t) + C.\end{aligned} \tag{3.3}$$

We divide the proof into two cases.

Case 1: $\delta > 1$ and $\mu > \frac{1}{1+\delta}$; or $\delta \le 1$ with $\mu > \frac{1+\delta}{4\delta}$.

Take $\alpha > \frac{1}{2}$. Combine (3.2) and (3.3),

$$\left(1 + \frac{\mu - \epsilon}{2\alpha - 1}\right)\xi(t) \le \frac{\alpha^2}{2\alpha - 1} g^{2\alpha-1}(t)g'(t) + C,$$

consequently

$$\xi(t) \le \frac{\alpha^2}{2\alpha - 1 + \mu - \epsilon} g^{2\alpha-1}(t)g'(t) + C, \quad \text{for any} \;\; t \ge 0. \tag{3.4}$$

According to $(H3)$, for any $0 < \delta' < \delta$, there exists $C > 0$ such that $g'(t) \ge (1+\delta')g(t) - C$ in $[0, \infty)$. Setting these estimates in (2.3), omitting the index λ and recalling that $f'(u) = g'(v) - g(v)$,

$$\begin{aligned} &\frac{\delta'\lambda}{1+\delta'}\int_\Omega e^\gamma g'(v)(g^\alpha(v) - 1)^2 - C\lambda \int_\Omega e^\gamma (g^\alpha(v) - 1)^2 \\ &\le \lambda \int_\Omega e^\gamma f'(u)(g^\alpha(v) - 1)^2 \\ &\le \frac{2\alpha^2\lambda}{\beta(2\alpha - 1 + \mu - \epsilon)}\int_\Omega e^\gamma g^{2\alpha}(v)g'(v) + C\lambda\int_\Omega e^\gamma g(v) + C\int_\Omega e^\gamma (g^\alpha(v) - 1)^2. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left[\frac{\delta'}{1+\delta'} - \frac{2\alpha^2}{\beta(2\alpha - 1 + \mu - \epsilon)}\right]\lambda\int_\Omega e^\gamma g'(v)g^{2\alpha}(v) \\ &\le \frac{2\delta' C}{1+\delta'}\int_\Omega e^\gamma g'(v)g^\alpha(v) + C\int_\Omega e^\gamma g(v) + C\int_\Omega e^\gamma (g^\alpha(v) - 1)^2. \end{aligned}$$

Choose δ' near δ such that

$$\text{either} \;\; \delta' > 1 \;\; \text{and} \;\; \mu > \frac{1}{1+\delta'} \quad \text{or} \quad \delta' < \delta \le 1 \;\; \text{with} \;\; \mu > \frac{1+\delta'}{4\delta'}.$$

Through direct computations, for $\epsilon > 0$ sufficiently small and $\beta = 2 - \epsilon$, there exists

$$\alpha \in \left(\frac{1}{2}, \frac{\delta'}{1+\delta'} + \frac{\sqrt{\delta'(1+\delta')(\mu - \epsilon) - \delta'}}{1+\delta'}\right)$$

such that

$$\left[\frac{\delta'}{1+\delta'} - \frac{2\alpha^2}{\beta(2\alpha - 1 + \mu - \epsilon)}\right] > 0. \tag{3.5}$$

For such α, we obtain

$$\lambda\int_\Omega e^\gamma g^{2\alpha}(v)g'(v) \le C, \quad \forall\, \lambda \in (0, \lambda^*). \tag{3.6}$$

Tending now δ' to δ and ϵ to 0, (3.6) holds true provided that

$$\alpha < \frac{\delta}{1+\delta} + \frac{\sqrt{\delta\mu(1+\delta) - \delta}}{1+\delta}. \tag{3.7}$$

Therefore

$$\int_\Omega e^\gamma g^{2\alpha+1}(v) \le C \int_\Omega e^\gamma g^{2\alpha}(v) g'(v) + C \le \widetilde{C},$$

which implies that $\|g(v)\|_{2\alpha+1} \le C$ for α verifying (3.7). Applying Lemma 3.1, we conclude that for $n < 2 + 4\alpha$ with α verifying (3.7), v_λ is uniformly bounded, hence u^* is a regular solution if n satisfies (1.4).

Case 2: $\delta \le 1$ and $\frac{1}{1+\delta} < \mu \le \frac{1+\delta}{4\delta}$.

Now we take $\alpha \in \left(\frac{1}{2}(1-\mu+\epsilon), \frac{1}{2}\right)$, the formulas (3.2) and (3.3) imply then

$$\left(1 + \frac{\mu - \epsilon}{2\alpha - 1}\right)\xi(t) \ge \frac{\alpha^2}{2\alpha - 1} g^{2\alpha - 1}(t) g'(t) + C.$$

The inequality (3.4) still holds true. Proceeding as for Case 1, we see that for $\delta' < \delta$ but nearby, $\epsilon > 0$ small and $\beta = 2 - \epsilon$, there exists

$$\alpha \in \left(\frac{1-\mu+\epsilon}{2}, \frac{\delta'}{1+\delta'} + \frac{\sqrt{\delta'(1+\delta')(\mu-\epsilon) - \delta'}}{1+\delta'}\right) \subset \left(\frac{1-\mu+\epsilon}{2}, \frac{1}{2}\right)$$

such that (3.5) is satisfied. Hence we conclude exactly as in *Case 1*. □

3.3. Proof of Theorem 1.5

Without loss of generality, assume again $g(0) = 1$. Take now $\phi(t) = te^{\alpha h(t)}$, where $\alpha > 0$ is a constant to be determined, then

$$\begin{aligned}\xi(t) &= \int_0^t \left[1 + s\alpha h'(s)\right]^2 e^{2\alpha h(s)} ds \\ &= \int_0^t \left[1 + 2s\alpha h'(s)\right] e^{2\alpha h(s)} ds + \int_0^t \alpha^2 s^2 h'^2(s) e^{2\alpha h(s)} ds \\ &= te^{2\alpha h(t)} + K(t).\end{aligned}$$

Thus, for $t \ge t_0$,

$$\begin{aligned}\frac{2K(t)}{\alpha} = 2\alpha \int_0^t s^2 h'^2(s) e^{2\alpha h(s)} ds &= C + \int_{t_0}^t s^2 h'(s) d\left(e^{2\alpha h(s)}\right) \\ &\le C + t^2 h'(t) e^{2\alpha h(t)} - \int_{t_0}^t e^{2\alpha h(s)} d\left(s^2 h'(s)\right),\end{aligned}$$

where the last integration is considered in the sense of Stieltjes. The monotonicity of $s^2 h'$ in $[t_0, \infty)$ yields

$$K(t) \le \frac{\alpha}{2} t^2 h'(t) e^{2\alpha h(t)} + C, \quad \forall\, t \ge t_0.$$

So we get

$$\xi(t) \le C + \left[t + \frac{\alpha}{2}t^2h'(t)\right]e^{2\alpha h(t)}, \quad \forall\, t \ge 0.$$

Using (2.3) (we drop the index λ),

$$\begin{aligned}
&\int_\Omega e^\gamma \left[e^{h(v)} + vh'(v)e^{h(v)} - ve^{h(v)} - 1\right] v^2 e^{2\alpha h(v)}\\
\le\; &\frac{2}{\beta}\int_\Omega e^\gamma \left(1 + ve^{h(v)}\right)\xi(v) + C\int_\Omega e^\gamma v^2 e^{2\alpha h(v)}\\
\le\; &\frac{2}{\beta}\int_\Omega e^\gamma \left(1 + ve^{h(v)}\right)\left[C + ve^{2\alpha h(v)} + \frac{\alpha}{2}v^2h'(v)e^{2\alpha h(v)}\right] + C\int_\Omega e^\gamma v^2 e^{2\alpha h(v)},
\end{aligned}$$

By Young's inequality,

$$\begin{aligned}
&\left(1 - \frac{\alpha}{\beta}\right)\int_\Omega e^\gamma v^3 h'(v)e^{(2\alpha+1)h(v)}\\
\le\; &C\int_\Omega e^\gamma \left[1 + v^2h'(v)e^{2\alpha h(v)} + v^3e^{(2\alpha+1)h(v)}\right].
\end{aligned} \tag{3.8}$$

Moreover, $g = o(g')$ at infinity yields $\lim_{t\to\infty} h'(t) = \infty$, hence

$$\frac{t^2h'(t)e^{2\alpha h(t)} + t^3e^{(2\alpha+1)h(t)}}{t^3h'(t)e^{(2\alpha+1)h(t)}} = \frac{1}{g(t)-1} + \frac{1}{h'(t)} \to 0 \ \text{ as } \ t \to \infty.$$

Fix $\beta \in (\alpha, 2)$, the inequality (3.8) implies

$$\int_\Omega \frac{[g(v)-1]^{2\alpha+1}}{v^{2\alpha}} = \int_\Omega ve^{(2\alpha+1)h(v)} \le C + \int_\Omega v^3h'(v)e^{(2\alpha+1)h(v)} \le C.$$

Recall that g is superlinear, we obtain $\|g(v)\|_1 \le C$. Consider again w satisfying $L(w) = \lambda g(v)$ in Ω and $w = 0$ on $\partial\Omega$, as $v \le w$ in Ω by maximum principle,

$$\int_\Omega \frac{(g(v)-1)^{2\alpha+1}}{w^{2\alpha}} \le C.$$

Following the proof of Lemma 2.1 in [17] (we just need a minor adjustment, say define $\Omega_1 = \{x \in \Omega : g(v) > w^T\}$ instead, here $T > 0$ is a suitable constant), we can obtain that if $2\alpha + 1 > n/2$, w is uniformly bounded in $L^\infty(\Omega)$, so does v. Taking $2 > \beta > \alpha > 7/4$, the result holds for $n \le 9$. □

3.4. Proof of Theorem 1.6

Here we choose $\phi(u) = (1-u)^{-\alpha} - 1$ in (2.4). For $2\lambda > \lambda^*$ and $\epsilon > 0$,

$$\left(p - \frac{2\alpha^2}{\beta(2\alpha+1)} - 2\epsilon\right)\int_\Omega \frac{e^\gamma}{(1-u)^{p+2\alpha+1}} \le C, \quad \forall\, \beta \in [1, 2).$$

We have used $f'(u)(1-u) \ge (p-\epsilon)f(u) - C$ in $[0,1)$ by (1.5). As $\epsilon > 0$ is arbitrary,

$$\int_\Omega \frac{1}{(1-u)^{p+2\alpha+1}} \le C$$

provided that

$$p > \frac{\alpha^2}{2\alpha+1}, \quad \text{i.e. when } \ \alpha < p + \sqrt{p(p+1)}.$$

Therefore $\|(1-u)^{-1}\|_q \le C$ if $q < 1 + 3p + 2\sqrt{p(p+1)}$. For any $\epsilon > 0$, as $f'(u)(1-u) \le (p+\epsilon)f(u) + C_\epsilon$ in $[0,1)$ by (1.5), we have $f(u) \le C(1-u)^{-p-\epsilon}$, consequently

$$g(v) = e^v f(1-e^{-v}) = \frac{f(u)}{1-u} \le C(1-u)^{-1-p-\epsilon},$$

hence $\|g(v)\|_r \le C$ when

$$r < \frac{1+3p+2\sqrt{p(p+1)}}{p+1+\epsilon}.$$

According to Lemma 3.1, the proof is done by taking $\epsilon \to 0^+$. □

4. Radial case

As we have mentioned, when $c = -\nabla\gamma$, the equation (P_λ) is rewritten as

$$-\text{div}(e^\gamma \nabla u) = \lambda e^\gamma f(u). \tag{4.1}$$

With the variational structure, the stability of minimal solutions u_λ is equivalent to

$$\int_\Omega e^\gamma |\nabla\psi|^2 \ge \lambda \int_\Omega e^\gamma f'(u_\lambda)\psi^2, \quad \forall\, \psi \in H_0^1(\Omega). \tag{4.2}$$

Moreover, for any C^1 functions ϕ and ξ satisfying $\phi(0) = \xi(0) = 0$ and $\xi' = \phi'^2$, the estimate (2.4) is replaced by

$$\int_\Omega e^\gamma f'(u_\lambda)\phi^2(u_\lambda) \le \int_\Omega e^\gamma f(u_\lambda)\xi(u_\lambda).$$

Taking now $\phi(t) = f(t) - f(0)$ and working as for Theorem 1 in [18], we have

Lemma 4.1. *When $c = \nabla\gamma$, the extremal solution $u^* \in H^2 \cap H_0^1(\Omega)$. More precisely,*

$$\int_\Omega f'(u_\lambda)f(u_\lambda) \le C, \quad \forall\, \lambda \in (0, \lambda^*]. \tag{4.3}$$

When $\Omega = B_1$ is the unit ball, $\gamma(x) = \gamma(r)$ with $r = |x|$, u_λ is radial by uniqueness of the minimal solution and satisfies

$$-u'' - \frac{n-1}{r}u' - \gamma' u' = \lambda f(u) \quad \text{in } (0,1], \tag{4.4}$$

with $u'(0) = 0$ and $u(1) = 0$. Our main result in this section is the regularity of the extremal solution u^* for any f satisfying (H) provided $n = 2$ and the optimal estimate for u' claimed in Theorem 1.2.

The method we use is similar to [5, 15], but the uniform boundedness of $\|u_\lambda\|_{C^1}$ is not enough to claim the regularity of u^*, because a singular u^* could be Lipschitz in many cases (see Remark 1.1). In fact, the estimate (4.3) is crucial for our proof.

As in [5, 15], since $u'_\lambda(r) \le 0$ by maximum principle or equation (4.4), the boundedness of $\|u_\lambda\|_{H_0^1}$ implies that for any $k \in \mathbb{N}$, $r > 0$, $\|u_\lambda\|_{C^k(\overline{B}_1 \setminus B_r)} \le C_{k,r}$, $\forall\, \lambda \in (0, \lambda^*]$. So we concentrate our attention near the origin. Derivating the equation (4.4) or (4.1) with respect to r,

$$-\mathrm{div}\,(e^\gamma \nabla u') = e^\gamma u' \left[\lambda f'(u) - \frac{n-1}{r^2} + \gamma''\right] \quad \text{in } (0, 1].$$

Using $\psi = r\eta(r)u'_\lambda(r)$ as test function in (4.2) with $\eta \in H_0^1(B_1) \cap C(\overline{B}_1)$, by similar calculation as for Lemma 2.1 in [5], we obtain

$$\int_{B_1} e^\gamma \Big[|\nabla(r\eta)|^2 - (n-1)\eta^2 + \gamma'' r^2\eta^2\Big] u_\lambda'^2 \ge 0, \quad \forall\, \lambda \in (0, \lambda^*]. \tag{4.5}$$

4.1. Proof of Theorem 1.1

For simplicity, we drop the index λ. All estimates below hold uniformly for λ. First as u_λ is radial, by maximum principle, we see that u is decreasing in r. Since f and f' are nondecreasing functions according to (H), the estimate (4.3) implies (as $n = 2$)

$$\pi r^2 f'(u(r)) f(u(r)) \le \int_{B_r} f'(u) f(u) \le C, \quad \forall\, r \in (0, 1].$$

By Lemma 2.1, we have

$$f(u(r)) \le \frac{C}{r} \quad \text{for all } r \in (0, 1]. \tag{4.6}$$

Let $r_0 \in (0, \frac{1}{2}]$. Let η be a radial function in $H_0^1(B_1) \cap C^0(\overline{B}_1)$ such that

$$\eta(r) = \begin{cases} r_0^{-1} & \text{if } r < r_0; \\ r^{-1} & \text{if } r_0 \le r \le \frac{1}{2}, \end{cases}$$

and η be a fixed C^1 function in $\overline{B}_1 \setminus B_{1/2}$, independent of r_0. The direct calculation yields

$$|\nabla(r\eta)|^2 - \eta^2 + \gamma'' r^2 \eta^2 = \begin{cases} \gamma'' r^2 r_0^{-2} & \text{if } r < r_0; \\ \gamma'' - r^{-2} & \text{if } r_0 < r \le \frac{1}{2}. \end{cases}$$

Using (4.5), as u is uniformly bounded in $H^1(B_1)$ by Proposition 2.4 and $r^2 r_0^{-2} \le 1$ in $[0, r_0]$, we get

$$\int_{r_0}^{\frac{1}{2}} \frac{u'(r)^2}{r} dr \le C.$$

Tending r_0 to 0, there holds

$$\int_0^1 \frac{u'(r)^2}{r} dr \le C. \tag{4.7}$$

Consider the following test function used in [15]: For any $r \le \frac{1}{2}$ and $0 < r_0 < r$,

$$\eta(s) = \begin{cases} (rr_0)^{-1} & \text{if } s < r_0; \\ (rs)^{-1} & \text{if } r_0 \le s < r; \\ s^{-2} & \text{if } r \le s \le \frac{1}{2}. \end{cases}$$

Applying again (4.5) and combining with (4.7), we obtain finally (with $r_0 \to 0$)

$$\int_0^r \frac{u'(s)^2}{s} ds \le Cr^2, \qquad \forall\, r \le 1. \tag{4.8}$$

As $(e^\gamma r u')' = -\lambda e^\gamma r f(u)$ with $n = 2$, so $e^\gamma r u'$ is nonincreasing in r. Then $u'(s) \le Cru'(r)/s$ for $s \in [r, 1]$, hence $u'(s) \le Cu'(r) \le 0$ for any $s \in [r, 2r]$ if $r \le \frac{1}{2}$. By (4.8), for any $0 < r \le \frac{1}{2}$,

$$C_1 r^2 \ge \int_0^{2r} \frac{u'(s)^2}{s} ds \ge \int_r^{2r} \frac{u'(s)^2}{s} ds \ge \frac{C_2}{r} \int_r^{2r} u'(r)^2 ds = C_3 u'(r)^2.$$

That means

$$|u'(r)| \le Cr \quad \text{in} \ \ [0, 1]. \tag{4.9}$$

However, we need to consider also $u''(r)$ as explained above. Let

$$G(r) = e^\gamma r u' \quad \text{and} \quad \Psi(r) = -2G(\sqrt{r}) - M \int_0^r (r - s) f\left(u(\sqrt{s})\right) ds$$

where M is a constant to be chosen. Using $G' = -\lambda e^\gamma r f(u)$,

$$\begin{aligned} \Psi''(r) &= \left[\lambda e^{\gamma(s)} f'(u(s)) \frac{u'(s)}{2s} + \lambda e^{\gamma(s)} f(u(s)) \frac{\gamma'(s)}{2s} - Mf(u(s))\right] \Big|_{s=\sqrt{r}} \\ &\le \left[\lambda e^{\gamma(s)} f(u(s)) \frac{\gamma'(s)}{2s} - Mf(u(s))\right] \Big|_{s=\sqrt{r}} \\ &\le C_0 f\left(u(\sqrt{r})\right) - Mf\left(u(\sqrt{r})\right). \end{aligned}$$

For the last line, we used $|\gamma'(s)|/s \le C$ in $[0, 1]$ since γ is a smooth function (so $\gamma'(0) = 0$). Fix $M > C_0 + 1$, Ψ is then concave in $[0, 1]$. On the other hand, by (4.6)

$$\Psi'(r) = \lambda e^{\gamma(\sqrt{r})} f\left(u(\sqrt{r})\right) - M \int_0^r f\left(u(\sqrt{s})\right) ds \ge C\lambda f(0) - CM\sqrt{r}.$$

There exists $r_1 > 0$ small enough such that $\Psi' \ge 0$ in $[0, r_1]$ with $\lambda \ge \frac{\lambda^*}{2}$. Using (4.4), (4.6) and (4.9), for $\lambda \ge \frac{\lambda^*}{2}$ and $r \le r_1$,

$$\begin{aligned} &- e^{\gamma(\sqrt{r})} \left[u''(\sqrt{r}) + \frac{u'(\sqrt{r})}{\sqrt{r}} + \gamma' u'(\sqrt{r})\right] - CM\sqrt{r} \\ &\le \Psi'(r) \le \frac{\Psi(r)}{r} \le -2e^{\gamma(\sqrt{r})} \frac{u'(\sqrt{r})}{\sqrt{r}} \le C. \end{aligned}$$

Applying one more time (4.9), we see that $u''(\sqrt{r}) \geq -C$ for any $\lambda \geq \frac{\lambda^*}{2}$ and $r \leq r_1$. Otherwise, by (4.4) and (4.9), $u''(r) \leq -u'(r)r^{-1} - \gamma'(r)u'(r) \leq C$, we claim then

$$\|u''\|_\infty \leq C, \quad \forall\, \lambda \geq \frac{\lambda^*}{2}.$$

Combining with (4.4) and (4.9), it means $\|\lambda f(u)\|_\infty \leq C$, no singularity will occur. $\square$

4.2. Proof of Theorem 1.2

As above, we drop the index λ and all estimations hold uniformly for λ. First, repeating the proof of Theorem 1.8, c) in [5], we obtain $f'(u(r)) \leq Cr^{-2}$ in $(0, 1]$. Using Lemma 2.1 with (4.5), $f(u(r)) \leq Cr^{-2}$ in $(0, 1]$. Consequently, by (4.4), for $n \geq 3$,

$$0 \leq -e^{\gamma} r^{n-1} u'(r) = \int_0^r e^{\gamma(s)} s^{n-1} f(u(s)) ds \leq C \int_0^r s^{n-3} ds \leq Cr^{n-2}.$$

Hence

$$|u'(r)| \leq \frac{C}{r}. \tag{4.10}$$

Let η be a radial function in $H_0^1(B_1) \cap C^0(\overline{B}_1)$ such that

$$\eta(r) = \begin{cases} r_0^{-\sqrt{n-1}} & \text{if } r < r_0; \\ r^{-\sqrt{n-1}} & \text{if } r_0 \leq r \leq r_1. \end{cases}$$

in $\overline{B}_{r_1}$ and be a fixed C^1 function in $\overline{B}_1 \setminus B_{r_1}$, here r_0 is any constant in $(0, r_1)$, $r_1 > 0$ is a small constant to be determined. Therefore

$$|\nabla(r\eta)|^2 - (n-1)\eta^2 + \gamma'' r^2 \eta^2 = \begin{cases} \left(\gamma'' r^2 + 2 - n\right) r_0^{-2\sqrt{n-1}} & \text{if } r < r_0; \\ \left(\gamma'' r^2 - 2\sqrt{n-1} + 1\right) r^{-2\sqrt{n-1}} & \text{if } r \in [r_0, r_1]. \end{cases}$$

We fix $r_1 > 0$ small enough such that

$$\max_{r \in [0, r_1]} \left\{\gamma'' r^2\right\} < \min\left(n - 2, 2\sqrt{n-1} - 1\right).$$

By (4.5), as $|\nabla(r\eta)|^2 - (n-1)\eta^2 + \gamma'' r^2 \eta^2 \leq 0$ for $r \in [0, r_0]$,

$$\int_{r_0}^{r_1} u'^2(r) r^{n-1-2\sqrt{n-1}} dr \leq C.$$

Tending r_0 to 0, we have

$$\int_0^{r_1} u'^2(r) r^{n-1-2\sqrt{n-1}} dr \leq C. \tag{4.11}$$

Now we take another test function used in [15],

$$\eta(r) = \begin{cases} r_0^{-\sqrt{n-1}-1} & \text{if } r < r_0; \\ r^{-\sqrt{n-1}-1} & \text{if } r_0 \leq r \leq r_1. \end{cases}$$

Combining (4.5) and (4.11), we conclude then

$$\int_0^{r_0} u'^2(r) r^{n-1} dr \le C r_0^{2+2\sqrt{n-1}}, \quad \forall\, r_0 \in [0, r_1].$$

By the monotonicity of $e^\gamma r^{n-1} u'$, similarly as for (4.9), it holds

$$|u'(r)| \le C r^{-\frac{n}{2}+1+\sqrt{n-1}}, \quad \forall\, r \in [0, 1].$$

Finally, combining with (4.10), we are done (in fact, $-\frac{n}{2} + 1 + \sqrt{n-1} \le -1$ for $n \ge 10$). □

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