Ruelle transfer operators for contact Anosov flows and decay of correlations

by Luchezar Stoyanov

Abstract. We prove strong spectral estimates for Ruelle transfer operators for arbitrary C^2 contact Anosov flows in any dimension. For such flows some of the consequences of the main result are: (a) exponential decay of correlations for Hölder continuous observables with respect to any Gibbs measure; (b) existence of a non-zero analytic continuation of the Ruelle zeta function with a pole at the entropy in a vertical strip containing the entropy in its interior; (c) a Prime Orbit Theorem with an exponentially small error. All these results apply for example to geodesic flows on arbitrary compact Riemann manifolds of negative curvature.

1 Introduction and Results

1.1 Introduction

The study of statistical properties of dynamical systems has a long history and has been the subject of a considerable interest due to their applications in statistical mechanics and thermodynamics. Many physical systems poses some kind of 'strong hyperbolicity' and are known to have or expected to have strong mixing properties. For example in the 70's, due to works by Sinai, Bowen and Ruelle, it was already known that for Anosov diffeomorphisms exponential decay of correlations takes place for Hölder continuous observables (see e.g. the survey article [ChY]). However the continuous case proved to be much more difficult and it took about twenty years until the breakthrough work of Dolgopyat [D], where he established exponential decay of correlations for Hölder continuous potentials in two major cases: (i) geodesic flows on compact surfaces of negative curvature (with respect to any Gibbs measure); (ii) transitive Anosov flows on compact Riemann manifolds with C^1 jointly non-integrable local stable and unstable foliations (with respect to the Sinai-Bowen-Ruelle measure).

Dolgopyat's work was followed by a considerable activity to establish exponential and other types of decay of correlations for various kinds of systems – see [BaL] for more information and historical remarks. See also [Ch1], [Ch2], [ChY], [BaG], [BaT], [DL], [GL], [L1], [M], [N], [Y1], [Y2], [T], [FT] and the references there. Liverani [L1] proved exponential decay of correlations for C^4 contact Anosov flows with respect to the measure determined by the Riemann volume. Some finer results were obtained later by Tsujii [T] (for C^3 contact Anosov flows) and recently by Nonnenmacher and Zworski [NZ] (for a class of C^∞ flows which includes the C^∞ contact Anosov flows); both papers dealing with the measure determined by the Riemann volume.

In this paper, as a consequence of the main result, we derive exponential decay of correlations for C^5 contact Anosov flows on Rieman manifolds M of any dimension and with respect to any Gibbs measure on M. It appears that so far the only results of this kind have been that of Dolgopyat [D] for geodesic flows on C^5 compact surfaces and the one in [St2] for Axiom A flows on basic sets (under additional assumptions including Lipschitz regularity of stable/unstable holonomy maps).

More recently the emphasis in studying decay of correlations appears to be in trying to establish such results for non-uniformly hyperbolic systems and systems with singularities, e.g. billiards. For example, very recently Baladi and Liverani [BaL] proved exponential decay of correlations for piecewise hyperbolic contact flows on three-dimensional manifolds. Many of these works used some ideas from [D], however most of them followed a different approach, namely the so called functional-analytic approach initiated by the work of Blank, Keller and Liverani [BKL] which involves the study of the so called Ruelle-Perron-Frobenius operators $\mathcal{L}_t g = \dfrac{g \circ \phi_{-t}}{|(\det d\phi_t)| \circ \phi_{-t}}$, $t \in \mathbb{R}$ (see e.g. the lectures of Liverani [L2] for a nice exposition of the main ideas).

A similar approach, however studying Ruelle-Perron-Frobenius operators acting on currents, was used in a very recent paper by Giulietti, Liverani and Pollicott [GLP] where they proved some remarkable results. For example, they established that for C^∞ Anosov flows the Ruelle zeta function is meromorphic in the whole complex plane.

In [D] Dolgopyat used a different approach and established some statistical properties (for the flows he considered) that appear to be much stronger than exponential decay of correlations. Indeed, using these properties, a certain technique developed in [D] involving estimates of Laplace transforms of correlations functions (following previous works of Pollicott [Po] and Ruelle [R3]), leads more or less automatically to exponential decay of correlations for Hölder continuous potentials. The approach in [D] involved studying spectral properties of the so called Ruelle transfer operators whose definition requires a Markov partition. Given an Anosov flow $\phi_t : M \longrightarrow M$ on a Riemann manifold M, consider a Markov partition consisting of rectangles $R_i = [U_i, S_i]$, where U_i and S_i are pieces of unstable/stable manifolds at some $z_i \in M$, the first return time function $\tau : R = \cup_{i=1}^{k_0} R_i \longrightarrow [0,\infty)$ and the standard Poincaré map $\mathcal{P} : R \longrightarrow R$ (see Sect. 2 for details). The *shift map* $\sigma : U = \cup_{i=1}^{k_0} U_i \longrightarrow U$, given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the projection along the leaves of local stable manifolds, defines a dynamical system which is essentially isomorphic to an one-sided Markov shift. Given a bounded function $f \in B(U)$, one defines the *Ruelle transfer operator* $L_f : B(U) \longrightarrow B(U)$ by $(L_f h)(x) = \sum_{\sigma(y)=x} e^{f(y)} h(y)$. Assuming that f is real-valued and Hölder continuous, let $P_f \in \mathbb{R}$ be such that the topological pressure of $f - P_f \tau$ with respect to σ is zero (cf. e.g. [PP]). Dolgopyat proved (for the type of flows he considered in [D]) that for small $|a|$ and large $|b|$ the spectral radius of the Ruelle operator $L_{f-(P_f+a+\mathbf{i}b)\tau} : C^\alpha(U) \longrightarrow C^\alpha(U)$ acting on α-Hölder continuous functions ($0 < \alpha \leq 1$) is uniformly bounded by a constant $\rho < 1$. Whenever the latter holds we will say that *Ruelle transfer operators related to f are eventually contracting on $C^\alpha(U)$.*

More general results of this kind were proved in [St2] for mixing Axiom A flows on basic sets under some additional regularity assumptions, amongst them – Lipschitzness of the so local stable holonomy maps[1] (see Sect. 2). Further results in this direction were established in [St3].

In this paper we consider for a given $\theta \in (0,1)$, the metric D_θ on $\widehat{U}$ defined by $D_\theta(x,y) = 0$ if $x = y$, $D_\theta(x,y) = 1$ if x,y belong to different U_i's and $D_\theta(x,y) = \theta^N$ if $\mathcal{P}^j(x)$ and $\mathcal{P}^j(y)$ belong to the same rectangle R_{i_j} for all $j = 0,1,\ldots,N-1$, and N is the largest integer with this property. Let $\widehat{U}$ be the set of all $x \in U$ whose orbits do not have common points with the boundary of R. Denote by $\mathcal{F}_\theta(\widehat{U})$ *the space of all functions $h : \widehat{U} \longrightarrow \mathbf{C}$ with Lipschitz constants* $|h|_\theta = \sup\{\frac{|h(x)-h(y)|}{D_\theta(x,y)} : x \neq y\,,\, x,y \in \widehat{U}\} < \infty$.

Our main result in this paper is that for any C^2 contact Anosov flow on a C^2 compact Riemann manifold M, for sufficiently large $\theta \in (0,1)$ and any real-valued function $f \in \mathcal{F}_\theta(\widehat{U})$ the Ruelle transfer operators related to f are eventually contracting on $\mathcal{F}_\theta(\widehat{U})$. A similar result holds for Hölder continuous functions on $\widehat{U}$ – see Corollary 1.2 below. Some immediate consequences of the main result are: (a) exponential decay of correlations for Hölder continuous observables with respect to any Gibbs measure on M (for C^5 contact Anosov flows); (b) existence of a non-zero analytic continuation of the Ruelle zeta function in a vertical strip containing the topological entropy h_T of the flow in its interior with a pole at h_T; (c) a Prime Orbit Theorem with an exponentially small error, thus proving a conjecture made recently in [GLP].

In the proof of the main result we use the general framework of the method of Dolgopyat [D]

[1]In general these are only Hölder continuous – see [Ha1], [Ha2].

and its development in [St2], however some significant new ideas have been implemented. The main problem is to deal with the lack of regularity of the local stable/unstable manifolds and related local stable/unstable holonomy maps[2] – as we mentioned earlier, in general these are only Hölder continuous. In [D] and [St2] these were assumed to be C^1 and Lipschitz, respectively. Since the definition of Ruelle operators itself involves sliding along local stable manifolds, it appears to be a significant problem to overcome the lack of regularity in general[3].

There are several novelties in the approach we use in this article that allow to deal with this difficulty: (a) making use of Pesin's theory of Lyapunov exponents; (b) using Liverani's Lemma B.7 in [L1][4] to estimate the so called temporal distance function[5] over cylinders using the smooth symplectic form defined by the contact form on M; (c) establishing and making use of the uniform D-Lipschitzness of the local (stable) holonomy maps on a large compact set of Lyapunov regular points. Here D is a certain metric (introduced in [St2]) defined by means of diameters of cylinders on local unstable manifolds. All these features are of fundamental importance in this article.

Here is the plan of the paper. The main results are stated in Sect. 1.2. In Sect. 1.3 we discuss certain points in the proof of Theorem 1.1. Sects. 2 and 3 contain some basic definitions and facts from hyperbolic dynamics and Pesin's theory of Lyapunov exponents, respectively.

In Sect. 4 we construct a compact subset P of the union of all rectangles R_i in the Markov partition consisting of Lyapunov regular points and state some properties concerning diameters of cylinders intersecting the set P. Several refinements of P are made later in Sect. 4 and Sect. 5 (these are called P_1, P_2, etc.). It turns out that choosing P appropriately, cylinders intersecting P have similar properties to these established in [St4] under some pinching conditions. These properties (Lemma 4.4) are proved in Sect. 9. In Sect. 4 we also state the Main Lemmas 4.2 and 4.3. We prove them in Sect. 8 using Liverani's Lemma 4.1. Sects. 5-7, which should be regarded as the central part of this article, are devoted to the proofs of Theorem 1.1 and Corollary 1.2.

1.2 Statement of results

Let $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ be a (pseudo-) Markov partition for the flow $\phi_t : M \longrightarrow M$ consisting of rectangles $R_i = [U_i, S_i]$, where U_i (resp. S_i) are subsets of $W^u_\epsilon(z_i)$ (resp. $W^s_\epsilon(z_i)$) for some $\epsilon > 0$ and $z_i \in M$ (cf. Sect. 2 for details). The first return time function $\tau : R = \cup_{i=1}^{k_0} R_i \longrightarrow [0,\infty)$ is essentially α_1-Hölder continous on R for some $\alpha_1 > 0$, i.e. there exists a constant $L > 0$ such that if $x, y \in R_i \cap \mathcal{P}^{-1}(R_j)$ for some i, j, where $\mathcal{P} : R \longrightarrow R$ is the standard Poincaré map, then $|\tau(x) - \tau(y)| \leq L\,(d(x,y))^{\alpha_1}$. The *shift map* $\sigma : U = \cup_{i=1}^{k_0} U_i \longrightarrow U$ is defined by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the projection along the leaves of local stable manifolds. Let $\widehat{U}$, D_θ, $\mathcal{F}_\theta(\widehat{U})$ and $|h|_\theta$ be defined as in Sect. 1.1. Define the norm $\|.\|_{\theta,b}$ on $\mathcal{F}_\theta(\widehat{U})$ by $\|h\|_{\theta,b} = \|h\|_0 + \frac{|h|_\theta}{|b|}$, where $\|h\|_0 = \sup_{x\in\widehat{U}} |h(x)|$.

Given a real-valued function $f \in \mathcal{F}_\theta(\widehat{U})$, set $g = g_f = f - P_f\tau$, where $P_f \in \mathbb{R}$ is the unique number such that the topological pressure $\mathrm{Pr}_\sigma(g)$ of g with respect to σ is zero (cf. [PP]).

We say that *Ruelle transfer operators related to f are eventually contracting on $\mathcal{F}_\theta(\widehat{U})$* if for every $\epsilon > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leq a_0$ and $|b| \geq b_0$, then

$$\|L^m_{f-(P_f+a+\mathbf{i}b)\tau} h\|_{\theta,b} \leq C\,\rho^m\,|b|^\epsilon\,\|h\|_{\theta,b}$$

[2]E.g. the local stable holonomy maps are defined by sliding along local stable manifolds.

[3]For this reason perhaps, many experts in the field have been predicting that this approach would not be working without strong additional regularity assumptions.

[4]See also Appendix D in [GLP] for an improved version of this lemma.

[5]Which is only Hölder continuos in general.

for any integer $m > 0$ and any $h \in \mathcal{F}_\theta(\widehat{U})$. This implies that the spectral radius of $L_{f-(P_f+a+\mathbf{i}b)\tau}$ on $\mathcal{F}_\theta(\widehat{U})$ does not exceed ρ.

The main result in this paper is the following.

Theorem 1.1. *Let $\phi_t : M \longrightarrow M$ be a C^2 contact Anosov flow, let $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ be a (pseudo-) Markov partition for ϕ_t as above and let $\sigma : U \longrightarrow U$ be the corresponding shift map. There exists $\hat{\theta} \in (0,1)$ such that for any $\theta \in [\hat{\theta}, 1)$ and any real-valued $f \in \mathcal{F}_\theta(\widehat{U})$ the Ruelle transfer operators related to f are eventually contracting on $\mathcal{F}_\theta(\widehat{U})$.*

In fact one can derive from the arguments in this paper that the conclusion of Theorem 1.1 holds for any C^2 Anosov flows satisfying a non-integrability condition (such as (LNIC) in [St2]) and some additional regularity condition (e.g. Lipschitzness of the local stable/unstable holonomy maps will suffice).

A result similar to Theorem 1.1 for Hölder continuous functions (with respect to the Riemann metric) looks a bit more complicated, since in general Ruelle transfer operators do not preserve any of the spaces $C^\alpha(\widehat{U})$. However, they preserve a certain 'filtration' $\cup_{0<\alpha\leq\alpha_0} C^\alpha(\widehat{U})$. Here $\alpha > 0$ and $C^\alpha(\widehat{U})$ is *the space of all α-Hölder complex-valued functions* on $\widehat{U}$. Then $|h|_\alpha$ is the smallest non-negative number so that $|h(x) - h(y)| \leq |h|_\alpha (d(x,y))^\alpha$ for all $x, y \in \widehat{U}$. Define the norm $\|.\|_{\alpha,b}$ on $C^\alpha(\widehat{U})$ by $\|h\|_{\alpha,b} = \|h\|_0 + \frac{|h|_\alpha}{|b|}$.

Corollary 1.2. *Under the assumptions of Theorem 1.1, there exists a constant $\alpha_0 > 0$ such that for any real-valued function $f \in C^\alpha(\widehat{U})$ the Ruelle transfer operators related to f are eventually contracting on $\cup_{0<\alpha\leq\alpha_0} C^\alpha(\widehat{U})$. More precisely, there exists a constant $\hat{\beta} \in (0,1]$ and for each $\epsilon > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leq a_0$ and $|b| \geq b_0$, then for every integer $m > 0$ and every $\alpha \in (0, \alpha_0]$ the operator $L^m_{f-(P_f+a+\mathbf{i}b)\tau} : C^\alpha(\widehat{U}) \longrightarrow C^{\alpha\hat{\beta}}(\widehat{U})$ is well-defined and $\|L^m_{f-(P_f+a+\mathbf{i}b)\tau} h\|_{\alpha\hat{\beta},b} \leq C\, \rho^m\, |b|^\epsilon\, \|h\|_{\alpha,b}$ for every $h \in C^\alpha(\widehat{U})$.*

The maximal constant $\alpha_0 \in (0,1]$ that one can choose above (which is determined by the minimal $\hat{\theta}$ one can choose in Theorem 1.1) is related to the regularity of the local stable/unstable foliations. Estimates for this constant can be derived from certain bunching condition concerning the rates of expansion/contraction of the flow along local unstable/stable manifolds (see [Ha1], [Ha2], [PSW]). In the proof of Corollary 1.2 in Sect. 7 below we give some rough estimate for α_0.

The above was first proved by Dolgopyat ([D]) in the case of geodesic flows on compact surfaces of negative curvature with $\alpha_0 = 1$ (then one can choose $\hat{\beta} = 1$ as well). The second main result in [D] concerns transitive Anosov flows on compact Riemann manifolds with C^1 jointly non-integrable local stable and unstable foliations. For such flows Dolgopyat proved that the conclusion of Theorem 1.1 with $\alpha_0 = 1$ holds for the Sinai-Bowen-Ruelle potential $f = \log\det(d\phi_\tau)_{|E^u}$. More general results were proved in [St2] for mixing Axiom A flows on basic sets (again for $\alpha_0 = 1$) under some additional regularity assumptions, including Lipschitzness of the local stable holonomy maps. Further results of this kind were established in [St3].

The Markov partition that appears in Theorem 1.1 and Corollary 1.2 can be chosen arbitrarily, as long as its size is sufficiently small.

We now state several immediate consequences of Theorem 1.1.

Corollary 1.3. *Let X be a C^2 compact Riemann manifold of strictly negative curvature and let $\phi_t : M = S^*(X) \longrightarrow M$ be the geodesic flow on X. There exists $\hat{\theta} \in (0,1)$ such that for any $\theta \in [\hat{\theta}, 1)$ and any $f \in \mathcal{F}_\theta(\widehat{U})$ the Ruelle transfer operators related to f are eventually contracting on $\mathcal{F}_\theta(\widehat{U})$.*

As another consequence of Theorem 1.1 and the procedure described in [D] one gets exponential decay of correlations for the flow $\phi_t : M \longrightarrow M$ with respect to any Gibbs measure.

Theorem 1.4. *Let $\phi_t : M \longrightarrow M$ be a C^5 contact Anosov flow on a C^5 manifold M, let F be a Hölder continuous function on M and let ν_F be the Gibbs measure determined by F on M. For every $\alpha > 0$ there exist constants $C = C(\alpha) > 0$ and $c = c(\alpha) > 0$ such that*

$$\left| \int_M A(x)B(\phi_t(x))\, d\nu_F(x) - \left(\int_M A(x)\, d\nu_F(x)\right)\left(\int_M B(x)\, d\nu_F(x)\right)\right| \leq Ce^{-ct}\|A\|_\alpha\, \|B\|_\alpha$$

for any two functions $A, B \in C^\alpha(M)$.

It appears that so far the only results concerning exponential decay of correlations for general Gibbs potentials have been that of Dolgopyat [D] for geodesic flows on compact surfaces and the one in [St2] for Axiom A flows on basic sets (under additional assumptions including Lipschitz regularity of stable/unstable holonomy maps). As we mentioned earlier, Liverani [L1] proves exponential decay of correlations for C^4 contact Anosov flows, and finer results (which imply exponential decay of correlations) were established by Tsujii [T] and Nonnenmacher and Zworski [NZ] (for C^3 and C^∞ contact Anosov flows, respectively), however all these three papers deal with the measure determined by the Riemann volume. In a recent paper Giulietti, Liverani and Pollicott [GLP] derive (amongst other things) exponential decay of correlations for contact Anosov flows with respect to the measure of maximal entropy (generated by the potential $F = 0$) under a bunching condition (which implies that the stable/unstable foliations are $\frac{2}{3}$-Hölder).

Next, consider the *Ruelle zeta function* $\zeta(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-1}$, $s \in \mathbf{C}$, where γ runs over the set of primitive closed orbits of $\phi_t : M \longrightarrow M$ and $\ell(\gamma)$ is the least period of γ. Denote by h_T the *topological entropy* of ϕ_t on M.

Using Theorem 1.1 and an argument of Pollicott and Sharp [PoS1], one derives the following[6].

Theorem 1.5. *Let $\phi_t : M \longrightarrow M$ be a C^2 contact Anosov flow on a C^2 compact Riemann manifold M. Then:*

(a) *The Ruele zeta function $\zeta(s)$ of the flow $\phi_t : M \longrightarrow M$ has an analytic and non-vanishing continuation in a half-plane $\mathrm{Re}(s) > c_0$ for some $c_0 < h_T$ except for a simple pole at $s = h_T$.*

(b) *There exists $c \in (0, h_T)$ such that*

$$\pi(\lambda) = \#\{\gamma : \ell(\gamma) \leq \lambda\} = \mathrm{li}(e^{h_T\lambda}) + O(e^{c\lambda})$$

as $\lambda \to \infty$, where $\mathrm{li}(x) = \displaystyle\int_2^x \frac{du}{\log u} \sim \frac{x}{\log x}$ *as $x \to \infty$.*

Part (b) in the above proves a conjecture made recently by Giulietti, Liverani and Pollicott [GLP]. Parts (a) and (b) were first established by Pollicott and Sharp [PoS1] for geodesic flows on compact surfaces of negative curvature (using [D]), and then similar results were proved in [St2] for mixing Axiom A flows on basic sets satisfying certain additional assumptions (as mentioned above). Recently, using different methods, it was proved in [GLP] that: (i) for volume preserving three dimensional Anosov flows (a) holds, and moreover, in the case of C^∞ flows, the Ruelle zeta function $\zeta(s)$ is meromorphic in $\mathbf{C}$ and $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 0$; (ii) (b) holds for geodesic flows on $\frac{1}{9}$-pinched compact Riemann manifolds of negative curvature. These were obtained as consequences of more general results in [GLP], one of the most remarkable being that for C^∞

[6] Instead of using the norm $\|\cdot\|_{1,b}$ as in [PoS1], in the present case one has to work with $\|\cdot\|_{\theta,b}$ for some $\theta \in (0, 1)$, and then one has to use the so called Ruelle's Lemma in the form proved in [W]. This is enough to prove the estimate (2.3) for $\zeta(s)$ in [PoS1], and from there the arguments are the same.

Anosov flows the Ruelle zeta function $\zeta(s)$ is meromorphic in $\mathbf{C}$. Very recently Dyatlov and Zworski [DZ] found a different proof that $\zeta(s)$ is meromorphic in $\mathbf{C}$ (for C^∞ Anosov flows with orientable stable and unstable bundles) using microlocal analysis.

Without going into details here, let us just mention that strong spectral estimates for Ruelle transfer operators as the ones described in Theorem 1.1 lead to a variety of deep results of various kinds – see e.g. [An], [PoS1] - [PoS4], [PeS1] - [PeS3] for some applications of the estimates in [D] and [St2]. Now using Theorem 1.1 above, one can prove some of these results for more general systems. For example, in [PeS3], using the spectral estimate in [St2] and under the assumptions there, a fine asymptotic was obtained for the number of closed trajectories in M with primitive periods lying in exponentially shrinking intervals $(x - e^{-\delta x}, x + e^{-\delta x})$, $\delta > 0$, $x \to +\infty$. Using the technique in [PeS3] and Theorem 1.1 above, one can now obtain similar results for any C^2 contact Anosov flow. Thus, quite remarkably, such a strong asymptotic takes place for any geodesic flow on a C^2 compact Riemann manifold of negative curvature.

Finally, we state a regularity result for local holonomy maps which is essentially used in the proof of Theorem 1.1 and involves a metric D on unstable manifolds weaker than the Riemann metric. It uses a (true) Markov partition $\{\widetilde{R}_i\}_{i=1}^{k_0}$ (see Sect. 2). Given z, z' in some rectangle $\widetilde{R}_i$ define the stable holonomy map $\mathcal{H}_z^{z'} : W^u_{\widetilde{R}}(z) \longrightarrow W^u_{\widetilde{R}}(z')$ using projection along stable leaves in $\widetilde{R} = \cup_{i=1}^{k_0} \widetilde{R}_i$. Then for any x, y in the same unstable leaf $W^u_{\widetilde{R}}(z)$ define $D(x, y) = \mathrm{diam}(\mathcal{C}(x, y))$, where $\mathcal{C}(x, y)$ is the smallest cylinder in $W^u_{\widetilde{R}}(z)$ containing x and y.

Theorem 1.6. *Let $\phi_t : M \longrightarrow M$ be a C^2 contact Anosov flow on a C^2 compact Riemann manifold M and let μ be a Gibbs measure on $\widetilde{R}$ defined by some potential $F \in \mathcal{F}_\theta(\widetilde{R})$. For any $\delta > 0$ there exists a compact subset Q of $\widetilde{R}$ consisting of Lyapunov regular points with $\mu(Q) > 1 - \delta$ such that the local stable holonomy maps are uniformly D-Lipschitz over Q. More precisely, there exists a constant $C > 0$ such that for any z, z' in any rectangle $\widetilde{R}_i$ and any cylinder $\mathcal{C}$ in $W^u_{\widetilde{R}}(z)$ with $\mathcal{C} \cap Q \neq \emptyset$ and $\mathcal{H}_z^{z'}(\mathcal{C}) \cap Q \neq \emptyset$ we have* $\mathrm{diam}(\mathcal{H}_z^{z'}(\mathcal{C})) \leq C\, \mathrm{diam}(\mathcal{C})$.

One can actually avoid the use of a Markov partition and use balls in Bowen's metric on local unstable manifolds to state a result equivalent to the above. Moreover it appears the argument in the proof can be adapted to deal with non-uniformly hyperbolic contact flows, including flows with singularities such as billiard flows. This will be discussed in details in a separate paper.

As Boris Hasselblatt remarked recently (personal communication), it appears unlikely that a result as in Theorem 1.6 will be true if D is replaced by the Riemann metric. It was shown by Hasselblatt and Wilkinson [HW] that for any $\alpha > 0$ there exists a large set of symplectic Anosov diffeomorphisms whose holonomy maps are almost everywhere no better than α-Hölder. On the other hand, there is a positive result for some hyperbolic Cantor sets: it was shown by Hasselblatt and Schmeling [HS] that for solenoids (under certain technical assumptions) the holonomies are Lipschitz almost everywhere.

1.3 A few comments on the proof of the main result

As we mentioned already, in the proof of Theorem 1.1 we use the general framework of Dolgopyat's method from [D] and its development in [St2]. As in [D], we deal with the normalized operators $L_{ab} = L_{f^{(a)} - \mathbf{i}\, b\tau}$, where $f^{(a)}(u) = f(u) - (P_f + a)\tau(u) + \ln h_a(u) - \ln h_a(\sigma(u)) - \ln \lambda_a$, $\lambda_a > 0$ being the largest eigenvalue of $L_{f-(P_f+a)\tau}$, and h_a a particular corresponding positive eigenfunction (see Sect. 5.1 for details). Their real parts $\mathcal{M}_a = L_{f^{(a)}}$ satisfy $\mathcal{M}_a 1 = 1$. Now instead of dealing with these operators on some $C^\alpha(U)$, we consider them on the space $\mathcal{F}_\theta(\widehat{U})$ of D_θ-Lipschitz functions on $\widehat{U}$. We choose $\theta \in (0, 1)$ so that $\tau \in \mathcal{F}_\theta(\widehat{U})$. The main benefit in working with D_θ is that the local stable holonomy maps are isometries in this metric, and every $h \in \mathcal{F}_\theta(\widehat{U})$ can be considered as a function on R which is constant on stable leaves. Such h has the same 'trace' on each $W^u_R(x)$, so we have the freedom to choose whichever unstable leaf is more convenient to work on.

Another simple thing that helps to avoid the lack of regularity is to approximate the partition $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ (which we call a *pseudo-Markov partition* below) by a true (at least according to the standard definition, see [B]) Markov partition $\{\widetilde{R}_i\}_{i=1}^{k_0}$, where each $\widetilde{R}_i$ is contained in a submanifold D_i of M of codimension one. We can take D_i so that $U_i \cup S_i \subset D_i$. The shift along the flow determines a bi-Hölder continuous bijection $\widetilde{\Psi} : R \longrightarrow \widetilde{R} = \cup_{i=1}^{k_0} \widetilde{R}_i$, and whenever we need to measure the 'size' of a cylinder $\mathcal{C}$ lying in some $W^u_R(x)$ we use $\mathrm{diam}(\widetilde{\Psi}(\mathcal{C}))$, instead of $\mathrm{diam}(\mathcal{C})$. Since the Poincaré map $\widetilde{\mathcal{P}} : \widetilde{R} = \cup_{i=1}^{k_0} \widetilde{R}_i \longrightarrow \widetilde{R}$ is essentially Lispchitz, estimates involving $\mathrm{diam}(\widetilde{\Psi}(\mathcal{C}))$ are much nicer. Roughly speaking, whenever we deal with Ruelle operators, measures and Gibbsian properties of measures, we work on R, and whenever we have to estimate distances and diameters in the Riemannian metric we use projections[7] in $\widetilde{R}$.

As in [D], the main result follows if we show that, given $f \in \mathcal{F}_\theta(\widehat{U})$, there exist constants $C > 0$, $\rho \in (0,1)$, $a_0 > 0$, $b_0 \geq 1$ and an integer $N \geq 1$ such that for $a, b \in \mathbb{R}$ with $|a| \leq a_0$ and $|b| \geq b_0$ and any $h \in \mathcal{F}_\theta(\widehat{U})$ with $\|h\|_{\theta,b} \leq 1$ we have

$$\int_U |L_{ab}^{mN} h|^2 \, d\nu \leq C\, \rho^m \tag{1.1}$$

for every positive integer m. Here ν is the Gibbs measure on U determined by $g = f - P_f\, \tau$, and we may assume that it is related to a Gibbs ($\mathcal{P}$-invariant) measure μ on R. In order to prove the analogous result in [D], Dolgopyat constructs for any choice of a and b, a family of (what we call below) *contraction operators* $\mathcal{N}_J$, and the proof of (1.1) goes as follows. Given h with (in his case) $\|h\|_{1,b} \leq 1$, define $h^{(m)} = L_{ab}^{Nm} h$, $H^{(0)} = 1$ and $H^{(m)} = \mathcal{N}_{J_m}(H^{(m-1)})$ for an appropriately chosen sequence of contraction operators $\mathcal{N}_{J_m}$, so that $|h^{(m)}| \leq H^{(m)}$ for all m. In [D] (and [St2]) the contraction operators indeed contract in the L^1 norm so that

$$\int_U (\mathcal{N}_J H)^2 \, d\nu \leq \rho \int_U H^2 \, d\nu \tag{1.2}$$

for some constant $\rho \in (0,1)$ independent of a, b, J and H. Thus,

$$\int_U |L_{ab}^{mN} h|^2 \, d\nu = \int_U |h^{(m)}|^2 \, d\nu \leq \int_U (H^{(m)})^2 \, d\nu \leq \rho^m. \tag{1.3}$$

In the present article our contraction operators do not satisfy (1.2). Moreover we cannot deal immediately with functions in $\mathcal{F}_\theta(\widehat{U})$; instead we fix a sufficiently small $\theta_1 \in (0,\theta)$ and assume initially $f, h \in \mathcal{F}_{\theta_1}(\widehat{U})$. The general case is dealt with using approximations. We construct a small compact subset K_0 of $\widehat{U}_1$ with $\nu(K_0) > 0$ such that the contraction operators have some kind of 'contraction features' on (or near) K_0. Then using $\nu(K_0) > 0$ and the strong mixing properties of $\mathcal{P} : R \longrightarrow R$ (which is essentially isomorphic to a Bernoulli shift, so it is a Kolmogorov automorphism), we derive that for some large integer $p_0 \geq 1$, if the sequence $H^{(m)}$ is defined as above, then $\int_U (H^{(pp_0)})^2 d\nu \leq C\, \rho^p$ $(p \geq 1)$ for some constants $C > 0$ and $\rho \in (0,1)$, independent of a, b and h, which is enough to derive an estimate similar to (1.3). This is one of the main features in the proof of the main result.

The non-integrability Lemma 4.3 plays a very important role in the proof of the main result. To prove it we use Liverani's Lemma which says that there exist constants $C_0 > 0$, $\vartheta > 0$ and

[7] In fact, sometimes it is more convenient to use the projections $\Psi_i : R_i \longrightarrow \check{R}_i = \cup_{z \in S_i} \check{U}_i(z)$, where $\check{U}_i(z)$ is the part of the true unstable manifold $W^u_{\epsilon_0}(z)$ corresponding to $W^u_{\widetilde{R}_i}(z)$ via the shift along the flow. This is particularly convenient when using Liverani's Lemma 4.1.

$\epsilon_0 > 0$ such that for any $z \in M$, any $u \in E^u(z)$ and $v \in E^s(z)$ with $\|u\|, \|v\| \leq \epsilon_0$ we have

$$|\Delta(\exp^u_z(u), \exp^s_z(v)) - d\omega_z(u, v)| \leq C_0 \left[\|u\|^2 \|v\|^\vartheta + \|u\|^\vartheta \|v\|^2\right], \tag{1.4}$$

where $d\omega_z$ is the symplectic form defined by the contact form on M and Δ is the temporal distance function (see Sect. 2). We want to use this when $z \in P$, $v \neq 0$ is fixed and $\|u\|$ is small. Then however the right-hand-side of (1.4) is only $O(\|u\|^\vartheta)$ which is not good enough. As Liverani suggests in Remark B.8 in [L1], one might be able to improve the estimate pushing the points $x = \exp^u_z(u)$ and $y = \exp^s_z(v)$ forwards or backwards along the flow[8]. We go forwards roughly until $\|d\phi_t(z)\cdot u\| \geq \|d\phi_t(z)\cdot v\|$ for some $t > 0$. Moreover we are only interested in directions $u \in E^u_1(z)$, where E^u_1 is the sub-bundle of E^u corresponding to the smallest positive Lyapunov exponent[9]. What we ultimately want is to get estimates of the temporal distance $|\Delta(\exp^u_z(u), \exp^s_z(v))|$ from below (for certain directions $u \in E^u_1$) and from above. The precise statements of these are given in Lemmas 4.2 and 4.3. Their proofs are contained in Sect. 8. Theorem 1.6 is proved in Sect. 5 (see Lemma 5.4 there) using Lemma 4.3.

As in [St2], here we work a lot with cylinders defined by the Markov partition. In [St2] we worked under a certain regularity assumption (the so called *regular distortion along unstable manifolds*). A range of examples of flows having this property was described in [St4]. However it seems unlikely that it holds for any (contact) Anosov flow. In the present article, using ideas from [St4] and [St3], we succeed to construct a compact set P_2 of Lyapunov regular points with $\mu(P_2) > 0$ such that the properties considered in [St2] hold whenever $B^u_T(x, \delta) \cap P_2 \neq \emptyset$ (see Lemmas 4.4, 4.5 and 5.3). This turns out to be enough for the proof of the main result, the most significant part of which is contained in Sects. 6-7.

Acknowledgements. Thanks are due to Boris Hasselblatt, Yakov Pesin, Mark Pollicott and Amie Wilkinson for providing me with various useful information, and to Vesselin Petkov for constant encouragement. Thanks are also due to Sebastian Gouëzel who discovered an error in the first version of the paper. Useful discussions with Mark Pollicott, Vesselin Petkov, Dima Dolgopyat, Sebastian Gouëzel and Carlangelo Liverani during the program "Hyperbolic dynamics, large deviations and fluctuations" at the Centre Bernoulli, EPFL, Lausanne (January-June 2013) are gratefully acknowledged.

2 Preliminaries

Throughout the whole paper M denotes a C^2 compact Riemann manifold, and $\phi_t : M \longrightarrow M$ ($t \in \mathbb{R}$) a C^2 contact Anosov flow on M. Thus, there exist constants $C > 0$ and $0 < \lambda < 1$ and a $d\phi_t$-invariant decomposition $T_xM = E^0(x) \oplus E^u(x) \oplus E^s(x)$ of T_xM ($x \in M$) into a direct sum of non-zero linear subspaces, where $E^0(x)$ is the one-dimensional subspace determined by the direction of the flow at x, such that $\|d\phi_t(u)\| \leq C\,\lambda^t\,\|u\|$ for all $u \in E^s(x)$ and $t \geq 0$, and $\|d\phi_t(u)\| \leq C\,\lambda^{-t}\,\|u\|$ for all $u \in E^u(x)$ and $t \leq 0$. We will denote by ω the *contact form* on M preserved by the flow. That is, ω is a C^2 one-form on M such that $\omega \wedge (d\omega)^n$ is nowhere zero and flow-invariant, i.e. $\omega(d\varphi_t(x)\cdot u) = \omega(u)$ for all $t \in \mathbb{R}$, $x \in M$ and $u \in T_xM$. We then must have $\dim(M) = 2n + 1$ for some $n \geq 1$.

For $x \in M$ and a sufficiently small $\epsilon > 0$ let

$$W^s_\epsilon(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } \ t \geq 0\ ,\ d(\phi_t(x), \phi_t(y)) \to_{t\to\infty} 0\}\ ,$$

[8] Liverani says that the best one can hope for is to get $o(\|u\|)$ in the right-hand-side of (1.4) if $\vartheta > \sqrt{3} - 1$, and this may be so, however we are interested in particular directions u and for these without any restrictions on ϑ we succeed to get a bit more.

[9] It is easy to see that for sufficiently small cylinders $\mathcal{C}$ in $W^u_{\epsilon_0}(z)$ intersecting P the largest distances in $\mathcal{C}$ occur roughly speaking in directions of E^u_1.

$$W_\epsilon^u(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } \; t \leq 0 \; , \; d(\phi_t(x), \phi_t(y)) \to_{t\to -\infty} 0 \; \}$$

be the (strong) *stable* and *unstable manifolds* of size ϵ. Then $E^u(x) = T_x W_\epsilon^u(x)$ and $E^s(x) = T_x W_\epsilon^s(x)$. Given $\delta > 0$, set $E^u(x;\delta) = \{u \in E^u(x) : \|u\| \leq \delta\}$; $E^s(x;\delta)$ is defined similarly.

It follows from the hyperbolicity of the flow on M that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in M$ and $d(x,y) < \epsilon_1$, then $W_{\epsilon_0}^s(x)$ and $\phi_{[-\epsilon_0,\epsilon_0]}(W_{\epsilon_0}^u(y))$ intersect at exactly one point $[x,y]$ (cf. [KH]). That is, there exists a unique $t \in [-\epsilon_0, \epsilon_0]$ such that $\phi_t([x,y]) \in W_{\epsilon_0}^u(y)$. Setting $\Delta(x,y) = t$, defines the so called *temporal distance function*[10] ([KB],[D], [Ch1], [L1]). For $x, y \in M$ with $d(x,y) < \epsilon_1$, define $\pi_y(x) = [x,y] = W_\epsilon^s(x) \cap \phi_{[-\epsilon_0,\epsilon_0]}(W_{\epsilon_0}^u(y))$. Thus, for a fixed $y \in M$, $\pi_y : W \longrightarrow \phi_{[-\epsilon_0,\epsilon_0]}(W_{\epsilon_0}^u(y))$ is the *projection* along local stable manifolds defined on a small open neighbourhood W of y in M. Choosing $\epsilon_1 \in (0, \epsilon_0)$ sufficiently small, the restriction $\pi_y : \phi_{[-\epsilon_1,\epsilon_1]}(W_{\epsilon_1}^u(x)) \longrightarrow \phi_{[-\epsilon_0,\epsilon_0]}(W_{\epsilon_0}^u(y))$ is called a *local stable holonomy map*[11]. Combining such a map with a shift along the flow we get another local stable holonomy map $\mathcal{H}_x^y : W_{\epsilon_1}^u(x) \longrightarrow W_{\epsilon_0}^u(y)$. In a similar way one defines local holonomy maps along unstable laminations.

We will say that A is an *admissible subset* of $W_\epsilon^u(z)$ if A coincides with the closure of its interior in $W_\epsilon^u(z)$. Admissible subsets of $W_\epsilon^s(z)$ are defined similarly.

Let D be a submanifold of M of codimension one such that $\mathrm{diam}(D) \leq \epsilon$ and D is transversal to the flow ϕ_t. Assuming that $\epsilon > 0$ is sufficiently small, the projection $\mathrm{pr}_D : \phi_{[-\epsilon,\epsilon]}(D) \longrightarrow D$ along the flow is well-defined and smooth. Given $x, y \in D$, set $\langle x,y\rangle_D = \mathrm{pr}_D([x,y])$. A subset $\widetilde{R}$ of D is called a *rectangle* if $\langle x,y\rangle_D \in \widetilde{R}$ for all $x, y \in \widetilde{R}$. The rectangle $\widetilde{R}$ is called *proper* if $\widetilde{R}$ coincides with the closure of its interior in D. For any $x \in \widetilde{R}$ define the stable and unstable leaves through x in $\widetilde{R}$ by $W^s_{\widetilde{R}}(x) = \mathrm{pr}_D(W_\epsilon^s(x) \cap \phi_{[-\epsilon,\epsilon]}(D)) \cap \widetilde{R}$ and $W^u_{\widetilde{R}}(x) = \mathrm{pr}_D(W_\epsilon^u(x) \cap \phi_{[-\epsilon,\epsilon]}(D)) \cap \widetilde{R}$. For a subset A of D we will denote by $\mathrm{Int}_D(A)$ the *interior* of A in D.

Let $\widetilde{\mathcal{R}} = \{\widetilde{R}_i\}_{i=1}^{k_0}$ be a family of proper rectangles, where each $\widetilde{R}_i$ is contained in a submanifold D_i of M of codimension one. We may assume that each $\widetilde{R}_i$ has the form $\widetilde{R}_i = \langle U_i, S_i\rangle_{D_i} = \{\langle x,y\rangle_{D_i} : x \in U_i, y \in S_i\}$, where $U_i \subset W_\epsilon^u(z_i)$ and $S_i \subset W_\epsilon^s(z_i)$, respectively, for some $z_i \in M$. Moreover, we can take D_i so that $U_i \cup S_i \subset D_i$. Set $\widetilde{R} = \cup_{i=1}^{k_0} \widetilde{R}_i$. We will denote by $\mathrm{Int}(\widetilde{R}_i)$ the *interior* of the set $\widetilde{R}_i$ in the topology of the disk D_i. The family $\widetilde{\mathcal{R}}$ is called *complete* if there exists $\chi > 0$ such that for every $x \in M$, $\phi_t(x) \in \widetilde{R}$ for some $t \in (0, \chi]$. The *Poincaré map* $\widetilde{\mathcal{P}} : \widetilde{R} \longrightarrow \widetilde{R}$ related to a complete family $\widetilde{\mathcal{R}}$ is defined by $\widetilde{\mathcal{P}}(x) = \phi_{\tilde{\tau}(x)}(x) \in \widetilde{R}$, where $\tilde{\tau}(x) > 0$ is the smallest positive time with $\phi_{\tilde{\tau}(x)}(x) \in \widetilde{R}$. The function $\tilde{\tau}$ is called the *first return time* associated with $\tilde{r}$. A complete family $\widetilde{\mathcal{R}} = \{\widetilde{R}_i\}_{i=1}^{k_0}$ of rectangles in M is called a *Markov family* of size $\chi > 0$ for the flow ϕ_t if: (a) $\mathrm{diam}(\widetilde{R}_i) < \chi$ for all i; (b) for any $i \neq j$ and any $x \in \mathrm{Int}_D(\widetilde{R}_i) \cap \widetilde{\mathcal{P}}^{-1}(\mathrm{Int}_D(\widetilde{R}_j))$ we have $W^s_{\widetilde{R}_i}(x) \subset \overline{\widetilde{\mathcal{P}}^{-1}(W^s_{\widetilde{R}_j}(\widetilde{\mathcal{P}}(x)))}$ and $\overline{\widetilde{\mathcal{P}}(W^u_{\widetilde{R}_i}(x))} \supset W^u_{\widetilde{R}_j}(\widetilde{\mathcal{P}}(x))$; (c) for any $i \neq j$ at least one of the sets $\widetilde{R}_i \cap \phi_{[0,\chi]}(\widetilde{R}_j)$ and $\widetilde{R}_j \cap \phi_{[0,\chi]}(\widetilde{R}_i)$ is empty. It is important to remark that both $\widetilde{\mathcal{P}}$ and $\tilde{\tau}$ are *essentially Lispchitz*, i.e. Lipschitz on each set of the form $\widetilde{R}_i \cap \widetilde{\mathcal{P}}^{-1}(\widetilde{R}_j)$.

The existence of a Markov family $\widetilde{\mathcal{R}}$ of an arbitrarily small size $\chi > 0$ for ϕ_t follows from the construction of Bowen [B].

Following [R2] and [D], we will now slightly change the Markov family $\widetilde{\mathcal{R}}$ to a *pseudo-Markov partition* $\mathcal{R} = \{R_i\}_{i=1}^k$ of *pseudo-rectangles* $R_i = [U_i, S_i] = \{[x,y] : x \in U_i, y \in S_i\}$, where U_i and S_i are as above. Set $R = \cup_{i=1}^{k_0} R_i$. Notice that $\mathrm{pr}_{D_i}(R_i) = \widetilde{R}_i$ for all i. Given $\xi = [x,y] \in R_i$, set $W^u_R(\xi) = W^u_{R_i}(\xi) = [U, y] = \{[x', y] : x' \in U_i\}$ and $W^s_R(\xi) = W^s_{R_i}(\xi) = [x, S_i] = \{[x, y'] : y' \in S_i\} \subset$

[10] In fact in [D] and [L1] a different definition for Δ is given, however in the important case (the only one considered below) when $x \in W_\epsilon^u(z)$ and $y \in W_\epsilon^s(z)$ for some $z \in M$, these definitions coincide with the present one.

[11] In a similar way one can define holonomy maps between any two sufficiently close local transversals to stable laminations; see e.g. [PSW].

$W^s_{\epsilon_0}(x)$. The corresponding *Poincaré map* $\mathcal{P} : R \longrightarrow R$ is defined by $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\phi_{\tau(x)}(x) \in R$. The function τ is the *first return time* associated with $\mathcal{R}$. The *interior* $\mathrm{Int}(R_i)$ of a rectangle R_i is defined by $\mathrm{pr}_D(\mathrm{Int}(R_i)) = \mathrm{Int}_D(\widetilde{R}_i)$. In a similar way one can define $\mathrm{Int}^u(A)$ for a subset A of some $W^u_{R_i}(x)$ and $\mathrm{Int}^s(A)$ for a subset A of some $W^s_{R_i}(x)$.

We may and will assume that the family $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ has the same properties as $\widetilde{\mathcal{R}}$, namely: (a') $\mathrm{diam}(R_i) < \chi$ for all i; (b') for any $i \neq j$ and any $x \in \mathrm{Int}(R_i) \cap \mathcal{P}^{-1}(\mathrm{Int}(R_j))$ we have $\mathcal{P}(\mathrm{Int}(W^s_{R_i}(x))) \subset \mathrm{Int}^s(W^s_{R_j}(\mathcal{P}(x)))$ and $\mathcal{P}(\mathrm{Int}(W^u_{R_i}(x))) \supset \mathrm{Int}(W^u_{R_j}(\mathcal{P}(x)))$; (c') for any $i \neq j$ at least one of the sets $R_i \cap \phi_{[0,\chi]}(R_j)$ and $R_j \cap \phi_{[0,\chi]}(R_i)$ is empty.

Notice that in general $\mathcal{P}$ and τ are only (essentially) Hölder continuous. However there is an obvious relationship between $\mathcal{P}$ and the (essentially) Lipschitz map $\widetilde{\mathcal{P}}$, and this will be used below.

From now on we will assume that $\widetilde{\mathcal{R}} = \{\widetilde{R}_i\}_{i=1}^{k_0}$ is a fixed Markov family for ϕ_t of size $\chi < \epsilon_0/2 < 1$ and that $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ is the related pseudo-Markov family. Set $U = \cup_{i=1}^{k_0} U_i$ and $\mathrm{Int}^u(U) = \cup_{j=1}^{k_0} \mathrm{Int}^u(U_j)$.

The *shift map* $\sigma : U \longrightarrow U$ is given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves. Notice that τ is constant on each stable leaf $W^s_{R_i}(x) = W^s_{\epsilon_0}(x) \cap R_i$. For any integer $m \geq 1$ and any function h on U set $h_m(u) = h(u) + h(\sigma(u)) + \ldots + h(\sigma^{m-1}(u))$, $u \in U$.

Denote by $\widehat{U}$ (or $\widehat{R}$) the *core* of U (resp. R), i.e. the set of those $x \in U$ (resp. $x \in R$) such that $\mathcal{P}^m(x) \in \mathrm{Int}(R) = \cup_{i=1}^{k_0} \mathrm{Int}(R_i)$ for all $m \in \mathbb{Z}$. It is well-known (see [B]) that $\widehat{U}$ is a residual subset of U (resp. R) and has full measure with respect to any Gibbs measure on U (resp. R). Clearly in general τ is not continuous on U, however τ is *essentially Hölder* on $\widehat{U}$. The same applies to $\sigma : U \longrightarrow U$. Throughout we will mainly work with the restrictions of τ and σ to $\widehat{U}$. Set $\widehat{U}_i = U_i \cap \widehat{U}$. For any $A \subset M$, let $\widehat{A}$ be *the set of all $x \in A$ whose trajectories do not pass through boundary points* of R.

Let $B(\widehat{U})$ be the *space of bounded functions* $g : \widehat{U} \longrightarrow \mathbf{C}$ with its standard norm $\|g\|_0 = \sup_{x \in \widehat{U}} |g(x)|$. Given a function $g \in B(\widehat{U})$, the *Ruelle transfer operator* $L_g : B(\widehat{U}) \longrightarrow B(\widehat{U})$ is defined by

$$(L_g h)(u) = \sum_{\sigma(v)=u} e^{g(v)} h(v).$$

Given $\alpha > 0$, let $C^\alpha(\widehat{U})$ denote the *space of essentially α-Hölder continuous functions* $h : \widehat{U} \longrightarrow \mathbf{C}$, i.e. such that there exists $L \geq 0$ with $|h(x) - h(y)| \leq L\,(d(x,y))^\alpha$ for all $i = 1, \ldots, k_0$ and all $x, y \in \widehat{U}_i$. The smallest $L > 0$ with this property is called the α-Hölder exponent of h and is denoted $|h|_\alpha$. Set $\|g\|_\alpha = \|g\|_0 + |g|_\alpha$.

The hyperbolicity of the flow implies the existence of constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma > 1$ such that

$$c_0 \gamma^m \, d(x,y) \leq d(\widetilde{\mathcal{P}}^m(x)), \widetilde{\mathcal{P}}^m(y)) \leq \frac{\gamma_1^m}{c_0} d(x,y) \tag{2.1}$$

for all $x, y \in \widetilde{R}$ such that $\widetilde{\mathcal{P}}^j(x), \widetilde{\mathcal{P}}^j(y)$ belong to the same $\widetilde{R}_{i_j}$ for all $j = 0, 1, \ldots, m$.

Throughout this paper $\alpha_1 > 0$ will be fixed constant be such that $\tau \in C^{\alpha_1}(\widehat{U})$ and the local stable/unstable holonomy maps are uniformly α_1-Hölder.

3 Lyapunov regularity

Set $f = \phi_1$ and denote by $\mathcal{L}'$ the set of all *Lyapunov regular points* of f (see [P1] or section 2.1 in [BP]). It is well-known that $\mathcal{L}'$ is dense in M and has full measure with respect to any f-invariant probability measure on M. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ be the *exponentials of the positive Lyapunov exponents* of f over $\mathcal{L}'$ (so we have $1 < \lambda_1$). **Fix an arbitrary constant** $\beta \in (0,1]$ such that $\lambda_j^\beta < \lambda_{j+1}$, $1 \leq j < k$. Take $\hat{\epsilon} > 0$ so small that

$$e^{8\hat{\epsilon}} < \lambda_1 \;,\; e^{8\hat{\epsilon}} < \lambda_j/\lambda_{j-1} \;\; (j = 2, \ldots, k). \tag{3.1}$$

Some further assumptions about $\hat{\epsilon}$ will be made later. Set

$$1 < \nu_0 = \lambda_1 e^{-8\hat{\epsilon}} \quad , \quad \mu_j = \lambda_j e^{-\hat{\epsilon}} < \lambda_j < \nu_j = \lambda_j e^{\hat{\epsilon}} \tag{3.2}$$

for all $j = 1, \ldots, k$.

Fix $\hat{\epsilon} > 0$ with the above properties. There exists a *subset $\mathcal{L}$ of $\mathcal{L}'$ of full measure* with respect to any f-invariant probability measure on M such that for $x \in \mathcal{L}$ we have an f-invariant decomposition $E^u(x) = E^u_1(x) \oplus E^u_2(x) \oplus \ldots \oplus E^u_k(x)$ into subspaces of constant dimensions $n_1, \ldots, n_k$ with $n_1 + n_2 + \ldots + n_k = n^u = \dim(E^u(x))$ such that for some *Lyapunov $\hat{\epsilon}$-regularity function* $R = R_{\hat{\epsilon}} : \mathcal{L} \longrightarrow (1, \infty)$, i.e. a function with

$$e^{-\hat{\epsilon}} \leq \frac{R(f(x))}{R(x)} \leq e^{\hat{\epsilon}} \quad , \quad x \in \mathcal{L}, \tag{3.3}$$

we have

$$\frac{1}{R(x)\, e^{n\hat{\epsilon}}} \leq \frac{\|df^n(x)\cdot v\|}{\lambda_i^n \|v\|} \leq R(x)\, e^{n\hat{\epsilon}} \quad , \quad x \in \mathcal{L}\;,\; v \in E^u_i(x) \setminus \{0\}\;,\; n \geq 0. \tag{3.4}$$

We have a similar decomposition in $E^s(x)$, $x \in \mathcal{L}$. Since the flow is assumed contact, we have $n^s = \dim(E^s(x)) = n^u$.

For $x \in \mathcal{L}$ and $1 \leq j \leq k$ set

$$\widehat{E}^u_j(x) = E^u_1(x) \oplus \ldots \oplus E^u_{j-1}(x) \quad , \quad \widetilde{E}^u_j = E^u_j(x) \oplus \ldots \oplus E^u_k(x).$$

Also set $\widehat{E}^u_1(x) = \{0\}$ and $\widehat{E}^u_{k+1}(x) = E^u(x)$. For any $x \in \mathcal{L}$ and any $u \in E^u(x)$ we will write $u = (u^{(1)}, u^{(2)}, \ldots, u^{(k)})$, where $u^{(i)} \in E^u_i(x)$ for all i. We will denote by $\|\cdot\|$ the norm on $E^u(x)$ generated by the Riemann metric, and we will also use the norm $|u| = \max\{\|u^{(i)}\| : 1 \leq i \leq k\}$.

Taking the regularity function $R(x)$ appropriately (see e.g. Sect. 8 in [LY2] or the Appendix in [BPS]), we may assume that

$$|u| \leq \|u\| \leq R(x)|u| \quad , \quad x \in \mathcal{L}\;,\; u \in E^u(x)\;,$$

and $\angle(\widehat{E}^u_j(x), \widetilde{E}^u_j(x)) \geq \frac{1}{R(x)}$ for all $x \in \mathcal{L}$ and $2 \leq j \leq k$.

It follows from the general theory of non-uniform hyperbolicity (see [P1], [BP]) that for any $j = 1, \ldots, k$ the invariant bundle $\{\widetilde{E}^u_j(x)\}_{x\in\mathcal{L}}$ is uniquely integrable over $\mathcal{L}$, i.e. there exists a continuous f-invariant family $\{\widetilde{W}^{u,j}_{\tilde{r}(x)}(x)\}_{x\in\mathcal{L}}$ of C^2 submanifolds $\widetilde{W}^{u,j}(x) = \widetilde{W}^{u,j}_{\tilde{r}(x)}(x)$ of M tangent to the bundle $\widetilde{E}^u_j$ for some Lyapunov $\hat{\epsilon}/2$*-regularity function* $\tilde{r} = \tilde{r}_{\hat{\epsilon}/2} : \mathcal{L} \longrightarrow (0,1)$. Moreover, with $\beta \in (0,1]$ as in the beginning of this section, for $j > 1$ it follows from Theorem 6.6 in [PS] and (3.1) that there exists an f-invariant family $\{\widehat{W}^{u,j}_{\tilde{r}(x)}(x)\}_{x\in\mathcal{L}}$ of $C^{1+\beta}$ submanifolds $\widehat{W}^{u,j}(x) = \widehat{W}^{u,j}_{\tilde{r}(x)}(x)$ of M tangent to the bundle $\widehat{E}^u_j$. (However this family is not unique in general.) For each $x \in \mathcal{L}$ and each $j = 2, \ldots, k$ fix an f-invariant family $\{\widehat{W}^{u,j}_{\tilde{r}(x)}(x)\}_{x\in\mathcal{L}}$ with the latter properties. Then we can find a Lyapunov $\hat{\epsilon}$-regularity function $r = r_{\hat{\epsilon}} : \mathcal{L} \longrightarrow (0,1)$ and for any $x \in \mathcal{L}$ a $C^{1+\beta}$ diffeomorphism $\Phi^u_x : E^u(x; r(x)) \longrightarrow \Phi_x(E^u(x; r(x)) \subset W^u_{\tilde{r}(x)}(x)$ such that

$$\Phi^u_x(\widehat{E}^u_j(x; r(x))) \subset \widehat{W}^{u,j}_{\tilde{r}(x)}(x) \quad , \quad \Phi^u_x(\widetilde{E}^u_j(x; r(x))) \subset \widetilde{W}^{u,j}_{\tilde{r}(x)}(x) \tag{3.5}$$

for all $x \in \mathcal{L}$ and $j = 2, \ldots, k$. Moreover, since for each $j > 1$ the submanifolds $\widehat{W}^{u,j}_{r(x)}(x)$ and $\exp^u_x(\widehat{E}^u_j(x; r(x)))$ of $W^u_{\tilde{r}(x)}(x)$ are tangent at x of order $1 + \beta$, we can choose Φ^u_x so that the diffeomorphism $\Psi^u_x = (\exp^u_x)^{-1} \circ \Phi^u_x : E^u(x : r(x)) \longrightarrow \Psi^u_x(E^u(x : r(x))) \subset E^u(x; \tilde{r}(x))$ is $C^{1+\beta}$-close to identity. Thus, replacing $R(x)$ with a larger regularity function if necessary, we may assume that

$$\|\Psi^u_x(u) - u\| \leq R(x)\|u\|^{1+\beta} \quad , \quad \|(\Psi^u)^{-1}_x(u) - u\| \leq R(x)\|u\|^{1+\beta} \tag{3.6}$$

for all $x \in \mathcal{L}$ and $u \in E^u(x; \tilde{r}(x))$, and also that

$$\|d\Phi^u_x(u)\| \leq R(x) \quad , \quad \|(d\Phi^u_x(u))^{-1}\| \leq R(x) \quad , \quad x \in \mathcal{L}\,,\ u \in E^u(x; r(x)). \tag{3.7}$$

Finally, again replacing $R(x)$ with a larger regularity function if necessary, we may assume that

$$\|\Phi^u_x(v) - \Phi^u_x(u) - d\Phi^u_x(u)\cdot(v - u)\| \leq R(x)\, \|v - u\|^{1+\beta}\ ,\ x \in \mathcal{L}\,,\ u, v \in E^u(x; r(x)), \tag{3.8}$$

and

$$\|d\Phi^u_x(u) - \mathrm{id}\| \leq R(x)\, \|u\|^{\beta} \quad , \quad x \in \mathcal{L}\,,\ u \in E^u(x; r(x)). \tag{3.9}$$

In a similar way one defines the maps Φ^s_x and we will assume that $r(x)$ is chosen so that these maps satisfy the analogues of the above properties.

For any $x \in \mathcal{L}$ consider the $C^{1+\beta}$ map (defined locally near 0)

$$\hat{f}_x = (\Phi^u_{f(x)})^{-1} \circ f \circ \Phi^u_x : E^u(x) \longrightarrow E^u(f(x)).$$

It is important to notice that $\hat{f}^{-1}_x(\widehat{E}^u_j(f(x); r(f(x))) \subset \widehat{E}^u_j(x; r(x))$ and $\hat{f}^{-1}_x(\widetilde{E}^u_j(f(x); r(f(x))) \subset \widetilde{E}^u_j(x; r(x))$ for all $x \in \mathcal{L}$ and $j > 1$.

Given $y \in \mathcal{L}$ and any integer $j \geq 1$ we will use the notation

$$\hat{f}^j_y = \hat{f}_{f^{j-1}(y)} \circ \ldots \circ \hat{f}_{f(y)} \circ \hat{f}_y \quad , \quad \hat{f}^{-j}_y = (\hat{f}_{f^{-j}(y)})^{-1} \circ \ldots \circ (\hat{f}_{f^{-2}(y)})^{-1} \circ (\hat{f}_{f^{-1}(y)})^{-1}\ ,$$

at any point where these sequences of maps are well-defined.

It is well known (see e.g. the Appendix in [LY1] or section 3 in [PS]) that there exist Lyapunov $\hat{\epsilon}$-regularity functions $\Gamma = \Gamma_{\hat{\epsilon}} : \mathcal{L} \longrightarrow [1, \infty)$ and $r = r_{\hat{\epsilon}} : \mathcal{L} \longrightarrow (0,1)$ and for each $x \in \mathcal{L}$ a norm $\|\cdot\|'_x$ on T_xM such that

$$\|v\| \leq \|v\|'_x \leq \Gamma(x)\|v\| \quad , \quad x \in \mathcal{L}\,,\ v \in T_xM, \tag{3.10}$$

and for any $x \in \mathcal{L}$ and any integer $m \geq 0$, assuming $\hat{f}^j_x(u), \hat{f}^j_x(v) \in E^u(f^j(x), r(f^j(x)))$ are well-defined for all $j = 1, \ldots, m$, the following hold:

$$\mu^m_j\, \|u - v\|'_x \leq \|\hat{f}^m_x(u) - \hat{f}^m_x(v)\|'_{f^m(x)} \quad , \quad u, v \in \widetilde{E}^u_j(x; r(x)), \tag{3.11}$$

$$\mu^m_1\, \|u - v\|'_x \leq \|\hat{f}^m_x(u) - \hat{f}^m_x(v)\|'_{f^m(x)} \quad , \quad u, v \in E^u(x; r(x)), \tag{3.12}$$

$$\mu^m_1\, \|v\|'_x \leq \|d\hat{f}^m_x(u)\cdot v\|'_{f^m(x)} \leq \nu^m_k\, \|v\|'_x \quad , \quad x \in \mathcal{L}\,,\ u \in E^u(x; r(x))\ ,\ v \in E^u(x), \tag{3.13}$$

$$\mu_j^m \|v\|'_x \leq \|d\hat{f}_x^m(0)\cdot v\|'_{f^m(x)} \leq \nu_j^m \|v\|'_x \quad , \quad x \in \mathcal{L}\,,\; v \in E_j^u(x). \tag{3.14}$$

Next, Taylor's formula (see also section 3 in [PS]) implies that there exists a Lyapunov $\hat{\epsilon}$-regularity function $D = D_{\hat{\epsilon}} : \mathcal{L} \longrightarrow [1,\infty)$ such that for any $i = \pm 1$ we have

$$\|\hat{f}_x^i(v) - \hat{f}_x^i(u) - d\hat{f}_x^i(u)\cdot(v-u)\| \leq D(x)\,\|v-u\|^{1+\beta}\;,\; x \in \mathcal{L}\,,\; u, v \in E^u(x; r(x)), \tag{3.15}$$

and

$$\|d\hat{f}_x^i(u) - d\hat{f}_x^i(0)\| \leq D(x)\,\|u\|^{\beta} \quad , \quad x \in \mathcal{L}\,,\; u \in E^u(x; r(x)). \tag{3.16}$$

Finally, we state here a Lemma from [St3] which will be used several times later.

Lemma 3.1. (Lemma 3.3 in [St3]) *There exist a Lyapunov $6\hat{\epsilon}$-regularity function $L = L_{6\hat{\epsilon}} : \mathcal{L} \longrightarrow [1,\infty)$ and a Lyapunov $7\hat{\epsilon}/\beta$-regularity function $r = r_{7\hat{\epsilon}/\beta} : \mathcal{L} \longrightarrow (0,1)$ such that for any $x \in \mathcal{L}$, any integer $p \geq 1$ and any $v \in E^u(z, r(z))$ with $\|\hat{f}_z^p(v)\| \leq r(x)$, where $z = f^{-p}(x)$, we have $\|w_p^{(1)} - v_p^{(1)}\| \leq L(x)|v_p|^{1+\beta}$, where $v_p = \hat{f}_z^p(v) \in E^u(x)$ and $w_p = d\hat{f}_z^p(0)\cdot v \in E^u(x)$. Moreover, if $|v_p| = \|v_p^{(1)}\| \neq 0$, then $1/2 \leq \|w_p^{(1)}\|/\|v_p^{(1)}\| \leq 2$.*

From now on we will assume that $r(x)$ **($x \in \mathcal{L}$) is a fixed regularity function** which satisfies the requirements in Lemma 3.1 and also the ones mentioned earlier.

4 Non-integrability of Anosov flows

4.1 Choice of constants, compact set of positive measure

In what follows we assume that $\widetilde{\mathcal{R}} = \{\widetilde{R}_i\}_{i=1}^{k_0}$ is a fixed Markov partition for φ_t on M of size $< 1/2$ and $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ is the related pseudo-Markov partition as in Sect. 2. We will use the notation associated with these from Sect. 2. Fix $0 < r_1 < r_2 < 1$ so that $\mathrm{diam}(\widetilde{R}_i) \leq r_2$,

$$\widetilde{B}^u(z, r_1) = \{y \in W^u_{\widetilde{R}_i}(z) : d(z,y) < \delta\} \subset \mathrm{Int}^u(W^u_{\widetilde{R}_i}(z))$$

for all $z \in S_i = W^s_{R_i}(z_i)$ and $i = 1, \ldots, k_0$. Then choose a small constant $\delta_1 > 0$ with $0 < \delta_1 < \frac{1}{100}\min\{r_1, \epsilon_0\}$, so that $\mathrm{dist}(z_i, \partial W^s_{R_i}(z_i)) > 100\delta_1$ for all $i = 1, \ldots, k_0$. Finally, fix constants $0 < \tau_0 < \hat{\tau}_0 < 1$ so that $\tau_0 \leq \tau(x) \leq \hat{\tau}_0$ for all $x \in R$ and $\tau_0 \leq \tilde{\tau}(x) \leq \hat{\tau}_0$ for all $x \in \widetilde{R}$.

For $x \in R_i$ and $\delta > 0$ set $B^u(x,\delta) = \{y \in W^u_{R_i}(x) : d(x,y) < \delta\}$, and in a similar way define $B^s(x,\delta)$. For brevity sometimes we will use the notation $U_i(z) = W^u_{R_i}(z)$ for $z \in R_i$. Given a non-empty subset A of some rectangle R_i, set $\mathcal{O}^u(A) = \cup_{z\in A} W^u_{R_i}(z)$.

Let $\alpha_1 > 0$ be as in Sect. 2 and let f be an α_1-Hölder continuous potential on R. Set $g = f - P_f\tau$, where $P_f \in \mathbb{R}$ is chosen so that the topological pressure of g with respect to the Poincaré map $\mathcal{P} : R = \cup_{i=1}^{k_0} R_i \longrightarrow R$ is 0. Let $\mu = \mu_g$ be the Gibbs measure on R determined by g; then $\mu(\widehat{R}) = 1$. We will assume that f (and therefore g) depends on forward coordinates only[12] i.e. it is constant on stable leaves of R_i for each i.

Since g is constant on stable leaves, it generates a *Gibbs measure* ν^u on U with respect to σ. Let $g^{(s)}$ be a function on R which is homological to g and constant on unstable leaves in R; then $g^{(s)}$ can be regarded as a function on $S = \cup_{i=1}^{k_0} S_i$ and determines a *Gibbs measure* ν^s on S.

[12]If the initial potential f_1 on R is α^2-Hölder, applying Sinai's Lemma (see e.g. [PP]) produces an α-Hölder potential f depending on forward coordinates only.

A sequence $i_p, i_{p+1}, \ldots, i_q$ of elements of $\{1, \ldots, k_0\}$ for some integers $p \leq q$, will be called *admissible* if $\widetilde{\mathcal{P}}(\mathrm{Int}(\widetilde{R}_{i_j})) \cap \mathrm{Int}(\widetilde{R}_{i_{j+1}}) \neq \emptyset$ for all $j = p, p+1, \ldots, q-1$. Given such a sequence, consider the *cylinder*

$$\mathcal{C}_R[i_p, i_{p+1}, \ldots, i_q] = \{x \in R_{i_p} : \mathcal{P}^j(x) \in R_{i_{p+j}} \ , \ 1 \leq j \leq q-p\}$$

in R. When $p = 0$, we can define similarly the usual 'unstable' cylinders in U_{i_0}: $\mathcal{C}^u[i_0, \ldots, i_q] = \{x \in U_{i_0} : \mathcal{P}^j(x) \in R_{i_j} \ , \ 1 \leq j \leq q\}$, and for $q = 0$ we define a 'stable' cylinder in S_{i_0}: $\mathcal{C}^s[i_p, \ldots, i_0] = \{x \in S_{i_0} : \mathcal{P}^j(x) \in R_{i_j} \ , \ p \leq j \leq 0\}$. Then (see Proposition A2.2 in [P2] or Sect. 2.3 in [Ch3]) there exist constants $0 < A_1 < A_2$ such that for every cylinder $\mathcal{C}_R$ as above with $p \leq 0 \leq q$ we have

$$A_1 \leq \frac{\mu(\mathcal{C}_R[i_p, i_{p+1}, \ldots, i_q])}{\nu^s(\mathcal{C}^s[i_p, \ldots, i_0])\, \nu^u(\mathcal{C}^u[i_0, \ldots, i_q])} \leq A_2.$$

Moreover we have $\nu^u(\mathcal{C}^u[i_0, \ldots, i_q]) = \mu(\mathcal{C}_R[i_0, \ldots, i_q])$, and $\nu^s(\mathcal{C}^s[i_p, \ldots, i_0]) = \mu(\mathcal{C}_R[i_p, \ldots, i_0])$.

The above shows that μ is almost the direct product of ν^u and ν^s. More precisely, let $\hat{\mu}$ be *the probability measure* on R such that $\hat{\mu} = \nu^u \times \nu^s$ on each R_i, where we use the natural (Borel measurable) isomorphism $R_i = [U_i, S_i] \approx U_i \times S_i$. It then follows from the above that for every bounded Borel measurable function H on R and every $i = 1, \ldots, k_0$ we have

$$A_1 \int_{U_i} \int_{S_i} H([x,y])\, d\nu^u(x)\, d\nu^s(y) \leq \int_R H\, d\mu \leq A_2 \int_{U_i} \int_{S_i} H([x,y])\, d\nu^u(x)\, d\nu^s(y). \tag{4.1}$$

For later convenience, for every i and every $z \in R_i$ we will denote by ν^u_z *the measure on* $W^u_{R_i}(z)$ determined by ν^u and the projection $\pi_z : U_i \longrightarrow W^u_{R_i}(z)$ along stable manifolds in R_i, i.e. $\nu^u_z(\pi_z(A)) = \nu^u(A)$ for every Borel measurable subset A of U_i. In a similar way we define[13] ν^s_z.

Given an unstable leaf $W = W^u_{\widetilde{R}_i}(z)$ in some rectangle $\widetilde{R}_i$ and an admissible sequence $\imath = i_0, \ldots, i_m$ of integers $i_j \in \{1, \ldots, k_0\}$, the set $C_W[\imath] = \{x \in W : \widetilde{\mathcal{P}}^j(x) \in \widetilde{R}_{i_j} \ , \ j = 0, 1, \ldots, m\}$ will be called a *cylinder of length* m in W (or an *unstable cylinder* in $\widetilde{R}$ in general). When $W = U_i$ we will simply write $C[\imath]$. In a similar way one defines cylinders $C_V[\imath]$, where $V = W^u_{R_i}(z)$ is an unstable leaf in some rectangle R_i.

Let $\mathrm{pr}_D : \cup_{i=1}^{k_0} \phi_{[-\epsilon,\epsilon]}(D_i) \longrightarrow \cup_{i=1}^{k_0} D_i$ be the *projection along the flow*, i.e. for all $i = 1, \ldots, k_0$ and all $x \in \phi_{[-\epsilon,\epsilon]}(D_i)$ we have $\mathrm{pr}_D(x) = \mathrm{pr}_{D_i}(x)$. For any $i = 1, \ldots, k_0$ and any $z \in S_i$ denote by $\check{U}_i(z)$ the *part of the unstable manifold* $W^u_{\epsilon_0}(z)$ such that $\mathrm{pr}_D(\check{U}_i(z)) = W^u_{\widetilde{R}_i}(z)$. The shift along the flow determines a bi-Hölder continuous bijections $\Psi_i : W^u_{R_i}(z) \longrightarrow \check{U}_i(z)$ and $\widetilde{\Psi}_i : W^u_{R_i}(z) \longrightarrow W^u_{\widetilde{R}_i}(z)$ for all i. These define bi-Hölder continuous bijections $\Psi_i : R_i \longrightarrow \check{R}_i = \cup_{z \in S_i} \check{U}_i(z)$ and $\widetilde{\Psi}_i : R_i \longrightarrow \widetilde{R}_i$ for all i. Finally, the maps Ψ_i and $\widetilde{\Psi}_i$ combine to define bi-Hölder continuous bijections

$$\Psi : R \longrightarrow \check{R} = \cup_{i=1}^{k_0} \check{R}_i \quad , \quad \widetilde{\Psi} : R \longrightarrow \widetilde{R}. \tag{4.2}$$

By Luzin's Theorem, there exists a compact subset P of $\mathcal{L} \cap \widehat{R}$ with $\mu(P)$ arbitrarily close to 1 such that the functions $R(x)$, $r(x)$ and $\tilde{r}(x)$ are continuous on P. Then there exist constants $r_0 > 0$, $R_0 > 1$, $\Gamma_0 > 1$, $D_0 > 0$ and $L_0 > 1$, such that

$$r(x), \tilde{r}(x) \geq r_0 \ , \ R(x) \leq R_0 \ , \ \Gamma(x) \leq \Gamma_0 \ , \ D(x) \leq D_0 \ , \ L(x) \leq L_0, \tag{4.3}$$

for all $x \in P$, where $D(x)$ and $L(x)$ are the regularity functions from (3.15)-(3.16) and Lemma 3.1, respectively. We choose $r_0 > 0$ so that $r_0 < \frac{1}{100}\delta_1$. Set $\widetilde{P} = \widetilde{\Psi}(P) \subset \widetilde{R}$. Choosing $r_0 > 0$

[13] In general ν^u_z and ν^s_z are not the conditional measures determined by μ.

sufficiently small and $R_0 > 0$, etc. sufficiently large, we will assume that (4.3) hold for all $x \in \widetilde{P}$ and all $x \in \phi_{[-1,1]}(P)$.

Since $\mu(P \cap \widehat{R})$ is close to 1, it contains a *compact subset* with measure close to 1. More precisely, given $\hat{\delta} > 0$ so small that

$$0 < \hat{\delta} < \min\{\frac{\delta_1}{100}, \frac{\tau_0}{2}\} \quad , \quad \hat{\delta} \leq \frac{\hat{\epsilon}}{\log \lambda_k}, \tag{4.4}$$

we will assume P is chosen so that

$$\mu(P) > 1 - \hat{\delta} \quad , \quad \mu(R_i \cap P) > (1 - \hat{\delta})\mu(R_i) \quad , \quad i = 1, \dots k_0. \tag{4.5}$$

Since the topological boundary $\partial \widetilde{R}_i$ of each rectangle $\widetilde{R}_i$ in D_i has μ-measure zero, shrinking P (and $\widetilde{P}$) slightly and replacing r_0 with a smaller number, we may and will assume that

$$\operatorname{dist}(P \cap R_i, \partial R_i) \geq r_0 \quad , \quad \operatorname{dist}(\widetilde{P} \cap \widetilde{R}_i, \partial \widetilde{R}_i) \geq r_0 \quad , \quad i = 1, \dots, k_0. \tag{4.6}$$

4.2 Non-integrability

The two-form $d\omega$ is C^1, so there exists a constant $C_0 > 0$ such that

$$|d\omega_x(u, v)| \leq C_0 \|u\| \, \|v\| \quad , \quad u, v \in T_x M \; , \; x \in M. \tag{4.7}$$

Moreover, there exists a constant $\theta_0 > 0$ such that for any $x \in M$ and any $u \in E^u(x)$ with $\|u\| = 1$ there exists $v \in E^s(x)$ with $\|v\| = 1$ such that $|d\omega_x(u, v)| \geq \theta_0$.

The main ingredient in this section is the following lemma of Liverani (Lemma B.7 in [L1]) which significantly strengthens a lemma of Katok and Burns ([KB]).

Lemma 4.1. ([L1]) *Let ϕ_t be a C^2 contact flow on M with a C^2 contact form ω. Then there exist constants $C_0 > 0$, $\vartheta > 0$ and $\epsilon_0 > 0$ such that for any $z \in M$, any $x \in W^u_{\epsilon_0}(z)$ and any $y \in W^s_{\epsilon_0}(z)$ we have*

$$|\Delta(x, y) - d\omega_z(u, v)| \leq C_0 \left[\|u\|^2 \, \|v\|^\vartheta + \|u\|^\vartheta \|v\|^2 \right] \; , \tag{4.8}$$

where $u \in E^u(z)$ and $v \in E^s(z)$ are such that $\exp^u_z(u) = x$ and $\exp^s_z(v) = y$.

Note. Actually Lemma B.7 in [L1] is more precise with a particular choice of the constant ϑ determined by the (uniform) Hölder exponents of the stable/unstable foliations and the corresponding local holonomy maps. However in this paper we do not need this extra information.

From now on we will assume that $C_0 > 0$, $\vartheta > 0$ and ϵ_0 satisfy (4.7) and (4.8).

We will show below that for Lyapunov regular points $x \in \mathcal{L}$ the estimate (4.8) can be improved what concerns the involvement of u for certain choices of u and v. More precisely, we will show that choosing v in a special way, $\Delta(x, y)$ becomes a C^1 function of $x = \exp^u_z(u)$ with a non-zero uniformly bounded derivative in a certain direction. However we can do this only for z and x in a subset of $\mathcal{L}$ of ('large') positive measure.

Since the maps $\mathcal{L} \ni x \mapsto E^s_i(x)$ and $\mathcal{L} \ni x \mapsto E^u_i(x)$ ($i = 1, \dots, k_0$) are measurable, using Luzin's Theorem, we may assume that the set P from Sect. 4.1 is chosen so that these maps are continuous on P. Then we can choose a *linear basis* $\{e^s_1(x), \dots, e^s_{n^s}(x)\}$ in $E^s(x)$ ($x \in P$) consisting of unit vectors depending continuously on $x \in P$ such that the first n_1 of these vectors span $E^s_1(x)$, the next n_2 vectors span $E^s_2(x)$, etc. Choose a similar linear basis $\{e^u_1(x), \dots, e^u_{n^u}(x)\}$ in $E^u(x)$ ($x \in P$).

Next, the characteristic function χ_P of P is in $L^1(R, \mu)$, so by Birkhoff's Theorem and (4.5),

$$h_n(x) = \frac{\chi_P(x) + \chi_P(\mathcal{P}(x)) + \ldots + \chi_P(\mathcal{P}^{n-1}(x))}{n} \to \int_R \chi_P \, d\mu = \mu(P) > 1 - \hat{\delta}$$

as $n \to \infty$ for almost all $x \in R$. Using Egorov's Theorem, there exists a compact subset P' of P with $\mu(P') > 1 - \hat{\delta}$, such that $h_n(x) \to 1$ as $n \to \infty$ uniformly for $x \in P'$. Thus, there exists an integer $n_0 \geq 1$ (depending on P') so that $h_n(x) \geq 1 - \hat{\delta}$ for all $x \in P'$ and all $n \geq n_0$. That is,

$$\sharp\{j : \, 0 \leq j \leq n-1 \, , \, \mathcal{P}^j(x) \in P\} \geq (1 - \hat{\delta})\, n \quad , \quad n \geq n_0, \tag{4.9}$$

for all $x \in P'$. **Fix a subset P' of P and an integer n_0** with the above property.

We will now state two Main Lemmas. Their proofs, both using Liverani's Lemma 4.1, are given in Sect. 8.

Lemma 4.2. (a) *There exist constants $C_1 > 0$ and $\beta_1 \in (0,1)$ with the following properties:*

(i) *For any $\hat{z}_0 \in S_1$, any cylinder $\mathcal{C}$ in $W^u_{R_1}(\hat{z}_0)$ with $\mathcal{C} \cap P' \neq \emptyset$ and any $z \in \mathcal{C}$ we have*

$$\frac{1}{C_1 \lambda_1^p e^{2\hat{\epsilon} p}} \leq \mathrm{diam}(\Psi(\mathcal{C})) \leq \frac{C_1 e^{2\hat{\epsilon} p}}{\lambda_1^p}, \tag{4.10}$$

where $p = [\tilde{\tau}_m(z)]$.

(ii) *For any $\hat{z}_0 \in S_1$, any cylinder $\mathcal{C}$ in $W^u_{R_1}(\hat{z}_0)$ with $\mathcal{C} \cap P' \neq \emptyset$, any $x_0, z_0 \in \mathcal{C}$ and any $y_0, b_0 \in W^s_{R_1}(z_0)$ and we have $|\Delta(x_0, y_0) - \Delta(x_0, b_0)| \leq C_1 \, \mathrm{diam}(\Psi(\mathcal{C}))\, (d(y_0, b_0))^{\beta_1}$. In particular,*

$$|\Delta(x_0, y_0)| \leq C_1 \, \mathrm{diam}(\Psi(\mathcal{C}))\, (d(y_0, z_0))^{\beta_1} \leq C_1 \, \mathrm{diam}(\Psi(\mathcal{C})).$$

(b) *Let $J_0 \geq 0$ be a fixed integer. There exists a constant $C_1' = C_1'(J_0) > 0$ such that for any $\hat{z}_0 \in S_1$, any cylinder $\mathcal{C}$ in $W^u_{R_1}(\hat{z}_0)$ with $\mathcal{C} \cap (\cup_{|j| \leq J_0} \mathcal{P}^j(P')) \neq \emptyset$, any $x_0, z_0 \in \mathcal{C}$ and any $y_0 \in W^s_{R_1}(z_0)$ we have $|\Delta(x_0, y_0)| \leq C_1' \, \mathrm{diam}(\Psi(\mathcal{C}))$.*

Set

$$\beta_0 = \frac{1}{\sqrt{1 + \theta_0^2/(64 C_0^2)}}.$$

Next, **fix a sufficiently large integer** $\ell_0 = \ell_0(\delta) \geq 1$ and **continuous families** $\eta_1(x), \eta_2(x), \ldots, \eta_{\ell_0}(x)$ $(x \in M)$ of unit vectors in $E^u_1(x)$ such that for any $x \in M$ and any $\xi \in E^u_1(x)$ with $\|\xi\| = 1$ there exists j with $\langle \xi, \eta_j(x) \rangle \geq 2\beta_0$.

Lemma 4.3. *Let ϕ_t be a C^2 contact Anosov flow on M. Let $\eta_1(x), \eta_2(x), \ldots, \eta_{\ell_0}(x)$ $(x \in M)$ be continuous families of unit vectors in $E^u_1(x)$ as above, and let $\kappa \in (0,1)$ be a constant. Then there exist constants $\epsilon'' > 0$, $0 < \delta'' < \delta'$ (depending on κ in general), $\delta_0 \in (0,1)$, $C_1 \geq 1$, and an integer $N_0 \geq 1$ such that for any integer $N \geq N_0$ there exist a compact subset $P_1 = P_1(N)$ of P' with $\mu(P_1) > 0$, a compact subset $\tilde{S}_0 = \tilde{S}_0(N)$ of S_1 with $\nu^s(\tilde{S}_0) > 0$ and $P_1 \subset P' \cap \mathcal{O}^u(\tilde{S}_0)$ and families of points $y_1^{(\ell)}(z) \in B^s(z, \delta') \cap \mathcal{P}^N(U_1(z))$, $y_2^{(\ell)}(z) \in B^s(z, \delta'') \cap \mathcal{P}^N(U_1(z))$ $(z \in \tilde{S}_0$; $\ell = 1, \ldots, \ell_0)$ with the following property:*

For any $\hat{z}_0 \in \tilde{S}_0$, any cylinder $\mathcal{C}$ in $\check{U}_1(\hat{z}_0)$ contained in $W^u_{\epsilon''}(\hat{z}_0)$, and any $x_0 \in \mathcal{C}$, $z_0 \in \mathcal{C} \cap \Psi(P_1)$ of the form $z_0 = \Phi^u_{\hat{z}_0}(u_0)$, $z_0 = \Phi^u_{\hat{z}_0}(w_0)$ such that

$$d(x_0, z_0) \geq \kappa \, \mathrm{diam}(\mathcal{C}) \tag{4.11}$$

and

$$\left\langle \frac{w_0 - u_0}{\|w_0 - u_0\|}, \eta_j(\hat{z}_0) \right\rangle \geq \beta_0 \tag{4.12}$$

for some $j = 1, \ldots, \ell_0$*, we have*

$$\frac{\beta_0 \delta_0 \kappa}{8R_0} \operatorname{diam}(\mathcal{C}) \leq |\Delta(x_0, \pi_{d_1}(z_0)) - \Delta(x_0, \pi_{d_2}(z_0))| \tag{4.13}$$

for any $d_1 \in B^s(y_1^{(j)}(\hat{z}_0), \delta'')$ *and* $d_2 \in B^s(z_1, \delta'')$*. In particular,* (4.13) *holds with* $d_1 = y_1^{(j)}(\hat{z}_0)$ *and* $d_2 = y_2^{(j)}(\hat{z}_0)$*.*

See Figure 1 on p. 60.

4.3 Regular distortion of cylinders

In [St4] we established some nice properties concerning diameters of cylinders for Axiom A flows on basic sets satisfying a pinching condition which we called *regular distortion along unstable manifolds*. In [St3] something similar was established for Anosov flows with Lipschitz local stable holonomy maps. It seems unlikely that any Anosov flow will have such properties, however it turns out that for general Anosov flows something similar holds for cylinders in R that intersect 'at both ends' a compact set of Lyapunov regular points. More precisely we have the following.

Lemma 4.4. *Assume that the compact subset* $\widetilde{P}$ *of* $\mathcal{L} \cap \widetilde{R}$ *satisfies* (4.3) *and* (4.6)*. Then:*

(a) *There exists a constant* $0 < \rho_1 < 1$ *such that for any unstable leaf* W *in* $\widetilde{R}$*, any cylinder* $C_W[\imath] = C_W[i_0, \ldots, i_m]$ *in* W *and any subcylinder* $C_W[\imath'] = C_W[i_0, i_1, \ldots, i_{m+1}]$ *of* $C_W[\imath]$ *of co-length* 1 *such that* $\mathcal{P}^{m+1}(C_W[\imath'] \cap \widetilde{P}) \cap \widetilde{P} \neq \emptyset$ *we have* $\rho_1 \operatorname{diam}(C_W[\imath]) \leq \operatorname{diam}(C_W[\imath'])$*.*

(b) *For any constant* $\rho' \in (0, 1)$ *there exists an integer* $q' \geq 1$ *such that for any unstable leaf* W *in* $\widetilde{R}$*, any cylinder* $C_W[\imath] = C_W[i_0, \ldots, i_m]$ *in* W *and any sub-cylinder* $C_W[\imath'] = C_W[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+q'}]$ *of* $C_W[\imath]$ *of co-length* q' *with* $\mathcal{P}^{m+q'}(C_W[\imath'] \cap \widetilde{P}) \cap \widetilde{P} \neq \emptyset$ *we have* $\operatorname{diam}(C_W[\imath']) \leq \rho' \operatorname{diam}(C_W[\imath])$*.*

(c) *There exist an integer* $q_0 \geq 1$ *and a constant* $\rho_1 \in (0, 1)$ *such that for any unstable leaf* W *in* $\widetilde{R}$*, any cylinder* $C_W[\imath] = C_W[i_0, \ldots, i_m]$ *in* W *and any* $z \in C_W[\imath] \cap \widetilde{P}$ *with* $\widetilde{\mathcal{P}}^m(z) \in \widetilde{P}$ *there exists* $x \in C_W[\imath]$ *such that if* $C_W[\imath'] = C_W[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+q_0}]$ *is the sub-cylinder of* $C_W[\imath]$ *of co-length* q_0 *containing* x *then* $d(z, y) \geq \rho_1 \operatorname{diam}(C_W[\imath])$ *for all* $y \in C_W[\imath']$*.*

This Lemma will be used essentially in the proof of the main result in Sects. 5-7 below. Its proof is given in Sect. 9.

As one can see from Lemma 4.4, there we have some relationship between diameters of cylinders $C_W[\imath]$ and certain sub-cylinders of theirs $C_W[\imath']$, however only when both $C_W[\imath']$ and its natural expansion $\widetilde{\mathcal{P}}^{m'}(C_W[\imath'])$, m' being the length of $C_W[\imath']$, intersect the compact set $\widetilde{P}$. For the applications the second condition is particularly inconvenient, and in the next lemma we succeed to get rid of it to some extend.

Lemma 4.5. *Let* Q *be a compact subset of* P *with* $\mu(Q) > 0$ *and let* $\delta, \rho' \in (0, 1)$*. Let* $\rho_1 \in (0, 1)$ *be as in Lemma 4.4. Then for any* $\hat{\rho} \in (0, \rho_1)$ *there exist a compact subset* Q' *of* Q *with* $\mu(Q') > (1 - \delta)\mu(Q)$ *and integers* $n' \geq 1$ *and* $q' \geq 1$ *with the following properties:*

(a) *For any unstable leaf* V *in* R*, any cylinder* $\mathcal{C}_V$ *in* V *of length* $\geq n'$ *and any sub-cylinder* $\mathcal{C}'_V$ *of* $\mathcal{C}_V$ *of co-length* 1 *such that* $\mathcal{C}'_V \cap Q' \neq \emptyset$ *we have* $\hat{\rho} \operatorname{diam}(\widetilde{\Psi}(\mathcal{C}_V)) \leq \operatorname{diam}(\widetilde{\Psi}(\mathcal{C}'_V))$*.*

(b) *There exists an integer $q' \geq 1$ such that for any cylinder $\mathcal{C}$ in R of length $\geq n'$ and any sub-cylinder $\mathcal{C}'$ of $\mathcal{C}$ of co-length q' with $\mathcal{C}' \cap Q' \neq \emptyset$ we have* $\mathrm{diam}(\widetilde{\Psi}(\mathcal{C}')) \leq \rho' \, \mathrm{diam}(\widetilde{\Psi}(\mathcal{C}))$.

Proof. Given $x \in R$ and an integer m denote by $C_m(x)$ the cylinder of length m in $W^u_{\widetilde{R}}(\widetilde{\Psi}(x))$ containing $\widetilde{\Psi}(x)$.

(a) Consider the functions $f_m(x) = \dfrac{\mathrm{diam}(C_{m+1}(x))}{\mathrm{diam}(C_m(x))}$ and $F_m(x) = \inf\{f_n(x) : n \geq m\}$, $x \in R$, $m \geq 1$. Clearly, $0 < f_m(x) \leq 1$ for all x and m and $\{F_m(x)\}$ is an increasing sequence for all $x \in Q$. Thus, $F(x) = \lim_{m\to\infty} F_m(x)$ exists for all $x \in Q$ and is a measurable function on Q. Assume that $F(x) < \rho_1$ on a subset Q_1 of Q with $\mu(Q_1) > 0$. By Egorov's theorem, shrinking slightly Q_1 if necessary, we may assume that Q_1 is compact and $F_m(x) \to F(x)$ uniformly on Q_1. Thus, there exists $m' \geq 1$ such that $F_m(x) < \rho_1$ for all $m \geq m'$ and all $x \in Q_1$. Thus, given $x \in Q_1$ there exists $m_x \geq m'$ such that $f_{m_x}(x) < \rho_1$. Since $Q_1 \subset Q \subset P \subset \widehat{R}$, x is an interior point in every cylinder containing it, so for every n, $\mathrm{diam}(C_n(x))$ is a continuous function of $x \in Q_1$ (and also on P). Thus, there exists an open subset W_x of R such that $f_{m_x}(y) < \rho_1$ for all $y \in W_x \cap Q_1$. Since Q_1 is compact, there exist $x_1, \ldots, x_p \in Q_1$ such that $Q_1 \subset \cup_{i=1}^p W_{x_i}$. Set $m_i = m_{x_i}$ for each $i = 1, \ldots, p$.

We may assume that the numbers m_i are all different. Otherwise we just have to drop some of the x_i's from the list $x_1, \ldots, x_p$. E.g. if $m_1 = m_p$, drop x_p and replace W_{x_1} by $W_{x_1} \cup W_{x_p}$. Then renaming the sets W_{x_i} if necessary, we may assume $1 < m_1 < m_2 < \ldots < m_p$. Since the map $\mathcal{P} : R \longrightarrow R$ is mixing of all orders with respect to the Gibbs measure μ (see e.g. [CFS] or [Rud]), we have

$$\lim_{m\to\infty} \mu\left(\mathcal{P}^{-m}(Q_1) \cap \mathcal{P}^{-m(m_1+1)}(P) \cap \mathcal{P}^{-m(m_2+1)}(P) \cap \ldots \cap \mathcal{P}^{-m(m_p+1)}(P)\right) = \mu(Q_1)(\mu(P))^p > 0.$$

Thus, taking $m \geq m'$ sufficiently large, there exists

$$y \in \mathcal{P}^{-m}(Q_1) \cap \mathcal{P}^{-m(m_1+1)}(P) \cap \mathcal{P}^{-m(m_2+1)}(P) \cap \ldots \cap \mathcal{P}^{-m(m_p+1)}(P).$$

Then $x = \mathcal{P}^m(y) \in Q_1$ and $\mathcal{P}^{m_i+1}(x) \in P$ for all $i = 1, \ldots, p$. We then have $x \in W_{x_i}$ for some i, so $f_{m_i}(x) < \rho_1$, i.e. $\mathrm{diam}(C_{m_i+1}(x)) < \rho_1 \, \mathrm{diam}(C_{m_i}(x))$. However, $\mathcal{P}^{m_i+1}(x) \in P$, so this is a contradiction with the choice of ρ_1 and Lemma 4.4(a). (Recall that $\widetilde{P} = \widetilde{\Psi}(P)$ and $Q_1 \subset Q \subset P$.)

This proves that $F(x) \geq \rho_1$ almost everywhere in Q. By Egorov's theorem there exists a compact subset Q' of Q with $\mu(Q') > (1-\delta)\mu(Q)$ such that $F(x) \geq \rho_1$ and $F_m(x) \to F(x)$ uniformly on Q'. Since $F(x) \geq \rho_1 > \hat{\rho}$ on Q', there exists $n' \geq 1$ such that for $m \geq n'$ we have $F_m(x) > \hat{\rho}$ for all $x \in Q'$. This implies $f_n(x) > \hat{\rho}$ for all $n \geq n'$ and all $x \in Q'$.

(b) The proof is very similar to that of part (a). ■

5 Construction of a 'contraction set'

5.1 Normalized Ruelle operators and the metric D_θ

Let the constants $C_0 > 0$, $c_0 > 0$, $1 < \gamma < \gamma_1$ be as in Sects. 2 and 4, so that (2.1) and (4.7) hold. **Fix a constant** θ such that

$$\max\left\{\frac{1}{\gamma^{\alpha_1}} \, , \, \frac{1}{(\lambda_1 e^{-2\hat{\epsilon}})^{\tau_0}}\right\} = \hat{\theta} \leq \theta < 1. \tag{5.1}$$

Recall the metric D_θ on $\widehat{U}$ and the space $\mathcal{F}_\theta(\widehat{U})$ from Sect. 1.1. In the same way we define the distance $D_\theta(x,y)$ for x, y in $\widehat{W}$ for an unstable leaf W of a rectangle R_i (or $\widetilde{R}_i$), where $\widehat{W} = W \cap \widehat{R}$, and the space $\mathcal{F}_\theta(\widehat{W})$. Lemma 5.2 below shows that $\tau \in \mathcal{F}_\theta(\widehat{U})$. For a non-empty subset A of U (or some $W^u_R(x)$) let $\mathrm{diam}_\theta(A)$ be the *diameter* of A with respect to D_θ.

Let $f \in \mathcal{F}_\theta(\widehat{U})$ be a fixed real-valued function and let $g = f - P_f\,\tau$, where $P_f \in \mathbb{R}$ is such that $\mathrm{Pr}_\sigma(g) = 0$. Since f is a Hölder continuous function on $\widehat{U}$, it can be extended to a Hölder continuos function on R which is constant on stable leaves.

Next, by Ruelle-Perron-Frobenius' Theorem (see e.g. chapter 2 in [PP]) for any real number a with $|a|$ sufficiently small, as an operator on $\mathcal{F}_\theta(\widehat{U})$, $L_{f-(P_f+a)\tau}$ has a *largest eigenvalue* λ_a and there exists a (unique) regular probability measure $\hat{\nu}_a$ on $\widehat{U}$ with $L^*_{f-(P_f+a)\tau}\hat{\nu}_a = \lambda_a\,\hat{\nu}_a$, i.e. $\int L_{f-(P_f+a)\tau} H\,d\hat{\nu}_a = \lambda_a \int H\,d\hat{\nu}_a$ for any $H \in \mathcal{F}_\theta(\widehat{U})$. Fix a corresponding (positive) eigenfunction $h_a \in \mathcal{F}_\theta(\widehat{U})$ such that $\int h_a\,d\hat{\nu}_a = 1$. Then $d\nu = h_0\,d\hat{\nu}_0$ defines a *σ-invariant probability measure* $\nu = \nu^u$ on U, which is in fact the Gibbs measure ν^u determined by g on U (see Sect. 4.1). Since $\mathrm{Pr}_\sigma(f - P_f\tau) = 0$, it follows from the main properties of pressure (cf. e.g. chapter 3 in [PP]) that $|\mathrm{Pr}_\sigma(f - (P_f + a)\tau)| \leq \|\tau\|_0\,|a|$. Moreover, for small $|a|$ the maximal eigenvalue λ_a and the eigenfunction h_a are Lipschitz in a, so there exist constants $a'_0 > 0$ and $C' > 0$ such that $|h_a - h_0| \leq C'\,|a|$ on $\widehat{U}$ and $|\lambda_a - 1| \leq C'|a|$ for $|a| \leq a'_0$.

For $|a| \leq a'_0$, as in [D], consider the function

$$f^{(a)}(u) = f(u) - (P_f + a)\tau(u) + \ln h_a(u) - \ln h_a(\sigma(u)) - \ln \lambda_a$$

and the operators $L_{ab} = L_{f^{(a)} - \mathbf{i}\,b\tau}, \mathcal{M}_a = L_{f^{(a)}} : \mathcal{F}_\theta(\widehat{U}) \longrightarrow \mathcal{F}_\theta(\widehat{U})$. One checks that $\mathcal{M}_a\,1 = 1$ and $|(L^m_{ab}h)(u)| \leq (\mathcal{M}^m_a|h|)(u)$ for all $u \in \widehat{U}$, $h \in \mathcal{F}_\theta(\widehat{U})$ and $m \geq 0$. It is also easy to check that $L^*_{f^{(0)}}\nu = \nu$, i.e. $\int L_{f^{(0)}} H\,d\nu = \int H\,d\nu$ for any $H \in \mathcal{F}_\theta(\widehat{U})$ (in fact, for any bounded continuous function H on $\widehat{U}$).

Since g has zero topological pressure with respect to the shift map $\sigma : U \longrightarrow U$, there exist constants $0 < c_1 \leq c_2$ such that for any cylinder $\mathcal{C} = \mathcal{C}^u[i_0, \ldots, i_m]$ of length m in U we have

$$c_1 \leq \frac{\nu(\mathcal{C})}{e^{g_m(y)}} \leq c_2 \quad , \quad y \in \mathcal{C}, \tag{5.2}$$

(see e.g. [PP] or [P2]).

We now state some basic properties of the metric D_θ that will be needed later.

Lemma 5.1. (a) *For any cylinder $\mathcal{C}$ in U the characteristic function $\chi_{\widehat{\mathcal{C}}}$ of $\widehat{\mathcal{C}}$ on $\widehat{U}$ is Lipschitz with respect to D_θ and $\mathrm{Lip}_\theta(\chi_{\mathcal{C}}) \leq 1/\mathrm{diam}_\theta(\mathcal{C})$.*

(b) *There exists a constant $C_2 > 0$ such that if $x, y \in \widehat{U}_i$ for some i, then $|\tau(x) - \tau(y)| \leq C_2\,D_\theta(x,y)$. That is, $\tau \in \mathcal{F}_\theta(\widehat{U})$. Moreover, we can choose $C_2 > 0$ so that $|\tau_m(x) - \tau_m(y)| \leq C_2\,D_\theta(\sigma^m(x), \sigma^m(y))$ whenever $x, y \in \widehat{U}_i$ belong to the same cylinder X of length m.*

(c) *There exist constants $C_2 > 0$ and $\beta' > 0$ such that if $x, y \in \widehat{U}_i$ for some i, then $(d(x,y))^{\alpha_1} \leq C_2\,D_\theta(x,y)$, and for any $z \in R$ and any cylinder $\mathcal{C}$ in $W^u_R(z)$ we have $\mathrm{diam}_\theta(\mathcal{C}) \leq C_2(\mathrm{diam}(\Psi(\mathcal{C})))^{\beta'}$.*

Proof. (a) is trivial.

(b) Assume $x \neq y$ and let $\mathcal{C}$ be the cylinder of largest length m containing both x and y. Then $D_\theta(x,y) = \theta^{m+1}$. On the other hand, (2.1) and (5.1) imply $|\tau(x) - \tau(y)| \leq |\tau|_{\alpha_1}\,(d(x,y))^{\alpha_1} \leq |\tau|_{\alpha_1}\,(d(\widetilde{\mathcal{P}}^m(x), \widetilde{\mathcal{P}}^m(y))/(c_0\gamma^m))^{\alpha_1} \leq \frac{\mathrm{Const}}{(\gamma^{\alpha_1})^m} \leq C_2\,D_\theta(x,y)$ for some global constant $C_2 > 0$. The above also shows that $(d(x,y))^{\alpha_1} \leq \mathrm{Const}\;\theta^m \leq C_2\,D_\theta(x,y)$, which proves part (b).

The proof of part (c) is similar. ■

It follows from Lemma 5.1 that $\tau \in \mathcal{F}_\theta(\widehat{U})$, so assuming $f \in \mathcal{F}_\theta(\widehat{U})$, we have $h_a \in \mathcal{F}_\theta(\widehat{U})$ for all $|a| \leq a_0'$. Then $f^{(a)} \in \mathcal{F}_\theta(\widehat{U})$ for all such a. Moreover, using the analytical dependence of h_a and λ_a on a and assuming that the constant $a_0' > 0$ is sufficiently small, there exists $T = T(a_0')$ such that

$$T \geq \max\{ \|f^{(a)}\|_0 \, , \, |f^{(a)}_{|\widehat{U}}|_\theta \, , |\tau_{|\widehat{U}}|_\theta \} \tag{5.3}$$

for all $|a| \leq a_0'$. Fix $a_0' > 0$ and $T > 0$ and with these properties. Taking the constant $T > 0$ sufficiently large, we have $|f^{(a)} - f^{(0)}| \leq T\,|a|$ on $\widehat{U}$ for $|a| \leq a_0'$.

The following Lasota-Yorke type inequality is similar to the one in [D], and in fact the same as the corresponding one in [St2] (although we now use a different metric) and the proof is also the same.

Lemma 5.2. *There exists a constant $A_0 > 0$ such that for all $a \in \mathbb{R}$ with $|a| \leq a_0'$ the following holds: If the functions h and H on $\widehat{U}$, the constant $B > 0$ and the integer $m \geq 1$ are such that $H > 0$ on $\widehat{U}$ and $|h(v) - h(v')| \leq B\, H(v')\, D_\theta(v, v')$ for any i and any $v, v' \in \widehat{U}_i$, then for any $b \in \mathbb{R}$ with $|b| \geq 1$ we have $|L_{ab}^m h(u) - L_{ab}^m h(u')| \leq A_0\, [B\, \theta^m\, (\mathcal{M}_a^m H)(u') + |b|\, (\mathcal{M}_a^m |h|)(u')]\, D_\theta(u, u')$ whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k_0$.* ■

5.2 First step – fixing N, a few compact sets of positive measure

Fix a constant $\omega_0 \in (0, 1)$ so that

$$1 - \omega_0 < \frac{c_1}{c_2} \rho_2^{q_0}, \tag{5.4}$$

where the constants c_1 and c_2 are as in (5.2) and $\rho_2 = e^{-\|g\|_0} < 1$. **Fix constants** $\rho_1 \in (0, 1)$ and $q_0 \geq 1$ such that Lemma 4.4(a), (b), (c) hold with $\rho' = \rho_1$ and $q' = q_0$. Then set $\hat{\rho} = \frac{\rho_1}{8}$. We will now use Lemma 4.3 with $\kappa = \hat{\rho}$.

In what follows we will use the entire set-up and notation from Sect. 4, e.g. the compact subset P of $\mathcal{L} \cap R$, $\widetilde{P} = \widetilde{\Psi}(P) \subset \widetilde{R}$, the numbers $r_0 > 0$, $R_0 > 0$, etc., satisfying (4.3) for all $x \in P$ and all $x \in \widetilde{P}$, etc. Let $\eta_1(x), \eta_2(x), \ldots, \eta_{\ell_0}(x)$ $(x \in M)$ be continuous families of unit vectors in $E_1^u(x)$ as in the text just before Lemma 4.3, and let $\epsilon'' \in (0, \epsilon')$, $0 < \delta'' < \delta' < \delta_1$, $\beta_0 \in (0, 1)$, $\delta_0 \in (0, 1)$, $C_1 \geq 1$ be constants, and $N_0 \geq 1$ be an integer, for any $N \geq N_0$ let $y_1^{(\ell)}(z) \in B^s(z, \delta') \cap \mathcal{P}^N(\check{U}_1(z))$, $y_2^{(\ell)}(z) \in B^s(z, \delta'') \cap \mathcal{P}^N(\check{U}_1(z))$ $(z \in \tilde{S}_0;\ \ell = 1, \ldots, \ell_0)$ be families of points satisfying the requirements of Lemmas 4.2 and 4.3 (with $\kappa = \hat{\rho}$). **Fix an integer** $n_0 \geq 1$ with (4.9) so large that for any $z \in S_1$ and any cylinder $\mathcal{C}$ of length $\geq n_0$ in $U_1(z)$ we have $\mathrm{diam}(\Psi(\mathcal{C})) \leq \epsilon''$. Let $U_0(z)$ be **the open cylinder of length** n_0 in $W^u_{R_1}(z)$ containing z and let $U_0 = U_0(z_1) \subset U_1$. Set

$$\hat{\delta}_0 = \frac{\beta_0 \delta_0 \hat{\rho}}{8 R_0}.$$

Let $E > 1$ be a constant – we will see later how large it should be, let $\epsilon_1 > 0$ be a constant with

$$0 < \epsilon_1 \leq \min\left\{ \frac{1}{32 C_0} \, , \, \frac{1}{4E} \right\}, \tag{5.5}$$

and let $N > N_0$ be an integer such that

$$\gamma^N \geq \frac{1}{C_0 \delta''} \quad , \quad \theta^N < \frac{\rho_1^2 \beta_0 \delta_0 \epsilon_1}{256 E} \quad , \quad \theta_2^N < \frac{\hat{\delta}_0 \hat{\rho} \epsilon_1}{64 E}, \tag{5.6}$$

where $\theta_2 = \max\{\theta, 1/\gamma^{\alpha_1 \beta_1}\}$, $\beta_1 > 0$ being the constant from Lemma 4.2.

Using Lemma 4.3, **fix a compact subset P_1 of P with $\mu(P_1) > 0$, a compact subset $\tilde{S}_0$ of S_1 with $\nu^s(\tilde{S}_0) > 0$ and $P_1 \subset \cup_{z\in\tilde{S}_0} B^u(z,\epsilon'') \cap P'$ and families of points** $y_1^{(\ell)}(z) \in B^s(z,\delta') \cap \mathcal{P}^N(U_1(z))$, $y_2^{(\ell)}(z) \in B^s(z,\delta'') \cap \mathcal{P}^N(U_1(z))$ ($z \in \tilde{S}_0$; $\ell = 1,\ldots,\ell_0$) with the properties described in Lemma 4.3. Some additional assumptions about E and N will be made later.

Denote by P_1' the *set of those points* $x \in P_1$ *such that* x *is a Borel density point* of $P_1 \cap W^u_{R_1}(x)$ for the measure ν^u. Then by Borel's Density Lemma (see e.g. [P2]), $\mu(P_1') = \mu(P_1) > 0$. Using the metric D_θ on $W^u_R(x) \cap \widehat{R}$, for every $x \in P_1'$ there exists an integer $m(x) \geq 0$ such that for $0 < \epsilon \leq \theta^{m(x)}$ and the ϵ-ball $B(x,\epsilon) = \{y : D_\theta(x,y) < \epsilon\}$ in $W^u_{R_1}(x)$ we have $\nu^u_x(B(x,\epsilon)\cap P_1) > \omega_0 \nu^u_x(B(x,\epsilon))$. Notice that $B(x,\epsilon)$ is actually a cylinder of length $m(x)$ in $W^u_{R_1}(x)$. Using Luzin's Theorem, there exists a *compact subset* P_2 of P_1' such that $\mu(P_2) > 0$ and $m(x)$ is continuous on P_2. Then $m_1 = \sup_{x\in P_2} m(x) < \infty$, and for any $z \in \tilde{S}_0$ and any cylinder $\mathcal{C}$ in $W^u_{R_1}(z)$ of length $\geq m_1$ such that $\mathcal{C} \cap P_2 \neq \emptyset$ we have

$$\nu^u_z(\mathcal{C} \cap P_1) \geq \omega_0\, \nu^u_z(\mathcal{C}). \tag{5.7}$$

Fix an integer $m_1 \geq 1$ and a compact subset P_2 of P_1 with the above properties.

The following lemma gives some kind of a relationship between ν^u_z-measures of sets of the form $\mathcal{C} \cap P_1$ and their diameters. Namely, we show (amongst other things) that when $\nu^u_z(\mathcal{C}\cap P_1)/\nu^u_z(\mathcal{C})$ is 'large', as in (5.7), then $\mathrm{diam}(\Psi(\mathcal{C}\cap P_1))/\mathrm{diam}(\Psi(\mathcal{C}))$ is also significant.

Lemma 5.3. *For every $z \in \tilde{S}_0$ and every cylinder $\mathcal{C}$ of length $q \geq m_1$ in $W^u_{R_1}(z)$ with $\mathcal{C} \cap P_2 \neq \emptyset$ the following hold:*

(a) *For every sub-cylinder $\mathcal{D}$ of $\mathcal{C}$ of co-length q_0 we have $\mathcal{D} \cap P_1 \neq \emptyset$.*

(b) *For any $x_0 \in \mathcal{C}$, $z_0 \in \mathcal{C}\cap P_1$ of the form $z_0 = \Phi^u_{\hat{z}_0}(u_0)$, $z_0 = \Phi^u_{\hat{z}_0}(w_0)$ with $d(\Psi(x_0),\Psi(z_0)) \geq \frac{1}{2}\mathrm{diam}(\Psi(\mathcal{C}))$ we have $\|u_0^{(1)} - w_0^{(1)}\| \geq \frac{2}{3}\|u_0 - w_0\|$ and $\|\tilde{u}_0^{(2)} - \tilde{w}_0^{(2)}\| \leq \frac{\beta_0}{3}\|u_0 - w_0\|$ where $\tilde{u}^{(2)} = u^{(2)} + \ldots + u^{(k)} \in \widetilde{E}^u_2$.*

(c) $\mathrm{diam}(\Psi(\mathcal{C}\cap P_1)) \geq \hat{\rho}\,\mathrm{diam}(\Psi(\mathcal{C}))$. *Moreover, there exist sub-cylinders $\mathcal{D}$ and $\mathcal{D}'$ of $\mathcal{C}$ of co-length q_0 such that $d(\Psi(y),\Psi(x)) \geq \hat{\rho}\,\mathrm{diam}(\Psi(\mathcal{C}))$ for all $y \in \mathcal{D}'$ and $x \in \mathcal{D}$.*

(d) *There exists an integer $q_1 \geq q_0$ such that for any sub-cylinder $\mathcal{C}'$ of $\mathcal{C}$ of co-length q_1 with $\mathcal{C}' \cap P_1 \neq \emptyset$ we have* $\mathrm{diam}(\Psi(\mathcal{C}')) \leq \frac{\hat{\delta}_0}{8C_1}\,\mathrm{diam}(\Psi(\mathcal{C}))$.

Proofs. (a) Let $\mathcal{D}$ be a sub-cylinder of $\mathcal{C}$ of co-length q_0. Let $y \in \mathcal{D}$. Then (5.2), (5.4) and (5.7) imply

$$\frac{\nu^u_z(\mathcal{D})}{\nu^u_z(\mathcal{C})} = \frac{\nu(\pi^{(U)}(\mathcal{D}))}{\nu(\pi^{(U)}(\mathcal{C}))} \geq \frac{c_1 e^{g_{q+q_0}(y)}}{c_2 e^{g_q(y)}} = \frac{c_1}{c_2} e^{g_{q_0}(\mathcal{P}^q(y))} \geq \frac{c_1}{c_2}\rho_2^{q_0} > 1 - \omega_0.$$

If $\mathcal{D}\cap P_1 = \emptyset$, then $\nu^u_z(\mathcal{C}\cap P_1) \leq \nu^u_z(\mathcal{C}\setminus\mathcal{D}) = \nu^u_z(\mathcal{C}) - \nu^u_z(\mathcal{D}) < \nu^u_z(\mathcal{C}) - (1-\omega_0)\nu^u_z(\mathcal{C}) = \omega_0\,\nu^u_z(\mathcal{C})$. However $\mathcal{C}$ satisfies (5.7), so this is a contradiction.

(b) follows easily from Lemma 4.2(a).

(c) Let $z_0 \in \mathcal{C}\cap P_2$, and let $\mathcal{D}$ be the sub-cylinder of $\mathcal{C}$ of co-length q_0 containing z_0. By Lemma 4.4, the choice of q_0, $\mathrm{Lip}(\widetilde{\Psi}\circ\Psi^{-1}) \leq 2$ and $\mathrm{Lip}(\Psi\circ\widetilde{\Psi}^{-1}) \leq 2$, it follows that $\mathrm{diam}(\Psi(\mathcal{D})) \leq \frac{\rho_1}{8}\mathrm{diam}(\Psi(\mathcal{C}))$.

Next, in a similar way it follows from Lemma 4.4(c) that there exists a sub-cylinder $\mathcal{D}'$ of $\mathcal{C}$ of co-length q_0 such that for any $y \in \mathcal{D}'$ we have $d(\Psi(y),\Psi(z_0)) \geq \frac{\rho_1}{4}\,\mathrm{diam}(\Psi(\mathcal{C}))$. By part (a), $\mathcal{D}'\cap P_1 \neq \emptyset$ and $\mathcal{D}\cap P_1 \neq \emptyset$. Then $d(\Psi(y),\Psi(z_0)) \geq \frac{\rho_1}{4}\mathrm{diam}(\Psi(\mathcal{C}))$ for all $y \in \mathcal{D}'$ and $d(\Psi(x),\Psi(z_0)) \leq \frac{\rho_1}{8}\mathrm{diam}(\Psi(\mathcal{C}))$ for all $x \in \mathcal{D}$. Thus, $d(\Psi(x),\Psi(y)) \geq d(\Psi(y),\Psi(z_0)) -$

$d(\Psi(x),\Psi(z_0)) \geq \frac{\rho_1}{4}\mathrm{diam}(\Psi(\mathcal{C})) - \frac{\rho_1}{8}\mathrm{diam}(\Psi(\mathcal{C})) = \frac{\rho_1}{8}\,\mathrm{diam}(\Psi(\mathcal{C}))$ for any $y \in \mathcal{D}'$ and any $x \in \mathcal{D}$, so this proves the second assertion.

(d) This follows from Lemma 4.4(b): take $q_1 = q_0^r$ for some sufficiently large integer $r \geq 1$. ■

From now on, **we will assume that** $P_2 \subset P_1$**,** $q_1 \geq q_0$ **and** $m_1 \geq 1$ **are fixed** with the properties in Lemma 5.3. Then for any cylinder $\mathcal{C}$ as in Lemma 5.3 we have $\mathrm{diam}(\widetilde{\Psi}(\mathcal{C} \cap P_1)) \geq \hat{\rho}\,\mathrm{diam}(\widetilde{\Psi}(\mathcal{C}))$.

As a consequence of the above we derive uniform D-Lipschitzness of the local stable holonomy maps on a compact subset of $\mathcal{L} \cap R$ of large measure. Theorem 1.6 is an immediate consequence of this.

Lemma 5.4. *For every $\delta > 0$ there exist a constant $A > 0$ and a compact subset Q of $\mathcal{L} \cap R$ with $\mu(Q) > 1 - \delta$ and the following property: if $\mathcal{C}_1$ and $\mathcal{C}_2$ are cylinders in $W^u_{R_i}(z')$ and $W^u_{R_i}(z'')$, respectively, for some $i = 1,\ldots,k_0$ and some $z', z'' \in S_i$ such that $\mathcal{C}_1 \cap Q \neq \emptyset$, $\mathcal{C}_2 \cap Q \neq \emptyset$ and $\pi_{z''}(\mathcal{C}_1) = \mathcal{C}_2$, then $\mathrm{diam}(\Psi(\mathcal{C}_1)) \leq A\,\mathrm{diam}(\Psi(\mathcal{C}_2))$. Moreover, we can choose Q so that $P_2 \subset Q$, where P_2 is the set from Lemma 5.3.*

Proof. Fix an integer $J_0 \geq 1$ so that $\mathrm{diam}(\mathcal{P}^j(W^s_R(z))) \leq \delta''/2$ for all $z \in R$ and all $j \geq J_0$. Given $\delta > 0$, since $\mu(P_2) > 0$, we can take the integer $j_1 \geq 1$ so large that the set $Q = \cup_{j=0}^{j_1}\mathcal{P}^{-jJ_0}(P_2)$ has $\mu(Q) > 1 - \delta$. We will now show that Q has the required property for some constant $A = A(P_2, J_0, j_1, \delta, m_1, \hat{\rho}) > 0$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be cylinders in $W^u_{R_i}(z')$ and $W^u_{R_i}(z'')$, respectively, for some $i = 1,\ldots,k_0$ and some $z', z'' \in S_i$ such that $\mathcal{C}_1 \cap Q \neq \emptyset$, $\mathcal{C}_2 \cap Q \neq \emptyset$ and $\mathcal{H}^{z''}_{z'}(\mathcal{C}_1) = \mathcal{C}_2$; then $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same length m. The case $m < \max\{m_1, j_1J_0\}$ is trivial, since then the diameters of $\Psi(\mathcal{C}_1)$ and $\Psi(\mathcal{C}_2)$ are bounded below and above by positive constants. Assume $m \geq \max\{m_1, j_1J_0\}$. Let $\mathcal{C}_1 \cap \mathcal{P}^{-j'J_0}(P_2) \neq \emptyset$ and $\mathcal{C}_2 \cap \mathcal{P}^{-jJ_0}(P_2) \neq \emptyset$ for some $j, j' = 0, 1, \ldots, j_1$.

We will consider in details the case $j' \leq j$ (the other case is similar). Assuming this, using (2.1), we can also assume that $j' = 0$; the general case is easily obtained from this one. Then $\mathcal{C}_1, \mathcal{C}_2 \subset R_1$ and $\mathcal{C}_1 \cap P_2 \neq \emptyset$. Since $P_2 \subset P_1$, it now follows that $z' \in \tilde{S}_0$ (see Lemma 4.3).

Take $x \in \mathcal{C}_1$, $z \in \mathcal{C}_1 \cap P_1$ such that for $x_0 = \Psi(x)$, $z_0 = \Psi(z)$ we have $d(x_0, z_0) \geq \frac{1}{2}\mathrm{diam}(\Psi(\mathcal{C}_1))$. Then by Lemma 5.3(b), $\|u_0^{(1)} - w_0^{(1)}\| \geq \frac{2}{3}\|u_0 - w_0\|$ and $\|\tilde{u}_0^{(2)} - \tilde{w}_0^{(2)}\| \leq \frac{\beta_0}{3}\|u_0 - w_0\|$, where $z_0 = \Phi^u_{z'}(u_0)$, $z_0 = \Phi^u_{z'}(w_0)$. Fix an $\ell = 1,\ldots,\ell_0$ such that $\left\langle \frac{u_0^{(1)} - w_0^{(1)}}{\|u_0^{(1)} - w_0^{(1)}\|}, \eta_\ell(z') \right\rangle \geq 2\beta_0$. Since $\eta_\ell(z') \in E^u_1(z')$, this implies

$$
\begin{aligned}
\left\langle \frac{u_0 - w_0}{\|u_0 - w_0\|}, \eta_\ell(z') \right\rangle &= \left\langle \frac{u_0^{(1)} - w_0^{(1)}}{\|u_0 - w_0\|}, \eta_\ell(z') \right\rangle + \left\langle \frac{\tilde{u}_0^{(2)} - \tilde{w}_0^{(2)}}{\|u_0 - w_0\|}, \eta_\ell(z') \right\rangle \\
&\geq \frac{2}{3}\left\langle \frac{u_0^{(1)} - w_0^{(1)}}{\|u_0^{(1)} - w_0^{(1)}\|}, \eta_\ell(z') \right\rangle - \frac{\beta_0}{3}\left|\left\langle \frac{u_0^{(1)} - w_0^{(1)}}{\|u_0^{(1)} - w_0^{(1)}\|}, \eta_\ell(z') \right\rangle\right| \geq \beta_0.
\end{aligned}
$$

Fix arbitrary $d_1 \in B^s(y_1^{(\ell)}, \delta'')$ and $d_2 \in B^s(z_1, \delta'')$. Then by Lemma 4.3, we have $\hat{\delta}_0\,\mathrm{diam}(\Psi(\mathcal{C}_1)) \leq |\Delta(x_0, \pi_{d_1}(z_0)) - \Delta(x_0, \pi_{d_2}(z_0))|$, where as before $\hat{\delta}_0 = \frac{\beta_0\delta_0\hat{\rho}}{8R_0}$. Clearly the above will be also true if we replace d_i ($i = 1, 2$) by the intersection point d'_i of $W^s_R(z)$ and $W^u_R(d_i)$. Moreover, an easy calculation shows that

$$|\Delta(x_0, \pi_{d'_1}(z_0)) - \Delta(x_0, \pi_{d'_2}(z_0))| = |\Delta(x, \pi_{d'_1}(z)) - \Delta(x, \pi_{d'_2}(z))|.$$

(See the proof of Lemma 5.5 below where this is done in details.) Consider the points $\tilde{x} = \pi_{z''}(x)$, $\tilde{z} = \pi_{z''}(z) \in \mathcal{C}_2$.

First, assume $j = 0$; then $\mathcal{C}_2 \cap P_2 \neq \emptyset$. By Lemmas 4.3 and 4.2 and the above,

$$\begin{aligned}\hat{\delta}_0 \operatorname{diam}(\Psi(\mathcal{C}_1)) &\leq |\Delta(x, \pi_{d_1'}(z)) - \Delta(x, \pi_{d_2'}(z))| \\ &= |\Delta(\tilde{x}, \pi_{d_1'}(\tilde{z})) - \Delta(\tilde{x}, \pi_{d_2'}(\tilde{z}))| \leq C_1 \operatorname{diam}(\Psi(\mathcal{C}_2)),\end{aligned}$$

so $\operatorname{diam}(\Psi(\mathcal{C}_1)) \leq (C_1/\hat{\delta}_0) \operatorname{diam}(\Psi(\mathcal{C}_2))$. Similarly, $\operatorname{diam}(\Psi(\mathcal{C}_2)) \leq (C_1/\hat{\delta}_0) \operatorname{diam}(\Psi(\mathcal{C}_1))$.

Next, assume $j \geq 1$. Since the length of $\mathcal{C}_2$ is $\geq jJ_0$, $\mathcal{P}^{jJ_0} : \mathcal{C}_2 \longrightarrow \mathcal{C} = \mathcal{P}^{jJ_0}(\mathcal{C}_2)$ is a homeomorphism. Then $\mathcal{C}_2 \cap \mathcal{P}^{-jJ_0}(P_2) \neq \emptyset$ implies that $\mathcal{C} \subset W^u_{R_1}(\hat{z}_0)$ for some $\hat{z}_0 \in \tilde{S}_0$, and $\mathcal{C} \cap P_1 \neq \emptyset$ (since $P_2 \subset P_1$). Moreover $jJ_0 \geq J_0$ and the choice of J_0 show that $\mathcal{P}^{jJ_0}(W^s_R(z)) \subset B^s(y, \delta''/2)$ for some $y \in W^u_R(\hat{z}_0)$, so in particular $\tilde{d}_1' = \mathcal{P}^{jJ_0}(d_1'), \tilde{d}_2' = \mathcal{P}^{jJ_0}(d_2') \in B^s(z_1, \delta'')$. Clearly, $\tilde{d}_i' \in W^u_{R_1}(\tilde{d}_i)$ for some $\tilde{d}_i \in S_1$, and moreover we have $\tilde{d}_i \in B^s(z_1, \delta'')$. Setting $\tilde{x}_0 = \Psi(\mathcal{P}^{jJ_0}(\tilde{x}))$ and $\tilde{z}_0 = \Psi(\mathcal{P}^{jJ_0}(\tilde{z})) \in \Psi(\mathcal{C})$, it follows from Lemma 4.2 that $|\Delta(\tilde{x}_0, \pi_{\tilde{d}_1}(\tilde{z}_0))| \leq C_1 \operatorname{diam}(\Psi(\mathcal{C}))$ and $|\Delta(\tilde{x}_0, \pi_{\tilde{d}_2}(\tilde{z}_0))| \leq C_1 \operatorname{diam}(\Psi(\mathcal{C}))$, so

$$\begin{aligned}|\Delta(\tilde{x}, \pi_{d_1'}(\tilde{z})) - \Delta(\tilde{x}, \pi_{d_2'}(\tilde{z}))| &= |\Delta(\mathcal{P}^{jJ_0}(\tilde{x}), \pi_{\tilde{d}_1}(\mathcal{P}^{jJ_0}(\tilde{z}))) - \Delta(\mathcal{P}^{jJ_0}(\tilde{x}), \pi_{\tilde{d}_2}(\mathcal{P}^{jJ_0}(\tilde{z})))| \\ &= |\Delta(\tilde{x}_0, \pi_{\tilde{d}_1}(\tilde{z}_0)) - \Delta(\tilde{x}_0, \pi_{\tilde{d}_2}(\tilde{z}_0))| \leq 2C_1 \operatorname{diam}(\Psi(\mathcal{C})).\end{aligned}$$

On the other hand (2.1) gives $\operatorname{diam}(\Psi(\mathcal{C})) \leq \frac{\gamma_1^{j_1 J_0}}{c_0} \operatorname{diam}(\Psi(\mathcal{C}_2))$, so

$$\begin{aligned}\hat{\delta}_0 \operatorname{diam}(\Psi(\mathcal{C}_1)) &\leq |\Delta(x, \pi_{d_1}(z)) - \Delta(x, \pi_{d_2}(z))| \\ &= |\Delta(\tilde{x}, \pi_{d_1}(\tilde{z})) - \Delta(\tilde{x}, \pi_{d_2}(\tilde{z}))| \leq 2C_1 \operatorname{diam}(\Psi(\mathcal{C})) \leq \frac{2C_1\gamma_1^{j_1 J_0}}{c_0} \operatorname{diam}(\Psi(\mathcal{C}_2)).\end{aligned}$$

Thus, we can take $A = 2C_1\gamma_1^{j_1 J_0}/(c_0\hat{\delta}_0)$.

The case $j' > j$ is considered similarly. ■

5.3 Main consequence of Lemma 4.3

Recall that $\hat{\delta}_0 = \dfrac{\beta_0\delta_0\hat{\rho}}{8R_0}$ where $\beta_0 > 0$ and $\delta_0 > 0$ are fixed constants with the properties described in Lemma 4.3.

Lemma 5.5. *For any $\hat{z}_0 \in \tilde{S}_0$, any $\ell = 1, \ldots, \ell_0$ and any $i = 1, 2$ there exists a (Hölder) continuous map $U_1 \ni x \mapsto v_i^{(\ell)}(\hat{z}_0, x) \in U_1$ (depending continuously on $\hat{z}_0$; i.e. $v_i^{(l)} : \tilde{S}_0 \times U_1 \longrightarrow U_1$ is continuous) such that $\sigma^N(v_i^{(l)}(\hat{z}_0, x)) = x$ for all $x \in \widehat{U}_1$ and the following property holds:*

For any cylinder $\mathcal{C}$ in $U(\hat{z}_0)$ of length $\geq m_1$ with $\mathcal{C} \cap P_2 \neq \emptyset$ there exist sub-cylinders $\mathcal{D}$ and $\mathcal{D}'$ of $\mathcal{C}$ of co-length q_1 with $\mathcal{D} \cap P_1 \neq \emptyset$, $\mathcal{D}' \cap P_1 \neq \emptyset$ and $\ell = 1, \ldots, \ell_0$ such that for any points $x \in \mathcal{D}$ and $z \in \mathcal{D}'$, setting $x' = \pi^{(U)}(x)$, $z' = \pi^{(U)}(z)$, we have $d(\Psi(x), \Psi(z)) \geq \frac{\hat{\rho}}{2} \operatorname{diam}(\Psi(\mathcal{C}))$ and

$$I_{N,\ell}(x', z') = \left|\varphi_\ell(\hat{z}_0, x') - \varphi_\ell(\hat{z}_0, z')\right| \geq \frac{\hat{\delta}_0}{2} \operatorname{diam}(\Psi(\mathcal{C})),$$

where

$$\varphi_\ell(\hat{z}_0, x) = \tau_N(v_1^{(l)}(\hat{z}_0, x)) - \tau_N(v_2^{(l)}(\hat{z}_0, x)).$$

Moreover, $I_{N,\ell}(x', z') \leq C_1 \operatorname{diam}(\Psi(\mathcal{C}))$ for any $x, z \in \mathcal{C}$, where $C_1 > 0$ is the constant from Lemma 4.2.

Proof. We will use the points $y_1^{(\ell)}(\hat{z}_0) \in B^s(\hat{z}_0, \delta') \cap \mathcal{P}^N(U_0(\hat{z}_0))$ and $y_2^{(\ell)}(\hat{z}_0) \in B^s(\hat{z}_0, \delta'') \cap \mathcal{P}^N(U_0(\hat{z}_0))$ from Lemma 4.3. Given $\ell = 1, \ldots, \ell_0$ and $i = 1, 2$, by the choice of the point $y_i^{(\ell)}(\hat{z}_0)$, there exists a cylinder $L_i^{(\ell)}(\hat{z}_0)$ of length N in $U_0(\hat{z}_0)$ so that $\mathcal{P}^N : \widehat{L}_i^{(\ell)}(\hat{z}_0) \longrightarrow \widehat{W}^u_{R_1}(y_i^{(\ell)}(\hat{z}_0))$ is a bijection; then it is a bi-Hölder homeomorphism. Consider its inverse and its Hölder continuous extension $\mathcal{P}^{-N} : W^u_{R_1}(y_i^{(\ell)}(\hat{z}_0)) \longrightarrow L_i^{(\ell)}(\hat{z}_0)$. Clearly we can choose the cylinder $L_i^{(\ell)}(\hat{z}_0)$ so that it depends continuously on $\hat{z}_0 \in \tilde{S}_0$, namely if W is an open subset of R_1 containing $L_i^{(\ell)}(\hat{z}_0)$, then for $\hat{z} \in \tilde{S}_0$ sufficiently close to $\hat{z}_0$ we have $L_i^{(\ell)}(\hat{z}) \subset W$. Set $M_i^{(\ell)}(\hat{z}_0) = \pi^{(U)}(L_i^{(\ell)}(\hat{z}_0)) \subset U_0$; this is then a cylinder of length N in U_0 depending continuously on $\hat{z}_0 \in \tilde{S}_0$. Define the maps

$$\tilde{v}_i^{(\ell)}(\hat{z}_0, \cdot) : U_1 \longrightarrow L_i^{(\ell)}(\hat{z}_0) \subset U_0(\hat{z}_0) \subset W^u_{R_1}(\hat{z}_0) \quad , \quad v_i^{(l)}(\hat{z}_0, \cdot) : U_1 \longrightarrow M_i^{(\ell)}(\hat{z}_0) \subset U_0$$

by $\tilde{v}_i^{(\ell)}(\hat{z}_0, x) = \mathcal{P}^{-N}(\pi_{y_i^{(\ell)}(\hat{z}_0)}(x))$ and $v_i^{(l)}(\hat{z}_0, x) = \pi^{(U)}(\tilde{v}_i^{(\ell)}(\hat{z}_0, x))$. Then

$$\mathcal{P}^N(\tilde{v}_i^{(\ell)}(\hat{z}_0, x)) = \pi_{y_i^{(\ell)}(\hat{z}_0)}(x) = W^s_{\epsilon_0}(x) \cap W^u_{R_1}(y_i^{(\ell)}(\hat{z}_0)), \tag{5.8}$$

and

$$\mathcal{P}^N(v_i^{(l)}(\hat{z}_0, x)) = W^s_{\epsilon_0}(x) \cap \mathcal{P}^N(M_i^{(\ell)}(\hat{z}_0)) = \pi_{d_i^{(\ell)}(\hat{z}_0)}(x), \tag{5.9}$$

where $d_i^{(\ell)}(\hat{z}_0) \in S_1$ is such that $\mathcal{P}^N(M_i^{(\ell)}(\hat{z}_0)) = W^u_{R_1}(d_i^{(\ell)}(\hat{z}_0))$ (and such points $d_i^{(\ell)}(\hat{z}_0)$ clearly exist). Next, there exist $x' \in M_i^{(\ell)}(\hat{z}_0)$ and $y' \in L_i^{(\ell)}(\hat{z}_0)$ such that $y' \in W^s_{\epsilon_0}(x')$, $\mathcal{P}^{N'}(x') = d_i^{(\ell)}(\hat{z}_0)$ and $\mathcal{P}^{N'}(y') = y_i^{(\ell)}(\hat{z}_0)$. Since stable leaves shrink exponentially fast, using (2.1) and (5.6) we get $d(d_i^{(\ell)}(\hat{z}_0), y_i^{(\ell)}(\hat{z}_0)) \leq \frac{1}{c_0\gamma^N} d(x', y') \leq \frac{\mathrm{diam}(\widetilde{R}_1)}{\gamma^N} < \delta''$. Thus, $d_1^{(\ell)}(\hat{z}_0), d_2^{(\ell)}(\hat{z}_0)$ satisfy the assumptions and therefore the conclusions of Lemma 4.3.

Set $d_1^{(\ell)} = d_1^{(\ell)}(\hat{z}_0)$ and $d_2^{(\ell)} = d_2^{(\ell)}(\hat{z}_0)$ for brevity. Let $\mathcal{C}$ be a cylinder in $U(\hat{z}_0)$ of length $\geq m_1$ with $\mathcal{C} \cap P_2 \neq \emptyset$. Then by Lemma 5.3(c), there exist $x_0, z_0 \in \Psi(\mathcal{C} \cap P_1)$ such that $d(x_0, z_0) \geq \hat{\rho}\,\mathrm{diam}(\Psi(\mathcal{C}))$. Moreover, by Lemma 5.3(d), if $\mathcal{D}$ and $\mathcal{D}'$ are the sub-cylinders of $\mathcal{C}$ of co-length q_1 containing $\Psi^{-1}(x_0)$ and $\Psi^{-1}(z_0)$, respectively, then $\mathrm{diam}(\Psi(\mathcal{D})) \leq \frac{\hat{\delta}_0}{8C_1}\,\mathrm{diam}(\Psi(\mathcal{C})) \leq \frac{\hat{\rho}}{8}\,\mathrm{diam}(\Psi(\mathcal{C}))$ and similarly $\mathrm{diam}(\Psi(\mathcal{D}')) \leq \frac{\hat{\delta}_0}{8C_1}\,\mathrm{diam}(\Psi(\mathcal{C}))$. Thus, for any $x \in \Psi(\mathcal{D})$ and $z \in \Psi(\mathcal{D}')$ we have $d(x, z) \geq \frac{\hat{\rho}}{2}\,\mathrm{diam}(\Psi(\mathcal{C}))$. Let $x_0 = \Phi^u_{\hat{z}_0}(u_0)$ and $z_0 = \Phi^u_{\hat{z}_0}(w_0)$, where $u_0, w_0 \in E^u(\hat{z}_0)$. By the choice of the constant β_0 and the family of unit vectors $\{\eta_\ell(\hat{z}_0)\}_{\ell=1}^{m_0}$, there exists some $\ell = 1, \ldots, \ell_0$ such that $\left\langle \frac{u_0^{(1)} - w_0^{(1)}}{\|u_0^{(1)} - w_0^{(1)}\|}, \eta_\ell(z') \right\rangle \geq 2\beta_0$. Then as in the proof of Lemma 5.4 we derive $\left\langle \frac{u_0 - w_0}{\|u_0 - w_0\|}, \eta_\ell(z') \right\rangle \geq \beta_0$, and it follows from Lemma 4.3 that

$$\hat{\delta}_0\,\mathrm{diam}(\Psi(\mathcal{C})) \leq \left| \Delta(x_0, \pi_{d_1^{(\ell)}}(z_0)) - \Delta(x_0, \pi_{d_2^{(\ell)}}(z_0)) \right|. \tag{5.10}$$

Consider $\tilde{x} = \Psi^{-1}(x_0), \tilde{z} = \Psi^{-1}(z_0) \in \mathcal{C} \cap P_1$ and the projections of x_0, z_0 to U_1 along stable leaves: $x'_0 = \pi^{(U)}(\tilde{x}) \in U_0$, $z'_0 = \pi^{(U)}(\tilde{z}) \in U_0$. We have

$$\begin{aligned} I_{N,\ell}(x'_0, z'_0) &= \left| [\tau_N(v_1^{(l)}(\hat{z}_0, x'_0)) - \tau_N(v_2^{(l)}(\hat{z}_0, x'_0))] - [\tau_N(v_1^{(l)}(\hat{z}_0, z'_0)) - \tau_N(v_2^{(l)}(\hat{z}_0, z'_0))] \right| \\ &= \left| [\tau_N(v_1^{(l)}(\hat{z}_0, x'_0)) - \tau_N(v_1^{(l)}(\hat{z}_0, z'_0))] - [\tau_N(v_2^{(l)}(\hat{z}_0, x'_0)) - \tau_N(v_2^{(l)}(\hat{z}_0, z'_0))] \right| \\ &= \left| \Delta(\mathcal{P}^N(v_1^{(l)}(\hat{z}_0, x'_0)), \mathcal{P}^N(v_1^{(l)}(\hat{z}_0, z'_0))) - \Delta(\mathcal{P}^N(v_2^{(l)}(\hat{z}_0, x'_0)), \mathcal{P}^N(v_2^{(l)}(\hat{z}_0, z'_0))) \right| \\ &= \left| \Delta(x'_0, \pi_{d_1^{(\ell)}}(z'_0)) - \Delta(x'_0, \pi_{d_2^{(\ell)}}(z'_0)) \right| = \left| \Delta(\tilde{x}, \pi_{d_1^{(\ell)}}(\tilde{z})) - \Delta(\tilde{x}, \pi_{d_2^{(\ell)}}(\tilde{z})) \right|. \end{aligned}$$

We claim that the latter is the same as the right-hand-side of (5.10). Indeed, let $\Delta(\tilde{x}, \pi_{d_1^{(\ell)}}(\tilde{z})) = s_1$ and $\Delta(\tilde{x}, \pi_{d_2^{(\ell)}}(\tilde{z})) = s_2$. Then $\phi_{s_1}([\tilde{x}, \pi_{d_1^{(\ell)}}(\tilde{z})]) \in W^s_{\epsilon_0}(\pi_{d_1^{(\ell)}}(\tilde{z}))$ and $\phi_{s_2}([\tilde{x}, \pi_{d_2^{(\ell)}}(\tilde{z})]) \in W^s_{\epsilon_0}(\pi_{d_2^{(\ell)}}(\tilde{z}))$.

Let $\phi_s(x_0) = \tilde{x}$ and $\phi_t(z_0) = \tilde{z}$. It is then straightforward to see (using $\pi_{d_1^{(\ell)}}(\tilde{z}), \pi_{d_2^{(\ell)}}(\tilde{z}) \in W^s_{\epsilon_0}(\tilde{z})$) that $\Delta(x_0, \pi_{d_1^{(\ell)}}(z_0)) = s + s_1 - t$ and $\Delta(x_0, \pi_{d_2^{(\ell)}}(z_0)) = s + s_2 - t$. Thus,

$$\left|\Delta(x_0, \pi_{d_1^{(\ell)}}(z_0)) - \Delta(x_0, \pi_{d_2^{(\ell)}}(z_0))\right| = |(s+s_1-t)-(s+s_2-t)| = \left|\Delta(\tilde{x}, \pi_{d_1^{(\ell)}}(\tilde{z})) - \Delta(\tilde{x}, \pi_{d_2^{(\ell)}}(\tilde{z}))\right|.$$

Combining this with (5.10) gives $I_{N,\ell}(x_0', z_0') \geq \hat{\delta}_0 \operatorname{diam}(\Psi(\mathcal{C}))$.

For arbitrary $x, z \in \Psi(\mathcal{C})$, setting $x' = \pi^{(U)}(x)$, the above calculation and Lemma 4.2 give $I_{N,\ell}(x', z') = \left|\Delta(x, \pi_{d_1^{(\ell)}}(z)) - \Delta(x, \pi_{d_2^{(\ell)}}(z))\right| \leq C_1 \operatorname{diam}(\Psi(\mathcal{C}))$. The same argument shows that for any $x \in \Psi(\mathcal{D})$ and x_0 as above we have

$$I_{N,\ell}(x', x_0') = \left|\Delta(x, \pi_{d_1^{(\ell)}}(x_0)) - \Delta(x, \pi_{d_2^{(\ell)}}(x_0))\right| \leq C_1 \operatorname{diam}(\Psi(\mathcal{D})) \leq \frac{\hat{\delta}_0}{8} \operatorname{diam}(\Psi(\mathcal{C})).$$

Similarly, for any $z \in \Psi(\mathcal{D}')$ and z_0 as above we have

$$I_{N,\ell}(z_0', z') = \left|\Delta(z_0, \pi_{d_1^{(\ell)}}(z)) - \Delta(z_0, \pi_{d_2^{(\ell)}}(z))\right| \leq C_1 \operatorname{diam}(\Psi(\mathcal{D}')) \leq \frac{\hat{\delta}_0}{8} \operatorname{diam}(\Psi(\mathcal{C})).$$

Then

$$\Delta(x, \pi_{d_i^{(\ell)}}(z)) = \Delta(x, \pi_{d_i^{(\ell)}}(x_0)) + \Delta(x_0, \pi_{d_i^{(\ell)}}(z)) = \Delta(x, \pi_{d_i^{(\ell)}}(x_0)) + \Delta(x_0, \pi_{d_i^{(\ell)}}(z_0)) + \Delta(z_0, \pi_{d_i^{(\ell)}}(z)),$$

yields $I_{N,\ell}(x', z') \geq I_{N,\ell}(x_0', z_0') - I_{N,\ell}(x', x_0') - I_{N,\ell}(z_0', z') \geq \hat{\delta}_0 \operatorname{diam}(\Psi(\mathcal{C})) - 2\,\frac{\hat{\delta}_0}{8} \operatorname{diam}(\Psi(\mathcal{C})) \geq \frac{\hat{\delta}_0}{2} \operatorname{diam}(\Psi(\mathcal{C}))$. This completes the proof of the lemma. ■

6 Contraction operators

We use the notation and the set-up from Sect. 5.

6.1 Some definitions

First, we define a compact subset K_0 of U_1 with $\nu(K_0) > 0$ which will play an important role below. Roughly speaking this is where the contraction exhibited by the contraction operators defined in Sect. 6.2 will actually occur.

Next, **fix constants** $\hat{\delta}_1 \in (0, \hat{\delta}/4)$ **with** $\hat{\delta}_1 < \frac{1}{2}\mu(P_2)$**,** $A > 0$ **and a compact subset** Q of $\mathcal{L} \cap R$ such that $\mu(Q) > 1 - \hat{\delta}_1$, $P_2 \subset Q$ and the property described in Lemma 5.4 holds with $\delta = \hat{\delta}_1$. We assume $\hat{\delta}_1 > 0$ is chosen so small that

$$\gamma\,(\gamma/\gamma_1)^{\hat{\delta}_1} > \gamma^{1/2}, \tag{6.1}$$

where $1 < \gamma < \gamma_1$ are the constants from (2.1), while $A > 0$ is chosen so large that for any $i = 1, \ldots, k_0$ and any $x, x' \in R_i$ we have $\operatorname{diam}(\Psi(W^u_{R_i}(x))) \leq A \operatorname{diam}(\Psi(W^u_{R_i}(x')))$.

We now need to repeat part of the procedures in Sect. 4. Using Birkhoff's Theorem again,

$$h_n(x) = \frac{\chi_Q(x) + \chi_Q(\mathcal{P}(x)) + \ldots + \chi_Q(\mathcal{P}^{n-1}(x))}{n} \to \int_R \chi_Q \, d\mu = \mu(Q) > 1 - \hat{\delta}_1$$

as $n \to \infty$ for almost all $x \in R$. Using Egorov's Theorem, fix a compact subset Q_1 of Q with $\mu(Q_1) > 1 - \hat{\delta}_1$, such that $h_n(x) \to 1$ as $n \to \infty$ uniformly for $x \in Q_1$. Thus, there exists an integer $n_1 \geq 1$ so that $h_n(x) \geq 1 - \hat{\delta}_1$ for all $x \in Q_1$ and all $n \geq n_1$. That is,

$$\sharp\{j : 0 \leq j \leq n-1 \,,\, \mathcal{P}^j(x) \in Q\} \geq (1 - \hat{\delta}_1)\, n \quad , \quad n \geq n_1,\ x \in Q_1. \tag{6.2}$$

Then $P_3 = P_2 \cap Q_1$ is a compact subset of P_2 with $\mu(P_3) > \frac{1}{2}\mu(P_2) > 0$.

Fix an integer d_0 such that

$$d_0 \geq \max\{\frac{2}{\log\gamma}\log(\frac{A}{\hat{\rho}^{q_1}\, c_0^2})\ ,\ 2n_1/\hat{\delta}_1\}, \tag{6.3}$$

where $q_1 \geq 1$ is the constant from Lemma 5.3(d).

Set $\nu_0'' = \frac{\mu(P_2)}{2A_2\nu^s(\tilde{S}_0)}$ and $S^\sharp = \{z \in \tilde{S}_0 : \nu^u_z(U_1(z)\cap P_2) \geq \nu_0''\}$. Then[14] by (4.1), $\nu^s(S^\sharp) \geq \frac{\mu(P_2)}{2(1-\nu_0'')} > 0$, and so $\mu(P_2 \cap \mathcal{O}(S^\sharp)) > 0$.

We need to refine slightly the set P_3 in a way similar to the constructions of P_1 and P_2. First, as we did with P_1, fix a compact subset P_4' of P_3 and an integer $m_2 \geq m_1$ such that for any $z \in \tilde{S}_0$ and any cylinder $\mathcal{C}$ in $W^u_R(z)$ of length $\geq m_2$ with $\mathcal{C} \cap P_4' \neq \emptyset$ we have

$$\nu^u_z(\mathcal{C} \cap P_3) \geq \omega_0\, \nu^u_z(\mathcal{C}). \tag{6.4}$$

Then as in Lemma 5.3, **fix a compact subset** P_4 of $P_4' \cap \mathcal{O}(S^\sharp)$ such that $\mu(P_4) > 0$ and for any unstable leaf W in R, any cylinder $\mathcal{C}$ in W of length $\geq m_2$ with $\mathcal{C} \cap P_4 \neq \emptyset$ we have $\mathrm{diam}(\Psi(\mathcal{C} \cap P_3)) \geq \hat{\rho}\,\mathrm{diam}(\Psi(\mathcal{C}))$.

Moreover, as in the proof of Lemma 5.3(a), for every $z \in \tilde{S}_0$ and any cylinder $\mathcal{C}$ in $W^u_R(z)$ of length $\geq m_2$ with $\mathcal{C} \cap P_4 \neq \emptyset$ it follows from (6.4) that for any sub-cylinder $\mathcal{D}$ of $\mathcal{C}$ of co-length $\leq q_0$ we have $\mathcal{D} \cap P_3 \neq \emptyset$.

Set $\nu_0''' = \frac{\mu(P_4)}{2A_2\nu^s(S^\sharp)}$ and $S_0^\sharp = \{z \in S^\sharp : \nu^u_z(U_0(z)\cap P_4) \geq \nu_0'''\}$. Then $\nu^s(S_0^\sharp) \geq \frac{\mu(P_4)}{2(1-\nu_0'')} > 0$ (as above), so $\mu(P_4 \cap \mathcal{O}(S_0^\sharp)) > 0$.

Fix an arbitrary $\hat{z}_0 \in S_0^\sharp$**.** From now on for brevity we will use the notation

$$v_i^{(l)}(x) = v_i^{(l)}(\hat{z}_0, x) \quad , \quad \varphi_\ell(x) = \varphi_\ell(\hat{z}_0, x) = \tau_N(v_1^{(l)}(\hat{z}_0, x)) - \tau_N(v_2^{(l)}(\hat{z}_0, x))$$

for any $x \in U_1$. **Fix a subset** K of $U_1(\hat{z}_0) \cap P_4$ such that for $K_0 = \pi^{(U)}(K) \subset U_1$ we have $\nu(K_0) > 0$ and $\sigma^j(K_0) \cap K_0 = \emptyset$ for all $j = 1, \ldots, d_0$. It follows easily from the above that such a set K (and so K_0) exists. It then follows that there exists a constant $\hat{\delta}_2 > 0$ such that

$$\mathrm{dist}(\sigma^j(K_0), K_0) \geq \hat{\delta}_2 \quad , \quad j = 1, \ldots, d_0. \tag{6.5}$$

Fix $\hat{\delta}_2 > 0$ **with this property.**

Set

$$\mu_0 = \mu_0(N) = \min\{\ \theta^{q_1} e^{-T/(1-\theta)}/6\ ,\ \frac{1}{8\, e^{2TN}}\ \sin^2(\hat{\delta}_0\hat{\rho}\epsilon_1/16)\ \}, \tag{6.6}$$

and

$$b_0 = \max\{\theta^{-m_2}\ ,\ (2C_0\gamma_1^{d_0}/(c_0\hat{\delta}_2))^{1/\alpha_1}\ ,\ (3C_2Te^{T/(1-\theta)}/(1-\theta))^{1/\beta'}\}, \tag{6.7}$$

where $\beta' > 0$ is as in Lemma 5.1(c).

Throughout the rest of Sect. 6, b will be a **fixed real number** with $|b| \geq b_0$.

---

[14]Indeed,

$$\begin{aligned} 2A_2\nu_0''\nu^s(\tilde{S}_0) = \mu(P_2) \quad &\leq \quad A_2 \int_{\tilde{S}_0} \nu^u_z(U_1(z)\cap P_2)\, d\nu^s(z) \\ &= \quad A_2 \int_{\tilde{S}_0\setminus S^\sharp} \nu^u_z(U_1(z)\cap P_2)\, d\nu^s(z) + A_2 \int_{S^\sharp} \nu^u_z(U_1(z)\cap P_2)\, d\nu^s(z) \\ &\leq \quad A_2\, \nu_0''\, (\nu^s(\tilde{S}_0) - \nu^s(S^\sharp)) + A_2\, \nu^s(S^\sharp) \leq A_2\, \nu_0''\nu^s(\tilde{S}_0) + A_2\,(1-\nu_0'')\nu^s(S^\sharp), \end{aligned}$$

so $A_2\nu_0''\nu^s(\tilde{S}_0) \leq (1-\nu_0'')\nu^s(S^\sharp)$ and therefore $\nu^s(S^\sharp) \geq \frac{\mu(P_2)}{2(1-\nu_0'')} > 0$.

6.2 Choice of cylinders, definition of the contraction operators

Let $\mathcal{C}_1, \ldots, \mathcal{C}_{\tilde{m}}$ be the maximal closed cylinders in $U_1(\hat{z}_0)$ with $\mathrm{diam}(\Psi(\mathcal{C}_m)) \leq \epsilon_1/|b|$ $(m = 1, \ldots, \tilde{m})$, and let $\mathcal{D}_1, \ldots, \mathcal{D}_{\tilde{j}}$ be the list of all closed cylinders in $U_1(\hat{z}_0)$ which are sub-cylinders of co-length q_1 of some $\mathcal{C}_m$. Here $q_1 \geq 1$ is the constant from Lemma 5.3(d). Set $\mathcal{C}'_m = \pi^{(U)}(\mathcal{C}_m) \subset U_1$, $\mathcal{D}'_j = \pi^{(U)}(\mathcal{D}_j) \subset U_1$. Then $\cup_{m=1}^{\tilde{m}} \mathcal{C}'_m = \cup_{j=1}^{\tilde{j}} \mathcal{D}'_j = U_1$. Next, let $\mathcal{D}_1, \ldots, \mathcal{D}_{j_0}$, for some $j_0 \leq \tilde{j}$, be the list of those $\mathcal{D}_j$ with $\mathcal{D}_j \cap P_4 \neq \emptyset$, and similarly, let $\mathcal{C}_1, \ldots, \mathcal{C}_{m_0}$, for some $m_0 \leq \tilde{m}$, be the list of those $\mathcal{C}_m$ such that $\mathcal{C}_m \cap P_4 \neq \emptyset$. Set

$$V_b = \cup_{j=1}^{j_0} \widehat{\mathcal{D}}'_j \subset \widehat{U}_1.$$

We have $\mathrm{diam}(\Psi(\mathcal{C}_m)) \leq \epsilon_1/|b|$ and $\mathrm{diam}(\Psi(\mathcal{P}^j(\mathcal{C}_m)) \leq \frac{\gamma_1^{d_0} \epsilon_1}{c_0 |b|}$ for $1 \leq j \leq d_0$. It then follows from (6.5), $\epsilon_1 \leq 1/2$ and $|b| \geq b_0$ that

$$\sigma^j(V_b) \cap V_b = \emptyset \quad , \quad j = 1, \ldots, d_0. \tag{6.8}$$

Now from the choice of the cylinders, (5.7) and Lemmas 4.4 and 5.3 we get:

$$\nu^u(\mathcal{C}_m \cap P_3) \geq \omega_0\, \nu^u(\mathcal{C}_m) \ , \ \mathrm{diam}(\Psi(\mathcal{C}_m \cap P_3)) \geq \hat{\rho}\, \mathrm{diam}(\Psi(\mathcal{C}_m)) \quad , \quad 1 \leq m \leq m_0, \tag{6.9}$$

$$\hat{\rho}\, \frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\Psi(\mathcal{C}_m)) \leq \frac{\epsilon_1}{|b|} \quad , \quad 1 \leq m \leq m_0, \tag{6.10}$$

$$\mathrm{diam}(\Psi(\mathcal{D}_j)) \leq \frac{\hat{\rho}}{4}\, \mathrm{diam}(\Psi(\mathcal{C}_m)) \quad \text{whenever} \quad \mathcal{D}_j \subset \mathcal{C}_m \ , \ 1 \leq j \leq j_0. \tag{6.11}$$

Let $X_{i,j}^{(\ell)} = v_i^{(l)}(\mathcal{D}'_j) \subset U_1$ for all $i = 1, 2$, $j = 1, \ldots, j_0$ and $\ell = 1, \ldots, \ell_0$. By Lemma 5.1(a), the *characteristic function* $\omega_{i,j}^{(\ell)} = \chi_{\widehat{X}_{i,j}^{(\ell)}} : \widehat{U} \longrightarrow [0,1]$ of $\widehat{X}_{i,j}^{(\ell)}$ belongs to $\mathcal{F}_\theta(\widehat{U})$ and $\mathrm{Lip}_\theta(\omega_{i,j}^{(\ell)}) \leq 1/\mathrm{diam}_\theta(X_{i,j}^{(\ell)})$.

A subset J of the set $\Xi(b) = \{\ (i,j,\ell)\ :\ 1 \leq i \leq 2\ ,\ 1 \leq j \leq j_0\ ,\ 1 \leq \ell \leq \ell_0\ \}$ will be called *representative* if for every $j = 1, \ldots, j_0$ there exists at most one pair (i, ℓ) such that $(i,j,\ell) \in J$, and for any $m = 1, \ldots, m_0$ there exists $(i,j,\ell) \in J$ such that $\mathcal{D}_j \subset \mathcal{C}_m$. Let $\mathsf{J}(\mathsf{b})$ be the *family of all representative subsets* J of $\Xi(b)$.

Given $J \in \mathsf{J}(\mathsf{b})$, define the function $\omega = \omega_J(b) : \widehat{U} \longrightarrow [0,1]$ by

$$\omega = 1 - \mu_0 \sum_{(i,j,\ell) \in J} \omega_{i,j}^{(\ell)}.$$

Clearly $\omega \in \mathcal{F}_\theta(\widehat{U})$ and $\frac{1}{2} \leq 1 - \mu_0 \leq \omega(u) \leq 1$ for any $u \in \widehat{U}$. Define the *contraction operator*

$$\mathcal{N} = \mathcal{N}_J(a,b) : \mathcal{F}_\theta(\widehat{U}) \longrightarrow \mathcal{F}_\theta(\widehat{U}) \quad \text{by} \quad (\mathcal{N}h) = \mathcal{M}_a^N(\omega_J \cdot h).$$

6.3 Main properties of the contraction operators

First, we derive an important consequence of Lemma 5.4 and the construction of K_0.

Lemma 6.1. *Let $\sigma^p(\mathcal{D}_j) \subset \mathcal{C}_{m'}$ for some $p \geq 0$, $j \leq j_0$ and $m' \leq m_0$, and let $\mathcal{D}_j \subset \mathcal{C}_m$ for some $m \leq m_0$. Then $p = 0$ and $m = m'$.*

Proof. Assume that $p > 0$. It then follows from (6.8) that $p > d_0$.

From the assumptions we get $\pi_{\hat{z}_0}(\mathcal{P}^p(\mathcal{D}_j)) \subset \mathcal{C}_{m'} \subset U_1(\hat{z}_0)$. Since $j \leq j_0$, there exists $z \in \mathcal{D}_j \cap P_4$. Then $P_4 \subset P_3 \subset Q_1$ implies $z \in Q_1$. Similarly, there exists $y \in \mathcal{C}_{m'} \cap P_4 \subset Q_1$.

Set $n = [2p\hat{\delta}_1]$. By (6.3) and $p > d_0$, we have $n \geq n_1$. Consider the sets

$$B = \{i : 0 \leq i \leq n-1 \ , \ \mathcal{P}^i(y) \in Q\} \quad , \quad C = \{i : 0 \leq i \leq p+n-1 \ , \ \mathcal{P}^i(z) \in Q\}.$$

By (6.2), $\sharp(B) \geq n(1-\hat{\delta}_1)$ and $\sharp(C) \geq (p+n)(1-\hat{\delta}_1)$. Notice that $(p+n)(1-\hat{\delta}_1) - p > n\hat{\delta}_1$. (This is $(p - p\hat{\delta}_1 + n - n\hat{\delta}_1) - p > n\hat{\delta}_1$, i.e. $n(1-2\hat{\delta}_1) > p\hat{\delta}_1$. Since $\hat{\delta}_1 < 1/8$ we have $n(1-2\hat{\delta}_1) > 3n/4$. Moreover, $n > 2p\hat{\delta}_1 - 1 > 4p\hat{\delta}_1/3$, since $p > 3/(2\hat{\delta}_1)$ by (6.3). Thus, $3n/4 > p\hat{\delta}_1$.) So, for the set $C' = \{i : 0 \leq i \leq n-1 \ , \ \mathcal{P}^{p+i}(z) \in Q\}$ we have $\sharp C' \geq \sharp C - p > n\hat{\delta}_1$, and therefore $\sharp B + \sharp C' > n(1-\hat{\delta}_1) + n\hat{\delta}_1 = n$. Thus, $B \cap C' \neq \emptyset$.

We will now consider two cases.

Case 1. The length of the cylinder $\mathcal{C}_{m'}$ is $\geq n$. Fix an arbitrary $i \in B \cap C'$. Then $\mathcal{P}^i(y) \in Q$ and $\mathcal{P}^{p+i}(z) \in Q$. The cylinders $\mathcal{D} = \mathcal{P}^{p+i}(\mathcal{D}_j)$ and $\mathcal{C} = \mathcal{P}^i(\mathcal{C}_{m'})$ belong to the same rectangle R_t, so $\mathcal{D} \subset W^u_{R_t}(x)$ and $\mathcal{C} \subset W^u_{R_t}(x')$ for some $x, x' \in S_t$. The assumptions imply $\pi_{x'}(\mathcal{D}) \subset \mathcal{C}$. Set $\mathcal{D}' = \pi_x(\mathcal{C})$. Then $\mathcal{D} \subset \mathcal{D}'$, so $\mathcal{D}' \cap Q \neq \emptyset$. Moreover, $\mathcal{C} \cap Q \neq \emptyset$ and $\pi_{x'}(\mathcal{D}') = \mathcal{C}$. It now follows from Lemma 5.4 that $\mathrm{diam}(\Psi(\mathcal{D})) \leq \mathrm{diam}(\Psi(\mathcal{D}')) \leq A\,\mathrm{diam}(\Psi(\mathcal{C}))$.

On the other hand, by (2.1), $\mathrm{diam}(\Psi(\mathcal{D})) \geq c_0\gamma^{p+i}\mathrm{diam}(\Psi(\mathcal{D}_j)) \geq c_0\gamma^{p+i}\frac{\hat{\rho}^{q_1}\epsilon_1}{|b|}$, by Lemma 4.4, and also $\mathrm{diam}(\Psi(\mathcal{C})) \leq \frac{\gamma_1^i}{c_0}\mathrm{diam}(\Psi(\mathcal{C}_{m'})) \leq \frac{\gamma_1^i\epsilon_1}{c_0|b|}$. Thus, $c_0\gamma^{p+i}\dfrac{\hat{\rho}^{q_1}\epsilon_1}{|b|} \leq A\,\dfrac{\gamma_1^i\epsilon_1}{c_0|b|}$, i.e. $\gamma^p(\gamma/\gamma_1)^i \leq \dfrac{A}{c_0^2\hat{\rho}^{q_1}}$. Since $i < n = [p\hat{\delta}_1]$, we have $i \leq p\hat{\delta}_1$, so $\gamma^p(\gamma/\gamma_1)^{p\hat{\delta}_1} \leq \dfrac{A}{c_0^2\hat{\rho}^{q_1}}$, i.e. $(\gamma(\gamma/\gamma_1)^{\hat{\delta}_1})^p \leq \dfrac{A}{c_0^2\hat{\rho}^{q_1}}$. This and (6.1) imply $\gamma^{p/2} \leq \frac{A}{c_0^2\hat{\rho}^{q_1}}$, so $p \leq \frac{2}{\log\gamma}\log(\frac{A}{c_0^2\hat{\rho}^{q_1}})$. However this is a contradiction with $p > d_0$ and (6.3). So, this case with the assumption $p > 0$ is impossible.

Case 2. The length i of $\mathcal{C}_{m'}$ is $< n$. Then $\mathcal{C} = \mathcal{P}^i(\mathcal{C}_{m'}) = W^u_{R_t}(x')$ and $\mathcal{D} = \mathcal{P}^{p+i}(\mathcal{D}_j) \subset W^u_{R_t}(x)$ for some rectangle R_t and some $x, x' \in S_t$. By the choice of the constant A, we have $\mathrm{diam}(\Psi(\mathcal{D})) \leq \mathrm{diam}(\Psi(W^u_{R_t}(x))) \leq A\,\mathrm{diam}(\Psi(W^u_{R_t}(x'))) = \mathrm{diam}(\Psi(\mathcal{C}))$, and we can now proceed as in Case 1.

Thus, we must have $p = 0$. From this $m' = m$ follows immediately. ■

Given $u, u' \in \widehat{U}$, we will denote by $\ell(u,u') \geq 0$ *the length of the smallest cylinder* $Y(u,u')$ *containing* u and u'.

Define the *'distance'* $\mathcal{D}(u,u')$ for $u, u' \in \widehat{U}$ by[15]: (i) $\mathcal{D}(u,u') = 0$ if $u = u'$; (ii) Let $u \neq u'$, and assume there exists $p \geq 0$ with $\sigma^p(Y(u,u')) \subset \mathcal{C}'_m$, $\ell(u,u') \geq p$, for some $m \leq m_0$. Take the maximal p with this property and the corresponding m and set $\mathcal{D}(u,u') = \frac{D_\theta(u,u')}{\mathrm{diam}_\theta(\mathcal{C}_m)}$; (iii) Assume $u \neq u'$, however there is no $p \geq 0$ with the property described in (ii). Then set $\mathcal{D}(u,u') = \theta^{-N}$. (The latter case is included for completeness – we do not use it below.)

Notice that $\mathcal{D}(u,u') \leq 1$ always. Some other properties of $\mathcal{D}$ are contained in the following, part (b) of which needs Lemma 6.1.

Lemma 6.2. *Assume that* $u, u' \in \widehat{U}$, $u \neq u'$, *and* $\sigma^N(v) = u$, $\sigma^N(v') = u'$ *for some* $v, v' \in \widehat{U}$ *with* $\ell(v,v') \geq N$. *Assume also that there exists* $p \geq 0$ *with* $\sigma^p(Y(u,u')) \subset \mathcal{C}'_m$, $\ell(u,u') \geq p$, *for some* $m \leq m_0$. *Then:*

(a) $\mathcal{D}(v,v') = \theta^N\,\mathcal{D}(u,u')$.

(b) *Assume in addition that* $\omega_J(v) < 1$ *and* $\omega_J(v') = 1$ *for some* $J \in \mathsf{J}(\mathsf{b})$. *Then* $p = 0$ *and* $|\omega_J(v) - \omega_J(v')| \leq \frac{\mu_0}{\theta^{q_1}}\,\mathcal{D}(u,u')$.

[15] Clearly $\mathcal{D}$ depends on the cylinders $\mathcal{C}_m$ and therefore on the parameter b as well.

Proof. (a) Let p be the maximal integer with the given property and let $m \le m_0$ correspond to p. Then $\sigma^{p+N}(Y(v,v')) \subset \mathcal{C}'_m$, $\ell(v,v') \ge p+N$, and $p+N$ is the maximal integer with this property. Thus, $\mathcal{D}(v,v') = \frac{D_\theta(v,v')}{\mathrm{diam}_\theta(\mathcal{C}_m)} = \theta^N \frac{D_\theta(u,u')}{\mathrm{diam}_\theta(\mathcal{C}_m)} = \theta^N \mathcal{D}(u,u')$.

(b) $\omega_J(v) < 1$ means that $v \in X^{(\ell)}_{i,j}$ for some $(i,j,\ell) \in J$, and so $v = v_i^{(l)}(u)$ for some $u \in \mathcal{D}'_j$. Then $u = \sigma^N(v)$. If $u' \in \mathcal{D}'_j$, then $v'' = v_i^{(l)}(u') \in X^{(\ell)}_{i,j}$ and $\sigma^N(v'') = u'$, so we must have $v'' = v'$, which implies $\omega_J(v') = \omega_J(v_i^{(l)}(u')) = 1$, a contradiction. This shows that $u' \notin \mathcal{D}'_j$, and so $D_\theta(u,u') \ge \mathrm{diam}_\theta(\mathcal{D}'_j) = \theta^{q_1} \mathrm{diam}_\theta(\mathcal{C}_m)$.

Since $j \le j_0$ (by $J \subset \Xi(b)$ and the definition of $\Xi(b)$), we have $\mathcal{D}_j \subset \mathcal{C}_{m'}$ for some $m' \le m_0$. We will now show that $m' = m$ and $p = 0$.

Since $u \in \mathcal{D}'_j$, $u' \notin \mathcal{D}'_j$ and $\ell(u,u') \ge p$, it follows that $\sigma^p(u) \in \sigma^p(\mathcal{D}'_j)$ and $\sigma^p(u') \notin \sigma^p(\mathcal{D}'_j)$. On the hand, by assumption, $\sigma^p(u), \sigma^p(u') \in \mathcal{C}'_m$. Thus, the cylinder $\sigma^p(\mathcal{D}'_j)$ must be contained in $\mathcal{C}'_m$. Since $j \le j_0$ and $m \le m_0$, Lemma 6.1 gives $p = 0$ and $m' = m$. In particular, $\mathcal{D}(u,u') = \frac{D_\theta(u,u')}{\mathrm{diam}_\theta(\mathcal{C}_m)}$. We now get $|\omega_J(v) - \omega_J(v')| = \mu_0 = \mu_0 \frac{D_\theta(u,u')}{D_\theta(u,u')} \le \mu_0 \frac{D_\theta(u,u')}{\theta^{q_1}\mathrm{diam}_\theta(\mathcal{C}_m)} = \frac{\mu_0}{\theta^{q_1}} \mathcal{D}(u,u')$. This proves the lemma. ∎

Given $E > 0$ as in Sect. 5.2, let $\mathcal{K}_E$ be *the set of all functions* $H \in \mathcal{F}_\theta(\widehat{U})$ *such that* $H > 0$ on $\widehat{U}$ and $\frac{|H(u)-H(u')|}{H(u')} \le E\, \mathcal{D}(u,u')$ for all $u,u' \in \widehat{U}$ for which there exists an integer $p \ge 0$ with $\sigma^p(Y(u,u')) \subset \mathcal{C}_m$ for some $m \le m_0$ and $\ell(u,u') \ge p$.

Using Lemmas 6.1 and 6.2 we will now prove the main lemma in this section.

Lemma 6.3. *For any* $J \in \mathsf{J}(\mathsf{b})$ *we have* $\mathcal{N}_J(\mathcal{K}_E) \subset \mathcal{K}_E$.

Proof. Let $J \in \mathsf{J}(\mathsf{b})$ and $H \in \mathcal{K}_E$. Set $\mathcal{N} = \mathcal{N}_J$ and $\omega = \omega_J$. We will show that $\mathcal{N}_J H \in \mathcal{K}_E$.

Let $u,u' \in \widehat{U}$ be such that there exists an integer $p \ge 0$ with $\sigma^p(Y(u,u')) \subset \mathcal{C}_m$ for some $m \le m_0$ and $\ell(u,u') \ge p$.

Given $v \in \widehat{U}$ with $\sigma^N(v) = u$, let $C[\imath] = C[i_0,\ldots,i_N]$ be the cylinder of length N containing v. Set $\widehat{C}[\imath] = C[\imath] \cap \widehat{U}$. Then $\sigma^N(\widehat{C}[\imath]) = \widehat{U}_i$. Moreover, $\sigma^N : \widehat{C}[\imath] \longrightarrow \widehat{U}_i$ is a homeomorphism, so there exists a unique $v' = v'(v) \in \widehat{C}[\imath]$ such that $\sigma^N(v') = u'$. Then $D_\theta(\sigma^j(v), \sigma^j(v'(v))) = \theta^{N-j} D_\theta(u,u')$ for all $j = 0,1,\ldots,N-1$. Also $D_\theta(v,v'(v)) = \theta^N D_\theta(u,u')$ and $\mathcal{D}(v,v'(v)) = \theta^N \mathcal{D}(u,u')$. Using (5.3), we get

$$
\begin{aligned}
|f^{(a)}_N(v) - f^{(a)}_N(v')| &\le \sum_{j=0}^{N-1} |f^{(a)}(\sigma^j(v)) - f^{(a)}(\sigma^j(v'))| \le \sum_{j=0}^{N-1} \mathrm{Lip}_\theta(f^{(a)})\, \theta^{N-j}\, D_\theta(u,u') \\
&\le \frac{T}{1-\theta} D_\theta(u,u').
\end{aligned}
$$

Using the above and the definition of $\mathcal{N} = \mathcal{N}_J$, and setting $v' = v'(v)$ for brevity, we get

$$
\begin{aligned}
&\frac{|(\mathcal{N}H)(u) - (\mathcal{N}H)(u')|}{\mathcal{N}H(u')} = \frac{|\sum_{\sigma^N v = u} e^{f^{(a)}_N(v)}\, \omega(v) H(v) - \sum_{\sigma^N v = u} e^{f^{(a)}_N(v'(v))}\, \omega(v'(v)) H(v'(v))|}{\mathcal{N}H(u')} \\
\le\;& \frac{|\sum_{\sigma^N v = u} e^{f^{(a)}_N(v)} [\omega(v)H(v) - \omega(v')H(v')]|}{\mathcal{N}H(u')} + \frac{\sum_{\sigma^N v = u} |e^{f^{(a)}_N(v)} - e^{f^{(a)}_N(v')}|\, \omega(v')H(v')}{\mathcal{N}H(u')} \\
\le\;& \frac{\sum_{\sigma^N v = u} e^{f^{(a)}_N(v) - f^{(a)}_N(v')} e^{f^{(a)}_N(v')}\, |\omega(v) - \omega(v')| H(v')}{\mathcal{N}H(u')} + \frac{\sum_{\sigma^N v = u} e^{f^{(a)}_N(v)}\, \omega(v) |H(v) - H(v')|}{\mathcal{N}H(u')} \\
&+ \frac{\sum_{\sigma^N v = u} |e^{f^{(a)}_N(v) - f^{(a)}_N(v')} - 1|\, e^{f^{(a)}_N(v')} \omega(v') H(v')}{\mathcal{N}H(u')}.
\end{aligned}
$$

By the definition of ω, either $\omega(v) = \omega(v')$ or one of these numbers is < 1 and the other is 1. Lemma 6.2 then implies $|\omega(v) - \omega(v')| \leq \frac{\mu_0}{\theta^{q_1}} \mathcal{D}(u,u')$. Apart from that $H \in \mathcal{K}_E$ implies $|H(v) - H(v')| \leq E H(v') \mathcal{D}(v,v') = E H(v') \theta^N \mathcal{D}(u,u')$, while $\left| e^{f_N^{(a)}(v) - f_N^{(a)}(v')} - 1 \right| \leq e^{T/(1-\theta)} \frac{T}{1-\theta} D_\theta(u,u')$. Thus,

$$
\begin{aligned}
\frac{|(\mathcal{N}H)(u) - (\mathcal{N}H)(u')|}{\mathcal{N}H(u')} &\leq e^{T/(1-\theta)} \frac{\mu_0}{\theta^{q_1}} \frac{\sum_{\sigma^N v = u} e^{f_N^{(a)}(v')} \mathcal{D}(u,u') H(v')}{\mathcal{N}H(u')} \\
&+ \frac{\sum_{\sigma^N v = u} e^{f_N^{(a)}(v) - f_N^{(a)}(v')} e^{f_N^{(a)}(v')} 2\omega(v') E H(v') \theta^N \mathcal{D}(u,u')}{\mathcal{N}H(u')} + e^{T/(1-\theta)} \frac{T}{1-\theta} D_\theta(u,u') \\
&\leq 2e^{T/(1-\theta)} \frac{\mu_0}{\theta^{q_1}} \mathcal{D}(u,u') + 2e^{T/(1-\theta)} E \theta^N \mathcal{D}(u,u') + e^{T/(1-\theta)} \mathrm{diam}_\theta(\mathcal{C}_m) \frac{T}{(1-\theta)} \mathcal{D}(u,u') \\
&\leq E\, \mathcal{D}(u,u'),
\end{aligned}
$$

using (6.6) and Lemma 5.1, and assuming $2e^{T/(1-\theta)}\theta^N \leq 1/3$ and $e^{T/(1-\theta)} C_2(\epsilon_1/|b|)^{\beta'} \frac{T}{(1-\theta)} \leq \frac{1}{3} \leq \frac{E}{3}$; the latter follows from $|b| \geq b_0$ and (6.7). Hence $\mathcal{N}H \in \mathcal{K}_E$. ■

6.4 Main properties of the operators L_{ab}^N

Here we assume $f \in \mathcal{F}_{\theta_1}(\widehat{U})$ for some θ_1 with $0 < (\theta_1)^{\alpha_1} \leq \min\{\theta, 1/\gamma_1\}$. Then using the proof of Lemma 5.1(c) and taking $C_2 > 0$ sufficiently large we have

$$\mathrm{diam}_{\theta_1}(\mathcal{C}) \leq C_2\, \mathrm{diam}(\Psi(\mathcal{C})) \tag{6.12}$$

for any cylinder $\mathcal{C}$ in $U_1(\hat{z}_0)$. Set

$$\theta_2 = \max\{\theta, 1/\gamma^{\alpha_1 \beta_1}\},$$

where $\beta_1 > 0$ is the constant from Lemma 4.2, and as before, $\alpha_1 > 0$ is chosen so that the local stable/unstable holonomy maps are uniformly α_1-Hölder.

Given points $u, u' \in U_1$ we will denote $\tilde{u} = \Psi(\pi_{\hat{z}_0}(u))$ and $\tilde{u}' = \Psi(\pi_{\hat{z}_0}(u'))$; these are then points on the true unstable manifold $W^u_{\epsilon_0}(\hat{z}_0)$. In this section we will frequently work under the following assumption for points $u, u' \in \widehat{U}_1$ contained in some cylinder $\mathcal{C}$ in U_1, an integer $p \geq 0$ and points $v, v' \in \widehat{U}_1$:

$$u, u' \in \mathcal{C}\,,\ \mathcal{C} \cap K_0 \neq \emptyset\,,\ \sigma^p(v) = v_i^{(l)}(u)\,,\ \sigma^p(v') = v_i^{(l)}(u')\,,\ \ell(v,v') \geq N, \tag{6.13}$$

for some $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$. From (6.13) we get $\ell(v,v') \geq p + N$ and $\sigma^{p+N}(v) = u$, $\sigma^{p+N}(v') = u'$. With the cylinder $\mathcal{C}$ we will use the notation $\tilde{\mathcal{C}} = \Psi(\pi_{\hat{z}_0}(\mathcal{C})) \subset W^u_{\epsilon_0}(\hat{z}_0)$. Notice that $C \cap K_0 \neq \emptyset$ implies $\pi_{\hat{z}_0}(\mathcal{C}) \cap K \neq \emptyset$, so in particular $\pi_{\hat{z}_0}(\mathcal{C}) \cap P_4 \neq \emptyset$.

The following estimate plays a central role in this section.

Lemma 6.4. *There exists a global constant $C_3 > 0$ independent of b and N such that if the points $u, u' \in \widehat{U}_1$, the cylinder $\mathcal{C}$ in U_1, the integer $p \geq 0$ and the points $v, v' \in \widehat{U}_1$ satisfy* (6.13) *for some $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$, and $w, w' \in \widehat{U}$ are such that $\sigma^N w = v$, $\sigma^N w' = v'$ and $\ell(w,w') \geq N$, then $|\tau_N(w) - \tau_N(w')| \leq C_3\, \theta_2^{p+N}\, \mathrm{diam}(\tilde{\mathcal{C}})$.*

Proof. Assume that the points u, u', v, v', w, w' and the cylinder $\mathcal{C}$ satisfy the assumptions in the lemma. Clearly, $\ell(w,w') \geq p + 2N$ and

$$\tau_N(w) - \tau_N(w') = [\tau_{p+2N}(w) - \tau_{p+2N}(w')] - [\tau_{p+N}(v) - \tau_{p+N}(v')]. \tag{6.14}$$

Recall the construction of the map $v_i^{(l)}$ from the proof of Lemma 5.5. In particular by (5.9), $\mathcal{P}^N(v^{(l)}(u)) = \pi_{d_i^{(\ell)}}(u)$, where we set $d_i^{(\ell)} = d_i^{(\ell)}(\hat{z}_0) \in W^s_{R_1}(\hat{z}_0)$ for brevity. Since $\sigma^p(v) = v_i^{(l)}(u)$ and $\sigma^p(v') = v_i^{(l)}(u')$, we have $\sigma^{p+N}(v) = u$ and $\sigma^{p+N}(v') = u'$, so $\mathcal{P}^{p+N}(v), \mathcal{P}^{p+N}(v') \in W^u_{R_1}(d')$ for some $d' \in W^s_{R_1}(\hat{z}_0)$. Moreover, $\mathcal{P}^p(v) \in W^s_{R_1}(v_i^{(l)}(u))$ and the choice of N imply (as in the proof of Lemma 5.5) that $d(d_i^{(\ell)}, d') < \delta''$, the constant from Lemma 4.3. Similarly, $\mathcal{P}^{p+2N}(w), \mathcal{P}^{p+2N}(w') \in W^u_{R_1}(d'')$ for some $d'' \in W^s_{R_1}(\hat{z}_0)$ with $d(d_i^{(\ell)}, d'') < \delta''$. Moreover, since the local stable/unstable holonomy maps are uniformly α_1-Hölder (by the choice of α_1), there exists a global constant $C_3' > 0$ such that $d(d', d'') \leq C_3'(d(\mathcal{P}^{p+N}(v), \mathcal{P}^{p+2N}(w)))^{\alpha_1}$. Using this and (2.1) for points on local stable manifolds, i.e. going backwards along the flow, we get

$$d(d', d'') \leq C_3'(d(\mathcal{P}^{p+N}(v), \mathcal{P}^{p+2N}(w)))^{\alpha_1} \leq C_3'(d(v, \mathcal{P}^N(w))/(c_0\gamma^{p+N}))^{\alpha_1} \leq C_3'\gamma^{-\alpha_1(p+N)}/c_0^{\alpha_1}.$$

Hence $(d(d', d''))^{\beta_1} \leq (C_3'/c_0^{\alpha_1})^{\beta_1}(1/\gamma^{p+N})^{\alpha_1\beta_1} \leq C_3''\theta_2^{p+N}$.

We are preparing to use Lemma 4.2. Set $\hat{u} = \pi_{\hat{z}_0}(u)$ and $\hat{u}' = \pi_{\hat{z}_0}(u')$. Then for $\Psi(\hat{u}) = \tilde{u}$ and $\Psi(\hat{u}') = \tilde{u}'$ we have $\tilde{u} = \phi_{t(u)}(\hat{u})$ and $\tilde{u}' = \phi_{t(u')}(\hat{u}')$ for some $t(u), t(u') \in \mathbb{R}$. So

$$\tau_{p+N}(v) - \tau_{p+N}(v') = \Delta(\mathcal{P}^{p+N}(v), \mathcal{P}^{p+N}(v')) = \Delta(\hat{u}, \pi_{d'}(\hat{u}')) = \Delta(\tilde{u}, \pi_{d'}(\tilde{u}')) + t(u) - t(u'),$$

and similarly

$$\tau_{p+2N}(w) - \tau_{p+2N}(w') = \Delta(\mathcal{P}^{p+2N}(w), \mathcal{P}^{p+2N}(w')) = \Delta(\hat{u}, \pi_{d''}(\hat{u}')) = \Delta(\tilde{u}, \pi_{d''}(\tilde{u}')) + t(u) - t(u').$$

This, (6.14), Lemma 4.2 and the above estimate yield

$$\begin{aligned} |\tau_N(w) - \tau_N(w')| &= |[\Delta(\tilde{u}, \pi_{d'}(\tilde{u}')) - t(u) + t(u')] - [\Delta(\tilde{u}, \pi_{d''}(\tilde{u}')) - t(u) + t(u')]| \\ &= |\Delta(\tilde{u}, \pi_{d'}(\tilde{u}')) - \Delta(\tilde{u}, \pi_{d''}(\tilde{u}'))| \leq C_1 \mathrm{diam}(\tilde{\mathcal{C}})\, (d(d', d''))^{\beta_1} \leq C_3\theta_2^{p+N}\, \mathrm{diam}(\tilde{\mathcal{C}}). \end{aligned}$$

This proves the lemma. ■

Let $P_0 > 0$ be a fixed constant (it is enough to take $P_0 = P + a_0$) and let $E_1 = 2C_4'' e^{C_4''}$, where $C_4'' = \frac{T_0}{1-\theta} + P_0 C_3$, and $C_3 > 0$ is the constant from Lemma 6.4. Assume N is so large that $\theta_2^N e^{C_4''} \leq 1/2$.

Denote by $\mathcal{K}_0$ *the set of all* $h \in \mathcal{F}_\theta(\widehat{U})$ *such that* $h \geq 0$ *on* $\widehat{U}$ *and for any* $u, u' \in \widehat{U}_1$ *contained in some cylinder* $\mathcal{C}$ *in* U_1, *any integer* $p \geq 0$ *and any points* $v, v' \in \widehat{U}_1$ *satisfying* (6.13) *for some* $i = 1, 2$ *and* $\ell = 1, \ldots, \ell_0$ *we have* $|h(v) - h(v')| \leq E_1\, \theta_2^{p+N}\, h(v')\, \mathrm{diam}(\tilde{\mathcal{C}})$.

We are going to show that the eigenfunctions $h_a \in \mathcal{K}_0$ for $|a| \leq a_0'$ (see Sect. 5.1). This will be derived from the following.

Lemma 6.5. *For any real constant* s *with* $|s| \leq P_0$ *we have* $L^N_{f-s\tau}(\mathcal{K}_0) \subset \mathcal{K}_0$.

Proof. We will use Lemma 6.4 and some standard estimates.

Assume that $u, u' \in \widehat{U}_1$, the cylinder $\mathcal{C}$ in U_1, the integer $p \geq 0$ and the points $v, v' \in \widehat{U}_1$ satisfy (6.13) for some $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$, and $w, w' \in \widehat{U}$ are such that $\sigma^N w = v$, $\sigma^N w' = v'$ and $\ell(w, w') \geq N$; then $w' = w'(w)$ is uniquely determined by w.

Using $f \in \mathcal{F}_{\theta_1}(\widehat{U})$, the choice of θ_1 and (6.12), we get

$$|f_N(w) - f_N(w')| \leq \frac{T_0}{1-\theta_1} D_{\theta_1}(v, v') \leq \frac{T_0}{1-\theta_1}\theta_1^{p+N}\, \mathrm{diam}_{\theta_1}(\mathcal{C}) \leq C_4'\theta_2^{p+N}\, \mathrm{diam}(\tilde{\mathcal{C}}),$$

where $T_0 = |f|_{\theta_1}$ and $C_4' = C_2 T_0/(1-\theta_1)$. This and Lemma 6.4 imply

$$|(f - s\tau)_N(w) - (f - s\tau)_N(w')| \leq C_4''\theta_2^{p+N}\, \mathrm{diam}(\tilde{\mathcal{C}})$$

for all $s \in \mathbb{R}$ with $|s| \leq P_0$, where $C_4'' > 0$ is as above.

Thus, given s with $|s| \leq P_0$ and $h \in \mathcal{K}_0$ we have:

$$|(L^N_{f-s\tau}h)(v) - (L^N_{f-s\tau}h)(v')| = |\sum_{\sigma^N w = v} e^{(f-s\tau)_N(w)}\, h(w) - \sum_{\sigma^N w=v} e^{(f-s\tau)_N(w'(w))}\, h(w'(w))|$$

$$\begin{aligned}
&\leq |\sum_{\sigma^N w=v} e^{(f-s\tau)_N(w)}\, [h(w) - h(w')]| + \sum_{\sigma^N w=v} |e^{(f-s\tau)_N(w)} - e^{(f-s\tau)_N(w')}|\, h(w') \\
&\leq \sum_{\sigma^N w=v} e^{(f-s\tau)_N(w)-(f-s\tau)_N(w')} e^{(f-s\tau)_N(w')}\, E_1 \theta_2^{p+2N} \mathrm{diam}(\tilde{\mathcal{C}})\, h(w') \\
&\quad + \sum_{\sigma^N w=v} \left| e^{(f-s\tau)_N(w)-(f-s\tau)_N(w')} - 1 \right| e^{(f-s\tau)_N(w')} h(w') \\
&\leq E_1 \theta_2^{p+2N} \mathrm{diam}(\tilde{\mathcal{C}})\, e^{C_4''}\, (L^N_{f-s\tau}h)(v') + e^{C_4''}\, C_4'' \theta_2^{p+N}\, \mathrm{diam}(\tilde{\mathcal{C}})\, (L^N_{f-s\tau}h)(v') \\
&\leq E_1\, \theta_2^{p+N}\, \mathrm{diam}(\tilde{\mathcal{C}})\, (L^N_{f-s\tau}h)(v'),
\end{aligned}$$

since $e^{C_4''}\, C_4'' \leq E_1/2$ and $\theta_2^N\, e^{C_4''} \leq 1/2$ by (5.6). Hence $L^N_{f-s\tau}h \in \mathcal{K}_0$. ■

Corollary 6.6. *For any real constant a with $|a| \leq a_0$ we have $h_a \in \mathcal{K}_0$.*

Proof. Let $|a| \leq a_0$. Since the constant function $h = 1 \in \mathcal{K}_0$, it follows from Lemma 6.5 that $L^{mN}_{f-(P+a)\tau} 1 \in \mathcal{K}_0$ for all $m \geq 0$. Now the statement follows from the Ruelle-Perron-Frobenius Theorem (see e.g. [PP]) and the fact that $\mathcal{K}_0$ is closed in $\mathcal{F}_\theta(\widehat{U})$. ■

6.5 The main estimate for L^N_{ab}

We will now define a class of pairs of functions similar to $\mathcal{K}_0$ however involving the parameter b.

Denote by $\mathcal{K}_b$ *the set of all pairs (h, H) such that $h \in \mathcal{F}_\theta(\widehat{U})$, $H \in \mathcal{K}_E$ and the following two properties hold:*

(i) $|h| \leq H$ on $\widehat{U}$,

(ii) for any $u, u' \in \widehat{U}_1$ contained in some cylinder $\mathcal{C}$ in U_1 with $\mathcal{C} \cap K_0 \neq \emptyset$, any integer $p \geq 0$ and any points $v, v' \in \widehat{U}_1$ satisfying (6.13) for some $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$ we have

$$|h(v) - h(v')| \leq E\, |b|\, \theta_2^{p+N}\, H(v')\, \mathrm{diam}(\tilde{\mathcal{C}}). \tag{6.15}$$

Recall that here $\tilde{\mathcal{C}} = \Psi(\pi_{\hat{z}_0}(\mathcal{C}))$.

Our aim in this section is to prove the following.

Lemma 6.7. *Choosing $E > 1$ and μ_0 as in Sect. 5.2 and assuming N is sufficiently large, for any $|a| \leq a_0'$, any $|b| \geq b_0$ and any $(h, H) \in \mathcal{K}_b$, there exists a representative set $J \in \mathsf{J}(\mathsf{b})$ such that $(L^N_{ab}h, \mathcal{N}_J H) \in \mathcal{K}_b$.*

To prove this we need the following lemma, whose proof is essentially the same as that of Lemma 14 in [D].

Lemma 6.8. *If $h \in \mathcal{F}_\theta(\widehat{U})$ and $H \in \mathcal{K}_E$ satisfy* (i), (ii), *then for any $j = 1, \ldots, j_0$, $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$ we have:*

(a) $\dfrac{1}{2} \leq \dfrac{H(v_i^{(l)}(u'))}{H(v_i^{(l)}(u''))} \leq 2$ *for all $u', u'' \in \widehat{\mathcal{D}}_j'$;*

(b) *Either for all* $u \in \widehat{\mathcal{D}}'_j$ *we have* $|h(v_i^{(l)}(u))| \le \frac{3}{4} H(v_i^{(l)}(u))$, *or* $|h(v_i^{(l)}(u))| \ge \frac{1}{4} H(v_i^{(l)}(u))$ *for all* $u \in \widehat{\mathcal{D}}'_j$.

Proof of Lemma 6.7. The constant $E_1 > 1$ from Sect. 6.4 depends only on C''_4, and we take N so large that $E_1 \theta^N \le 1/4$; then $C''_4 \theta^N \le 1/2$ holds, too.

Let $|a| \le a'_0$, $|b| \ge b_0$ and $(h, H) \in \mathcal{K}_b$. We will construct a representative set $J \in \mathsf{J}(\mathsf{b})$ such that $(L^N_{ab} h, \mathcal{N}_J H) \in \mathcal{K}_b$.

Consider for a moment an arbitrary (at this stage) representative set J. We will first show that $(L^N_{ab} h, \mathcal{N}_J H)$ has property (ii).

Assume that the points u, u', the cylinder $\mathcal{C}$ in U_1, the integer $p \ge 0$ and the points $v, v' \in \widehat{U}_1$ satisfy (6.13) for some $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$.

For any w, w' with $\sigma^N w = v$, $\sigma^N(w') = v'$ and $\ell(w, w') \ge N$ we have

$$\begin{aligned} f^{(a)}_N(w) &= f_N(w) - (P+a)\tau_N(w) + (\ln h_a - \ln h_a \circ \sigma)_N(w) - N\lambda_a \\ &= f_N(w) - (P+a)\tau_N(w) + \ln h_a(w) - \ln h_a(v) - N\lambda_a. \end{aligned}$$

Since $h_a \in \mathcal{K}_0$ by Corollary 6.6,

$$|\ln h_a(w) - \ln h_a(w')| \le \frac{|h_a(w) - h_a(w')|}{\min\{|h_a(w)|, |h_a(w')|\}} \le E_1\, \theta_2^{p+2N} \operatorname{diam}(\tilde{\mathcal{C}}),$$

and similarly, $|\ln h_a(v) - \ln h_a(v')| \le E_1\, \theta_2^{p+N} \operatorname{diam}(\tilde{\mathcal{C}})$. Using this and Lemma 6.4, as in the proof of Lemma 6.5 we get

$$|f^{(a)}_N(w) - f^{(a)}_N(w')| \le C''_4 \theta_2^{p+2N} \operatorname{diam}(\tilde{\mathcal{C}}) + 2E_1\, \theta_2^{p+N} \operatorname{diam}(\tilde{\mathcal{C}}) \le (C''_4 + 2E_1)\, \theta_2^{p+N} \operatorname{diam}(\tilde{\mathcal{C}}).$$

By the choice of N, this implies, $|f^{(a)}_N(w) - f^{(a)}_N(w')| \le 1$.

Hence for any a and b with $|a| \le a'_0$ and $|b| \ge b_0$ we have:

$$\begin{aligned} &|(L^N_{ab} h)(v) - (L^N_{ab} h)(v')| = \Big| \sum_{\sigma^N w = v} e^{(f^{(a)}_N - \mathbf{i} b \tau_N)(w)}\, h(w) - \sum_{\sigma^N w = v} e^{(f^{(a)}_N - \mathbf{i} b \tau_N)(w'(w))}\, h(w'(w)) \Big| \\ \le\ & \Big| \sum_{\sigma^N w = v} e^{(f^{(a)}_N - \mathbf{i} b \tau_N)(w)} [h(w) - h(w')] \Big| + \sum_{\sigma^N w = v} |e^{(f^{(a)}_N - \mathbf{i} b \tau_N)(w)} - e^{(f^{(a)}_N - \mathbf{i} b \tau_N)(w')}|\, |h(w')| \\ \le\ & \sum_{\sigma^N w = v} e^{(f^{(a)}_N(w) - f^{(a)}_N(w')} e^{f^{(a)}_N(w')}\, E|b|\theta_2^{p+2N} \operatorname{diam}(\tilde{\mathcal{C}})\, H(w') \\ &+ \sum_{\sigma^N w = v} |e^{(f^{(a)}_N - \mathbf{i} b \tau_N)(w) - (f^{(a)}_N - \mathbf{i} b \tau_N)(w')} - 1|\, e^{f^{(a)}_N(w')} H(w') \\ \le\ & e\, E|b|\, \theta_2^{p+2N} \operatorname{diam}(\tilde{\mathcal{C}})\, (\mathcal{M}^N_a H)(v') + e\, (C''_4 + 2E_1 + C_3|b|)\theta_2^{p+N} \operatorname{diam}(\tilde{\mathcal{C}})\, (\mathcal{M}^N_a H)(v') \\ \le\ & [2e\, E\theta_2^N + 2e(C''_4 + 2E_1 + C_3)]\, |b|\, \theta_2^{p+N} \operatorname{diam}(\tilde{\mathcal{C}})\, (\mathcal{N}_J H)(v') \le E|b|\, \theta_2^{p+N} \operatorname{diam}(\tilde{\mathcal{C}})\, (\mathcal{N}_J H)(v'), \end{aligned}$$

assuming $2e\theta^N \le 1/2$ and $2e(C''_4 + 2E_1 + C_3) \le E/2$. Thus, $(L^N_{ab} h, \mathcal{N}_J H)$ has property (ii).

So far the choice of J was not important. We will now construct a representative set J so that $(L^N_{ab} h, \mathcal{N}_J H)$ has property (i), namely

$$|L^N_{ab} h|(u) \le (\mathcal{N}_J H)(u) \tag{6.16}$$

for all $u \in \widehat{U}$.

Define the functions $\psi_\ell, \gamma_\ell^{(1)}, \gamma_\ell^{(2)} : \widehat{U}_1 \longrightarrow \mathbf{C}$ by

$$\psi_\ell(u) = e^{(f_N^{(a)}+\mathbf{i}b\tau_N)(v_1^{(l)}(u))} h(v_1^{(l)}(u)) + e^{(f_N^{(a)}+\mathbf{i}b\tau_N)(v_2^{(l)}(u))} h(v_2^{(l)}(u)),$$

$$\gamma_\ell^{(1)}(u) = (1-\mu_0)\, e^{f_N^{(a)}(v_1^{(l)}(u))} H(v_1^{(l)}(u)) + e^{f_N^{(a)}(v_2^{(l)}(u))} H(v_2^{(l)}(u)),$$

while $\gamma_\ell^{(2)}(u)$ is defined similarly with a coefficient $(1-\mu_0)$ in front of the second term.

Recall the functions $\varphi_\ell(u) = \varphi_\ell(\hat{z}_0, u)$, $u \in U_1$, from Sect. 5.3.

Notice that (6.16) is trivially satisfied for $u \notin V_b$ for any choice of J.

Consider an arbitrary $m = 1, \ldots, m_0$. We will construct $j \leq j_0$ with $\mathcal{D}_j \subset \mathcal{C}_m$, and a pair (i, ℓ) for which (i, j, ℓ) will be included in J.

Case 1. There exist $j \leq j_0$ with $\mathcal{D}_j \subset \mathcal{C}_m$, $i = 1,2$ and $\ell \leq \ell_0$ such that the first alternative in Lemma 6.8(b) holds for $\widehat{\mathcal{D}}_j$, i and ℓ. For such j, choose $i = i_j$ and $\ell = \ell_j$ with this property and include (i, j, ℓ) in J. Then $\mu_0 \leq 1/4$ implies $|\psi_\ell(u)| \leq \gamma_\ell^{(i)}(u)$ for all $u \in \widehat{\mathcal{D}}'_j$, and regardless how the rest of J is defined, (6.16) holds for all $u \in \widehat{\mathcal{D}}'_j$, since

$$\begin{aligned}
\left|(L_{ab}^N h)(u)\right| &\leq \Big|\sum_{\sigma^N v = u,\ v \neq v_1^{(l)}(u), v_2^{(l)}(u)} e^{(f_N^{(a)}+\mathbf{i}b\tau_N)(v)} h(v)\Big| + |\psi_\ell(u)| \\
&\leq \sum_{\sigma^N v = u,\ v \neq v_1^{(l)}(u), v_2^{(l)}(u)} e^{f_N^{(a)}(v)} |h(v)| + \gamma_\ell^{(i)}(u) \leq \sum_{\sigma^N v = u,\ v \neq v_1^{(l)}(u), v_2^{(l)}(u)} e^{f_N^{(a)}(v)} \omega(v) H(v) \\
&\quad + [e^{f_N^{(a)}(v_1^{(l)}(u))} \omega_J(v_1^{(l)}(u)) H(v_1^{(l)}(u)) + e^{f_N^{(a)}(v_2^{(l)}(u))} \omega_J(v_2^{(l)}(u)) H(v_2^{(l)}(u))] \\
&\leq (\mathcal{N}_J H)(u). \qquad (6.17)
\end{aligned}$$

Case 2. For all $j \leq j_0$ with $\mathcal{D}_j \subset \mathcal{C}_m$, $i = 1,2$ and $\ell \leq \ell_0$ the second alternative in Lemma 6.8(b) holds for $\widehat{\mathcal{D}}_j$, i and ℓ, i.e.

$$|h(v_i^{(l)}(u))| \geq \frac{1}{4}\, H(v_i^{(l)}(u)) > 0 \qquad (6.18)$$

for any $u \in \widehat{\mathcal{C}}'_m$.

Let $u, u' \in \widehat{\mathcal{C}}'_m$, and let $i = 1,2$. Using (6.18) and the assumption that $(h, H) \in \mathcal{K}_b$, and in particular property (ii) with $p = 0$, $v = v_i^{(l)}(u)$ and $v' = v_i^{(l)}(u')$, and assuming e.g. $\min\{|h(v_i^{(l)}(u))|, |h(v_i^{(l)}(u'))|\} = |h(v_i^{(l)}(u'))|$, we get

$$\frac{|h(v_i^{(l)}(u)) - h(v_i^{(l)}(u'))|}{\min\{|h(v_i^{(l)}(u))|, |h(v_i^{(l)}(u'))|\}} \leq \frac{E|b|\,\theta_2^N H(v_i^{(l)}(u'))}{|h(v_i^{(l)}(u'))|} \mathrm{diam}(\Psi(\mathcal{C}_m)) \leq 4E|b|\,\theta_2^N \frac{\epsilon_1}{|b|} = 4E\theta_2^N \epsilon_1.$$

So, the angle between the complex numbers $h(v_i^{(l)}(u))$ and $h(v_i^{(l)}(u'))$ (regarded as vectors in $\mathbb{R}^2$) is $< 8E\theta_2^N \epsilon_1 < \pi/6$ by (5.6). In particular, for each $i = 1,2$ we can choose a real continuous function $\theta_i^{(m)}(u)$, $u \in \widehat{\mathcal{C}}'_m$, with values in $[0, \pi/6]$ and a constant $\lambda_i^{(m)}$ such that

$$h(v_i^{(l)}(u)) = e^{\mathbf{i}(\lambda_i^{(m)} + \theta_i^{(m)}(u))} |h(v_i^{(l)}(u))| \quad , \quad u \in \widehat{\mathcal{C}}'_m.$$

Fix an arbitrary $u_0 \in \widehat{\mathcal{C}}'_m$ and set $\lambda^{(m)} = |b|\varphi_\ell(u_0)$. Replacing e.g $\lambda_2^{(m)}$ by $\lambda_2^{(m)} + 2r\pi$ for some integer r, we may assume that $|\lambda_2^{(m)} - \lambda_1^{(m)} + \lambda^{(m)}| \leq \pi$.

Using the above, $\theta \leq 2\sin\theta$ for $\theta \in [0, \pi/3]$, and some elementary geometry yields $|\theta_i^{(m)}(u) - \theta_i^{(m)}(u')| \leq 2\sin|\theta_i^{(m)}(u) - \theta_i^{(m)}(u')| < 16E\theta_2^N \epsilon_1$ for all $u, u' \in \widehat{\mathcal{C}}'_m$.

The difference between the arguments of the complex numbers $e^{\mathbf{i}\, b\, \tau_N(v_1^{(l)}(u))} h(v_1^{(l)}(u))$ and $e^{\mathbf{i}\, b\, \tau_N(v_2^{(l)}(u))} h(v_2^{(l)}(u))$ is given by the function

$$\begin{aligned}\Gamma_\ell(u) &= [b\, \tau_N(v_2^{(l)}(u)) + \theta_2^{(m)}(u) + \lambda_2^{(m)}] - [b\, \tau_N(v_1^{(l)}(u)) + \theta_1^{(m)}(u) + \lambda_1^{(m)}] \\ &= (\lambda_2^{(m)} - \lambda_1^{(m)}) + |b| \varphi_\ell(u) + (\theta_2^{(m)}(u) - \theta_1^{(m)}(u)).\end{aligned}$$

It follows from Lemma 5.5 that there exist $j, j' \leq j_0$ with $j \neq j'$, $\mathcal{D}_j, \mathcal{D}_{j'} \subset \mathcal{C}_m$, and $\ell = 1, \ldots, \ell_0$ such that for all $u \in \widehat{\mathcal{D}}'_j$ and $u' \in \widehat{\mathcal{D}}'_{j'}$ we have

$$\frac{\hat{\delta}_0 \hat{\rho} \epsilon_1}{|b|} \leq \hat{\delta}_0 \operatorname{diam}(\Psi(\mathcal{C}_m)) \leq |\varphi_\ell(u) - \varphi_\ell(u')| \leq C_1 \operatorname{diam}(\Psi(\mathcal{C}_m)) \leq C_1 \frac{\epsilon_1}{|b|}. \tag{6.19}$$

Fix $\ell_m = \ell$ with this property. Then for $u \in \widehat{\mathcal{D}}'_j$ and $u' \in \widehat{\mathcal{D}}'_{j'}$ we have

$$\begin{aligned}|\Gamma_\ell(u) - \Gamma_\ell(u')| &\geq |b|\, |\varphi_\ell(u) - \varphi_\ell(u')| - |\theta_1^{(m)}(u) - \theta_1^{(m)}(u')| - |\theta_2^{(m)}(u) - \theta_2^{(m)}(u')| \\ &\geq \hat{\delta}_0 \hat{\rho} \epsilon_1 - 32 E \theta_2^N \epsilon_1 > 2\epsilon_3,\end{aligned}$$

since $32 E \theta_2^N < \hat{\delta}_0 \hat{\rho}/2$ by (5.6), where $\epsilon_3 = \dfrac{\hat{\delta}_0 \hat{\rho} \epsilon_1}{4}$.

Thus, $|\Gamma_\ell(u) - \Gamma_\ell(u')| \geq 2\epsilon_3$ for all $u \in \widehat{\mathcal{D}}'_j$ and $u' \in \widehat{\mathcal{D}}'_{j'}$. Hence either $|\Gamma_\ell(u)| \geq \epsilon_3$ for all $u \in \widehat{\mathcal{D}}'_j$ or $|\Gamma_\ell(u')| \geq \epsilon_3$ for all $u' \in \widehat{\mathcal{D}}'_{j'}$.

Assume for example that $|\Gamma_\ell(u)| \geq \epsilon_3$ for all $u \in \widehat{\mathcal{D}}'_j$. On the other hand, (6.19) and the choice of ϵ_1 imply that for any $u \in \widehat{\mathcal{C}}'_m$ we have

$$|\Gamma_\ell(u)| \leq |\lambda_2^{(m)} - \lambda_1^{(m)} + \lambda^{(m)}| + |b|\, |\varphi_\ell(u) - \varphi_\ell(u_0)| + |\theta_2^{(m)}(u) - \theta_1^{(m)}(u)| \leq \pi + C_1 \epsilon_1 + 16 E \theta_2^N \epsilon_1 < \frac{3\pi}{2}.$$

Thus, $\epsilon_3 \leq |\Gamma_\ell(u)| < \frac{3\pi}{2}$ for all $u \in \widehat{\mathcal{D}}'_j$.

Hence, we see that for $u \in \widehat{\mathcal{D}}'_j$ the difference $\Gamma_\ell(u)$ between the arguments of the complex numbers $e^{\mathbf{i}\, b\, \tau_N(v_1^{(l)}(u))} h(v_1^{(l)}(u))$ and $e^{\mathbf{i}\, b\, \tau_N(v_2^{(l)}(u))} h(v_2^{(l)}(u))$, defined as a number in the interval $[0, 2\pi)$, satisfies $\Gamma_\ell(u) \geq \epsilon_3$ for all $u \in \widehat{\mathcal{D}}'_j$.

Now as in [D] (and [St1]) we derive[16] that $|\psi_\ell(u)| \leq \gamma_\ell^{(1)}(u)$ for all $u \in \widehat{\mathcal{D}}'_j$. Now set $j_m = j$, $\ell_m = \ell$ and e.g. $i_m = 1$, and include (i_m, j_m, ℓ_m) in J. Then $\mathcal{D}_{j_m} \subset \mathcal{C}_m$ and as in the proof of (6.17) we deduce that (6.16) holds on $\widehat{\mathcal{D}}'_{j_m}$.

This completes the construction of the set $J = \{(i_m, j_m, \ell_m) : m = 1, \ldots, m_0\} \in \mathsf{J}(\mathsf{b})$ and also the proof of (6.16) for all $u \in V_b$. As we mentioned in the beginning of the proof, (6.16) always holds for $u \in \widehat{U} \setminus V_b$. ■

[16] Given such u, consider the points $\rho_1 = e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v_1^{(l)}(u))} h(v_1^{(l)}(u))$ and $\rho_2 = e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v_2^{(l)}(u))} h(v_2^{(l)}(u))$ in the complex plane $\mathbf{C}$, and let φ be the smaller angle between ρ_1 and ρ_2. Then $\epsilon_3 \leq \varphi \leq 3\pi/2$. Moreover, (5.3), (6.18) and Lemma 6.8(a) imply $\frac{|\rho_1|}{|\rho_2|} = e^{f_N^{(a)}(v_1^{(l)}(u)) - f_N^{(a)}(v_2^{(l)}(u))} \frac{|h(v_1^{(l)}(u))|}{|h(v_2^{(l)}(u))|} \leq L = 16\, e^{2T\, N}$. By symmetry $\frac{|\rho_2|}{|\rho_1|} \leq L$. As in [D] (see also [St1]) this yields $|\rho_1 + \rho_2| \leq \min(\ |\rho_1| + (1-t)|\rho_2|\ ,\ (1-t)|\rho_1| + |\rho_2|\ )$, where $t = \frac{2}{L} \sin^2 \frac{\epsilon_3}{4} = \frac{1}{8 e^{2TN}} \sin^2 \left(\frac{\hat{\delta}_0 \hat{\rho} \epsilon_1}{16} \right)$. Since $\mu_0 \leq t$ by (6.6), we have $|\psi_\ell(u)| \leq \gamma_\ell^{(i)}(u)$ for all $u \in \widehat{\mathcal{D}}'_j$ and $i = 1, 2$.

7 Proofs of the Main Results

Here we prove Theorem 1.1 and Corollary 1.2. The main step is to obtain L^1-contraction estimates for large powers of the contraction operators using the properties of these operators on the (small) compact set K_0 with $\nu(K_0) > 0$ and the strong mixing properties of the shift map $\mathcal{P} : R \longrightarrow R$.

Lemma 7.1. *Given the number N, there exist an integer $p_0 \geq 1$ and a constant $C_5' > 0$ such that for any integer $t \geq 1$ and any sequence of integers $i_0 = 0 < i_1 < \ldots < i_t$ we have*

$$\nu\left(\cap_{j=0}^{t} \sigma^{-i_j p_0 N}(U \setminus K_0)\right) \leq C_5' \, \rho^{t+1}, \tag{7.1}$$

where $\rho = 1 - \tilde{\nu}_0 \in (0,1)$, $\tilde{\nu}_0 = \frac{1}{2}\mu((\pi^{(U)})^{-1}(K_0))$.

Proof of Lemma 7.1. We will use the fact that $(\widehat{R}, \mathcal{P}^{-1}, \mu)$ is isomorphic to a Bernoulli shift and is therefore a Kolmogorov automorphism (see e.g. [Ro], [CFS] or [Rud]).

First, let us make the elementary remark, that if $W \subset \widehat{U}$ is ν-measurable, setting

$$\widetilde{W} = \{x \in R : \pi^{(U)}(x) \in W\} = (\pi^{(U)})^{-1}(W),$$

we get a μ-measurable set with $\nu(W) = \mu(\widetilde{W})$ (see Sect. 4.1), and $\mathcal{P}^{-n}(\widetilde{W}) = \widetilde{\sigma^{-n}(W)}$ for all $n \geq 0$, so $\nu(\sigma^{-n}(W)) = \mu(\mathcal{P}^{-n}(\widetilde{W}))$.

Setting $X = \bigcap_{j=0}^{t} \sigma^{-i_j p_0 N}(U \setminus K_0)$ for some t, we have

$$\widetilde{X} = \cap_{j=0}^{t} \left(\sigma^{-i_j p_0 N}(U \setminus K_0)\right)^{\sim} = \cap_{j=0}^{t} \mathcal{P}^{-i_j p_0 N}(\widetilde{U \setminus K_0}),$$

and the above implies $\nu(X) = \mu(\widetilde{X}) = \mu\left(\cap_{j=0}^{t} \mathcal{P}^{-i_j p_0 N}(\widetilde{U \setminus K_0})\right)$, so it is enough to estimate the right-hand-side of the latter.

Consider the partition $\widetilde{\Gamma} = \{\widetilde{K_0}, \widetilde{U \setminus K_0}\}$ of $\widehat{R}$, and let $\epsilon = \tilde{\nu}_0 = \frac{\mu(\widetilde{K_0})}{2}$. It follows from the K-property of $\mathcal{P}^{-1}$ (see e.g. Sect. 4.7 in [Rud]) that there exists an integer $p_0 \geq 1$ such that for any integer $t \geq 2$ and any sequence of t integers $0 = i_1' < i_2' < \ldots < i_t'$ with $i_{j+1}' - i_j' \geq p_0$ for all $j = 0, 1, \ldots, t-1$, and any $B_1, \ldots B_t \in \widetilde{\Gamma}$ we have

$$\left| \frac{\mu(B_1 \cap \mathcal{P}^{-i_2'}(B_2) \cap \ldots \cap \mathcal{P}^{-i_t'}(B_t))}{\mu(\mathcal{P}^{-i_2'}(B_2) \cap \ldots \cap \mathcal{P}^{-i_t'}(B_t))} - \mu(B_1) \right| < \epsilon,$$

so

$$\mu\left(B_1 \cap \mathcal{P}^{-i_2'}(B_2) \cap \ldots \cap \mathcal{P}^{-i_t'}(B_t)\right) \leq (\mu(B_1) + \epsilon)\, \mu\left(\mathcal{P}^{-i_2'}(B_2) \cap \ldots \cap \mathcal{P}^{-i_t'}(B_t)\right).$$

We will use this with $B_j = \widetilde{U \setminus K_0}$ and $i_j' = i_j p_0 N$ for all $j = 0, 1, \ldots, t$. Since $\mu(\widetilde{U \setminus K_0}) + \epsilon = 1 - \mu(\widetilde{K_0}) + \epsilon = 1 - \tilde{\nu}_0 = \rho$, it follows that

$$\begin{aligned} \mu(\cap_{j=0}^{t} \mathcal{P}^{-i_j p_0 N}(\widetilde{U \setminus K_0})) &= \mu(\widetilde{U \setminus K_0} \cap \cap_{j=1}^{t} \mathcal{P}^{-i_j p_0 N}(\widetilde{U \setminus K_0})) \\ &\leq \rho\, \mu(\cap_{j=1}^{t} \mathcal{P}^{-i_j p_0 N}(\widetilde{U \setminus K_0})) = \rho\, \mu(\cap_{i=0}^{t-1} \mathcal{P}^{-(i_{i+1} - i_1) p_0 N}(\widetilde{U \setminus K_0})). \end{aligned}$$

Continuing by induction proves (7.1). ■

For any $J \in \mathsf{J}(\mathsf{b})$ set $W_J = \cup\{\widehat{\mathcal{D}}_j' : (i,j,\ell) \in J \text{ for some } i, \ell\} \subset V_b$. Using Lemma 6.3 and the class of functions $\mathcal{K}_E$ we will now prove the following important estimates[17].

[17] This should be regarded as the analogue of Lemma 12 in [D] (and Lemma 5.8 in [St2]).

Lemma 7.2. (a) *There exists a global constant* $C_5'' > 0$ *such that for any* $H \in \mathcal{K}_E$ *and any* $J \in \mathsf{J}(\mathsf{b})$ *we have*

$$\int_{V_b} H^2\, d\nu \leq C_5'' \int_{W_J} H^2\, d\nu. \tag{7.2}$$

(b) *For any* $H \in \mathcal{K}_E$ *and any* $J \in \mathsf{J}(\mathsf{b})$ *we have*

$$\int_{V_b} (\mathcal{N}_J H)^2\, d\nu \leq \rho_3 \int_{V_b} L^N_{f^{(0)}}(H^2)\, d\nu, \tag{7.3}$$

where $\rho_3 = \rho_3(N) = \dfrac{e^{a_0 NT}}{1 + \frac{\mu_0 e^{-NT}}{C_5''}} < 1$, *assuming that* $a_0 > 0$ *is sufficiently small.*

Proofs. (a) Let $H \in \mathcal{K}_E$ and $J \in \mathsf{J}(\mathsf{b})$. Consider an arbitrary $m = 1, \ldots, m_0$. There exists $(i_m, j_m, \ell_m) \in J$ such that $\mathcal{D}_{j_m} \subset \mathcal{C}_m$. As in the proof of Lemma 5.3 we have $\frac{\nu(\mathcal{D}'_{j_m})}{\nu(\mathcal{C}'_m)} \geq 1 - \omega_0$ by (6.4). Since $H \in \mathcal{K}_E$, for any $u, u' \in \mathcal{C}'_m$ we have $\frac{|H(u) - H(u')|}{H(u')} \leq E\mathcal{D}(u, u') \leq E$, so $H(u)/H(u') \leq 1 + E \leq 2E$. Thus, if $M_1 = \max_{\mathcal{C}'_m} H$ and $M_2 = \min_{\mathcal{C}'_m} H$ we have $M_1/M_2 \leq 2E$. This gives

$$\int_{\mathcal{C}'_m} H^2\, d\nu \leq M_1^2 \nu(\mathcal{C}'_m) \leq \frac{4E^2}{1 - \omega_0} \int_{\mathcal{D}'_{j_m}} H^2\, d\nu.$$

Hence

$$\int_{V_b} H^2\, d\nu \leq \sum_{m=1}^{m_0} \int_{\mathcal{C}'_m} H^2\, d\nu \leq \frac{4E^2}{1 - \omega_0} \sum_{m=1}^{m_0} \int_{\mathcal{D}'_{j_m}} H^2\, d\nu \leq C_5'' \int_{W_J} H^2\, d\nu,$$

with $C_5'' = \frac{4E^2}{1 - \omega_0}$, since $\cup_{m=1}^{m_0} \widehat{\mathcal{D}}'_{j_m} = W_J$. This proves (7.2).

(b) Let again $H \in \mathcal{K}_E$ and $J \in \mathsf{J}(\mathsf{b})$. By Lemma 6.3, $\mathcal{N}_J H \in \mathcal{K}_E$, while the Cauchy-Schwartz inequality implies $(\mathcal{N}_J H)^2 = (\mathcal{M}_a^N \omega H)^2 \leq (\mathcal{M}_a^N \omega_J^2)\, (\mathcal{M}_a^N H^2) \leq (\mathcal{M}_a^N \omega_J)\, (\mathcal{M}_a^N H^2) \leq \mathcal{M}_a^N H^2$. Notice that if $u \notin W_J$, then $\omega_J(u) = 1$. Let $u \in W_J$; then $u \in \widehat{\mathcal{D}}'_j$ for some (unique) $j \leq j_0$, and there exists a unique $(i(j), j, \ell(j)) \in J$. Set $i = i(j)$, $\ell = \ell(j)$ for brevity. Then $v_i^{(l)}(u) \in \widehat{X}_{i,j}^{(\ell)}$, so $\omega_{i,j}^{(\ell)}(v_i^{(l)}(u)) = 1$, and therefore $\omega(v_i^{(\ell)}(u)) \leq 1 - \mu_0\, \omega_{i,j}^{(\ell)}(v_i^{(l)}(u)) = 1 - \mu_0$. In fact, if $\sigma^N(v) = u$ and $\omega(v) < 1$, then $\omega_{i',j'}^{(\ell')}(v) = 1$ for some $(i', j', \ell') \in J$, so $v \in X_{i',j'}^{(\ell')}$. Then $u = \sigma^N(v) \in \sigma^N(X_{i',j'}^{(\ell')}) = \widehat{\mathcal{D}}'_{j'}$. Thus, we must have $j' = j$, and since for a given j, there is only one element $(i(j), j, \ell(j))$ in J, we must have also $i' = i(j)$ and $\ell' = \ell(j)$. Assuming e.g. that $i = 1$, this implies $v = v_1^{(l)}(u)$, and therefore

$$(\mathcal{M}_a^N \omega_J)(u) = \sum_{\sigma^N v = u} e^{f_N^{(a)}(v)} - \mu_0\, e^{f_N^{(a)}(v_1^{(l)}(u))} \leq (\mathcal{M}_a^N\ 1)(u) - \mu_0\, e^{-NT} = 1 - \mu_0\, e^{-NT}.$$

This holds for all $u \in W_J$, so $(\mathcal{N}_J H)^2 \leq (1 - \mu_0 e^{-NT})\, (\mathcal{M}_a^N H^2)$ on W_J. Using part (a) we get:

$$\begin{aligned}
\int_{V_b} (\mathcal{N}_J H)^2\, d\nu &= \int_{V_b \setminus W_J} (\mathcal{N}_J H)^2\, d\nu + \int_{W_J} (\mathcal{N}_J H)^2\, d\nu \\
&\leq \int_{V_b \setminus W_J} (\mathcal{M}_a^N H)^2\, d\nu + (1 - \mu_0 e^{-NT}) \int_{W_J} (\mathcal{M}_a^N H)^2\, d\nu \\
&\leq \int_{V_b} (\mathcal{M}_a^N H)^2\, d\nu - \mu_0 e^{-NT} \int_{W_J} (\mathcal{N}_J H)^2\, d\nu \\
&\leq \int_{V_b} (\mathcal{M}_a^N H)^2\, d\nu - \frac{\mu_0 e^{-NT}}{C_5''} \int_{V_b} (\mathcal{N}_J H)^2\, d\nu.
\end{aligned}$$

Using this and $(\mathcal{M}_a^N H)^2 \leq (\mathcal{M}_a^N 1)^2 (\mathcal{M}_a^N H^2) \leq \mathcal{M}_a^N H^2 = L_{f^{(0)}}^N (e^{f_N^{(a)} - f_N^{(0)}} H^2) \leq e^{a_0 NT} (L_{f^{(0)}}^N H^2)$, we get $(1 + \mu_0 e^{-NT}/C_5'') \int_{V_b} (\mathcal{N}_J H)^2 \, d\nu \leq \int_{V_b} (\mathcal{M}_a^N H)^2 \, d\nu \leq e^{a_0 NT} \int_{V_b} L_{f^{(0)}}^N H^2 \, d\nu$. Thus (7.3) holds with $\rho_3 = \dfrac{e^{a_0 NT}}{1 + \frac{\mu_0 e^{-NT}}{C_5''}} > 0$. Taking $a_0 = a_0(N) > 0$ sufficiently small, we have $\rho_3 < 1$. ■

We can now prove that iterating sufficiently many contraction operators provides an L^1-contraction on U.

Lemma 7.3. *Given the number N, there exist constants $C_5 \geq 1$, $\rho_4 = \rho_4(N) \in (0,1)$, $a_0 = a_0(N) > 0$ and $b_0 = b_0(N) \geq 1$ such that for any $|a| \leq a_0$ and $|b| \geq b_0$ and any sequence $J_1, J_2, \ldots, J_r \ldots$ of representative subsets of $\Xi(b)$, setting $H^{(0)} = 1$ and $H^{(r+1)} = \mathcal{N}_{J_r}(H^{(r)})$ $(r \geq 0)$ we have*

$$\int_U (H^{(pp_0)})^2 \, d\nu \leq C_5 \, \rho_4^p \tag{7.4}$$

for all integers $p \geq 1$.

Proof of Lemma 7.3. Set $\omega_r = \omega_{J_r}$, $W_r = W_{J_r}$ and $\mathcal{N}_r = \mathcal{N}_{J_r}$. Since $H^{(0)} = 1 \in \mathcal{K}_E$, it follows from Lemma 6.3 that $H^{(r)} \in \mathcal{K}_E$ for all $r \geq 1$.

Let $\rho = 1 - \tilde{\nu}_0$ and p_0 be as in Lemma 7.1. Set

$$\rho_3 = \frac{e^{a_0 NT}}{1 + \frac{\mu_0 e^{-NT}}{C_5''}} < 1 \quad , \quad R = e^{a_0 NT} > 1 \quad , \quad h = \rho_3 \, \chi_{V_b} + R \, \chi_{U \setminus V_b},$$

and notice that ρ_3 is as in Lemma 7.2. The latter and $L_{f^{(a)}}^N ((h \circ \sigma^N) \, H) = h \, (L_{f^{(a)}}^N H)$ yield

$$\begin{aligned}
\int_U (H^{(m)})^2 \, d\nu &= \int_{V_b} (H^{(m)})^2 \, d\nu + \int_{U \setminus V_b} (H^{(m)})^2 \, d\nu \\
&\leq \rho_3 \int_{V_b} L_{f^{(0)}}^N (H^{(m-1)})^2 \, d\nu + e^{a_0 NT} \int_{U \setminus V_b} L_{f^{(0)}}^N (H^{(m-1)})^2 \, d\nu \\
&= \int_U h \, (L_{f^{(0)}}^N (H^{(m-1)})^2) \, d\nu = \int_U L_{f^{(0)}}^N ((h \circ \sigma^N) \, (H^{(m-1)})^2) \, d\nu \\
&= \int_U (h \circ \sigma^N) \, (H^{(m-1)})^2 \, d\nu.
\end{aligned}$$

Continuing by induction and using $H^{(0)} = 1$, we get

$$\int_U (H^{(m)})^2 \, d\nu \leq \int_U (h \circ \sigma^{mN}) \, (h \circ \sigma^{(m-1)N}) \ldots (h \circ \sigma^{2N}) \, (h \circ \sigma^N) \, d\nu. \tag{7.5}$$

Fix $\delta \in (0,1)$. We will see later how small δ should be. Fix an arbitrary integer $p \geq 1$ and set $m = [\delta p]$, $m' = p - m$.

Let W be the set of those $x \in U$ such that $x \in \sigma^{-jp_0 N}(U \setminus V_b)$ for at least m' different $j = 0, 1, \ldots, p-1$. Since $K_0 \subset V_b$, for such j we have $x \in \sigma^{-jp_0 N}(U \setminus K_0)$. We can choose m' different numbers $j_1, \ldots, j_{m'} = 0, 1, \ldots, p-1$ in $\begin{pmatrix} p \\ m' \end{pmatrix}$ different ways, so it follows from Lemma 7.1 that

$$\nu(W) \leq \begin{pmatrix} p \\ m' \end{pmatrix} C_5' \, \rho^{m'+1} \leq C_5' \begin{pmatrix} p \\ m \end{pmatrix} \rho^{p - \delta p}. \tag{7.6}$$

Next, using Stirling's formula, and writing $a_p \sim b_p$ whenever $a_p/b_p \to$ const > 0 as $p \to \infty$, we get[18]

$$\begin{aligned}\binom{p}{m} &= \frac{p!}{m!\,(p-m)!} \sim \frac{\sqrt{2\pi p}\,(p/e)^p}{\sqrt{2\pi m}\,(m/e)^m\,\sqrt{2\pi(p-m)}\,((p-m)/e)^{p-m}}\\ &\sim \sqrt{\frac{p}{m(p-m)}}\,\frac{1}{(m/p)^m(1-m/p)^{p-m}} \le \text{Const}\,\left(\frac{1}{\delta^\delta(1-\delta)^{1-\delta}}\right)^p.\end{aligned}$$

This and (7.6) imply that there exists a constant $C_5'''' = C_5''''(\delta) > 0$ such that

$$\nu(W) \le C_5''''\left(\frac{\rho^{1-\delta}}{\delta^\delta(1-\delta)^{1-\delta}}\right)^p.$$

Now we can choose $\delta \in (0, 1/2)$ so that

$$\rho' = \frac{\rho^{1-\delta}}{\delta^\delta(1-\delta)^{1-\delta}} < 1. \tag{7.7}$$

This is possible since for $f(\delta) = \frac{\rho^{1-\delta}}{\delta^\delta(1-\delta)^{1-\delta}}$ we have

$$\log f(\delta) = (1-\delta)\,\log\rho - \delta\,\log\delta - (1-\delta)\,\log(1-\delta) \to \log\rho < 0$$

as $\delta \to 0$.

Fix $\delta > 0$ with (7.7). Notice that if $x \in U \setminus W$, then $x \in \sigma^{-jp_0N}(V_b)$ for at least $p - m' + 1$ integers $j = 0, 1, \ldots, p$, so $(h \circ \sigma^{jp_0N})(x) = \rho_3$ for that many $j's$. Thus, (7.5) with $m = pp_0$ gives

$$\begin{aligned}\int_U (H^{(pp_0)})^2\,d\nu &\le \int_{U\setminus W}\prod_{j=1}^{pp_0}(h\circ\sigma^{jN})\,d\nu + \int_W\prod_{j=1}^{pp_0}(h\circ\sigma^{jN})\,d\nu\\ &\le \rho_3^{p-m'+1}R^{pp_0-p+m'-1} + R^{pp_0}\nu(W) \le \rho_3^{p\delta}R^{pp_0} + C_5''''\,(\rho' R^{p_0})^p\\ &\le \left(\rho_3^\delta\,R^{p_0}\right)^p + C_5''''\,(\rho' R^{p_0})^p.\end{aligned}$$

Now choose $a_0 = a_0(N, p_0, \delta) > 0$ so small that $\rho' R^{p_0} = \rho' e^{p_0 a_0 NT} < 1$ and

$$\rho_3^\delta R^{p_0} = \left(\frac{e^{a_0NT}}{1 + \frac{\mu_0 e^{-NT}}{C_5''}}\right)^\delta e^{p_0a_0NT} = \frac{e^{a_0(p_0+\delta)NT}}{(1 + \frac{\mu_0e^{-NT}}{C_5''})^\delta} < 1.$$

Setting $\rho_4 = \max\{\rho_3^\delta R^{p_0}, \rho' R^{p_0}\} \in (0, 1)$, and $C_5 = 1 + C_5''''$, we obtain (7.4). This completes the proof of the lemma. ■

Proof of Theorem 1.1. Let $\hat\theta \le \theta < 1$, where $\hat\theta$ is as in (5.1), and let $N \ge N_0$. Choose $a_0 = a_0(N)$, $b_0 = b_0(N)$, $\rho_4 = \rho_4(N) \in (0, 1)$, $p_0 \ge 1$ and $C_5 > 0$ as in Lemmas 7.1 and 7.3. Take $\theta_1 \in (0, \theta]$ and $\theta_2 \in [\theta, 1)$ as in Sect. 6.4 and assume again $f \in \mathcal{F}_{\theta_1}(\widehat{U})$. Recall the set $\mathcal{K}_b$ of pairs (h, H) from Sect. 6.5. For any $h \in \mathcal{F}_\theta(\widehat{U})$ we set $\|h\|_\theta = \|h\|_0 + |h|_\theta$.

Let $|a| \le a_0$ and $|b| \ge b_0$, and let $h \in \mathcal{F}_{\theta_1}(\widehat{U})$ be such that $\|h\|_{\theta_1,b} \le 1$. Then $|h(u)| \le 1$ for all $u \in \widehat{U}$ and $|h|_{\theta_1} \le |b|$. Assume that the points u, u', the cylinder $\mathcal{C}$ in U_1, the integer $p \ge 0$

[18] Assuming $0 \le \delta \le 1/2$ and $1 \le m \le p/2$, we have $\frac{p}{m(p-m)} \le \frac{p}{p/2} = 2$. Also, notice that $\frac{(m/p)^m}{\delta^{p\delta}} \ge \frac{(\frac{p\delta-1}{p})^{p\delta}}{\delta^{p\delta}} = (1 - \frac{1}{p\delta})^{p\delta} \to e^{-1}$ as $p \to \infty$. Finally, $(1 - m/p)^{p-m} \ge \frac{1}{2}(1-\delta)^{p(1-\delta)}$.

and the points $v, v' \in \widehat{U}_1$ satisfy (6.13) for some $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$. Then, using (6.12) and $|h|_{\theta_1} \leq |b|$ we get

$$\begin{aligned} |h(v) - h(v')| &\leq |b|\, D_{\theta_1}(v, v') = |b|\, \theta_1^{p+N}\, D_{\theta_1}(u, u') \leq |b|\, \theta_1^{p+N}\, \mathrm{diam}_{\theta_1}(\mathcal{C}) \\ &\leq |b|\, \theta_1^{p+N}\, C_2\, \mathrm{diam}(\Psi(\mathcal{C})) \leq E|b|\, \theta_2^{p+N}\, \mathrm{diam}(\Psi(\mathcal{C})), \end{aligned}$$

since $C_2 \leq E$. Thus, $(h, 1) \in \mathcal{K}_b$. Set $h^{(m)} = L_{ab}^{mN} h$ for $m \geq 0$. Define the sequence of functions $\{H^{(m)}\}$ recursively by $H^{(0)} = 1$ and $H^{(m+1)} = \mathcal{N}_{J_m} H^{(m)}$, where $J_m \in \mathsf{J}(\mathsf{b})$ is chosen using Lemma 6.7 so that $(h^{(m)}, H^{(m)}) \in \mathcal{K}_b$ for all $m \geq 0$. Now Lemma 7.3 implies $\int_U (H^{(m)})^2 d\nu \leq C_5\, \rho_4^{m/p_0 - 1} = C_6'\, \rho_5^m$ for all integers $m \geq 1$, where $\rho_5 = (\rho_4)^{1/p_0} \in (0,1)$ and $C_6' = C_5/\rho_4 > 0$. Hence

$$\int_U |L_{ab}^{mN} h|^2\, d\nu = \int_U |h^{(m)}|^2\, d\nu \leq \int_U (H^{(m)})^2\, d\nu \leq C_6'\, \rho_5^m.$$

From this it follows that for any $h \in \mathcal{F}_{\theta_1}(\widehat{U})$ we have $\int_U |L_{ab}^{mN} h|^2\, d\nu \leq C_6'\, \rho_5^m\, \|h\|_{\theta_1, b}^2$, and so

$$\int_U |L_{ab}^{mN} h|\, d\nu \leq C_6\, \rho_6^m\, \|h\|_{\theta_1, b}, \tag{7.8}$$

where $\rho_6 = \sqrt{\rho_5}$.

Next, we apply an approximation procedure to deal with functions $h \in \mathcal{F}_\theta(\widehat{U})$. Fix an arbitrary $\epsilon > 0$. Assume that $\|h\|_{\theta, b} \leq 1$; then $\|h\|_0 \leq 1$ and $|h|_\theta \leq |b|$. So, using Lemma 5.2 with $H = 1$, we get

$$|L_{ab}^r h|_\theta \leq A_0[|b|\theta^r + |b|] \leq 2A_0|b| \tag{7.9}$$

for any integer $r \geq 0$.

There exists $\beta_2 > 0$ with $\theta = \theta_1^{\beta_2}$. Take the smallest integer p so that $\theta^p \leq 1/|b|^2$. It is known (see e.g. the end of Ch. 1 in [PP]) that there exists $h' \in \mathcal{F}_{\theta_1}(\widehat{U})$ which is constant on cylinders of length p so that $\|h - h'\|_0 \leq |h|_\theta\, \theta^p$. Then $\|h - h'\|_0 \leq \frac{1}{|b|}$ and so $\|h'\|_0 \leq 2$, and it follows easily from this that $|h'|_{\theta_1} \leq \frac{4}{\theta_1^p} \leq \frac{4}{\theta^{p/\beta_2}} \leq C_7'|b|^s$, where $s = \frac{2}{\beta_2} > 0$ and $C_7' = 4/\theta^{1/\beta_2}$. This and (7.8) imply

$$\int_U |L_{ab}^{mN} h'|\, d\nu \leq C_7''\, \rho_6^m |b|^{s-1}. \tag{7.10}$$

Moreover, as in (7.9) for h, we get

$$|L_{ab}^r h'|_\theta \leq A_0[C_7'|b|^s \theta_1^r + 2|b|] \leq E|b|^s \tag{7.11}$$

for any integer $r \geq 0$, assuming $A_0(C_7' + 2) \leq E$.

Next, recall from the Perron-Ruelle-Frobenius Theorem (see e.g. [PP]) that there exist global constants $C_7 \geq 1$ and $\rho_7 \in (0,1)$ such that

$$\|L_{f^{(0)}}^n w - h_0 \int_U w\, d\nu\| \leq C_7\, \rho_7^n\, \|w\|_\theta \tag{7.12}$$

for all $w \in \mathcal{F}_\theta(\widehat{U})$ and all integers $n \geq 0$. The same estimate holds (we will assume with the same constants C_7 and ρ_7) for $h \in \mathcal{F}_{\theta_1}$ replacing h_0 with the corresponding eigenfunction h_0' and $\|w\|_\theta$ by $\|w\|_{\theta_1}$. Take $\tilde{p} > 1$, $\tilde{q} > 1$ so that $1/\tilde{p} + 1/\tilde{q} = 1$ and $s/\tilde{q} \leq \epsilon$. Then for any integer $r \geq 1$ we have

$$\begin{aligned} |L_{ab}^{2mN} h'| &= |L_{ab}^{mN}\left(|L_{ab}^{mN} h'|\right) \leq \mathcal{M}_a^{mn} |L_{ab}^{mN} h'| = L_{f^{(0)}}^{mN}\left(e^{f_{mN}^{(a)} - f_{mN}^{(0)}}\, |L_{ab}^{mN} h'|\right) \\ &\leq \left(L_{f^{(0)}}^{mN}\left(e^{f_{mN}^{(a)} - f_{mN}^{(0)}}\right)^{\tilde{p}}\right)^{1/\tilde{p}} \left(L_{f^{(0)}}^{mN} |L_{ab}^{mN} h'|^{\tilde{q}}|\right)^{1/\tilde{q}}. \end{aligned}$$

For the first term in the product (5.3) implies $\left(L^{mN}_{f^{(0)}}\left(e^{f^{(a)}_{mN}-f^{(0)}_{mN}}\right)^{\tilde p}\right)^{1/\tilde p} \leq e^{T|a_0|mN}$. For the second term, using (7.11), $\tilde q > 1$, $|h| \leq 1$ and (7.12) with $w = |L^{mN}_{ab} h'|$, we get

$$L^{mN}_{f^{(0)}} |L^{mN}_{ab} h'|^{\tilde q} \leq L^{mN}_{f^{(0)}} |L^{mN}_{ab} h'| \leq \|h'_0\| \int_U |L^{mN}_{ab} h'|\, d\nu + C_7\, \rho_7^{mN}\, \|L^{mN}_{ab} h'\|_{\theta_1}.$$

By (7.11), $\|L^{mN}_{ab} h'\|_{\theta_1} \leq E|b|^s$, so by (7.8),

$$L^{mN}_{f^{(0)}} |L^{mN}_{ab} h'|^{\tilde q} \leq 8C_6 \|h'_0\|\, \rho_6^m |b|^s + EC_7 |b|^s \rho_7^{mN} \leq C'_8 |b|^s \rho_8^m.$$

Thus, taking $a_0 > 0$ sufficiently small and using $s/\tilde q \leq \epsilon$, we obtain

$$|L^{2mN}_{ab} h'| \leq C_8 |b|^\epsilon\, \rho_9^m \tag{7.13}$$

for all integers $m \geq 0$, where $\rho_9 = \rho_8^{1/\tilde q}$.

We have $\rho_9 = e^{-\beta_3}$ for some $\beta_3 > 0$. Taking $2mN = n[\log|b|]$, we get

$$|L^{n[\log|b|]}_{ab} h'| \leq C_8\, e^{-(n/2N)\beta_3 \log|b|}\, |b|^\epsilon = C_8\, \frac{1}{|b|^{n\beta_3/(2N)-\epsilon}} \leq \frac{C_8}{|b|},$$

if we assume $n \geq n_3$, where n_3 is an integer with $n_3\beta_3/(2N) \geq \epsilon + 1$. Thus,

$$\|L^{n_3[\log|b|]}_{ab} h\|_0 \leq \|L^{n_3[\log|b|]}_{ab} h'\|_0 + \|L^{n_3[\log|b|]}_{ab} (h - h')\|_0 \leq \frac{C_8}{|b|} + \frac{1}{|b|} = \frac{C'_9}{|b|}.$$

Then, taking $n_4 > n_3$ sufficiently large and using (7.9) and Lemma 5.2 with $B = 2A_0|b|$ and $H = 1$, we get

$$\begin{aligned} |L^{n_4[\log|b|]}_{ab} h|_\theta &= |L^{(n_4-n_3)[\log|b|]}_{ab} (L^{n_3[\log|b|]}_{ab} h)|_\theta \\ &\leq A_0 \left[2A_0|b|\, \theta^{(n_4-n_3)[\log|b|]} + |b|\, \|L^{n_3[\log|b|]}_{ab} h\|_0\right] \leq C''_9. \end{aligned}$$

Hence $\|L^{n_4[\log|b|]}_{ab} h\|_{\theta,b} \leq \frac{C_9}{|b|} \leq \frac{1}{|b|^{1/2}}$, assuming $|b| \geq b_0$ and b_0 is sufficiently large.

Let $0 \leq m \leq n_4[\log|b|]$. Then $m/n_4 \leq \log|b|$, so $e^{m/n_4} \leq |b|$, i.e. $1/|b| \leq e^{-m/n_4} = \rho_{10}^m$, where $\rho_{10} = e^{-1/n_4} \in (0,1)$. This, $\|h - h'\|_0 \leq 1/|b|$ and (7.13) yield $|L^m_{ab} h| \leq |L^m_{ab} h'| + \frac{1}{|b|} \leq C_8 |b|^\epsilon\, \rho_9^{m/(2N)} + \rho_{10}^m \leq C'_{10} |b|^\epsilon \rho_{11}^m$. Then as above $|L^{2m}_{ab} h|_\theta = |L^m_{ab}(L^m_{ab} h)|_\theta \leq A_0\, [2A_0|b|\, \theta^m + |b|\, \|L^m_{ab} h\|_0] \leq C''_{10} |b|^{1+\epsilon} \rho_{12}^m$, so $\|L^{2m}_{ab} h\|_{\theta,b} \leq C_{10} |b|^\epsilon \rho_{12}^m$ for all integers $0 \leq m \leq n_4[\log|b|]$.

The above implies that for all $h \in \mathcal{F}_\theta(\widehat U)$ and all integers $0 \leq m \leq n_4[\log|b|]$ we have

$$\|L^{n_4[\log|b|]}_{ab} h\|_{\theta,b} \leq \frac{\|h\|_{\theta,b}}{|b|^{1/2}} \quad , \quad \|L^m_{ab} h\|_{\theta,b} \leq C_{10} |b|^\epsilon \rho_{12}^{m/2} \|h\|_{\theta,b}. \tag{7.14}$$

Finally, for any integer $n \geq 0$, writing $n = rn_4[\log|b|] + m$, where $0 \leq m < n_4[\log|b|]$, and using (7.14) we get $\|L^n_{ab} h\|_{\theta,b} = \|L^{rn_4[\log|b|]}_{ab} (L^m_{ab} h)\|_{\theta,b} \leq (1/|b|^{1/2})^r \|L^m_{ab} h\|_{\theta,b} \leq \frac{C_{10}|b|^\epsilon \rho_{12}^{m/2}}{|b|^{r/2}}$. Set $\rho_{13} = \max\{\sqrt{\rho_{12}}, e^{-1/(2n_4)}\}$. As above for $n' = n_4[\log|b|]$ we have $e^{n'/n_4} \leq |b|$, so $\frac{1}{|b|^{r/2}} \leq e^{-rn'/(2n_4)} \leq \rho_{13}^{rn_4[\log|b|]}$. Combining this with the previous estimates gives $\|L^n_{ab} h\|_{\theta,b} \leq C_{10} |b|^\epsilon \rho_{13}^n$.

From this, using an argument from [D] (see also Sect. 3 in [St1]) it follows that there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leq a_0$ and $|b| \geq b_0$, then $\|L^m_{f-(P_f+a+\mathbf{i}b)\tau} h\|_{\theta,b} \leq C\, \rho^m\, |b|^\epsilon\, \|h\|_{\theta,b}$ for any integer $m > 0$ and any $h \in \mathcal{F}_\theta(\widehat U)$.

This completes the proof under the assumption that $f \in \mathcal{F}_{\theta_1}(\widehat{U})$. The case $f \in \mathcal{F}_\theta(\widehat{U})$ follows by using another approximation procedure (following [D]), so we omit the details. ■

Proof of Corollary 1.2. Let again $\hat{\theta}$ be as in (5.1). Given $\epsilon > 0$, choose the constants $C > 0$, $\rho \in (0,1)$, $a_0 > 0$ and $b_0 \geq 1$ as in Theorem 1.1. Let $\hat{\theta} \leq \theta < 1$. As in the proof of Lemma 5.1, $(d(x,y))^\alpha \leq \text{Const}\ D_\theta(x,y)$ will always hold assuming $1/\gamma^\alpha \leq \theta$, i.e. $\alpha \geq \frac{|\log\theta|}{\log\gamma}$. Here $1 < \gamma < \gamma_1$ are the constants from (2.1). Then for such α we have $|h|_\theta \leq \text{Const}\ |h|_\alpha$.

Set $\alpha_0 = \frac{|\log\hat{\theta}|}{\log\gamma} > 0$. Let again $\alpha_1 \in (0,1]$ be such that the local stable holonomy maps on $\widetilde{R}$ are uniformly α_1-Hölder, i.e. there exists a constant $C_{11} > 0$ such that for any $z, z' \in \widetilde{R}_i$ for some $i = 1, \ldots, k_0$ and any $x, y \in W^u_{\widetilde{R}}(z)$ for the projections $x', y' \in W^u_{\widetilde{R}}(z')$ of x, y along stable leaves we have $d(x', y') \leq C_{11}\,(d(x,y))^{\alpha_1}$.

Let $\alpha \in (0, \alpha_0]$; then $\alpha = \frac{|\log\theta|}{\log\gamma}$ for some $\theta \in [\hat{\theta}, 1)$. Then $|h|_\theta \leq C'_{12}|h|_\alpha$ for any $h \in C^\alpha(\widehat{U})$.

Assume that for a given $h \in C^\alpha(\widehat{U})$ we have $\|h\|_{\alpha,b} \leq 1$; then $\|h\|_0 \leq 1$ and $|h|_\alpha \leq |b|$, so $|h|_\theta \leq C'_{12}|b|$ and therefore $\|h\|_{\theta,b} \leq C'_{12} + 1$. By Theorem 1.1, $\|L^n_{ab} h\|_{\theta,b} \leq 2C'_{12}C_{10}|b|^\epsilon \rho_{13}^n$, so

$$|L^n_{ab} h|_0 \leq C_{12}|b|^\epsilon \rho_{13}^n \quad , \quad n \geq 0. \tag{7.15}$$

Next, one needs to repeat part of the arguments from the proof of Theorem 1.1 above.

First, one needs a version of Lemma 5.2 for functions $w \in C^\alpha(\widehat{U})$. Using standard arguments, it follows easily that, given an integer $m \geq 0$ and $u, u' \in U_i$ for some $i = 1, \ldots, k_0$, if $\sigma^m(v) = u$, $\sigma^m(v') = u'$ and $v' = v'(v)$ belongs to the cylinder of length m containing v, then

$$\begin{aligned} |w(\sigma^j v) - w(\sigma^j(v'))| &\leq |w|_\alpha (d(\sigma^j v), \sigma^j(v'))^\alpha \leq \frac{|w|_\alpha}{C_0^\alpha \gamma^{\alpha(m-j)}} (d(\widetilde{\mathcal{P}}^{m-j}(\sigma^j v), \widetilde{\mathcal{P}}^{m-j}(\sigma^j v'))^\alpha \\ &\leq C'_{13} \frac{|w|_\alpha}{\gamma^{\alpha(m-j)}} (d(u,u'))^{\alpha\alpha_1}. \end{aligned} \tag{7.16}$$

Thus, for all integers $m \geq 0$ we have

$$|w_m(v) - w_m(v')| \leq C''_{13}|w|_\alpha (d(u,u'))^{\alpha\alpha_1}. \tag{7.17}$$

This is true for $w = f$, $w = \tau$. Now using standard arguments, for $|a| \leq a_0$ and $w \in C^\alpha(\widehat{U})$ we get $L^m_{f-(P+a)\tau} w \in C^{\alpha\alpha_1}(\widehat{U})$ for all $w \in C^\alpha(\widehat{U})$ and all integers $m \geq 0$. Since $w = 1 \in C^\alpha(\widehat{U})$, it now follows from Perron-Ruelle-Frobenius Theorem that the eigenfunction $h_a \in C^{\alpha\alpha_1}(\widehat{U})$ and so $f^{(a)} \in C^{\alpha\alpha_1}(\widehat{U})$ for all $|a| \leq a_0$. Moreover, taking a_0 sufficiently small, we may assume that $\|h_a\|_{\alpha\alpha_1} \leq C'_{14} = \text{Const}$ for all $|a| \leq a_0$. From (7.17) we also get $|f^{(a)}_m(v) - f^{(a)}_m(v')| \leq C''_{13}(d(u,u'))^{\alpha\alpha_1^2}$, and $|f^{(a)}(v) - f^{(a)}(v')| \leq C''_{14}\,\rho_{14}^m (d(u,u'))^{\alpha\alpha_1^2}$ for some constant $\rho_{14} \in (0,1)$.

For $h \in C^\alpha(\widehat{U})$ we have $|h(v) - h(v')| \leq C''_{13} \frac{|h|_\alpha}{\gamma^{\alpha m}} (d(u,u'))^{\alpha\alpha_1}$ by (7.16), and using again some standard estimates, we obtain

$$|L^m_{ab} h(u) - L^m_{ab} h(u')| \leq C_{14}\,[\rho_{14}^m\,|h|_\alpha + |b|\,\|h\|_0]\ (d(u,u'))^{\alpha\alpha_1^2}. \tag{7.18}$$

Since $|h|_\alpha \leq |b|$ and $\|h\|_0 \leq 1$, this gives $|L^m_{ab} h|_{\alpha\alpha_1^2} \leq \text{Const}\ |b|$ for all $m \geq 0$. Using (7.15) and (7.18) with h replaced by $L^m_{ab} h$ and α replaced by $\alpha\alpha_1^2 \leq \alpha_0$, we get

$$\begin{aligned} |(L^{2m}_{ab} h)(u) - (L^{2m}_{ab} h)(u')| &\leq \text{Const}\ \left[\rho_{14}^m\,|L^m_{ab} h|_{\alpha\alpha_1^2} + |b|\,\|L^m_{ab} h\|_0\right]\ (d(u,u'))^{\alpha\alpha_1^4} \\ &\leq C'_{15}\,[\rho_{14}^m\,|b| + |b|\,|b|^\epsilon \rho_{13}^m]\ (d(u,u'))^{\alpha\alpha_1^4} \end{aligned}$$

Thus, $\|L^{2m}_{ab} h\|_{\alpha\alpha_1^4, b} \leq C_{15}|b|^\epsilon \rho_{14}^m$ for all $m \geq 0$ and all $h \in C^\alpha(\widehat{U})$ with $\|h\|_{\alpha,b} \leq 1$. Since $L^m_{f-(P_f+a+\mathbf{i}b)\tau} h = \frac{1}{h_a}\,L^m_{ab}(h_a h)$, it is now easy to get $\|L^m_{f-(P+a-\mathbf{i}b)} h\|_{\alpha\alpha_1^4, b} \leq C_{16}|b|^\epsilon \rho_{14}^m\,\|h\|_{\alpha,b}$ for all $m \geq 0$ and all $h \in C^\alpha(\widehat{U})$. Setting $\hat{\beta} = \alpha_1^4$ proves the assertion. ■

8 Temporal distance estimates on cylinders

Here we prove Lemmas 4.2 and 4.3.

8.1 A simple lemma

Notice that in Lemma 4.1 the exponential maps are used to parametrize $W^u_\epsilon(z)$ and $W^s_\epsilon(z)$. The particular choice of the exponential maps is not important, however it is important that these maps are C^2. So, we cannot use the maps Φ^u_z and Φ^s_z defined in Sect. 3. In order to use Lemma 4.1 we will need in certain places to replace the local liftings $\hat{f}^p_z$ of the iterations f^p of the map f by slightly different maps.

For $x \in \mathcal{L}$ consider the C^2 map

$$\tilde{f}_x = (\exp^u_{f(x)})^{-1} \circ f \circ \exp^u_x : E^u(x; r(x)) \longrightarrow E^u(f(x), \tilde{r}(f(x)))$$

(assuming $r(x)$ is chosen small enough). As with the maps $\hat{f}$, for $y \in \mathcal{L}$ and an integer $j \geq 1$ we will use the notation $\tilde{f}^j_y = \tilde{f}_{f^{j-1}(y)} \circ \ldots \circ \tilde{f}_{f(y)} \circ \tilde{f}_y$ and $\tilde{f}^{-j}_y = (\tilde{f}_{f^{-j}(y)})^{-1} \circ \ldots \circ (\tilde{f}_{f^{-2}(y)})^{-1} \circ (\tilde{f}_{f^{-1}(y)})^{-1}$ at any point where these sequences of maps are well-defined. In a similar way one defines the maps $\tilde{f}_x$ and their iterations on $E^s(x; r(x))$.

Following the notation in Sect. 3 and using the fact that the flow ϕ_t is contact, the negative Lyapunov exponents over $\mathcal{L}$ are $-\log\lambda_1 > -\log\lambda_2 > \ldots > -\log\lambda_k$. Fix $\hat{\epsilon} > 0$ as in Sect. 3, assuming in addition that

$$\hat{\epsilon} \leq \frac{\log\lambda_1}{100} \min\{\beta, \vartheta\} \quad , \quad \hat{\epsilon} < \frac{\log\lambda_1 (\log\lambda_2 - \log\lambda_1)}{4\log\lambda_1 + 2\log\lambda_2}.$$

For $x \in \mathcal{L}$ we have an f-invariant decomposition $E^s(x) = E^s_1(x) \oplus E^s_2(x) \oplus \ldots \oplus E^s_k(x)$ into subspaces of dimensions $n_1, \ldots, n_k$, where $E^s_i(x)$ ($x \in \mathcal{L}$) is the df-invariant subbundle corresponding to the Lyapunov exponent $-\log\lambda_i$. Thus, for some *Lyapunov $\hat{\epsilon}$-regularity function* $R = R_{\hat{\epsilon}} : \mathcal{L} \longrightarrow (1, \infty)$, which we may assume coincides with the one in Sect. 3, we have

$$\frac{1}{R(x)\, e^{m\hat{\epsilon}}} \leq \frac{\|df^m(x) \cdot v\|}{\lambda_i^{-m}\|v\|} \leq R(x)\, e^{m\hat{\epsilon}} \quad , \quad x \in \mathcal{L}\ ,\ v \in E^s_i(x) \setminus \{0\}\ ,\ m \geq 0\ . \tag{8.1}$$

For the contact form ω it is known (see e.g. Sect. in [KH] or Appendix B in [L1]) that ω vanishes on the tangent bundle of every stable/unstable manifold of a point on M, while $d\omega$ vanishes on the tangent bundle of every weak stable/unstable manifold. For Lyapunov regular points we get a bit of extra information.

Lemma 8.1. *For every $x \in \mathcal{L}$ and every $u = (u^{(1)}, \ldots, u^{(k)}) \in E^u(x; r(x))$ and $v = (v^{(1)}, \ldots, v^{(k)}) \in E^s(x; r(x))$ we have*

$$d\omega_x(u, v) = \sum_{i=1}^k d\omega_x(u^{(i)}, v^{(i)}). \tag{8.2}$$

Proof. It is enough to show that $d\omega_x(u^{(i)}, v^{(j)}) = 0$ if $i \neq j$. Let e.g. $i < j$. Using (3.4), (4.7), (8.1) and the fact that $d\omega$ is df-invariant, for $m \geq 0$ and $x_m = f^m(x)$ we get

$$|d\omega_x(u^{(i)}, u^{(j)})| = |d\omega_{x_m}(df^m(x) \cdot u^{(i)}, df^m(x) \cdot v^{(j)})| \leq C_0 R^2(x) \|u^{(i)}\|\, \|v^{(j)}\|\, \frac{(\lambda_i e^{2\hat{\epsilon}})^m}{\lambda_j^m}.$$

Since $\lambda_i e^{2\hat{\epsilon}} < \lambda_j$, the latter converges to 0 as $m \to \infty$, so $d\omega_x(u^{(i)}, v^{(j)}) = 0$.

The case $i > j$ is considered similarly by taking $m \to -\infty$. ■

8.2 Proof of Lemma 4.2(a)(i)

Let $\hat{z}_0 \in S_1$. We will consider cylinders $\mathcal{C}$ in $\check{U}(\hat{z}_0) = \Psi(W^u_{R_1}(\hat{z}_0))$ with $\mathcal{C} \cap \Psi(P') \neq \emptyset$ (instead of considering cylinders $\mathcal{C}$ in $W^u_{R_1}(\hat{z}_0)$ with $\mathcal{C} \cap P' \neq \emptyset$) with corresponding obvious changes in the estimates we need to prove.

Let $\mathcal{C}$ be such a cylinder and let m be the length of $\mathcal{C}$. The case $m < 2n_0$ is trivial, so we will assume $m \geq 2n_0$.

Fix an arbitrary $z_0 \in \mathcal{C} \cap \Psi(P')$. Given $x_0 \in \mathcal{C}$, write $x_0 = \Phi^u_{z_0}(\xi_0) = \exp^u_{z_0}(\tilde{\xi}_0)$ for some $\xi_0, \tilde{\xi}_0 \in E^u(z_0)$ with $\tilde{\xi}_0 = \Psi^u_{z_0}(\xi_0)$. Then $\|\xi_0\| \leq R_0 \operatorname{diam}(\mathcal{C})$. Set $\mathcal{C}' = \widetilde{\Psi} \circ \Psi^{-1}(\mathcal{C}) \subset \widetilde{R}$, $T = \tilde{\tau}_m(z_0)$ and $p = [T]$, so that $p \leq T < p+1$.

Since m is the length of $\mathcal{C}'$, $\widetilde{\mathcal{P}}^m(\mathcal{C}')$ contains a whole unstable leaf of a proper rectangle $\widetilde{R}_j$. Moreover, $z_0 \in \mathcal{C} \cap P'$, and the choice of P' shows that there exists an integer m' with $m \leq m' \leq m(1+\hat{\delta})$ such that $z = \widetilde{\mathcal{P}}^{m'}(z_0) \in P$. Let $z \in \widetilde{R}_i$. By the choice of P, this implies $B^u(z, r_0) \subset W^u_{\widetilde{R}_i}(z)$. In particular, for every point $b' \in B^u(z, r_0)$ there exists $b \in \mathcal{C}$ with $\widetilde{\mathcal{P}}^{m'}(b) = b'$. Set $p' = [\tilde{\tau}_{m'}(z_0)] + 1$. Since $\tilde{\tau}$ takes values in $[0,1]$, the definition of the set P shows that $p \leq p' \leq p(1+\hat{\delta})$ and $z_{p'} = f^{p'}(z_0) \in \phi_{[-1,1]}(P)$, so $r(z_{p'}) \geq r_0$ by (4.3). Clearly, $p' \geq \tilde{\tau}_{m'}(z_0)$. Then for every $b \in W^u_{r_0}(z_{p'})$ there exists $b \in \mathcal{C}$ with $f^{p'}(b) = b'$. Consider an arbitrary $\zeta_{p'} \in E^u(z_{p'}; r_0/R_0)$ such that $\|\zeta^{(1)}_{p'}\| \geq r_0/R_0$, and set $\zeta = \hat{f}^{-p'}_{z_{p'}}(\zeta_{p'})$. Then $x = \Phi^u_z(\zeta) \in \mathcal{C}$, so $\operatorname{diam}(\mathcal{C}) \geq d(z_0, x) \geq \|\zeta\|/R_0 \geq \|\zeta^{(1)}\|/R_0$. On the other hand, Lemma 3.5 in [St4] (see Lemma 9.1 below) gives $\|\zeta^{(1)}\| \geq \frac{1}{\Gamma_0}\|\zeta^{(1)}\|'_{z_0} \geq \frac{\|\zeta^{(1)}_{p'}\|}{\Gamma_0 \mu_1^{p'}} \geq \frac{r_0/R_0}{\Gamma_0 \mu_1^{(1+\hat{\delta})p}} \geq \frac{r_0}{R_0 \Gamma_0 \mu_1^p e^{\hat{\epsilon} p}} = \frac{r_0}{R_0 \Gamma_0 \lambda_1^p e^{2\hat{\epsilon} p}}$, hence $\operatorname{diam}(\mathcal{C}) \geq \frac{c_3}{\lambda_1^p e^{2\hat{\epsilon} p}}$, where $c_3 = \frac{r_0}{R_0^2 \Gamma_0} \geq 1$.

This proves the left-hand-side inequality in (4.10) with $C_1 = 1/c_3$. The other inequality in (4.10) follows by a similar (in fact, easier) argument. We omit the details.

8.3 Proof of Lemma 4.2(a)(ii)

Let again $\mathcal{C}$ be a cylinder in $\check{U}(\hat{z}_0)$ with $\mathcal{C} \cap \Psi(P') \neq \emptyset$, and let $x_0, z_0 \in \mathcal{C}$, $y_0, b_0 \in W^s_{R_1}(z_0)$. For the length m of $\mathcal{C}$ we will assume $m > 4n_0$; the other case is trivial.

It is enough to consider the case when $z_0 \in \Psi(P')$. Indeed, assuming the statement is true in this case, let $z \in \mathcal{C} \cap \Psi(P')$. Set $\{y\} = W^u_R(y_0) \cap W^s_R(z)$ and $\{b\} = W^u_R(b_0) \cap W^s_R(z)$. Since the local unstable holonomy maps are uniformly Hölder, there exist (global) constants $C' > 0$ and $\beta' > 0$ such that $d(y, b) \leq C'(d(y_0, b_0))^{\beta'}$. Thus, using the assumption,

$$|\Delta(x_0, y) - \Delta(x_0, b)| \leq C_1 \operatorname{diam}(\mathcal{C})(d(y,b))^{\beta_1} \leq C_1 (C')^{\beta_1} \operatorname{diam}(\mathcal{C})(d(y_0, b_0))^{\beta' \beta_1}.$$

A similar estimate holds for $|\Delta(z_0, y) - \Delta(z_0, b)|$, so

$$\begin{aligned} |\Delta(x_0, y_0) - \Delta(x_0, b_0)| &= |(\Delta(x_0, y) - \Delta(z_0, y) - (\Delta(x_0, b) - \Delta(z_0, b))| \\ &\leq |\Delta(x_0, y) - \Delta(x_0, b)| + |\Delta(z_0, y) - \Delta(z_0, b)| \\ &\leq 2C_1 (C')^{\beta_1} \operatorname{diam}(\mathcal{C})(d(y_0, b_0))^{\beta' \beta_1}. \end{aligned}$$

So, from now on we assume that $z_0 \in \mathcal{C} \cap \Psi(P')$. Moreover, we can assume that $\mathcal{C}$ is the **smallest cylinder containing** x_0 and z_0; otherwise we will replace $\mathcal{C}$ by a smaller cylinder. Write $x_0 = \Phi^u_{z_0}(\xi_0) = \exp^u_{z_0}(\tilde{\xi}_0)$ for some $\xi_0, \tilde{\xi}_0 \in E^u(z_0)$ with $\tilde{\xi}_0 = \Psi^u_{z_0}(\xi_0)$. Then $\|\xi_0\|, \|\tilde{\xi}_0\| \leq R_0 \operatorname{diam}(\mathcal{C})$. Similarly, write $y_0 = \exp^s_{z_0}(\tilde{v}_0) = \Phi^s_{z_0}(v_0)$ and also $b_0 = \exp^s_{z_0}(\tilde{\eta}_0) = \Phi^s_{z_0}(\eta_0)$ for some $v_0, \tilde{v}_0, \eta_0, \tilde{\eta}_0 \in E^s(z_0)$ with $\tilde{v}_0 = \Psi^s_{z_0}(v_0)$ and $\tilde{\eta}_0 = \Psi^s_{z_0}(\eta_0)$. By (3.6),

$$\|\tilde{v}_0 - v_0\| \leq R_0 \|v_0\|^{1+\beta} \quad , \quad \|\tilde{\xi}_0 - \xi_0\| \leq R_0 \|\xi_0\|^{1+\beta} \quad , \quad \|\tilde{\eta}_0 - \eta_0\| \leq R_0 \|\eta_0\|^{1+\beta}. \tag{8.3}$$

8.3.1 Pushing forward

Set $p = [\tilde{\tau}_m(z_0)]$; then (4.10) holds. Set $q = [p/2]$. We will in fact assume that $q = p/2$; the difference with the case when p is odd is insignificant. For any integer $j \geq 0$ set $z_j = f^j(z_0)$, $x_j = f^j(x_0)$, $y_j = f^j(y_0)$ and also $\hat{\xi}_j = d\hat{f}^j_{z_0}(0)\cdot \xi_0$, $\xi_j = \hat{f}^j_{z_0}(\xi_0)$, $\tilde{\xi}_j = \tilde{f}^j_{z_0}(\tilde{\xi}_0)$, $\hat{v}_j = d\hat{f}^j_{z_0}(0)\cdot v_0$, $v_j = \hat{f}^j_{z_0}(v_0)$, $\tilde{v}_j = \tilde{f}^j_{z_0}(\tilde{v}_0)$, $b_j = f^j(b_0)$, $\hat{\eta}_j = d\hat{f}^j_{z_0}(0)\cdot \eta_0$, $\eta_j = \hat{f}^j_{z_0}(\eta_0)$, $\tilde{\eta}_j = \tilde{f}^j_{z_0}(\tilde{\eta}_0)$.

Since $p \geq 4n_0$, we have $q \geq 2n_0$. Notice that $\tilde{\xi}_0 = \Psi^u_{z_0}(\xi_0)$, $\tilde{v}_0 = \Psi^s_{z_0}(v_0)$, and also $\tilde{\xi}_j = \Psi^u_{z_j}(\xi_j)$, $\Phi^u_{z_j}(\xi_j) = x_j$, $\tilde{v}_j = \Psi^s_{z_j}(v_j)$, $\tilde{\eta}_j = \Psi^s_{z_j}(\eta_j)$, so by (3.6),

$$\|\xi_j - \tilde{\xi}_j\| \leq R(z_j)\|\xi_j\|^{1+\beta} \quad , \quad \|v_j - \tilde{v}_j\| \leq R(z_j)\|v_j\|^{1+\beta} \quad , \quad \|\eta_j - \tilde{\eta}_j\| \leq R(z_j)\|\eta_j\|^{1+\beta}. \tag{8.4}$$

Moreover, $\exp^u_{z_j}(\tilde{\xi}_j) = f^j(\exp^u_{z_0}(\xi_0)) = f^j(x_0) = x_j$, $\exp^s_{z_j}(\tilde{v}_j) = y_j$ and $\exp^s_{z_j}(\tilde{\eta}_j) = b_j$, so Lemma 4.1 implies

$$|\Delta(x_j, y_j) - d\omega_{z_j}(\tilde{\xi}_j, \tilde{v}_j)| \leq C_0 \left[\|\tilde{\xi}_j\|^2\, \|\tilde{v}_j\|^{\vartheta} + \|\tilde{\xi}_j\|^{\vartheta}\|\tilde{v}_j\|^2\right] \tag{8.5}$$

and similarly $|\Delta(x_j, b_j) - d\omega_{z_j}(\tilde{\xi}_j, \tilde{\eta}_j)| \leq C_0 \left[\|\tilde{\xi}_j\|^2\, \|\tilde{\eta}_j\|^{\vartheta} + \|\tilde{\xi}_j\|^{\vartheta}\|\tilde{\eta}_j\|^2\right]$ for every integer $j \geq 0$. From (8.4) one gets

$$|d\omega_{z_j}(\tilde{\xi}_j, \tilde{v}_j) - d\omega_{z_j}(\xi_j, v_j)| \leq 2C_0 R(z_j)\, \|\xi_j\|\, \|v_j\|(\|\xi_j\|^{\beta} + \|v_j\|^{\beta}),$$

$$|d\omega_{z_j}(\tilde{\xi}_j, \tilde{\eta}_j) - d\omega_{z_j}(\xi_j, \eta_j)| \leq 2C_0 R(z_j)\, \|\xi_j\|\, \|\eta_j\|(\|\xi_j\|^{\beta} + \|\eta_j\|^{\beta}),$$

and also[19] $\|\tilde{\xi}_j\| \leq 2\|\xi_j\|$, $\|\tilde{v}_j\| \leq 2\|v_j\|$ and $\|\tilde{\eta}_j\| \leq 2\|\eta_j\|$.

Using these, it follows from (8.5) that

$$\begin{aligned} |\Delta(x_j, y_j) - d\omega_{z_j}(\xi_j, v_j)| \;&\leq\; 2C_0 R(z_j)\, \|\xi_j\|\, \|v_j\|(\|\xi_j\|^{\beta} + \|v_j\|^{\beta}) \\ &\quad + 8C_0 \left[\|\xi_j\|^2\, \|v_j\|^{\vartheta} + \|\xi_j\|^{\vartheta}\|v_j\|^2\right]. \end{aligned} \tag{8.6}$$

and similarly

$$\begin{aligned} |\Delta(x_j, b_j) - d\omega_{z_j}(\xi_j, \eta_j)| \;&\leq\; 2C_0 R(z_j)\, \|\xi_j\|\, \|\eta_j\|(\|\xi_j\|^{\beta} + \|\eta_j\|^{\beta}) \\ &\quad + 8C_0 \left[\|\xi_j\|^2\, \|\eta_j\|^{\vartheta} + \|\xi_j\|^{\vartheta}\|\eta_j\|^2\right]. \end{aligned} \tag{8.7}$$

for every integer $j \geq 0$.

We will be estimating $|\Delta(x_0, y_0) - d\omega_{z_0}(\xi_0, v_0)|$. Since Δ is f-invariant and $d\omega$ is df-invariant we have $\Delta(x_0, y_0) = \Delta(x_j, y_j)$ and $d\omega_{z_0}(\xi_0, v_0) = d\omega_{z_j}(\hat{\xi}_j, \hat{v}_j)$, and also $\Delta(x_0, b_0) = \Delta_{z_j}(x_j, b_j)$ and $d\omega_{z_0}(\xi_0, \eta_0) = d\omega_{z_j}(\hat{\xi}_j, \hat{\eta}_j)$ for all j. (Notice that $d\hat{f}_x(0) = df(x)$ for all $x \in M$.)

Since $z_0 \in P'$ and $q \geq n_0$, by (4.9) there exist at least $q - \hat{\delta} q$ numbers $j = 1, \ldots, q$ with $f^j(z_0) \in P$. Fix an arbitrary integer ℓ with

$$(1 - \hat{\delta})q \leq \ell \leq q \quad , \quad z_\ell = f^\ell(z_0) \in P. \tag{8.8}$$

It then follows from Lemma 3.1, the choice of L_0 and $\|\xi_\ell\| \leq r(z_\ell)$ (since $\ell \leq q = p/2$; see also Sect. 8.3.2 below) that

$$\|\hat{\xi}^{(1)}_\ell - \xi^{(1)}_\ell\| \leq L_0 \|\xi_\ell\|^{1+\beta}. \tag{8.9}$$

[19] Indeed, from (8.4), $\|\tilde{\xi}_j\| \leq \|\xi_j\|(1 + R(z_j)\|\xi_j\|^{\beta}) \leq \|\xi_j\|(1 + R_0 e^{(p'-j)\hat{\epsilon}} r_0/\mu_1^{p'-j}) \leq \|\xi_j\|(1 + \frac{R_0 r_0}{(e^{-\hat{\epsilon}}\mu_1)^{n_0}}) \leq 2\|\xi_j\|$, assuming $n_0 \geq 1$ is sufficiently large. Similarly, $\|\tilde{v}_j\| \leq 2\|v_j\|$ and $\|\tilde{\eta}_j\| \leq 2\|\eta_j\|$.

Apart from that, using Lemma 9.7(b) below, backwards for stable manifolds, with $a = d\hat{f}_{z_\ell}^{-\ell}(0) \cdot v_\ell \in E^s(z_0)$, $b = d\hat{f}_{z_\ell}^{-\ell}(0) \cdot \eta_\ell \in E^s(z_0)$, since $v_0 = \hat{f}_{z_\ell}^{-\ell}(v_\ell)$ and $\eta_0 = \hat{f}_{z_\ell}^{-\ell}(\eta_\ell)$, it follows that

$$\|(a^{(1)} - b^{(1)}) - (v_0^{(1)} - \eta_0^{(1)})\| \leq L_0 \left[\|v_0 - \eta_0\|^{1+\beta} + \|\eta_0\|^\beta \|v_0 - \eta_0\|\right] \leq 2L_0\|v_0 - \eta_0\|.$$

Thus,

$$\|d\hat{f}_{z_\ell}^{-\ell}(0) \cdot (v_\ell^{(1)} - \eta_\ell^{(1)}) - (v_0^{(1)} - \eta_0^{(1)})\| \leq 2L_0\|v_0 - \eta_0\|. \tag{8.10}$$

In what follows we denote by Const a global constant (depending on constant like C_0, L_0, R_0 however independent of the choice of the cylinder $\mathcal{C}$, the points x_0, z_0, y_0, b_0, etc.) which may change from line to line.

Using (8.9), (8.10) and the above remarks, we obtain

$$\begin{aligned}
& |d\omega_{z_\ell}(\xi_\ell, v_\ell - \eta_\ell)| \\
\leq\ & |d\omega_{z_\ell}(\xi_\ell^{(1)}, v_\ell^{(1)} - \eta_\ell^{(1)})| + C_0 \sum_{i=2}^k \|\xi_\ell^{(i)}\| \,(\|v_\ell^{(i)}\| + \|\eta_\ell^{(i)}\|) \\
\leq\ & |d\omega_{z_\ell}(\hat{\xi}_\ell^{(1)}, v_\ell^{(1)} - \eta_\ell^{(1)})| + \text{Const } \|\xi_\ell\|^{1+\beta}\|v_\ell^{(1)} - \eta_\ell^{(1)}\| + C_0 \sum_{i=2}^k \|\xi_\ell^{(i)}\| \,(\|v_\ell^{(i)}\| + \|\eta_\ell^{(i)}\|) \\
\leq\ & |d\omega_{z_\ell}(d\hat{f}_{z_0}^{\ell}(0) \cdot \xi_0^{(1)}, v_\ell^{(1)} - \eta_\ell^{(1)})| + \text{Const } \|\xi_\ell\|^{1+\beta}\|v_\ell^{(1)} - \eta_\ell^{(1)}\| + C_0 \sum_{i=2}^k \|\xi_\ell^{(i)}\| \,(\|v_\ell^{(i)}\| + \|\eta_\ell^{(i)}\|) \\
=\ & |d\omega_{z_0}(\xi_0^{(1)}, d\hat{f}_{z_\ell}^{-\ell}(0) \cdot (v_\ell^{(1)} - \eta_\ell^{(1)}))| + \text{Const } \|\xi_\ell\|^{1+\beta}\|v_\ell^{(1)} - \eta_\ell^{(1)}\| + C_0 \sum_{i=2}^k \|\xi_\ell^{(i)}\| \,(\|v_\ell^{(i)}\| + \|\eta_\ell^{(i)}\|) \\
\leq\ & |d\omega_{z_0}(\xi_0^{(1)}, v_0^{(1)} - \eta_0^{(1)})| + 2C_0L_0\|\xi_0\| \,\|v_0 - \eta_0\| + \text{Const } \|\xi_\ell\|^{1+\beta}\|v_\ell^{(1)} - \eta_\ell^{(1)}\| \\
& + C_0 \sum_{i=2}^k \|\xi_\ell^{(i)}\| \,(\|v_\ell^{(i)}\| + \|\eta_\ell^{(i)}\|) \\
\leq\ & \text{Const } \text{diam}(\mathcal{C})\, \|v_0 - \eta_0\| + \text{Const } \|\xi_\ell\|^{1+\beta}\|v_\ell^{(1)} - \eta_\ell^{(1)}\| + C_0 \sum_{i=2}^k \|\xi_\ell^{(i)}\| \,(\|v_\ell^{(i)}\| + \|\eta_\ell^{(i)}\|).
\end{aligned} \tag{8.11}$$

8.3.2 Estimates for $\|\xi_\ell\|$, $\|v_\ell\|$ and $\|\eta_\ell\|$

We will now use the choice of ℓ to estimate $\|\xi_\ell\|$, $\|v_\ell\|$ and $\|\eta_\ell\|$ by means of $\|\xi_0\|$, $\|v_0\|$ and $\|\eta_0\|$. We will first estimate $\|\xi_q\|$, $\|v_q\|$ and $\|\eta_q\|$.

Using the definition of ξ_j, $p = 2q$, $z \in P$ and (3.11) we get $\|\xi_q\| \leq \|\xi_q\|'_{z_q} \leq \frac{\|\xi_p\|'_{z_p}}{\mu_1^{p-q}} \leq \frac{\Gamma(z_p) e^{q\hat{\epsilon}}\|\xi_p\|}{\lambda_1^q} \leq \frac{\Gamma_0 e^{2q\hat{\epsilon}}\|\xi_p\|}{\lambda_1^q}$. Since $\Phi^u_{z_p}(\xi_p) = x_p$ and $d(x_p, z_p) \leq \text{diam}(\widetilde{R}_i)$, we get $\|\xi_p\| \leq R(z_p) d(x_p, z_p) \leq R_0 e^{p\hat{\epsilon}} r_1 < R_0 e^{p\hat{\epsilon}}$. Thus,

$$\|\xi_q\| \leq \frac{R_0 \Gamma_0 e^{4q\hat{\epsilon}}}{\lambda_1^q}\ . \tag{8.12}$$

Using (3.11) again (on stable manifolds) and $\|v_0\| \leq 2\delta'/R_0 < 1$, we get

$$\|v_q\| = \|v_q\|'_{z_q} \leq \frac{\|v_0\|'_z}{\mu_1^q} \leq \frac{\Gamma_0 e^{q\hat{\epsilon}}\|v_0\|}{\lambda_1^q} \leq \frac{\Gamma_0 e^{q\hat{\epsilon}}}{\lambda_1^q}. \tag{8.13}$$

Similarly, $\|\eta_q\| \leq \frac{\Gamma_0 e^{q\hat{\epsilon}}}{\lambda_1^q}$.

Next, it follows from (4.10) that $(\lambda_1 e^{2\hat{\epsilon}})^{2q} \geq c_3/\mathrm{diam}(\mathcal{C})$, so

$$q \geq \frac{1}{2\log(\lambda_1 e^{2\hat{\epsilon}})} \log \frac{c_3}{\mathrm{diam}(\mathcal{C})}. \tag{8.14}$$

This and (8.12) give

$$\begin{aligned} \|\xi_q\| &\leq R_0\Gamma_0(\lambda_1 e^{-4\hat{\epsilon}})^{-q} = R_0\Gamma_0 e^{-q\log(\lambda_1 e^{-4\hat{\epsilon}})} \leq R_0\Gamma_0\, e^{-\frac{\log(\lambda_1 e^{-4\hat{\epsilon}})}{2\log(\lambda_1 e^{2\hat{\epsilon}})}\log\left(\frac{c_3}{\mathrm{diam}(\mathcal{C})}\right)} \\ &= R_0\Gamma_0 \left(\frac{c_3}{\mathrm{diam}(\mathcal{C})}\right)^{-\frac{\log\lambda_1 - 4\hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}}} \leq \frac{R_0\Gamma_0}{c_3}\,(\mathrm{diam}(\mathcal{C}))^{\frac{\log\lambda_1 - 4\hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}}}, \end{aligned} \tag{8.15}$$

since $\frac{\log\lambda_1 - 4\hat{\epsilon}}{\log\lambda_1 + 2\hat{\epsilon}} < 1$. Similarly, (8.13) yields $\|v_q\| \leq \frac{\Gamma_0}{c_3}\,(\mathrm{diam}(\mathcal{C}))^{\frac{\log\lambda_1 - \hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}}}$. The same estimate holds for $\|\eta_q\|$.

We need similar estimates, however with q replaced by ℓ. As in (8.15) one obtains

$$\|\xi_\ell\| \quad\leq\quad \|\xi_\ell\|'_{z_\ell} \leq \|\xi_q\|'_{z_q} \leq \Gamma_0\|\xi_q\| \leq \frac{R_0\Gamma_0^2}{c_3}\,(\mathrm{diam}(\mathcal{C}))^{\frac{\log\lambda_1 - 4\hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}}}.$$

Similarly, since $q - \ell \leq \hat{\delta}q$ by (8.8), using (4.4) we get $\lambda_k^{q-\ell} \leq \lambda_k^{\hat{\delta}q} < e^{\hat{\epsilon}q}$, and therefore

$$\|v_\ell\| \leq \Gamma(z_\ell)e^{(q-\ell)\hat{\epsilon}}\lambda_k^{q-\ell}\|v_q\| \leq \Gamma_0 e^{3q\hat{\epsilon}}\|v_q\| \leq \Gamma_0^2(\lambda_1 e^{-4\hat{\epsilon}})^{-q} \leq \frac{\Gamma_0^2}{c_3}\,(\mathrm{diam}(\mathcal{C}))^{\frac{\log\lambda_1 - 4\hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}}},$$

and again the same estimate holds for $\|\eta_\ell\|$. Thus, taking the constant $C'' > 0$ so large that $C'' \geq R_0\Gamma_0^2/c_3$, we get $\|v_\ell\|, \|\eta_\ell\|, \|\xi_\ell\| \leq C''\,(\mathrm{diam}(\mathcal{C}))^{\frac{\log\lambda_1 - 4\hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}}}$. Using these we get the following estimates for the terms in (8.11):

$$\|\xi_\ell\|\,\|v_\ell\|(\|\xi_\ell\|^\beta + \|v_\ell\|^\beta) \leq 2(C'')^3\,(\mathrm{diam}(\mathcal{C}))^{(2+\beta)\frac{\log\lambda_1 - 4\hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}}} \leq 2(C'')^3(\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}},$$

where we choose

$$0 < \hat{\beta} = \min\left\{\frac{1}{4}\min\{\beta, \vartheta\}\,,\ \frac{\log\lambda_2 - \log\lambda_1}{2\log\lambda_1}\right\}, \tag{8.16}$$

and we use the assumption $\hat{\epsilon} \leq \frac{\log\lambda_1}{100}\min\{\beta, \vartheta\}$. Then $(2+\beta)\frac{\log\lambda_1 - 4\hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}} \geq 1 + \hat{\beta}$ and also $(2+\vartheta)\frac{\log\lambda_1 - 4\hat{\epsilon}}{2\log\lambda_1 + 4\hat{\epsilon}} \geq 1 + \hat{\beta}$ which is used in the next estimate. Similarly,

$$\|\xi_\ell\|^{1+\beta}\|v_\ell\| \leq (C'')^3(\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}},$$

and

$$\|\xi_\ell\|^2\,\|v_\ell\|^\vartheta + \|\xi_\ell\|^\vartheta\|v_\ell\|^2 \leq 2(C'')^3(\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}}. \tag{8.17}$$

Next, for any $\xi = \xi^{(1)} + \xi^{(2)} + \ldots + \xi^{(k)} \in E^u(z)$ or $E^s(z)$ for some $z \in M$ set $\check{\xi}^{(2)} = \xi^{(2)} + \ldots + \xi^{(k)}$, so that $\xi = \xi^{(1)} + \check{\xi}^{(2)}$. Using Lemma 3.5 in [St4] (see Lemma 9.1 below), $p - \ell = 2q - \ell \geq q$ and the fact that $\|\xi_\ell\| \leq \|\xi_p\| \leq R_0 r_1 \leq R_0$, we get $\|\check{\xi}_\ell^{(2)}\|'_{z_\ell} \leq \frac{\Gamma_0\|\check{\xi}_\ell^{(2)}\|}{\mu_2^q} \leq \frac{\Gamma_0\|\xi_\ell\|}{\mu_2^q} \leq \frac{\Gamma_0 R_0}{\mu_2^q}$. Similarly, using Lemma 3.5 in [St4] (backwards for the map f^{-1} on stable manifolds), $z_0 \in P_1 \subset P$, $v_0 = v_{j,1}(z_0) \in E^s(z_0, r'_0)$ and the fact that $\|v_0\| \leq \delta' < 1$, we get $\|\check{v}_\ell^{(2)}\|'_{z_\ell} \leq \frac{\Gamma_0\|v_0\|}{\mu_2^{q(1-\hat{\delta})}} \leq \frac{\Gamma_0}{\mu_2^{q(1-\hat{\delta})}}$.

Hence for $i \geq 2$ we have $\|\xi_\ell^{(i)}\| \leq |\check{\xi}_\ell^{(2)}| \leq \|\check{\xi}_\ell^{(2)}\| \leq \frac{\Gamma_0 R_0}{\mu_2^q}$, and similarly $\|v_\ell^{(i)}\| \leq \dfrac{\Gamma_0}{\mu_2^{q(1-\hat{\delta})}}$. Using these estimates, (8.14), $\mu_2 = \lambda_2 e^{-\hat{\epsilon}}$, (4.4) and the assumptions about $\hat{\epsilon}$, we get

$$\begin{aligned}
\|\xi_\ell^{(i)}\| \, \|v_\ell^{(i)}\| &\leq \Gamma_0^2 R_0 \, (\lambda_2 e^{-2\hat{\epsilon}})^{-2q} = \Gamma_0^2 R_0 \, e^{-2q \log(\lambda_2 e^{-2\hat{\epsilon}})} \leq \Gamma_0^2 R_0 \, e^{\frac{-\log(\lambda_2 e^{-2\hat{\epsilon}})}{\log(\lambda_1 e^{2\hat{\epsilon}})} \log \frac{c_3}{\mathrm{diam}(\mathcal{C})}} \\
&\leq \Gamma_0^2 R_0 \left(\frac{\mathrm{diam}(\mathcal{C})}{c_3} \right)^{\frac{\log \lambda_2 - 2\hat{\epsilon}}{\log \lambda_1 + 2\hat{\epsilon}}} \leq C''(\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}},
\end{aligned}$$

using $\hat{\beta} \leq \frac{\log \lambda_2 - \log \lambda_1}{2 \log \lambda_1}$ by (8.16) and assuming $C'' \geq \Gamma_0^2 R_0/(c_3)^{\log \lambda_2/\log \lambda_1}$. Then

$$\frac{\log \lambda_2 - 2\hat{\epsilon}}{\log \lambda_1 + 2\hat{\epsilon}} - 1 = \frac{\log \lambda_2 - 2\hat{\epsilon} - \log \lambda_1 - 2\hat{\epsilon}}{\log \lambda_1 + 2\hat{\epsilon}} \geq \frac{\log \lambda_2 - \log \lambda_1 - 4\hat{\epsilon}}{\log \lambda_1 + 2\hat{\epsilon}} \geq \hat{\beta}.$$

8.3.3 Final estimate

Using (8.11) and the above estimates for $\|\xi_\ell\|$, $\|v_\ell\|$, $\|\xi_\ell^{(i)}\| \, \|v_\ell^{(i)}\|$, we obtain

$$|d\omega_{z_\ell}(\xi_\ell, v_\ell - \eta_\ell)| \leq \text{Const } \mathrm{diam}(\mathcal{C}) \, \|v_0 - \eta_0\| + \text{Const } (\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}}.$$

Next, using (8.6) and (8.7) with $j = \ell$ and the previous estimate we get

$$\begin{aligned}
|\Delta(x_0, y_0) - \Delta(x_0, b_0)| &= |\Delta(x_\ell, y_\ell) - \Delta(x_\ell, b_\ell)| \\
&\leq |d\omega_{z_\ell}(\xi_\ell, v_\ell - \eta_\ell)| + \text{Const } (\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}} \\
&\leq \text{Const } \mathrm{diam}(\mathcal{C}) \, \|v_0 - \eta_0\| + \text{Const } (\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}}. \qquad (8.18)
\end{aligned}$$

Case 1. $\mathrm{diam}(\mathcal{C}) \leq \|v_0 - \eta_0\|^{\vartheta/2}$. Then (8.18) immediately implies $|\Delta(x_0, y_0) - \Delta(x_0, b_0)| \leq$ Const $\mathrm{diam}(\mathcal{C}) \, \|v_0 - \eta_0\|^{\hat{\beta}\vartheta/2}$.

Case 2. $\mathrm{diam}(\mathcal{C}) \geq \|v_0 - \eta_0\|^{\vartheta/2}$. Set $\{X'\} = W_R^u(y_0) \cap W_R^s(x_0)$ and $X = \phi_{\Delta(x_0,y_0)}(X')$. Then $X \in W_{\epsilon_0}^u(y_0)$ and it is easy to see that $|\Delta(x_0, y_0) - \Delta(x_0, b_0)| = |\Delta(X, b_0)|$. We have $X = \exp_{y_0}^u(\tilde{t})$ and $b_0 = \exp_{y_0}^s(\tilde{s})$ for some $\tilde{t} \in E^u(y_0)$ and $\tilde{s} \in E^s(y_0)$. Clearly $\|\tilde{t}\| \leq$ Const . Using Liverani's Lemma (Lemma 4.1) we get $|\Delta(X, b_0)| \leq C_0[|d\omega_{y_0}(\tilde{t}, \tilde{s})| + \|\tilde{t}\|^2 \|\tilde{s}\|^\vartheta + \|\tilde{t}\|^\vartheta \|\tilde{s}\|^2] \leq$ Const $\|\tilde{s}\|^\vartheta$. However, $\|\tilde{s}\| \leq$ Const $d(y_0, b_0) \leq$ Const $\|v_0 - \eta_0\|$, so $|\Delta(X, b_0)| \leq$ Const $\|v_0 - \eta_0\|^\vartheta \leq$ Const $\mathrm{diam}(\mathcal{C})\|v_0 - \eta_0\|^{\vartheta/2}$. This proves the lemma. ■

8.4 Proof of Lemma 4.2(b)

This is done by using some of the arguments in Sect. 8.3 with only very small modifications[20], so we omit the details.

8.5 Proof of Lemma 4.3

8.5.1 Set-up – choice of some constants and initial points

We begin with another simple technical lemma.

Lemma 8.3. *Changing the position of the point z_1 in R_1, there exist a constant $\epsilon' \in (0, \epsilon_0]$ and a compact subset $\tilde{S}$ of $S_1 \cap P$ with $\nu^s(\tilde{S}) > 0$ with the following properties:*

[20] The main bit is the prove (4.10) under the assumption $\mathcal{C} \cap \mathcal{P}^j(P') \neq \emptyset$ for some $|j| \leq j_0$, and this is done as in Sect. 8.2.

(i) *For all* $x \in \tilde{S}$ *and all* $\epsilon \in (0, \epsilon']$ *we have*

$$\nu_x^u(B^u(x,\epsilon) \cap P') \geq \frac{1}{2}\, \nu_x^u(B^u(x,\epsilon)). \tag{8.19}$$

(ii) *the point* z_1 *with* $\{z_1\} = U_1 \cap S_1$ *is a* ν^s*-density point of* $\tilde{S}$ *and* $z_1 \in \mathrm{Int}^s(S_1)$.

Proof. Set $\nu_0' = \frac{\mu(P')}{2A_2 \nu^s(S_1)}$ and $\tilde{S}' = \{z \in S_1 \cap P' : \nu_z^u(U_1(z) \cap P') \geq \nu_0'\}$. Then, as for $\nu^s(S^\sharp)$ in Sect. 6.1, we derive $\nu^s(\tilde{S}') \geq \frac{\mu(P')}{2(1-\nu_0')} > 0$. For $z \in \tilde{S}'$, by Borel's Density Theorem, the set A_z of ν_z^u-density points x of the subset $U_1(z) \cap P'$ has $\nu_z^u(A_z) = \nu_z^u(W^u_{R_1}(z) \cap P') \geq \nu_0'$. Set $P'' = \cup_{z \in \tilde{S}'} A_z$. Then by (4.1),

$$\mu(P'') \geq A_1 \int_{\tilde{S}'} \nu_z^u(P' \cap U_1(z))\, d\nu^s(z) \geq A_1 \nu_0'\, \nu^s(\tilde{S}') > 0.$$

This implies that $\nu^u(\{z \in U_1 : \nu^s(W^s_{R_1}(z) \cap P'') > 0\}) > 0$. Changing the point z_1 if necessary (thus, the stable leaf S_1 as well), we may assume that z_1 is one such point, i.e. $\nu^s(S_1 \cap P'') > 0$.

Then $\nu^s(\tilde{S}' \cap P'') > 0$ and by the definition of P'', for every $x \in \tilde{S}' \cap P''$ there exists (a minimal number) $\epsilon(x) > 0$ such that (8.19) holds for all $\epsilon \in (0, \epsilon(x)]$. By Luzin's Theorem, there exists a compact subset $\tilde{S}$ of $\tilde{S}' \cap P'$ with $\nu^s(\tilde{S}) > 0$ such that $\epsilon(x)$ is a continuous function of $x \in \tilde{S}$. In particular, there exists $\epsilon' > 0$ such that (8.19) holds for all $x \in \tilde{S}$ and all $\epsilon \in (0, \epsilon']$. ■

Since $\eta_j(z_1) \in E_1^u(z_1)$, by Lemma 8.1 and the choice of $\theta_0 > 0$ (see Sect. 4.2), there exists $\check{v}_j \in E_1^s(z_1)$ with $d\omega_{z_1}(\eta_j(z_1), \check{v}_j) \geq \theta_0$ and $\|\check{v}_j\| = 1$. Choose $\delta' > 0$ such that

$$(\delta')^\beta < \frac{\beta_0 \kappa \theta_0}{128 L_0 C_0 R_0}, \tag{8.20}$$

where $\beta \in (0, 1]$ is the constant from Sect. 3, and also $B^s(z_1, \delta') \subset \mathrm{Int}^s(S_1)$.

Let $j = 1, \ldots, \ell_0$. **Fix an arbitrary** s_0 with $0 < s_0 < \delta'/(2R_0)$ and set $v_{j,1} = s_0 \check{v}_j \in E_1^s(z_1)$ and $y_1^{(j)} = \Phi^s_{z_1}(v_{j,1})$. Then $\|v_{j,1}\| = s_0 < \delta'/(2R_0)$. Setting

$$\delta_0 = \frac{s_0 \theta_0}{16} > 0, \tag{8.21}$$

we have $d\omega_{z_1}(\eta_j(z_1), v_{j,1}) \geq 16\delta_0$ for all j. Next, choose a small constant δ'' with

$$0 < \delta'' < \min\left\{\frac{\delta'}{3R_0}, \frac{\beta_0 \delta_0 \kappa}{100 R_0^3 L_0 C_0^2}\right\} \tag{8.22}$$

so small that

$$|d\omega_{z_1}(\eta_j(z_1), v)| \geq 8\delta_0 \quad , \quad v \in E_1^s(z_1)\,,\ \|v - v_{j,1}\| \leq \delta''\,, \tag{8.23}$$

for all $j = 1, 2, \ldots, \ell_0$. We will impose extra conditions on δ'' below.

Take an arbitrary t_0 with $0 < t_0 < \delta''/R_0$ and set $v_{j,2} = t_0 \check{v}_j \in E_1^s(z_1)$ and $y_2^{(j)} = \Phi^s_{z_1}(v_{j,2})$. Then $\|v_{j,2}\| = t_0 < \delta''/R_0$.

Next, for any $z \in \tilde{S} \cap B^s(z_1, \delta'')$ and any $z' \in B^u(z, \epsilon')$ the *local unstable holonomy map* $\mathcal{H}_z^{z'} : W^s_{\epsilon'}(z) \longrightarrow W^s_{\epsilon_0}(z')$ is well defined and uniformly (Hölder) continuous, so by (3.7) for $z' \in P \cap B^u(z, \epsilon')$ the same applies to $\widehat{\mathcal{H}}_z^{z'} = (\Phi^s_{z'})^{-1} \circ \mathcal{H}_z^{z'} \circ \Phi^s_z : E^s(z; \epsilon') \longrightarrow E^s(z'; r_0')$, taking $\epsilon' \in (0, r_0')$ sufficiently small. Thus, there exist constants $C' > 0$ and $\beta'' > 0$ (depending on the sets $\tilde{S}$ and P) so that for z, z' as above we have $\|\widehat{\mathcal{H}}_z^{z'}(u) - \widehat{\mathcal{H}}_z^{z'}(v)\| \leq C' \|u - v\|^{\beta''}$ for all $u, v \in E^s(z; \epsilon')$. Using (8.23), we can take $\epsilon'' \in (0, \epsilon')$ so small that $(2\epsilon'')^{\beta''} < \delta''$ and

$$|d\omega_{z'}(\eta_j(z'), w)| = |d\omega_{z'}(\eta_j(z'), w^{(1)})| \geq 4\delta_0 \quad , \quad j = 1, \ldots, \ell_0, \tag{8.24}$$

whenever $z' \in B^u(z,\epsilon'') \cap P$ for some $z \in \tilde{S} \cap B^s(z_1,\delta'')$ and $w = \widehat{\mathcal{H}}_z^{z'}(v)$ for some $v \in E^s(z,\epsilon')$ with $\Phi_z^s(v) \in B^s(y_1^{(j)},\delta'')$.

For later use, we will also assume that ϵ'' is chosen so small that for every $z \in S_1$, every cylinder in $W^u_{\epsilon''}(z)$ has length $> \frac{4n_0}{\min_{x\in\tilde{R}}\tilde{\tau}(x)} > 4n_0$, where n_0 is the integer with (4.9).

It follows from the uniform hyperbolicity of the flow and the fact that it is mixing (being a contact flow) that there exists an integer $N_0 \geq 1$ such that for any integer $N \geq N_0$ and any $z \in R$ we have $\mathcal{P}^N(B^u(z,\epsilon')) \cap B^s(y_1^{(j)},\delta'') \neq \emptyset$ and $\mathcal{P}^N(B^u(z,\epsilon')) \cap B^s(y_2^{(j)},\delta'') \neq \emptyset$ (see e.g. Proposition 18.3.10 in [KH] for the case of diffeomorphism; the case of flows is similar).

Fix an arbitrary $N \geq N_0$. It follows from the above that there exists $\delta_2 = \delta_2(N) \in (0,\delta'')$ and for each $i = 1,2$ and each $j = 1,\ldots,\ell_0$ a continuous map

$$\tilde{S}_0 = \tilde{S} \cap B^s(z_1,\delta_2) \ni z \mapsto y_i^{(j)}(z) \in \mathcal{P}^N(B^u(z,\epsilon'')) \cap B^s(y_i^{(j)},\delta''). \tag{8.25}$$

Then $y_i^{(j)}(z) = \Phi_z^s(v_{j,i}(z))$ for some $v_{j,i}(z) \in (\Phi_z^s)^{-1}(B^s(y_i^{(j)},\delta''))$. Set

$$P_1 = \cup_{z\in\tilde{S}_0} B^u(z,\epsilon'') \cap P' \subset \mathcal{O}^u(\tilde{S}_0).$$

Clearly $\nu^s(\tilde{S}_0) > 0$ and $\mu(P_1) > 0$. For $z' \in B^u(z,\epsilon'') \cap P_1$ set

$$v_{j,i}(z,z') = \widehat{\mathcal{H}}_z^{z'}(v_{j,i}(z)). \tag{8.26}$$

Notice that then

$$\Phi^s_{z'}(v_{j,i}(z,z')) = \pi_{y_i^{(j)}(z)}(z'). \tag{8.27}$$

It follows from $|\|v_{j,1}(z)\| - \|v_{j,1}\|| \leq R_0\delta'' < \frac{s_0}{100C_0}$ and $\|v_{j,1}\| = s_0$ that

$$\frac{s_0}{2} \leq \|v_{j,1}(z,z')\| \leq 2s_0 \quad , \quad z \in \tilde{S}_0 \ , \ z' \in B^u(z,\epsilon'') \cap P_1 \ , \ j = 1,\ldots,\ell_0. \tag{8.28}$$

8.5.2 Estimates for $|d\omega_{z_0}(\xi_0^{(1)}, v_0^{(1)})|$

Fix an arbitrary $\hat{z}_0 \in \tilde{S}_0$ and let $\kappa \in (0,1)$. Let $\mathcal{C}$ be a cylinder in $U_1(\hat{z}_0)$ contained in $W^u_{\epsilon''}(\hat{z}_0)$, and let $x_0 = \Phi^u_{\hat{z}_0}(u_0) \in \mathcal{C}$, $z_0 = \Phi^u_{\hat{z}_0}(w_0) \in \mathcal{C} \cap P_1$ satisfy (4.11) and (4.12). Denote by m the *length* of $\mathcal{C}$. Then by the choice of ϵ'', we have $m > 4n_0$, where n_0 is the integer with (4.9). Fix an arbitrary $j = 1,\ldots,\ell_0$. Let

$$x_0 = \Phi^u_{z_0}(\xi_0) \quad , \quad v_0 = v_{j,1}(\hat{z}_0, z_0) \in E^u(z_0; r_0/R_0)$$

for some $\xi_0 \in E^u(z_0; r_0')$; then $\|\xi_0\| \leq R_0 \mathrm{diam}(\mathcal{C})$. Since $z_0 \in P_1 \subset P$ and $\|w_0\| \leq \epsilon'' << r_0'$, the map $\Xi = (\Phi^u_{z_0})^{-1} \circ \Phi^u_{\hat{z}_0} : E^u(\hat{z}_0; r_0/R_0^2) \longrightarrow E^u(z_0)$ is well-defined. We have $d(\Phi^u_{z_0})^{-1}(z_0) = \mathrm{id}$, $\Xi(w_0) = 0$, $\Xi(u_0) = \xi_0$ and $d\Xi(w_0) = d(\Phi^u_{z_0})^{-1}(z_0) \circ d\Phi^u_{\hat{z}_0}(w_0) = d\Phi^u_{\hat{z}_0}(w_0)$. Now (3.8) implies[21]

$$\|\xi_0 - d\Phi^u_{\hat{z}_0}(w_0)\cdot(u_0 - w_0)\| \leq 10R_0^3\|u_0 - w_0\|^{1+\beta}. \tag{8.29}$$

[21] To prove (8.29), using C^2 coordinates in $W^u_{r_0}(\hat{z}_0)$, we can identify $W^u_{r_0}(\hat{z}_0)$ with an open subset V of $\mathbb{R}^{n^u}$ and regard $\Phi^u_{\hat{z}_0}$ and $\Phi^u_{z_0}$ as $C^{1+\beta}$ maps on V whose derivatives and their inverses are bounded by R_0. By Taylor's formula (3.8), $\Phi^u_{\hat{z}_0}(u_0) - \Phi^u_{\hat{z}_0}(w_0) = d\Phi^u_{\hat{z}_0}(w_0)\cdot(u_0 - w_0) + \eta$, for some $\eta \in \mathbb{R}^{n^u}$ with $\|\eta\| \leq R_0\|u_0 - w_0\|^{1+\beta}$. Hence $d(\Phi^u_{z_0})^{-1}(z_0)\cdot(\Phi^u_{\hat{z}_0}(u_0) - \Phi^u_{\hat{z}_0}(w_0)) = d\Phi^u_{\hat{z}_0}(w_0)\cdot(u_0 - w_0) + \eta$. Since $\hat{z}_0 \in \tilde{S}_0 \subset P$, by (3.9), $\|d\Phi^u_{\hat{z}_0}(w_0) - \mathrm{id}\| = \|d\Phi^u_{\hat{z}_0}(w_0) - d\Phi^u_{\hat{z}_0}(0)\| \leq R_0\|w_0\|^\beta$, so $\|d\Phi^u_{\hat{z}_0}(w_0)\| \leq 2R_0$. Using Taylor's formula again,

$$\Xi(u_0) - \Xi(w_0) = (\Phi^u_{z_0})^{-1}(\Phi^u_{\hat{z}_0}(u_0)) - (\Phi^u_{z_0})^{-1}(\Phi^u_{\hat{z}_0}(w_0)) = d(\Phi^u_{z_0})^{-1}(z_0)\cdot(\Phi^u_{\hat{z}_0}(u_0) - \Phi^u_{\hat{z}_0}(w_0)) + \zeta$$

for some ζ with $\|\zeta\| \leq R_0\|\Phi^u_{\hat{z}_0}(u_0) - \Phi^u_{\hat{z}_0}(w_0)\|^{1+\beta} \leq R_0\left(2R_0\|w_0 - u_0\| + R_0\|w_0 - u_0\|^{1+\beta}\right)^{1+\beta} \leq 9R_0^3\|u_0 - w_0\|^{1+\beta}$. Thus, $\xi_0 = \Xi(u_0) - \Xi(w_0) = d\Phi^u_{\hat{z}_0}(w_0)\cdot(u_0 - w_0) + \eta + \zeta$, where $\|\eta + \zeta\| \leq (R_0 + 9R_0^3)\|u_0 - w_0\|^{1+\beta} \leq 10R_0^3\|u_0 - w_0\|^{1+\beta}$.

Next, by (4.12) the direction of w_0-u_0 is close to $\eta_j(\hat{z}_0)$. More precisely, let $w_0-u_0 = t\eta_j(\hat{z}_0)+u$ for some $t \in \mathbb{R}$ and $u \perp \eta_j(\hat{z}_0)$. Then for $s = t/\|w_0-u_0\|$ we have $\frac{w_0-u_0}{\|w_0-u_0\|} = s\eta_j(\hat{z}_0) + \frac{u}{\|w_0-u_0\|}$, so $s = \left\langle \frac{w_0-u_0}{\|w_0-u_0\|}, \eta_j(\hat{z}_0)\right\rangle \geq \beta_0$, and therefore $t = s\|w_0-u_0\| \geq \beta_0\|w_0-u_0\|$. Moreover,

$$
\begin{aligned}
\|u\|^2 &= \|w_0 - w_0 - t\eta_j(\hat{z}_0)\|^2 = \|w_0-u_0\|^2 - 2t\langle w_0-u_0, \eta_j(\hat{z}_0)\rangle + t^2 \\
&= \|w_0-u_0\|^2\left(1 - 2s\left\langle \frac{w_0-u_0}{\|w_0-u_0\|}, \eta_j(\hat{z}_0)\right\rangle + s^2\right) = \|w_0-u_0\|^2(1-2s^2+s^2) \\
&= \|w_0-u_0\|^2(1-s^2) \leq (1-\beta_0^2)\|w_0-u_0\|^2 ,
\end{aligned}
$$

and therefore $\|u\| \leq \sqrt{1-\beta_0^2}\,\|w_0-u_0\|$.

Since $v_0 = v_{j,1}(\hat{z}_0, z_0) = \widehat{\mathcal{H}}^{z_0}_{\hat{z}_0}(v_{j,1}(\hat{z}_0))$, it follows from (8.24) with $z' = z_0$ and $w = v_0$ that $|d\omega_{z_0}(\eta_j(z_0), v_0)| \geq 4\delta_0$, while (8.28) gives $s_0/2 \leq \|v_0\| \leq 2s_0 \leq \delta'/R_0$. Using $d\Phi^u_{\hat{z}_0}(0) = \mathrm{id}$ and (3.9), we have $\|d\Phi^u_{\hat{z}_0}(w_0) - \mathrm{id}\| \leq R_0\|w_0\|^\beta \leq R_0(R_0\epsilon'')^\beta \leq R_0^2(\epsilon'')^\beta$. Moreover, for the constant β_0 from Sect. 4.2 we have $\beta_0^2(1+\theta_0^2/(64C_0)^2) = 1$, so $\beta_0^2\theta_0^2 = (64C_0)^2(1-\beta_0^2)$, and therefore $4C_0\sqrt{1-\beta_0^2} = \beta_0\theta_0/16$. The above, (8.29), $|d\omega_{z_0}(\eta_j(z_0), v_0))| \geq 4\delta_0$, (8.21), (8.22), $\|v_0^{(1)}\| \leq |v_0| \leq \|v_0\|$, Lemma 8.1 and the fact that $\eta_j(z_0) \in E^u_1(z_0)$ imply

$$
\begin{aligned}
&|d\omega_{z_0}(\xi_0^{(1)}, v_0^{(1)})| \\
= \;& |d\omega_{z_0}(\xi_0, v_0^{(1)})| \geq |d\omega_{z_0}(d\Phi^u_{\hat{z}_0}(w_0)\cdot(u_0-w_0), v_0^{(1)})| - |d\omega_{z_0}(\xi_0 - d\Phi^u_{\hat{z}_0}(w_0)\cdot(u_0-w_0), v_0^{(1)})| \\
\geq \;& t|d\omega_{z_0}(d\Phi^u_{\hat{z}_0}(w_0)\cdot\eta_j(z_0), v_0^{(1)})| - |d\omega_{z_0}(d\Phi^u_{\hat{z}_0}(w_0)\cdot u, v_0^{(1)})| - 10C_0R_0^3\|u_0-w_0\|^{1+\beta}\|v_0^{(1)}\| \\
\geq \;& \beta_0\|u_0-w_0\|\,[\,|d\omega_{z_0}(\eta_j(z_0), v_0^{(1)})| - |d\omega_{z_0}(d\Phi^u_{\hat{z}_0}(w_0)\cdot\eta_j(z_0) - \eta_j(z_0), v_0^{(1)})|\,] \\
& -C_0(1+R_0^2(\epsilon'')^\beta)\sqrt{1-\beta_0^2}\,\|u_0-w_0\|\|v_0^{(1)}\| - 10C_0R_0^3\|u_0-w_0\|^{1+\beta}\|v_0^{(1)}\| \\
\geq \;& \|u_0-w_0\|\,[\,\beta_0|d\omega_{z_0}(\eta_j(z_0), v_0)| - \beta_0C_0R_0^2(\epsilon'')^\beta\|v_0\| - 2C_0\sqrt{1-\beta_0^2}\,\|v_0\| - 10C_0R_0^3(2\epsilon'')^\beta\|v_0\|\,] \\
\geq \;& \|u_0-w_0\|\,[\,4\beta_0\delta_0 - 2\beta_0C_0R_0^2\delta''s_0 - 4C_0\sqrt{1-\beta_0^2}\,s_0 - 20C_0R_0^3\delta''s_0\,] \\
\geq \;& \|u_0-w_0\|\,[\,4\beta_0\delta_0 - \beta_0\delta_0 - \beta_0\delta_0 - \beta_0\delta_0\,] = \|u_0-w_0\|\,\beta_0\delta_0.
\end{aligned}
$$

Combining this with (4.11) and (3.7) gives

$$
|d\omega_{z_0}(\xi_0^{(1)}, v_0^{(1)})| \geq \frac{\beta_0\delta_0\kappa}{R_0}\,\mathrm{diam}(\mathcal{C})\,. \tag{8.30}
$$

Next, set $\tilde{\xi}_0 = \Psi^u_{z_0}(\xi_0) \in E^u(z_0)$. Then

$$
\exp^u_{z_0}(\tilde{\xi}_0) = \Phi^u_{z_0}(\xi_0) = x_0, \tag{8.31}
$$

and

$$
\frac{\kappa}{R_0}\,\mathrm{diam}(\mathcal{C}) \leq \|\xi_0\| \leq R_0\mathrm{diam}(\mathcal{C}). \tag{8.32}
$$

Setting $\tilde{v}_0 = \Psi^s_{z_0}(v_0) \in E^s(z_0)$, $y_0 = \exp^s_{z_0}(\tilde{v}_0)$ and using $v_0 = v_{j,1}(\hat{z}_0, z_0)$, (8.25) and (8.27), we get

$$
y_0 = \exp^s_{z_0}(\tilde{v}_0) = \Phi^s_{z_0}(v_0) = \pi_{y_1^{(j)}(\hat{z}_0)}(z_0) \in B^s(y_1^{(j)}, \delta'') \subset B^s(z_1, r_0'). \tag{8.33}
$$

We will now prove that

$$
|\Delta(x_0, y_0)| \geq \frac{\beta_0\delta_0\kappa}{2R_0}\,\mathrm{diam}(\mathcal{C}). \tag{8.34}
$$

From this and Lemma 4.2(a)(ii), (4.13) follows easily for $d_1 \in B^s(y_1^{(j)}(\hat{z}_0), \delta'')$ and $d_2 \in B^s(z_1, \delta'')$, using the choice of δ''.

It follows from (3.6), $\|v_0\| \leq r_0/R_0$ and $\|\xi_0\| \leq r_0/R_0$ that $\|\tilde{v}_0 - v_0\| \leq R_0\|v_0\|^{1+\beta}$ and $\|\tilde{\xi}_0 - \xi_0\| \leq R_0\|\xi_0\|^{1+\beta}$, and in particular $\|\tilde{v}_0\| \leq 2\|v_0\|$ and $\|\tilde{\xi}_0\| \leq 2\|\xi_0\| \leq 2R_0\mathrm{diam}(\mathcal{C})$.

As in Sect. 8.3.1, set $p = [\tilde{\tau}_m(z_0)]$, $q = [p/2]$, and for $j \geq 0$ define $z_j = f^j(z_0)$, $x_j = f^j(x_0)$, $y_j = f^j(y_0)$, $\hat{\xi}_j = d\hat{f}^j_{z_0}(0) \cdot \xi_0$, etc. in the same way. By the choice of $\epsilon'' > 0$ we have $p \geq 4n_0$, so $q \geq 2n_0$, and all estimates in Sect. 8.3.1 hold without change. Fix ℓ with (8.8); then (8.9) and (8.10) hold again.

We need an estimate from below for $|d\omega_{z_\ell}(\xi_\ell, v_\ell)|$ similar to (8.11). Instead of using Lemma 9.7 this time it is enough to use Lemma 3.1. Since $v_\ell = \hat{f}^\ell_{z_\ell}(v_0) \in E^s(z_\ell)$ and $z_0 \in P$ implies $L(z_0) \leq L_0$, for $w = d\hat{f}^{-\ell}_{z_\ell}(0) \cdot v_\ell$, using Lemma 3.1, we get

$$\|v_0^{(1)} - w^{(1)}\| \leq L_0(z)|v_0|^{1+\beta} \leq L_0\|v_0\|^{1+\beta}. \tag{8.35}$$

As in the proof of (8.11) we will now use the estimates in Sect. 8.3.2. It follows from Lemma 8.1, (8.9) and (8.35) that

$$\begin{aligned}
|d\omega_{z_\ell}(\xi_\ell, v_\ell)| &\geq |d\omega_{z_\ell}(\xi_\ell^{(1)}, v_\ell^{(1)})| - \sum_{i=2}^{k} |d\omega_{z_\ell}(\xi_\ell^{(i)}, v_\ell^{(i)})| \\
&\geq |d\omega_{z_\ell}(\hat{\xi}_\ell^{(1)}, v_\ell^{(1)})| - C_0 L_0 \|\xi_\ell\|^{1+\beta}\|v_\ell\| - C_0 \sum_{i=2}^{k} \|\xi_\ell^{(i)}\|\,\|v_\ell^{(i)}\| \\
&= |d\omega_{z_0}(d\hat{f}^{-\ell}_{z_\ell}(0)\cdot \hat{\xi}_\ell^{(1)}, d\hat{f}^{-\ell}_{z_\ell}(0)\cdot v_\ell^{(1)})| - C_0 L_0 \|\xi_\ell\|^{1+\beta}\|v_\ell\| - C_0 \sum_{i=2}^{k} \|\xi_\ell^{(i)}\|\,\|v_\ell^{(i)}\| \\
&= |d\omega_{z_0}(\xi_0^{(1)}, w^{(1)})| - C_0 L_0 \|\xi_\ell\|^{1+\beta}\|v_\ell\| - C_0 \sum_{i=2}^{k} \|\xi_\ell^{(i)}\|\,\|v_\ell^{(i)}\| \\
&\geq |d\omega_{z_0}(\xi_0^{(1)}, v_0^{(1)})| - C_0 L_0 R_0\,\mathrm{diam}(\mathcal{C})\|v_0\|^{1+\beta} - \mathrm{Const}\;(\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}}.
\end{aligned}$$

Combining this with (8.6) and (8.30) gives

$$\begin{aligned}
|\Delta(x_0, y_0)| &= |\Delta(x_\ell, y_\ell)| \geq |d\omega_{z_\ell}(\xi_\ell, v_\ell)| - 2C_0 R_0\,\|\xi_\ell\|\,\|v_\ell\|(\|\xi_\ell\|^\beta + \|v_\ell\|^\beta) \\
&\quad - 8C_0\left[\|\xi_\ell\|^2\,\|v_\ell\|^\vartheta + \|\xi_\ell\|^\vartheta\|v_\ell\|^2\right] \\
&\geq |d\omega_{z_0}(\xi_0^{(1)}, v_0^{(1)})| - C_0 L_0 R_0\,\mathrm{diam}(\mathcal{C})\|v_0\|^{1+\beta} - \mathrm{Const}\;(\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}} \\
&\geq \frac{\beta_0\delta_0\kappa}{R_0}\,\mathrm{diam}(\mathcal{C}) - C_0 L_0 R_0\,\mathrm{diam}(\mathcal{C})\|v_0\|^{1+\beta} - C'''(\mathrm{diam}(\mathcal{C}))^{1+\hat{\beta}}
\end{aligned}$$

for some constant $C''' > 0$. Now assume $(2\epsilon'')^{\hat{\beta}} \leq \frac{\beta_0\delta_0\kappa}{4R_0 C'''}$, and recall that $\|v_0\| \leq \delta'$ and $\mathrm{diam}(\mathcal{C}) \leq 2\epsilon''$. By (8.28), $\|v_0\| \leq 2s_0$, while (8.20) implies $\|v_0\|^\beta \leq (\delta'')^\beta < (\delta')^\beta < \frac{\beta_0\kappa\theta_0}{128 L_0 C_0 R_0}$. Thus, using (8.21), $C_0 L_0 R_0\,\mathrm{diam}(\mathcal{C})\|v_0\|^{1+\beta} \leq C_0 L_0 R_0 \mathrm{diam}(\mathcal{C})\, 2s_0\, \frac{\beta_0\kappa\theta_0}{128 L_0 C_0 R_0^2} \leq \mathrm{diam}(\mathcal{C})\,\frac{\beta_0\delta_0\kappa}{4R_0}$, and therefore $\Delta(x_0, y_0) \geq \frac{\beta_0\delta_0\kappa}{2R_0}\,\mathrm{diam}(\mathcal{C})$. This proves (8.34). ■

9 Regular distortion for Anosov flows

In this section we prove Lemma 4.1.

9.1 Expansion along E_1^u

Let again M be a C^2 complete Riemann manifold and ϕ_t be a C^2 Anosov flow on M. Set $\hat{\mu}_2 = \lambda_1 + \frac{2}{3}(\lambda_2 - \lambda_1)$ and $\hat{\nu}_1 = \lambda_1 + \frac{1}{3}(\lambda_2 - \lambda_1)$. Then $\hat{\mu}_2 < \mu_2 e^{-\hat{\epsilon}}$ and $\lambda_1 < \nu_1 < \hat{\nu}_1 < \hat{\mu}_2 < \mu_2 < \lambda_2$. Assume $\hat{\epsilon} > 0$ is so small that $e^{\hat{\epsilon}} < \frac{2\lambda_2}{\lambda_2 + \hat{\mu}_2}$.

For a non-empty set $X \subset E^u(x)$ set $\ell(X) = \sup\{\|u\| : u \in X\}$. Given $z \in \mathcal{L}$ and $p \geq 1$, setting $x = f^p(z)$, define $\widehat{B}_p^u(z, \delta) = \{u \in E^u(z) : \|\hat{f}_z^p(u)\| \leq \delta\}$.

For any $v = v^{(1)} + v^{(2)} + \ldots + v^{(k)} \in E^u(x)$ with $v^{(j)} \in E_j^u$, set $\tilde{v}^{(2)} = v^{(2)} + \ldots + v^{(k)} \in \widetilde{E}_2^u(x)$.

Lemma 9.1. *There exists a regularity function $\hat{r}(x)$ ($x \in \mathcal{L}$) with $\hat{r}(x) \leq r(x)$ for all $x \in \mathcal{L}$ such that for any $x \in \mathcal{L}$ and any $V = V^{(1)} + \widetilde{V}^{(2)} \in E^u(x; \hat{r}(x))$, setting $y = f^{-1}(x)$ and $U = \hat{f}_x^{-1}(V)$, we have $\|\widetilde{U}^{(2)}\|'_y \leq \frac{\|\widetilde{V}^{(2)}\|'_x}{\hat{\mu}_2}$ and $\|U^{(1)}\|'_y \geq \frac{\|V^{(1)}\|'_x}{\hat{\nu}_1}$. Moreover, if $V, W \in E^u(x; \hat{r}(x))$ and $W^{(1)} = V^{(1)}$, then for $S = \hat{f}_x^{-1}(W)$ we have $\|\widetilde{U}^{(2)} - \widetilde{S}^{(2)}\|'_y \leq \frac{\|\widetilde{V}^{(2)} - \widetilde{W}^{(2)}\|'_x}{\hat{\mu}_2}$, and, if $\widetilde{W}^{(2)} = \widetilde{V}^{(2)} \in E^u(x; \hat{r}(x))$ and $S = \hat{f}_x^{-1}(W)$ again, then $\|U^{(1)} - S^{(1)}\|'_y \geq \frac{\|V^{(1)} - W^{(1)}\|'_x}{\hat{\nu}_1}$.*

Proof. The first estimates follow from Lemma 3.5 in [St4]. The proofs of the others use the same standard arguments, so we omit them. ■

Next, for any $y \in \mathcal{L}$, $\epsilon \in (0, r(y)]$ and $p \geq 1$ set $\widehat{B}_p^{u,1}(y, \epsilon) = \widehat{B}_p^u(u; \epsilon) \cap E_1^u(y)$.

Replacing the regularity function $\hat{r}$ with a smaller one, we may assume that $L(x)(\hat{r}(x))^\beta \leq \frac{1}{100 n_1}$ for all $x \in \mathcal{L}$, where $n_1 = \dim(E_1^u(x))$.

The proof of the following lemma is almost the same as the proof of Proposition 3.2 in [St3], the only new part is (9.1). We omit the details.

Lemma 9.2 *Let $z \in \mathcal{L}$ and $x = f^p(z)$ for some integer $p \geq 1$, and let $\epsilon \in (0, \tilde{r}(x)]$. Then $\ell(\widehat{B}_p^u(z, \epsilon)) \leq 2\Gamma^2(x) R(z)\, \ell(\widehat{B}_p^{u,1}(z, \epsilon))$. Moreover for any $\epsilon' \in (0, \epsilon]$ there exists $u \in \widehat{B}_p^{u,1}(z, \epsilon')$ with*

$$\|u\| \geq \frac{\epsilon'}{2\epsilon \Gamma^2(x) R(z)} \ell(\widehat{B}_p^u(z, \epsilon)) \quad \text{and} \quad \|\hat{f}_z^p(u)\| \geq \epsilon'/2. \tag{9.1}$$

To prove the main result in this section, it remains to compare diameters of sets of the form $\widehat{B}_p^{u,1}(y, \epsilon)$.

Lemma 9.3. *There exist a regularity function $\hat{r}(x) < 1$ ($x \in \mathcal{L}$) such that:*

(a) $\ell\left(\widehat{B}_p^{u,1}(f^{-p}(x), \epsilon)\right) \leq 16 n_1 \frac{\epsilon}{\delta}\, \ell\left(\widehat{B}_p^{u,1}(f^{-p}(x), \delta)\right)$ *for any $x \in \mathcal{L}$, any $0 < \delta \leq \epsilon \leq \hat{r}(x)$ and any integer $p \geq 1$.*

(b) $\ell\left(\widehat{B}_p^{u,1}(f^{-p}(x), \delta)\right) \leq \rho\, \ell\left(\widehat{B}_p^{u,1}(f^{-p}(x), \epsilon)\right)$ *for any $x \in \mathcal{L}$, any $0 < \epsilon \leq \hat{r}(x)$, any $\rho \in (0, 1)$, any δ with $0 < \delta \leq \frac{\rho\, \epsilon}{16 n_1}$ and any integer $p \geq 1$.*

Theorem 9.4. *There exist a regularity function $\hat{r}(x) < 1$ ($x \in \mathcal{L}$) such that:*

(a) *For any $x \in \mathcal{L}$ and any $0 < \delta \leq \epsilon \leq \hat{r}(x)$ we have $\ell\left(\widehat{B}_p^u(z, \epsilon)\right) \leq \frac{32 n_1 \Gamma^2(x) R(z) \epsilon}{\delta}\, \ell\left(\widehat{B}_p^u(z, \delta)\right)$ for any integer $p \geq 1$, where $z = f^{-p}(x)$.*

(b) *For any $x \in \mathcal{L}$, any $0 < \epsilon \leq \hat{r}(x)$, any $\rho \in (0, 1)$ and any δ with $0 < \delta \leq \frac{\rho \epsilon}{32 n_1 \Gamma^2(x) R(z)}$ we have $\ell\left(\widehat{B}_p^u(z, \delta)\right) \leq \rho\, \ell\left(\widehat{B}_p^u(z, \epsilon)\right)$ for all integers $p \geq 1$, where $z = f^{-p}(x)$.*

(c) *For any $x \in \mathcal{L}$, any $0 < \epsilon' < \epsilon \leq \hat{r}(x)/2$, any $0 < \delta < \frac{\epsilon'}{100n_1}$ and any integer $p \geq 1$, setting $z = f^{-p}(x)$, there exists $u \in \widehat{B}^{u,1}_p(z,\epsilon')$ such that for every $v \in E^u(z)$ with $\|\hat{f}^p_z(u) - \hat{f}^p_z(v)\| \leq \delta$ we have $\|v\| \geq \frac{\epsilon'}{4\epsilon\Gamma^2(x)R(z)}\, \ell(\widehat{B}^u_p(z,\epsilon))$.*

Using Lemma 9.3, we will now prove Theorem 9.4. The proof of Lemma 9.3 is given in the next sub-section. In fact, part (c) above is a consequence of Lemmas 3.1 and 9.2 and does not require Lemma 9.3.

Proof of Theorem 9.4. Choose the function $\hat{r}(x)$ as in Lemma 9.3.

(a) Let $0 < \delta < \epsilon \leq \tilde{r}(x)$. Given an integer $p \geq 1$, set $z = f^{-p}(x)$. Then Lemmas 9.2 and 9.3 imply $\ell(\widehat{B}^u_p(z,\epsilon)) \leq 2\Gamma^2(x)R(z)\, \ell(\widehat{B}^{u,1}_p(z,\epsilon)) \leq 32n_1\Gamma^2(x)R(z)\frac{\epsilon}{\delta}\, \ell(\widehat{B}^u_p(z,\delta))$.

(b) Let $x \in \mathcal{L}$ and $0 < \epsilon \leq \hat{r}(x)$. Given $\rho \in (0,1)$, set $\rho' = \frac{\rho}{2\Gamma^2(x)R(z)} < \rho$. By Lemma 9.3(b) and Lemma 9.2, if $0 < \delta \leq \frac{\rho'\epsilon}{16n_1}$ then

$$\ell(\widehat{B}^u_p(z,\delta)) \leq 2\Gamma^2(x)R(z)\, \ell(\widehat{B}^{u,1}_p(z,\delta)) \leq 2\Gamma^2(x)R(z)\, \rho'\, \ell(\widehat{B}^u_p(z,\epsilon)) = \rho\, \ell(\widehat{B}^u_p(z,\epsilon)),$$

which completes the proof.

(c) Given $x \in \mathcal{L}$, $z = f^{-p}(x)$, let ϵ', ϵ and δ be as in the assumptions. Let $u \in \widehat{B}^{u,1}_p(z,\epsilon')$ be such that $\|u\|$ is the maximal possible. By Lemma 9.2, for $U = \hat{f}^p_z(u) \in E^u_1(x)$ we have $\epsilon'/2 \leq \|U\| \leq \epsilon'$. Setting $W = d\hat{f}^p_z(0)\cdot u \in E^u_1(x)$, Lemmas 3.1 and 9.1, and the assumption $L(x)(\hat{r}(x))^\beta \leq \frac{1}{100n_1}$ give $\|W - U\| \leq L(x)|U|^{1+\beta} \leq \frac{\|U\|}{100n_1}$, so $\|W\| \leq \frac{101\epsilon'}{100}$.

Let $v = (v^{(1)}, \tilde{v}^{(2)}) \in E^u(z)$ be such that for $V = \hat{f}^p_z(v)$ we have $\|V - U\| \leq \delta$. Then $|V - U| \leq \delta$, so $\|V^{(1)} - U^{(1)}\| \leq \delta$ and $\|\widetilde{V}^{(2)}\| \leq \delta$. Set $S = d\hat{f}^p_z(0)\cdot v$; then $S^{(1)} = d\hat{f}^p_z(0)\cdot v^{(1)}$. By Lemmas 3.1 and 9.1, $\|S^{(1)} - V^{(1)}\| \leq \frac{|V|}{100n_1} \leq \frac{\|V\|}{100n_1} \leq \frac{\epsilon'+\delta}{100n_1}$, so

$$\|S^{(1)} - W^{(1)}\| \leq \|S^{(1)} - V^{(1)}\| + \|V^{(1)} - U^{(1)}\| + \|U^{(1)} - W^{(1)}\| \leq \frac{\epsilon'+\delta}{100n_1} + \delta + \frac{\epsilon'}{100n_1} < \frac{\epsilon'}{30n_1}.$$

Choose an orthonormal basis $e_1, \ldots, e_{n_1}$ in $E^u_1(x)$ such that $W = W^{(1)} = c_1 e_1$ for some $c_1 \in [\epsilon'/3, \epsilon']$. Let $S^{(1)} = \sum_{i=1}^{n_1} d_i e_i$. Then the above implies $|d_1 - c_1| \leq \frac{\epsilon'}{30n_1}$ and $|d_i| \leq \frac{\epsilon'}{30n_1}$ for all $i = 2, \ldots, n_1$.

Notice that for any $i = 1, \ldots, n_1$, $u' = d\hat{f}^{-p}_x(0)\cdot(\epsilon' e_i/2) \in \widehat{B}^{u,1}_p(z,\epsilon')$. Indeed, by Lemma 3.1, $\|\hat{f}^p_z(u') - d\hat{f}^p_z(0)\cdot u'\| \leq \frac{\|\epsilon' e_i/2\|}{100n_1} = \frac{\epsilon'}{200n_1}$, so $\|\hat{f}^p_z(u')\| \leq \|d\hat{f}^p_z(0)\cdot u'\| + \frac{\epsilon'}{200n_1} = \frac{\epsilon'}{2} + \frac{\epsilon'}{200n_1} < \epsilon'$. By the choice of u, this implies $\|u'\| \leq \|u\|$, so $\|d\hat{f}^{-p}_x(0)\cdot e_i\| \leq \frac{2\|u\|}{\epsilon'}$ for all $i = 1, \ldots, n_1$.

The above yields

$$\|d_1\, d\hat{f}^{-p}_z(0)\cdot e_1\| \geq \|c_1 d\hat{f}^{-p}_z(0)\cdot e_1\| - \|(d_1 - c_1) d\hat{f}^{-p}_z(0)\cdot e_1\| \geq \|u\| - \frac{\epsilon'}{30n_1}\cdot\frac{2\|u\|}{\epsilon'} = \|u\|\left(1 - \frac{1}{15n_1}\right).$$

Moreover, for $i \geq 2$ we have $\|d_i\, d\hat{f}^{-p}_z(0)\cdot e_i\| \leq \frac{\epsilon'}{30n_1}\cdot\frac{2\|u\|}{\epsilon'} = \frac{\|u\|}{15n_1}$. Hence

$$\begin{aligned}\|v^{(1)}\| &= \|d\hat{f}^{-p}_x(0)\cdot S^{(1)}\| = \|\sum_{i=1}^{n_1} d_i\, d\hat{f}^{-p}_z(0)\cdot e_i\| \\ &\geq \|d_1\, d\hat{f}^{-p}_z(0)\cdot e_1\| - \sum_{i=2}^{n_1} \|d_i\, d\hat{f}^{-p}_z(0)\cdot e_i\| \geq \|u\|\left(1 - \frac{1}{15n_1}\right) - n_1\frac{\|u\|}{15n_1} > \frac{\|u\|}{2}.\end{aligned}$$

Combining this with Lemma 9.2 gives, $\|v\| \geq |v| \geq \|v^{(1)}\| > \frac{\|u\|}{2} \geq \frac{\epsilon'}{4\epsilon\Gamma^2(x)R(z)}\, \ell(\widehat{B}^u_p(z,\epsilon))$. ■

What we actually need later is the following immediate consequence of Theorem 9.4 which concerns sets of the form $B^u_T(z,\epsilon) = \{y \in W^u_\epsilon(z) : d(\phi_T(y), \phi_T(z)) \leq \epsilon\}$, where $z \in \mathcal{L}$, $\epsilon, T > 0$.

Corollary 9.5. *There exist an $\hat{\epsilon}$-regularity function $\hat{r}(x) < 1$ ($x \in \mathcal{L}$) and a global constant $L_1 \geq 1$ such that:*

(a) *For any $x \in \mathcal{L}$ and $0 < \delta \leq \epsilon \leq \hat{r}(x)$ we have* $\mathrm{diam}\,(B^u_T(z,\epsilon)) \leq L_1\, \Gamma^2(x) R(z) \frac{\epsilon}{\delta}\, \mathrm{diam}\,(B^u_T(z,\delta))$ *for all $T > 0$, where $z = \phi_{-T}(x)$.*

(b) *For any $x \in \mathcal{L}$, any $0 < \epsilon \leq \hat{r}(x)$, any $\rho \in (0,1)$ and any δ with $0 < \delta \leq \frac{\rho\epsilon}{L_1\Gamma^2(x)R(z)}$ we have* $\mathrm{diam}\,(B^u_T(z,\delta)) \leq \rho\, \mathrm{diam}\,(B^u_T(z,\epsilon))$ *for all $T > 0$, where $z = \phi_{-T}(x)$.*

(c) *For any $x \in \mathcal{L}$, any $0 < \epsilon' < \epsilon \leq \hat{r}(x)$, any $0 < \delta \leq \frac{\epsilon'}{100 n_1}$ and any $T > 0$, for $z = \phi_{-T}(x)$ there exists $z' \in B^u_T(z,\epsilon')$ so that $d(z,y) \geq \frac{\epsilon'}{L_1\epsilon\Gamma^2(x)R(z)}$* $\mathrm{diam}(B^u_T(z,\epsilon))$ *for all $y \in B^u_T(z',\delta)$.* ■

9.2 Linearization along E^u_1

Here we prove Lemma 9.3 using arguments similar to these in the proofs of Theorem 3.1 and Lemma 3.2 in [St4]. We use the notation from Sect. 9.1. Let $\hat{r}(x)$, $x \in \mathcal{L}$, be as in Lemma 9.1.

Proposition 9.6. *There exist regularity functions $\hat{r}_1(x) \leq \hat{r}(x)$ and $L(x)$, $x \in \mathcal{L}$, such that:*

(a) *There exists $F_x(u) = \lim_{p\to\infty} d\hat{f}^p_{f^{-p}(x)}(0) \cdot \hat{f}^{-p}_x(u) \in E^u_1(x;\hat{r}(x))$ for every $x \in \mathcal{L}$ and every $u \in E^u_1(x;\hat{r}_1(x))$. Moreover, $\|F_x(u) - u\| \leq L(x)\, \|u\|^{1+\beta}$ for any $u \in E^u_1(x,\hat{r}_1(x))$ and any integer $p \geq 0$.*

(b) *The maps $F_x : E^u_1(x;\hat{r}_1(x)) \longrightarrow F_x(E^u_1(x;\hat{r}_1(x))) \subset E^u_1(x;\hat{r}(x))$ ($x \in \mathcal{L}$) are uniformly Lipschitz. More precisely, $\|F_x(u) - F_x(v) - (u - v)\| \leq C_1\, [\|u - v\|^{1+\beta} + \|v\|^\beta \cdot \|u - v\|]$ for all $x \in \mathcal{L}$ and $u, v \in E^u_1(x;\hat{r}_1(x))$.*

(c) *For any $x \in M$ and any integer $q \geq 1$ we have $d\hat{f}^q_{x_q}(0) \circ F_{x_q}(v) = F_x \circ \hat{f}^q_{x_q}(v)$, where $x_q = f^{-q}(x)$, for any $v \in E^u_1(x_q;\hat{r}_1(x_q))$ with $\|\hat{f}^q_{x_q}(v)\| \leq \hat{r}_1(x)$.*

As in [St4] this is derived from the following lemma. Part (b) below is a bit stronger than what is required here, however we need it in this form for the proof of Lemma 4.2 in Sect. 8.

Lemma 9.7. *There exist regularity functions $\hat{r}_1(x)$ and $L(x)$, $x \in \mathcal{L}$ with the following properties:*

(a) *If $x \in M$, $z = f^p(x)$ and $\|\hat{f}^p_z(v)\| \leq r(x)$ for some $v \in E^u_1(z;\hat{r}_1(z))$ and some integer $p \geq 1$, then $\|d\hat{f}^p_z(0) \cdot v\| \leq 2\|\hat{f}^p_z(v)\|$ and $\|d\hat{f}^p_z(0) \cdot v - \hat{f}^p_z(v)\| \leq L(x)\, \|\hat{f}^p_x(v)\|^{1+\beta}$. Similarly, if $\|d\hat{f}^p_z(0) \cdot v\| \leq \hat{r}_1(x)$ for some $v \in E^u_1(z)$ and some integer $p \geq 1$, then $\|\hat{f}^p_x(v)\| \leq 2\|d\hat{f}^p_x(0) \cdot v\|$ and $\|\hat{f}^p_x(v) - d\hat{f}^p_x(0) \cdot v\| \leq L(x)\, \|d\hat{f}^p_x(0) \cdot v\|^{1+\beta}$.*

(b) *For any $x \in \mathcal{L}$ and any $p \geq 1$, the map $F^p_x = d\hat{f}^p_z(0) \circ (\hat{f}^p_x)^{-1}$, where $z = f^{-p}(x)$, is such that $\|[(F^p_x(a))^{(1)} - (F^p_x(b))^{(1)}] - [a^{(1)} - b^{(1)}]\| \leq L(x)\, [\|a - b\|^{1+\beta} + \|b\|^\beta \cdot \|a - b\|]$ for all $a, b \in E^u(x;\hat{r}_1(x))$.*

Proof of Lemma 9.7. Set $\hat{r}_1(x) = \hat{r}(x)/2$, $x \in \mathcal{L}$. Part (a) follows from Lemma 3.1 (see also the Remark after the lemma).

The proofs of the other parts are almost one-to-one repetitions of arguments from the proof of Lemma 3.2 in [St4], so we omit them. ■

Proof of Proposition 9.6. This is done following the arguments from the proof of Theorem 3.1 in [St3]. We omit the details again. ■

For $z \in \mathcal{L}$, $\epsilon \in (0, \hat{r}_1(z)]$ and an integer $p \geq 0$ set $\widetilde{B}^{u,1}_p(z,\epsilon) = F_z(\widehat{B}^{u,1}_p(z,\epsilon)) \subset E^u_1(z;\hat{r}(z))$. Then, using Proposition 9.6(c) we get

$$d\hat{f}_x^{-1}(0)(\widetilde{B}^{u,1}_{p+1}(x,\delta)) \subset \widetilde{B}^{u,1}_p(f^{-1}(x),\delta) \quad , \quad x \in \mathcal{L}\ ,\ p \geq 1. \tag{9.2}$$

Indeed, if $\eta \in \widetilde{B}^{u,1}_{p+1}(x,\delta)$, then $\eta = F_x(v)$ for some $v \in \widehat{B}^{u,1}_{p+1}(x,\delta)$, and then clearly $w = \hat{f}_x^{-1}(v) \in \widehat{B}^{u,1}_p(x,\delta)$. Setting $y = f^{-1}(x)$, by Proposition 9.6(c), $\eta = F_x(v) = F_x(\hat{f}_y(w)) = d\hat{f}_y(0)\cdot(F_y(w))$, so $d\hat{f}_x^{-1}(0)\cdot\eta = F_y(w) \in \widetilde{B}^{u,1}_p(y,\delta)$. Moreover, locally near 0 we have an equality in (9.2), i.e. if $\delta' \in (0,\delta)$ is sufficiently small, then $d\hat{f}_x^{-1}(0)(\widetilde{B}^{u,1}_{p+1}(x,\delta)) \supset \widetilde{B}^{u,1}_p(f^{-1}(x),\delta')$.

To prove part (a) of Lemma 9.3 we have to establish the following lemma which is similar to Lemma 4.4 in [St4] (see also the Appendix in [St4]) and the proof uses almost the same argument.

Lemma 9.8. *Let $x \in \mathcal{L}$ and $0 < \delta \leq \epsilon \leq \hat{r}_1(x)$. Then $\ell(\widetilde{B}^{u,1}_p(f^{-p}(x),\epsilon)) \leq 4n_1\frac{\epsilon}{\delta}\,\ell(\widetilde{B}^{u,1}_p(f^{-p}(x),\delta))$ for any integer $p \geq 0$, where $n_1 = \dim(E^u_1(x))$.*

Proof of Lemma 9.8. Choose an orthonormal basis $e_1, e_2, \ldots, e_{n_1}$ in $E^u_1(x)$ and set $u_i = \frac{\delta}{4}e_i$.

Consider an arbitrary integer $p \geq 1$ and set $z = f^{-p}(x)$. Given $v \in \widetilde{B}^{u,1}_p(z,\epsilon)$, we have $v = F_z(w)$ for some $w \in \widehat{B}^{u,1}_p(z,\epsilon)$. Then $\|\hat{f}^p_z(w)\| \leq \epsilon$. Now it follows from Proposition 9.6 that $\|d\hat{f}^p_z(0)\cdot v\| = \|d\hat{f}^p_z(0)\cdot F_z(w)\| = \|F_x(\hat{f}^p_z(w))\| \leq 2\|\hat{f}^p_z(w)\| \leq 2\epsilon$. So, $u = d\hat{f}^p_z(0)\cdot v \in E^u_1(x,2\epsilon)$. We have $u = \sum_{s=1}^{n_1} c_s\, u_s$ for some real numbers c_s and $u = \sum_{s=1}^{n_1}\frac{\delta c_s}{4}e_s$, so $\sqrt{\sum_{s=1}^{n_1} c_s^2} = \frac{4}{\delta}\|u\|$ and therefore $|c_s| \leq 4\epsilon\,\frac{1}{\delta}$ for all $s = 1,\ldots,n_1$.

We have $v_j = d\hat{f}_x^{-p}(0)\cdot u_j \in \widetilde{B}^{u,1}_p(z,\delta)$. Indeed, since $\|u_j\| \leq \frac{\delta}{4}$, we have $u_j = F_x(u'_j)$ for some $u'_j \in E^u_1(x,\delta/2)$. Set $v'_j = \hat{f}_x^{-p}(u'_j)$; then $\|\hat{f}^p_z(v'_j)\| \leq \frac{\delta}{2}$, so $v'_j \in \widehat{B}^{u,1}_p(x,\delta/2)$ and therefore $v_j = F_z(v'_j) \in \widetilde{B}^{u,1}_p(z,\delta)$. Using Proposition 9.6, we get $d\hat{f}_x^{-p}(0)\cdot u_j = d\hat{f}_x^{-p}(0)\cdot F_x(u'_j) = F_z(\hat{f}_x^{-p}(u'_j)) = F_z(v'_j) = v_j$. It now follows that

$$\|v\| = \|d\hat{f}_x^{-p}(0)\cdot u\| = \|\sum_{s=1}^{n_1} c_s\, d\hat{f}_x^{-p}(0)\cdot u_s\| \leq n_1\,4\epsilon\,\frac{1}{\delta}\max_{1\leq s\leq n_1}\|v_s\| \leq 4n_1\frac{\epsilon}{\delta}\,\ell(\widetilde{B}^{u,1}_p(z,\delta)).$$

Therefore $\ell(\widetilde{B}^{u,1}_p(z,\epsilon)) \leq 4n_1\frac{\epsilon}{\delta}\,\ell(\widetilde{B}^{u,1}_p(z,\delta))$. ■

Lemma 9.3(b) is a consequence of the following.

Lemma 9.9. *Let $x \in \mathcal{L}$ and let $0 < \epsilon \leq \hat{r}_1(x)$ and $\rho \in (0,1)$. Then for any δ with $0 < \delta \leq \frac{\rho\epsilon}{4n_1}$ we have $\ell(\widetilde{B}^{u,1}_p(f^{-p}(x),\delta)) \leq \rho\,\ell(\widetilde{B}^{u,1}_p(f^{-p}(x),\epsilon))$ for any integer $p \geq 0$.*

Proof of Lemma 9.9. As in the proof of Lemma 4.1(b) in [St4], we have to repeat the argument in the proof of Lemma 9.8. We omit the details. ■

9.3 Proof of Lemma 4.4

Using Corollary 9.5, the proof is not much different from that of Theorem 4.2 in [St3] (and the proof of Proposition 3.3 in [St2]). We sketch some of the details for completeness.

Let c_0, γ and γ_1 be the constants from (2.1), and let $B > 0$ be a *Lipschitz constant* for the *projection along the flow* $\psi : \cup_{i=1}^{k_0}\phi_{[-\epsilon,\epsilon]}(D_i) \longrightarrow \cup_{i=1}^{k_0} D_i$. Next, assuming that the constant $\hat{\epsilon} > 0$

from Sect. 3 is chosen so that $e^{\hat{\epsilon}}/\gamma < 1$, fix an integer $p_0 \geq 1$ such that $\frac{2\Gamma_0 R_0 e^{2\hat{\epsilon}} r_1}{r_0} < (\mu_1 e^{\hat{\epsilon}})^{p_0}$ and $\frac{1}{c_0(\gamma e^{-\hat{\epsilon}})^{p_0}} < \frac{r_0}{2}$, and set $r_0' = r_0 e^{-(p_0+1)\hat{\epsilon}}$.

First, as in the proof of Theorem 4.2 in [St3], we observe that for any $z \in \widetilde{R}_j$ with $\widetilde{\mathcal{P}}^{p_0+1}(z) \in \widetilde{P}$ for some j, if $C_V[\imath']$ ($\imath' = [i_0, \ldots, i_{p_0+1}]$ with $i_0 = j$) is the cylinder of length $p_0 + 1$ in $V = W^u_{\widetilde{R}}(z)$ containing z then for $\imath = [i_0, \ldots, i_{p_0}]$ we have $C_V[\imath] \subset B_V(z, r_0')$ and $r(z) \geq r_0'$.

(a), (b) These are done exactly as the proofs of parts (a) and (b) of Theorem 4.2 in [St3].

(c) Let again $m > p_0$, and let $\imath = [i_0, i_1, \ldots, i_m]$ and be an admissible sequence. Let $z \in \widetilde{R}_{i_0} \cap \widetilde{P}$ be such that $z' = \widetilde{\mathcal{P}}^m(z) \in \widetilde{P}$ and $\widetilde{\mathcal{P}}^j(z) \in \widetilde{R}_{i_j}$ for all $j = 0, 1, \ldots, m$. Set $\widetilde{\mathcal{P}}^{m-p_0}(z) = z''$, $V = W^u_{\widetilde{R}}(z'')$, $\mathcal{C} = C[\imath]$. If $z' = \phi_T(z)$ and $z'' = \phi_t(z)$, then $\phi_{T-t}(z'') = z'$, so $T - t = \tilde{\tau}_{p_0}(z'') < p_0$. Thus, $r(z'') \geq r(z') e^{-p_0 \hat{\epsilon}} \geq r_0 e^{-p_0 \hat{\epsilon}} > r_0'$. As in part (a), for the cylinder $\tilde{\mathcal{C}} = C_V[i_{m-p_0}, i_{m-p_0+1}, \ldots, i_m]$ in V, we have $z'' \in B_V(z'', c_0 r_0/\gamma_1^{p_0}) \subset \tilde{\mathcal{C}} = \widetilde{\mathcal{P}}^{m-p_0}(\mathcal{C}) \subset B_V(z'', r_0')$. Setting $\epsilon' = c_0 r_0/\gamma_1^{p_0} < \epsilon = r_0'$, it follows from Corollary 9.5(c) that for $0 < \delta = \frac{\epsilon'}{100 n_1}$ there exists $x \in B^u_t(z, \epsilon')$ such that for every $y \in W^u_\delta(z)$ with $d(\phi_t(y), \phi_t(x)) \leq \delta$ we have

$$d(z, y) \geq \frac{\epsilon'}{L_1 \epsilon \Gamma_0^2 R_0} \operatorname{diam}(B^u_T(z, r_0')) \geq \frac{c_0 r_0}{L_1 B r_0' \gamma_1^{p_0} \Gamma_0^2 R_0} \operatorname{diam}(\mathcal{C}), \tag{9.3}$$

since $\mathcal{C} \subset B^u_t(z, B r_0')$.

Take the integer $q_0 \geq 1$ so large that $\frac{1}{c_0 \gamma^{p_0+q_0}} < \frac{\delta}{B} = \frac{\epsilon'}{100 B n_1}$. Let $\mathcal{C}' = C[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+q_0}]$ be the sub-cylinder of $\mathcal{C}$ of co-length q_0 containing x. Then for the cylinder

$$\tilde{\mathcal{C}}' = C_V[i_{m-p_0}, i_{m-p_0+1}, \ldots, i_m, i_{m+1}, \ldots, i_m, i_{m+q_0}] \subset V$$

we have $\widetilde{\mathcal{P}}^{m-p_0}(x) \in \tilde{\mathcal{C}}'$ and $\operatorname{diam}(\tilde{\mathcal{C}}') < \frac{1}{c_0 \gamma^{p_0+q_0}} < \delta$. Since for any $y \in \mathcal{C}'$ we have $\widetilde{\mathcal{P}}^{m-p_0}(y) \in \tilde{\mathcal{C}}'$, it follows that $d(\widetilde{\mathcal{P}}^{m-p_0}(x), \widetilde{\mathcal{P}}^{m-p_0}(y)) < \delta$ and therefore $d(\phi_t(x), \phi_t(y)) < \delta$. Thus, y satisfies (9.3). This proves the assertion with $\rho_1 = \frac{c_0 r_0}{L_1 B r_0' \gamma_1^{p_0} \Gamma_0^2 R_0}$. ■

References

[An] N. Anantharaman, Precise counting results for closed orbits of Anosov flows, *Ann. Scient. Éc. Norm. Sup.* **33** (2000), 33-56.

[BaG] V. Baladi and S. Gouëzel, Banach spaces for piecewise cone hyperbolic maps, *J. Modern Dynamics* **4** (2010), 91-137.

[BaL] V. Baladi and C. Liverani, Exponential decay of correlations for piecewise cone hyperbolic contact flows, *Comm. Math. Phys.* **314** (2012), 689-773.

[BaT] V. Baladi and M. Tsujii, Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms, *Ann. Inst. Fourier* **57** (2007), 127-154.

[BP] L. Barreira and Ya. Pesin, *Lyapunov exponents and smooth ergodic theory*, University Lecture Series 23, American Mathematical Society, Providence, RI, 2001.

[BPS] L. Barreira, Ya. Pesin and J. Schmeling, Dimension and product structure of hyperbolic measures, *Ann. of Math.* **149** (1999), 755-783.

[BKL] M. Blank, G. Keller and C. Liverani, Ruelle-Perron-Frobenius spectrum for Anosov maps, *Nonlinearity* **15** (2002), 1905-1973.

[B] R. Bowen, Symbolic dynamics for hyperbolic flows, *Amer. J. Math.* **95** (1973), 429-460.

[Ch1] N. Chernov, Markov approximations and decay of correlations for Anosov flows, *Ann. of Math.* **147** (1998), 269-324.

[Ch2] N. Chernov, Invariant measures for hyperbolic dynamical systems, In: *Handbook of Dynamical Systems*, Ed. by A. Katok and B. Hasselblatt, Vol. 1A, pp. 321-407, North-Holland, Amsterdam, 2002.

[Ch3] N. Chernov, A stretched exponential bound on time correlations for billiard flows, *J. Stat. Phys.* **127** (2007), 21-50.

[ChY] N. Chernov and L.-S. Young, Decay of correlations for Lorentz gases and hard balls, *Encycl. of Math. Sc., Math. Phys.* II, Vol. 101, Ed. Szasz (2001), 89-120.

[CFS] I.P. Cornfeld, S.V. Fomin and Ya.G, Sinai, *Ergodic theory*, Springer-Verlag, Berlin, 1982.

[DL] M. Demers and C. Liverani, Stability of statistical properties in two-dimensional piecewise hyperbolic maps, *Trans. Amer. Math. Soc.* **360** (2008), 4777-4814.

[D] D. Dolgopyat, On decay of correlations in Anosov flows, *Ann. of Math.* **147** (1998), 357-390.

[DZ] S. Dyatlov and M. Zworski, Dynamical zeta functions for Anosov flows via microlocal analysis, Preprint 2013, arXiv:1306.4203.

[FT] F. Faure and M. Tsujii, Band structure of the Ruelle spectrum of contact Anosov flows, Preprint arXiv:1301.5525.

[GLP] P. Giulietti, C. Liverani and M. Pollicott, Anosov flows and dynamical zeta functions , *Ann. of Math.* **178** (2013), 687-773.

[GL] S. Gouëzel and G. Liverani, Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties, *J. Diff. Geom.* **79** (2008), 433-477.

[Ha1] B. Hasselblatt, Regularity of the Anosov splitting and horospheric foliations, *Ergod. Th.& Dynam. Sys.* **14** (1994), 645-666.

[Ha2] B. Hasselblatt, Regularity of the Anosov splitting II, *Ergod. Th.& Dynam. Sys.* **17** (1997), 169-172.

[HW] B. Hasselblatt and A. Wilkinson, Prevalence of Non-Lipschitz Anosov foliations, *Ergod. Th. & Dynam. Sys.* **19** (1999), 643-656.

[HS] B. Hasselblatt and J. Schmeling, Dimension product structure of hyperbolic sets, *Electronic Res. Announcements of the Amer. Math. Soc.* **10** 2004, 88-96.

[KB] A. Katok and Burns, Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems, *Ergod. Th. & Dynam. Sys.* **14** (1994), 757-785.

[KH] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, Cambridge 1995.

[LY1] F. Ledrappier and L.-S. Young, The metric theory of diffeomorphisms: Part I: Characterization of measures satisfying Pesin's entropy formula, *Ann. of Math.* **122** (1985), 540-574.

[LY2] F. Ledrappier and L.-S. Young, The metric theory of diffeomorphisms: Part II: Relations between entropy, exponents and dimension, *Ann. of Math.* **122** (1985), 509-539.

[L1] C. Liverani, On contact Anosov flows, *Ann. of Math.* **159** (2004), 1275-1312.

[L2] C. Liverani, Invariant measures and their properties. A functional analytic point of view, In: *Dynamical Systems. Part II: Topological Geometrical and Ergodic Properties of Dynamics.* Pubblicazioni della Classe di Scienze, Scuola Normale Superiore, Pisa. Centro di Ricerca Matematica "Ennio De Giorgi" : Proceedings. Published by the Scuola Normale Superiore in Pisa (2004).

[M] I. Melbourne, Rapid decay of correlations for nonuniformly hyperbolic flows, *Trans. Amer. Math. Soc.* **359** (2007), 2421-2441.

[N] F. Naud, Expanding maps on Cantor sets and analytic continuation of zeta functions, *Ann. Sci. Ecole Norm. Sup.* **38** (2005), 116-153.

[NZ] S. Nonnenmacher and M. Zworski, Decay of correlations for normally hyperbolic trapping, Preprint 2013 (arXiv: 1302:4483).

[PP] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque* **187-188**, (1990).

[P1] Ya. Pesin, Characteristic exponents and smooth ergodic theory, *Russian Mathematical Surveys* **32** (1977), 55-114.

[P2] Ya. Pesin, *Dimension theory in dynamical systems: Contemporary views and applications*, The University of Chicago Press, Chicago, 1997.

[PeS1] V. Petkov and L. Stoyanov, Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function, *Analysis and PDE* **3** (2010), 427-489.

[PeS2] V. Petkov and L. Stoyanov, Correlations for pairs of closed trajectories in open billiards, *Nonlinearity* **22**, 2657-2679 (2009).

[PeS3] V. Petkov and L. Stoyanov, Distribution of periods of closed trajectories in exponentially shrinking intervals, *Comm. Math. Phys.* **310** (2012), 675-704.

[Po] M. Pollicott, On the rate of mixing for Axiom A flows, *Invent. Math.* **81** (1985), 413-426.

[PoS1] M. Pollicott and R. Sharp, Exponential error terms for growth functions of negatively curved surfaces, *Amer. J. Math.* **120** (1998), 1019-1042.

[PoS2] M. Pollicott and R. Sharp, Asymptotic expansions for closed orbits in homology classes, *Geom. Dedicata* **87** (2001), 123-160.

[PoS3] M. Pollicott and R. Sharp, Correlations for pairs of closed geodesics, *Invent. Math.* **163** (2006), 1-24.

[PoS4] M. Pollicott and R. Sharp, Correlations of length spectra for negatively curved manifolds, *Comm. Math. Phys.* **319** (2013), 515-533.

[PS] C. Pugh and M. Shub, Ergodic attractors, *Trans. Amer. Math. Soc.* **312** (1989), 1-54.

[PSW] C. Pugh, M. Shub and A. Wilkinson, Hölder foliations, *Duke Math. J.* **86** (1997), 517-546; Correction: **105** (2000), 105-106.

[Ro] V. A. Rokhlin, Lectures on entropy theory of transformations with invariant measures, *Russ. Math. Surveys* **22** (1967), 1-52.

[Rud] D. Rudolph, *Fundamentals of measurable dynamics: ergodic theory on Lebesgue spaces*, Clarendon Press, Oxford, 1990.

[R1] D. Ruelle, *Thermodynamic formalism*, Addison-Wesley, Reading, Mass., 1978.

[R2] D. Ruelle, Flows which do not mix exponentially, *C.R.Acad. Sci. Paris* **296** (1983), 191-193.

[R3] D. Ruelle, Resonances for Axiom A flows, *J. Diff. Geometry* **25** (1987), 99-116.

[Si] Ya. Sinai, Gibbs measures in ergodic theory, *Russ. Math. Surveys* **27** (1972), 21-69.

[St1] L. Stoyanov, Spectrum of the Ruelle operator and exponential decay of correlations for open billiard flows, *Amer. J. Math.* **123** (2001), 715-759.

[St2] L. Stoyanov, Spectra of Ruelle transfer operators for Axiom A flows, *Nonlinearity*, **24** (2011), 1089-1120.

[St3] L. Stoyanov, Regular decay of ball diameters and spectra of Ruelle operators for contact Anosov flows, *Proc. Amer. Math. Soc.* **140** (2012), 3463-3478.

[St4] L. Stoyanov, Pinching conditions, linearization and regularity of Axiom A flows, *Discr. Cont. Dyn. Sys. A* **33** (2013), 391-412.

[T] M. Tsujii, Quasi-compactness of transfer operators for contact Anosov flows, *Nonlinearity* **23** (2010), 1495-1545.

[W] P. Wright, Ruelle's lemma and Ruelle zeta functions, *Asympt. Analysis* **80** (2012), 223-236.

[Y1] L.-S. Young, Statistical properties of systems with some hyperbolicity including certain billiards, *Ann. Math.* **147** (1998), 585-650.

[Y2] L.-S. Young, Recurrence times and rates of mixing, *Israel J. Math.* **110** (1999), 153-188.

University of Western Australia, Crawley WA 6009, Australia
E-mail address: luchezar.stoyanov@uwa.edu.au

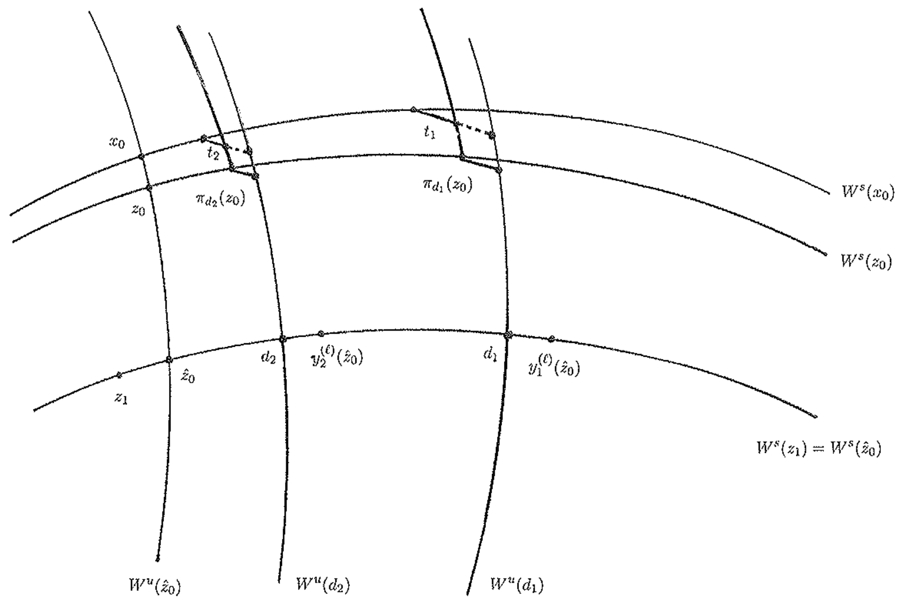

Figure 1 : $|\Delta(x_0, \pi_{d_1}(z_0)) - \Delta(x_0, \pi_{d_2}(z_0))| = |t_1 - t_2|$