

TILTING MODULES OVER TAME HEREDITARY ALGEBRAS

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ABSTRACT. We give a complete classification of the infinite dimensional tilting modules over a tame hereditary algebra R . We start our investigations by considering tilting modules of the form $T = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ where \mathcal{U} is a union of tubes, and $R_{\mathcal{U}}$ denotes the universal localization of R at \mathcal{U} in the sense of Schofield and Crawley-Boevey. Here $R_{\mathcal{U}}/R$ is a direct sum of the Prüfer modules corresponding to the tubes in \mathcal{U} . Over the Kronecker algebra, large tilting modules are of this form in all but one case, the exception being the Lukas tilting module L whose tilting class $\text{Gen } L$ consists of all modules without indecomposable preprojective summands. Over an arbitrary tame hereditary algebra, T can have finite dimensional summands, but the infinite dimensional part of T is still built up from universal localizations, Prüfer modules and (localizations of) the Lukas tilting module. We also recover the classification of the infinite dimensional cotilting R -modules due to Buan and Krause.

In this paper, we continue our study of tilting modules arising from universal localization started in [5]. More precisely, we consider tilting modules over a ring R that have the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ where \mathcal{U} is a set of finitely presented R -modules of projective dimension one, and $R_{\mathcal{U}}$ denotes the universal localization of R at \mathcal{U} in the sense of Schofield. We have seen in [5] that over certain rings this construction leads to a classification of all tilting modules. For example, over a Dedekind domain, every tilting module is equivalent to a tilting module of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ for some set of simple R -modules \mathcal{U} . Aim of this paper is to prove a similar result for finite dimensional tame hereditary algebras.

Universal localizations of a tame hereditary algebra R were already investigated by Crawley-Boevey in [14]. He showed that the normalized defect provides a rank function ρ as studied by Schofield in [34], and that the ρ -torsion modules are precisely the finite dimensional regular modules. He also described the shape of the universal localization $R_{\mathcal{U}}$ at a set \mathcal{U} of quasi-simple modules, proving that there are substantially different situations depending on whether \mathcal{U} does contain a complete clique (that is, all quasi-simples belonging to a certain tube) or not. In particular, $R_{\mathcal{U}}$ will be an infinite dimensional R -module whenever \mathcal{U} contains a complete clique.

We now want to employ these results to give a classification of the large tilting modules over a tame hereditary algebra R . By *large* we mean tilting modules T that are not equivalent to finite dimensional ones, that is, there is no finite dimensional tilting module T' such that $\text{Gen } T = \text{Gen } T'$.

Recall that by a result of Bazzoni and Herbera [8] a large tilting module T is determined up to equivalence by a set of finite dimensional modules \mathcal{S} , in the sense that its tilting class $\text{Gen } T$ coincides with the class of modules $X \in \text{Mod } R$ such that $\text{Ext}_R^1(\mathcal{S}, X) = 0$.

The set \mathcal{S} can be chosen to consist of the finite dimensional modules in ${}^{\perp}(T^{\perp})$, and then it turns out that $\mathcal{S} = \text{add}(\mathbf{p} \cup \mathbf{t}')$ where \mathbf{p} denotes the class of indecomposable preprojective modules, and

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$\mathbf{t}' \subset \mathbf{t}$ is a subset of the class of all finite dimensional indecomposable regular modules (Theorem 2.7).

Notice that, as a consequence, the lattice of large tilting modules has a largest and a smallest element. Indeed, the largest tilting class in $\text{Mod } R$ not generated by a finite dimensional tilting module is the class \mathbf{p}^\perp of modules without indecomposable preprojective summands, which is generated by the Lukas tilting module \mathbf{L} (see [24] and 1.4), while the smallest one is the class of all divisible modules \mathbf{t}^\perp , and the corresponding tilting module is the direct sum $\mathbf{W} = \bigoplus_{S \in \mathcal{U}} S[\infty] \oplus G$ of all Prüfer modules and the generic module (see [31] and 1.3), or in other words, it is the tilting module $R_{\mathbf{t}} \oplus R_{\mathbf{t}}/R$ arising from universal localization at the set of all quasi-simple modules.

Moreover, from the description of \mathcal{S} we also deduce that a large tilting module over the Kronecker algebra must have the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ for some set of quasi-simple R -modules \mathcal{U} in all but one case, the exception being the Lukas tilting module \mathbf{L} (Corollary 2.8).

In the general case, the situation is more involved due to the possible presence of finite dimensional summands in T coming from non-homogeneous tubes. On the other hand, there are at most finitely many such indecomposable summands up to isomorphism (Lemma 3.1). This allows to reduce the classification problem to a situation similar to the Kronecker case. More precisely, we show that T is equivalent to a tilting module of the form $Y \oplus M$ where Y is finite dimensional, while M has no finite dimensional indecomposable direct summands and is a tilting module over a suitable universal localization R' of R . Since R' will again be a tame hereditary algebra, this will enable us to conclude that M is either the Lukas tilting module over R' , or it arises from universal localization at a union of tubes over R' . Notice that the finite dimensional part Y can be described explicitly. It is a regular multiplicity-free exceptional module whose indecomposable summands are arranged in disjoint wings, and the number of summands from each wing equals the number of quasi-simple modules in that wing. A module satisfying these properties will be called a *branch module*.

Summarizing, we obtain two disjoint families of large tilting modules as described below.

Theorem A (cf. Theorem 5.6) *Let R be a finite dimensional tame hereditary algebra, and let $\mathbf{t} = \bigcup_{\lambda \in \mathfrak{T}} \mathbf{t}_\lambda$ where the \mathbf{t}_λ are the tubes in the Auslander-Reiten quiver of R .*

(1) *For every branch module Y there is a tilting module*

$$T_{(Y, \emptyset)} = Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$$

where \mathcal{U} is a suitable set of quasi-simple modules determined by Y .

(2) *For every branch module Y and every non-empty subset $\Lambda \subseteq \mathfrak{T}$ there is a tilting module*

$$T_{(Y, \Lambda)} = Y \oplus R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$$

where \mathcal{U}, \mathcal{V} are suitable sets of quasi-simple modules determined by Y and Λ .

Every large tilting module is equivalent to precisely one module in this list.

Observe that there are only finitely many branch modules up to isomorphism (Lemma 3.1). So, if $\mathcal{Y} = \{Y_1, \dots, Y_t\}$ is a complete irredundant set of branch modules, and $\mathcal{P}(\mathfrak{T})$ denotes the power set of \mathfrak{T} , then the large tilting modules are parametrized, up to equivalence, by the elements of $\mathcal{Y} \times \mathcal{P}(\mathfrak{T})$.

Combining this with decomposition results from [32], we obtain the following structure result.

Theorem B. (cf. Theorem 4.5 and Corollary 5.8) *Let R be a finite dimensional tame hereditary algebra. Let T be a tilting R -module which is not equivalent to a finite dimensional tilting module. Then T has a unique decomposition*

$$T = \bigoplus_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda(T) \oplus \overline{T}$$

where $T' = \bigoplus_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda(T)$ is a torsion module, hence a direct sum of Prüfer modules and finite dimensional regular modules, and \overline{T} is a torsion-free module.

More precisely, for each tube \mathfrak{t}_λ of rank r , the summand $\mathfrak{t}_\lambda(T)$ is given as follows:

- (i) if ${}^\perp(T^\perp)$ contains some modules from \mathfrak{t}_λ , but no complete ray, then $\mathfrak{t}_\lambda(T)$ is a branch module which is a direct sum of at most $r - 1$ modules from \mathfrak{t}_λ ;
- (ii) if ${}^\perp(T^\perp)$ contains some rays from \mathfrak{t}_λ , then $\mathfrak{t}_\lambda(T)$ has precisely r pairwise non-isomorphic indecomposable summands: these are the s Prüfer modules corresponding to the $s \leq r$ rays from \mathfrak{t}_λ contained in ${}^\perp(T^\perp)$, and $r - s$ modules from \mathfrak{t}_λ ;
- (iii) $\mathfrak{t}_\lambda(T) = 0$ whenever $\mathfrak{t}_\lambda \cap {}^\perp(T^\perp) = \emptyset$.

Moreover, the torsion-free summand \overline{T} is given as follows:

- (i) if ${}^\perp(T^\perp)$ contains no complete ray, then there is a set \mathcal{U} of quasi-simple R -modules containing no complete cliques such that \overline{T} is a tilting module over the universal localization $R_{\mathcal{U}}$ which is equivalent to the Lukas tilting $R_{\mathcal{U}}$ -module $\mathbf{L} \otimes R_{\mathcal{U}}$;
- (ii) if ${}^\perp(T^\perp)$ contains some rays, then there is a set \mathcal{V} of quasi-simple R -modules containing complete cliques such that \overline{T} is a projective generator over the universal localization $R_{\mathcal{V}}$.

In particular, we see that a large tilting module T is equivalent to some $T_{(Y, \emptyset)}$ if ${}^\perp(T^\perp)$ contains no complete ray, and it is equivalent to some $T_{(Y, \Lambda)}$ with $\Lambda \neq \emptyset$ if ${}^\perp(T^\perp)$ contains some rays. Indeed, Λ consists of those $\lambda \in \mathfrak{T}$ for which \mathfrak{t}_λ has some ray in ${}^\perp(T^\perp)$. Moreover, in the first case the torsion part T' of T coincides with Y up to multiplicities, while in the second case T' also has Prüfer modules as infinite dimensional summands. In fact, any combination of Prüfer modules $S[\infty]$ can occur in the torsion part as long as the corresponding quasi-simples S are not regular composition factors of the Auslander-Reiten translate $\tau^- Y$. Notice furthermore that in both cases the torsion-free part \overline{T} of T is determined by a suitable localization of the Lukas tilting module. For details we refer to Remark 5.7.

Recall that the large cotilting modules over R have been classified by Buan and Krause in [10, 11], given Bazzoni's result [7] that establishes the pure-injectivity of cotilting modules. By using the fact that every cotilting module over a finite dimensional algebra is equivalent to the dual of a tilting module [39], we can now recover this classification. Let us remark that the other direction does not work: one cannot use the classification of cotilting modules for studying the tilting modules, as duals of (large) cotilting modules need not even be tilting, cf. 1.4.

The paper is organized as follows. In Section 1, we collect some preliminaries on infinite dimensional modules, tilting theory, and universal localization. In Section 2, we prove that a large tilting module T is determined by a set $\mathcal{S} = \text{add}(\mathbf{p} \cup \mathbf{t}')$ as described above, and we settle the cases where $\mathbf{t}' = \emptyset$ (then T is equivalent to the Lukas tilting module) or \mathbf{t}' is a union of tubes (then T arises from universal localization). Section 3 is devoted to the finite dimensional summands of T . In Section 4,

we show that T has a canonical decomposition as above. The description of the torsion-free part \overline{T} is achieved in Section 5, where we also prove our classification and discuss the cases when T is noetherian over its endomorphism ring or (Σ) -pure-injective. In the Appendix we deal with the classification of cotilting modules.

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1. PRELIMINARIES

Throughout this note, let R be a finite dimensional tame hereditary (w.l.o.g. indecomposable) algebra over a field k . We denote by $\text{Mod } R$ (respectively, $R\text{Mod}$) the category of all right (respectively, left) R -modules and by $\text{mod } R$ (respectively, $R\text{mod}$) the category of finitely generated right (respectively, left) R -modules. Let $D : \text{mod } R \rightarrow R\text{mod}$ be the usual duality. Given a subcategory $\mathcal{S} \subset \text{mod } R$, the subcategory of $R\text{mod}$ consisting of the dual modules $D(S)$ with $S \in \mathcal{S}$ will be denoted by \mathcal{S}^* .

We adopt terminology and notation from [32, 31]. In particular, we denote by $\mathbf{p}, \mathbf{t}, \mathbf{q}$ the classes of indecomposable preprojective, regular, and preinjective right R -modules of finite length, respectively. The corresponding classes of left R -modules will be denoted by ${}_R\mathbf{p}, {}_R\mathbf{t}, {}_R\mathbf{q}$. An arbitrary R -module will be called *regular* if it has neither preinjective nor preprojective direct summands.

We fix a complete irredundant set of quasi-simple (i.e. simple regular) modules \mathbb{U} , and for each $S \in \mathbb{U}$, we denote by $S[m]$ the module of regular length m on the ray

$$S = S[1] \subset S[2] \subset \cdots \subset S[m] \subset S[m+1] \subset \cdots$$

and let $S[\infty] = \varinjlim S[m]$ be the corresponding *Prüfer module*. The *adic module* $S[-\infty]$ corresponding to $S \in \mathbb{U}$ is defined dually as the inverse limit along the coray ending at S .

We write $\mathbf{t} = \bigcup_{\lambda \in \overline{\mathfrak{T}}} \mathbf{t}_\lambda$, where \mathbf{t}_λ denotes the class of indecomposable modules in a tube of the Auslander-Reiten quiver of $\text{mod } R$. The tubes in $R\text{mod}$ will be denoted by ${}_R\mathbf{t}_\lambda$. It is well known that almost all tubes are homogeneous, that is, they contain a unique quasi-simple module up to isomorphism. In order to deal with the (at most three) non-homogeneous tubes, we consider the equivalence relation \sim on \mathbb{U} generated by

$$S \sim S' \text{ if } \text{Ext}_R^1(S, S') \neq 0.$$

According to [14], we call the equivalence classes of this relation *cliques*. In other words, two quasi-simple modules belong to the same clique iff they are in the same tube. The order of the clique is the *rank* of the tube.

We will need a combinatorial description of the *extension closure* of a set of quasi-simples $\mathcal{U} \subset \mathbb{U}$, that is, of the smallest subcategory $\mathcal{W} \subset \text{mod } R$ that contains \mathcal{U} and is closed under extensions. Given a tube \mathfrak{t}_λ of rank $r > 1$ and a module $X \in \mathfrak{t}_\lambda$ of regular length $m < r$, we consider the full subquiver \mathcal{W}_X of \mathfrak{t}_λ which is isomorphic to the Auslander-Reiten-quiver $\Theta(m)$ of the linearly oriented quiver of type \mathbb{A}_m with X corresponding to the projective-injective vertex of $\Theta(m)$. Following [33, 3.3], we call \mathcal{W}_X the *wing* of \mathfrak{t}_λ with *vertex* X . The following result is straightforward.

Lemma 1.1. Let \mathfrak{t}_λ be a tube of rank $r > 1$, and let $\mathcal{U} = \{U_1, \dots, U_m\} \subset \mathbb{U}$ be a set of $m < r$ quasi-simples in \mathfrak{t}_λ where $U_{i+1} = \tau^- U_i$ for all $1 \leq i < m$. Then the extension closure \mathcal{W} of \mathcal{U} consists of all finite direct sums of modules in the wing $\mathcal{W}_{U_1[m]} = \{U_i[k] \mid 1 \leq i \leq m, 1 \leq k \leq m - i + 1\}$. \square

Let us introduce some further notation. Let $\mathcal{M} \subset \text{Mod } R$ be a class of modules. Denote by $\text{Add } \mathcal{M}$ (respectively, $\text{add } \mathcal{M}$) the class consisting of all modules isomorphic to direct summands of (finite) direct sums of elements of \mathcal{M} . The class consisting of all modules isomorphic to direct summands of products of modules of \mathcal{M} is denoted by $\text{Prod } \mathcal{M}$. The class consisting of the right R -modules which are epimorphic images of arbitrary direct sums of elements in \mathcal{M} is denoted by $\text{Gen } \mathcal{M}$. Dually, we define $\text{Cogen } \mathcal{M}$ as the class of all submodules of arbitrary direct products of elements in \mathcal{M} . We further write

$$\begin{aligned} \mathcal{M}^o &= \{X_R \mid \text{Hom}_R(M, X) = 0 \text{ for each } M \in \mathcal{M}\} \\ \mathcal{M}^\perp &= \{X_R \mid \text{Ext}_R^1(M, X) = 0 \text{ for each } M \in \mathcal{M}\} \\ \mathcal{M}^\wedge &= \{X_R \mid \text{Ext}_R^i(M, X) = 0 \text{ for each } i \geq 0, M \in \mathcal{M}\} \\ \mathcal{M}^\tau &= \{{}_R X \mid \text{Tor}_1^R(M, X) = 0 \text{ for each } M \in \mathcal{M}\} \end{aligned}$$

and define dually ${}^o\mathcal{M}$, ${}^\perp\mathcal{M}$, ${}^\wedge\mathcal{M}$, ${}^\tau\mathcal{M}$. If \mathcal{M} contains a unique module M , then we shall denote these subcategories by $\text{Add } M$, M^o , M^\perp , etc.

Finally, we denote by G the *generic module*. It is the unique indecomposable infinite dimensional module which has finite length over its endomorphism ring. In the notation of 1.3 and 1.4, it is the unique indecomposable in $\mathcal{F} \cap \mathcal{D}$, that is, the unique indecomposable torsion-free divisible module, cf. [32, 5.3 and p.408].

We now collect some tools we will freely use when working with infinite dimensional modules.

- Lemma 1.2.** (1) If $M \in \text{Mod } R$ and X is a finitely generated indecomposable module in $\text{Add } M$, then X is isomorphic to a direct summand of M .
- (2) Every finite-dimensional R -module is *endofinite*, that is, it has finite length as a module over its endomorphism ring. Every direct sum of copies of finitely many endofinite modules is endofinite. Every dual $D(M)$ of an endofinite module M is endofinite.
- (3) Suppose M is endofinite. Then $\text{Add } M = \text{Prod } M$. In particular, if $M \in X^\perp$ for some $X \in \text{Mod } R$, then $\text{Add } M \subset X^\perp$.
- (4) Every endofinite R -module M is *pure-injective*, that is, pure-exact sequences starting at M split.

- (5) Every indecomposable pure-injective R -module is isomorphic to a module in the following list:
- the finitely generated indecomposable modules,
 - the Prüfer modules $S[\infty]$, $S \in \mathbb{U}$,
 - the adic modules $S[-\infty]$, $S \in \mathbb{U}$,
 - the generic module G .
- (6) Let M and N be infinite dimensional indecomposable pure-injective modules. Then
- $\text{Ext}_R^1(M, P) \neq 0$ for every $P \in \mathbf{p}$.
 - $\text{Ext}_R^1(Q, M) \neq 0$ for every $Q \in \mathbf{q}$.
 - $\text{Ext}_R^1(M, N) \neq 0$ if and only if there are $S \sim S'$ such that $M = S[\infty]$ and $N = S'[-\infty]$.

Proof: (1) Since X is a finitely generated module, being (isomorphic to) a direct summand of a direct sum $M^{(I)}$ of copies of M means that X is (isomorphic to) a direct summand in a finite subsum $M^{(I_0)}$. Now the claim follows from the fact that X has a local endomorphism ring.

The first statement in (2) is clear because every finite-dimensional R -module is finitely generated over its endomorphism ring, which is again a finite-dimensional k -algebra. For the other statements on endofinite modules, we refer to [16]. Details on pure-injective modules can be found in [22, Chapter 7]. The classification of the indecomposable pure-injective R -modules is contained in [15]. Statement (6) is shown in [10, 2.5 and 2.7] \square

Recall from [13] that a module T is *tilting* provided that $\text{Gen } T = T^\perp$, or equivalently, T satisfies

(T1) $\text{proj.dim}(T) \leq 1$;

(T2) $\text{Ext}_R^1(T, T^{(\kappa)}) = 0$ for any cardinal κ ;

(T3) There is an exact sequence $0 \rightarrow R \rightarrow T_0 \rightarrow T_1 \rightarrow 0$ with $T_0, T_1 \in \text{Add}(T)$.

Note that every tilting module T satisfies $\text{Add } T = T^\perp \cap {}^\perp(T^\perp)$. Moreover, T gives rise to a torsion pair with torsion class T^\perp and torsion-free class T^o . The class T^\perp is called a *tilting class*. Tilting modules having the same tilting classes are said to be *equivalent*. *Cotilting modules* and *cotilting classes* are defined dually, and equivalence of cotilting modules is defined correspondingly.

By [19, 5.1.12], two tilting modules T, T' are equivalent if and only if $\text{Add } T = \text{Add } T'$, while two cotilting modules C, C' are equivalent if and only if $\text{Prod } C = \text{Prod } C'$.

Here are some examples of infinite-dimensional tilting or cotilting modules.

Example 1.3. *The Reiten-Ringel tilting module.* It is shown in [31] that the module

$$\mathbf{W} = \bigoplus_{S \in \mathbb{U}} S[\infty] \oplus G$$

is an infinite dimensional tilting module whose tilting class $\text{Gen } \mathbf{W} = \mathbf{W}^\perp$ coincides with the class

$$\mathcal{D} = {}^o\mathbf{t} = \mathbf{t}^\perp$$

of all *divisible* modules, and moreover, \mathbf{W} is a cotilting module whose cotilting class $\text{Cogen } \mathbf{W} = {}^\perp\mathbf{W}$ coincides with the class

$$\mathcal{C} = {}^\perp\mathbf{q} = \mathbf{q}^o$$

of all modules without indecomposable preinjective direct summands. \square

Example 1.4. *The Lukas tilting module.* Based on a construction due to F. Lukas [28, 2.1], Kerner and Trlifaj showed in [24] that there is a countably infinitely generated \mathfrak{p} -filtered tilting module $\mathbf{L} \in \mathfrak{p}^\perp$ whose tilting class

$$\text{Gen } \mathbf{L} = \mathfrak{p}^\perp = {}^o\mathfrak{p}$$

coincides with the class of all modules without indecomposable preprojective direct summands. The corresponding torsion-free class \mathbf{L}° coincides with the class of preprojective modules in the sense of [32, Section 2]. In particular, \mathbf{L}° is contained in the class

$$\mathcal{F} = \mathfrak{t}^\circ = {}^\perp\mathfrak{t}$$

of all *torsion-free* modules, which is a cotilting class with cotilting module $D({}_R\mathbf{W})$, cf. [3, Prop.7]. Here ${}_R\mathbf{W}$ denotes the Reiten-Ringel tilting module in the category $R\text{Mod}$. The torsion class corresponding to the torsion-free class \mathcal{F} is the class $\text{Gen } \mathfrak{t}$ of all *torsion* modules.

Note that the dual $D({}_R\mathbf{W})$ of the cotilting module ${}_R\mathbf{W}$ is not tilting as it does not satisfy condition (T2). Indeed, G and the adic modules are summands of $D({}_R\mathbf{W})$, but no countable direct sum of copies of an adic module belongs to G^\perp , see [29, Prop.1 and Remark on p.265]. \square

Next, let us recall Schofield's notion of universal localization [34, Theorem 4.1].

Theorem 1.5. Let Σ be a set of morphisms between finitely generated projective right R -modules. Then there are a ring R_Σ and a morphism of rings $\lambda: R \rightarrow R_\Sigma$ such that

- (1) λ is Σ -*inverting*, i.e. if $\alpha: P \rightarrow Q$ belongs to Σ , then $\alpha \otimes_R 1_{R_\Sigma}: P \otimes_R R_\Sigma \rightarrow Q \otimes_R R_\Sigma$ is an isomorphism of right R_Σ -modules, and
- (2) λ is *universal Σ -inverting*, i.e. if S is a ring such that there exists a Σ -inverting morphism $\psi: R \rightarrow S$, then there exists a unique morphism of rings $\bar{\psi}: R_\Sigma \rightarrow S$ such that $\bar{\psi}\lambda = \psi$.

The morphism $\lambda: R \rightarrow R_\Sigma$ is an epimorphism in the category of rings with $\text{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$. It is called the *universal localization of R at Σ* .

Let now \mathcal{U} be a set of finitely presented right R -modules. For each $U \in \mathcal{U}$, consider a morphism α_U between finitely generated projective right R -modules such that

$$0 \rightarrow P \xrightarrow{\alpha_U} Q \rightarrow U \rightarrow 0$$

We will denote by $\lambda_{\mathcal{U}}: R \rightarrow R_{\mathcal{U}}$ the universal localization of R at the set $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$, and we will call it the *universal localization of R at \mathcal{U}* . Note that $R_{\mathcal{U}}$ does not depend on the choice of Σ .

Example 1.6. *Tilting modules arising from universal localization.* Let now $\mathcal{U} \subset \mathbb{U}$ be a set of quasi-simple modules. Then, as shown in [5, 4.7], the module

$$T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$$

is a tilting module with tilting class \mathcal{U}^\perp . In particular, if $\mathcal{U} = \mathbb{U}$, then $T_{\mathcal{U}}$ is equivalent to the Reiten-Ringel tilting module $\mathbf{W} = \bigoplus_{S \in \mathbb{U}} S[\infty] \oplus G$. \square

More generally, if \mathcal{U} is a union of cliques, then $R_{\mathcal{U}}$ is a torsion-free module, and $R_{\mathcal{U}}/R$ is a direct sum of the Prüfer modules corresponding to the quasi-simples in \mathcal{U} , as we are going to see below in Propositions 1.8 and 1.10 (compare also [39, 2.4]).

We first collect some facts on universal localization which we will also need later. Recall that, given a set of R -modules \mathcal{U} , the *torsion pair generated by \mathcal{U}* is the pair $(\mathcal{T}_{\mathcal{U}}, \mathcal{U}^{\circ})$ where $\mathcal{T}_{\mathcal{U}} = {}^{\circ}(\mathcal{U}^{\circ})$.

Proposition 1.7. Let \mathcal{U} be a set of quasi-simple modules and let \mathcal{W} be the extension closure of \mathcal{U} . Let further t be the torsion radical associated to the torsion pair $(\mathcal{T}_{\mathcal{U}}, \mathcal{U}^{\circ})$ generated by \mathcal{U} . The following statements hold true.

- (1) \mathcal{W} is a full exact abelian subcategory of $\text{mod } R$.
- (2) $R_{\mathcal{U}}$ coincides with $R_{\mathcal{W}}$, the universal localization of R at \mathcal{W} .
- (3) The torsion pair $(\mathcal{T}_{\mathcal{U}}, \mathcal{U}^{\circ})$ generated by \mathcal{U} coincides with the torsion pair $(\mathcal{T}_{\mathcal{W}}, \mathcal{W}^{\circ})$ generated by \mathcal{W} .
- (4) $\mathcal{U}^{\wedge} = \mathcal{W}^{\wedge}$ is the essential image of the restriction functor $\text{Mod } R_{\mathcal{U}} \rightarrow \text{Mod } R$. In other words, an R -module X is an $R_{\mathcal{U}}$ -module if and only if $X \in \mathcal{U}^{\wedge}$.
- (5) $\mathcal{T}_{\mathcal{U}} = \text{Gen } \mathcal{W} = \{X \in \text{Mod } R \mid X \otimes_R R_{\mathcal{U}} = 0\}$.
- (6) $R_{\mathcal{U}}/R$ is a directed union of finite extensions of modules in \mathcal{U} .
- (7) For every $A \in \text{Mod } R$ there is a short exact sequence

$$0 \rightarrow A/tA \rightarrow A \otimes_R R_{\mathcal{U}} \rightarrow A \otimes_R R_{\mathcal{U}}/R \rightarrow 0$$

where $A \otimes_R R_{\mathcal{U}} \in \mathcal{U}^{\wedge}$ and $A \otimes_R R_{\mathcal{U}}/R \in {}^{\wedge}(\mathcal{U}^{\wedge}) = \mathcal{T}_{\mathcal{U}} \cap {}^{\perp}(\mathcal{U}^{\perp})$.

Proof: (1), (2) We adopt Schofield's terminology from [37]. Since \mathcal{U} is a Hom-perpendicular set, \mathcal{W} is well-placed, cf. [37, p.4]. Then $\mathcal{W} = {}^{\wedge}(\mathcal{U}^{\wedge}) \cap \text{mod } R$ is the well-placed closure of \mathcal{U} , and $R_{\mathcal{U}} = R_{\mathcal{W}}$, cf. [37, 2.3].

(3), (4) We claim $\mathcal{U}^{\circ} = \mathcal{W}^{\circ}$. The inclusion ' \supset ' follows from $\mathcal{U} \subset \mathcal{W}$. Conversely, ${}^{\circ}(\mathcal{U}^{\circ})$ contains \mathcal{U} , and also its extension closure \mathcal{W} , hence $\mathcal{U}^{\circ} \subset \mathcal{W}^{\circ}$. Similarly, we prove $\mathcal{U}^{\perp} = \mathcal{W}^{\perp}$. We then deduce $\mathcal{U}^{\wedge} = \mathcal{W}^{\wedge}$. For the second statement see [1, 1.7].

(5) $\{X \in \text{Mod } R \mid X \otimes_R R_{\mathcal{U}} = 0\}$ is closed under extensions, direct sums and epimorphic images, hence it is a torsion class containing \mathcal{U} and thus also $\mathcal{T}_{\mathcal{U}}$, which in turn contains $\text{Gen } \mathcal{W}$. The converse inclusions follow from [36, 5.1 and 5.5].

(6) is a consequence of [34, Theorem 12.6], [35, Theorem 3] and [14, Lemma 4.4]. Another proof can be found in [37, Theorem 2.6].

(7) is contained in [25, page 2349]. We give a direct proof for the reader's convenience. Applying $A \otimes_R -$ on the short exact sequence $0 \rightarrow R \rightarrow R_{\mathcal{U}} \rightarrow R_{\mathcal{U}}/R \rightarrow 0$, we obtain an exact sequence $A \rightarrow A \otimes_R R_{\mathcal{U}} \rightarrow A \otimes_R R_{\mathcal{U}}/R \rightarrow 0$, which gives rise to the short exact sequence $0 \rightarrow A/tA \rightarrow A \otimes_R R_{\mathcal{U}} \rightarrow A \otimes_R R_{\mathcal{U}}/R \rightarrow 0$ because tA is the kernel of the canonical map $A \rightarrow A \otimes_R R_{\mathcal{U}}$, cf. (5) and [36, 5.5]. Since $A \otimes_R R_{\mathcal{U}}$ is an $R_{\mathcal{U}}$ -module, it follows from (4) that $A \otimes_R R_{\mathcal{U}} \in \mathcal{U}^{\wedge}$.

Let us show that $A \otimes_R R_{\mathcal{U}}/R \in {}^{\wedge}(\mathcal{U}^{\wedge})$. Take $M \in \mathcal{U}^{\wedge}$. First of all, note that $A \otimes_R R_{\mathcal{U}}/R$ is generated by $R_{\mathcal{U}}/R$, which belongs to $\mathcal{T}_{\mathcal{U}}$ by (5) and (6). Thus $A \otimes_R R_{\mathcal{U}}/R \in \mathcal{T}_{\mathcal{U}}$, and since $M \in \mathcal{U}^{\circ}$, we have $\text{Hom}_R(A \otimes R_{\mathcal{U}}/R, M) = 0$.

Next, note that $A/tA \otimes_R R_{\mathcal{U}} \cong A \otimes_R R_{\mathcal{U}}$ by (5). Then $\text{Hom}_R(A \otimes_R R_{\mathcal{U}}, M) \cong \text{Hom}_R(A/tA, M)$ because M is an $R_{\mathcal{U}}$ -module by (4), and we have the exact sequence

$$0 \rightarrow \text{Ext}_R^1(A \otimes_R R_{\mathcal{U}}/R, M) \rightarrow \text{Ext}_R^1(A \otimes_R R_{\mathcal{U}}, M) \xrightarrow{\psi} \text{Ext}_R^1(A/tA, M) \rightarrow 0$$

Now, if we prove that ψ is injective, we obtain that $\text{Ext}_R^1(A \otimes_R R_{\mathcal{U}}/R, M) = 0$, as desired.

Given an extension

$$\epsilon : 0 \rightarrow M \rightarrow X \rightarrow A \otimes_R R_{\mathcal{U}} \rightarrow 0,$$

its image under ψ is given by pullback

$$\begin{array}{ccccccccc} \epsilon : & 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & A \otimes_R R_{\mathcal{U}} & \longrightarrow & 0 \\ & & & \parallel & & \uparrow & & \uparrow & & \\ \psi(\epsilon) : & 0 & \longrightarrow & M & \longrightarrow & Z & \longrightarrow & A/tA & \longrightarrow & 0 \end{array}$$

Observe that X is an $R_{\mathcal{U}}$ -module because \mathcal{U}^\wedge is closed under extensions. Therefore we obtain the commutative diagram in $\text{Mod } R_{\mathcal{U}}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & A \otimes_R R_{\mathcal{U}} & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \cong \uparrow & & \\ 0 & \longrightarrow & M & \longrightarrow & Z \otimes_R R_{\mathcal{U}} & \longrightarrow & A/tA \otimes_R R_{\mathcal{U}} & \longrightarrow & 0 \end{array}$$

showing that ϵ is uniquely determined by $\psi(\epsilon)$. Thus ψ is injective.

Finally, we show that ${}^\wedge(\mathcal{U}^\wedge) = \mathcal{T}_{\mathcal{U}} \cap {}^\perp(\mathcal{U}^\perp)$. The inclusion ‘ \supset ’ follows from the definition. For ‘ \subset ’, consider $X \in {}^\wedge(\mathcal{U}^\wedge)$. If $A \in \mathcal{U}^\circ$, then there is an embedding $0 \rightarrow A \rightarrow A \otimes_R R_{\mathcal{U}} \in \mathcal{U}^\wedge$, hence $0 \rightarrow \text{Hom}_R(X, A) \rightarrow \text{Hom}_R(X, A \otimes_R R_{\mathcal{U}}) = 0$, which proves $X \in {}^\circ(\mathcal{U}^\circ) = \mathcal{T}_{\mathcal{U}}$. Moreover, if $A \in \mathcal{U}^\perp$, then by [5, 4.7] there is an epimorphism $R_{\mathcal{U}}^{(\alpha)} \rightarrow A \rightarrow 0$ where $R_{\mathcal{U}}^{(\alpha)} \in \mathcal{U}^\wedge$, thus $0 = \text{Ext}_R^1(X, R_{\mathcal{U}}^{(\alpha)}) \rightarrow \text{Ext}_R^1(X, A) \rightarrow 0$, showing $X \in {}^\perp(\mathcal{U}^\perp)$. \square

Proposition 1.8. Let $\mathcal{U} \subseteq \mathbb{U}$ be a set of quasi-simple modules. The following statements hold true.

- (1) The R -module $R_{\mathcal{U}}$ is torsion-free, and the R -module $R_{\mathcal{U}}/R$ is torsion regular.
- (2) The R -module $R_{\mathbb{U}}$ is torsion-free and divisible, and it is a direct sum of $\alpha = \dim_{\text{End}_R G} G$ copies of the generic module G . Moreover, $R_{\mathbb{U}}$ is a simple artinian ring isomorphic to the ring of $\alpha \times \alpha$ -matrices over the division ring $\text{End}_R G$, and G is the unique indecomposable $R_{\mathbb{U}}$ -module.
- (3) For any module V in the extension closure of \mathcal{U} there is an isomorphism of k - $\text{End}_R V$ -bimodules

$$\text{Hom}_R(V, R_{\mathcal{U}}/R) \cong \text{Ext}_R^1(V, R).$$

Proof: (1) and (2): Let $\mathcal{U} \subseteq \mathbb{U}$. First of all, we show that $R_{\mathcal{U}}/R$ is a torsion regular module, that is, it belongs to $\text{Gen } \mathbf{t}$ and has no summands in $\mathbf{p} \cup \mathbf{q}$. By Proposition 1.7(6), we can write $R_{\mathcal{U}}/R$ as a directed union $\varinjlim N_i$ with the N_i ’s finite extensions of elements in \mathcal{U} . Then $R_{\mathcal{U}}/R \in \text{Gen } \mathbf{t}$. Moreover, if $P \in \mathbf{p}$ (respectively, $Q \in \mathbf{q}$) were a direct summand of $R_{\mathcal{U}}/R$, then P (respectively, Q) would be a direct summand of some regular module N_i , a contradiction.

If $\mathcal{U} = \mathbb{U}$, then the fact that $R_{\mathbb{U}}$ is an $R_{\mathbb{U}}$ -module yields by Proposition 1.7(4) that $\text{Hom}_R(\mathbb{U}, R_{\mathbb{U}}) = \text{Ext}_R^1(\mathbb{U}, R_{\mathbb{U}}) = 0$, that is, $R_{\mathbb{U}}$ is a torsion-free divisible module. So [32, 5.4 and 5.6] imply that $R_{\mathbb{U}}$ is a direct sum of $-\delta(R)$ copies of G , where δ denote the defect (cf.[32, p.333]). The ring $R_{\mathbb{U}}$ is simple

artinian by [14, Lemma 4.4]. The R -module G is an $R_{\mathbb{U}}$ -module because it is a torsion-free divisible R -module, and it is the only simple $R_{\mathbb{U}}$ -module because it is indecomposable over R and hence over $R_{\mathbb{U}}$. Now $R_{\mathbb{U}} \cong \text{End}_{R_{\mathbb{U}}} R_{\mathbb{U}} \cong \text{End}_R R_{\mathbb{U}} \cong \text{End}_R(G^{(\alpha)}) \cong M_{\alpha}(\text{End}_R G)$, and $\text{End}_R G$ is a division ring by [32, 5.3]. Finally, $\alpha = \dim_{\text{End}_R G} G$ by the Theorem of Wedderburn-Artin.

In the general case, we have that $R_{\mathcal{U}}$ is torsion-free because $R_{\mathcal{U}} \subset R_{\mathbb{U}}$ by Proposition 1.11(4).

(3) follows from Proposition 1.7(4) by applying $\text{Hom}_R(V, -)$ to the exact sequence $0 \rightarrow R \rightarrow R_{\mathcal{U}} \rightarrow R_{\mathcal{U}}/R \rightarrow 0$. \square

Lemma 1.9. Let \mathbf{t}_{λ} be a tube of rank r . If X and Y are indecomposable regular modules in \mathbf{t}_{λ} of regular length at most r , then $\text{End}_R X$ and $\text{End}_R Y$ are isomorphic division rings. Moreover,

- (1) if $X \subseteq Y$, then $f(X) \subseteq X$ for all $f \in \text{End}_R Y$ and the map $\text{End}_R Y \rightarrow \text{End}_R X$, given by $f \mapsto f|_X$ is an isomorphism.
- (2) if $X \cong Y/K$ for some (unique regular) R -submodule K of Y , then the map $\text{End}_R Y \rightarrow \text{End}_R X$ given by $f \mapsto \bar{f}$ is an isomorphism where \bar{f} is the induced map on Y/K by f .

Proof: Let $\{U_1, \dots, U_r\} \subseteq \mathbb{U}$ be the set of r quasi-simples in \mathbf{t}_{λ} where $U_{i+1} = \tau^{-1}U_i$ for all $1 \leq i < r$. That $\text{End}_R(U_i[j])$ is a division ring for $1 \leq i \leq r$, $1 \leq j \leq r$ follows from the fact that \mathbf{t} is an abelian category and that every indecomposable regular module has unique regular composition series. By the same reason, the maps in (1) and (2) are well-defined injective morphisms of k -algebras.

Fix $i \in \{1, \dots, r\}$ and $1 \leq s < r$. Then the exact sequence

$$0 \rightarrow U_i \rightarrow U_i[s+1] \rightarrow U_{i+1}[s] \rightarrow 0$$

induces the following ones

$$0 \rightarrow \text{Hom}_R(U_i, U_i) \rightarrow \text{Hom}_R(U_i, U_i[s+1]) \rightarrow \text{Hom}_R(U_i, U_{i+1}[s]) = 0$$

$$0 = \text{Hom}_R(U_{i+1}[s], U_i[s+1]) \rightarrow \text{Hom}_R(U_i[s+1], U_i[s+1]) \rightarrow \text{Hom}_R(U_i, U_i[s+1]) \rightarrow 0$$

$$0 \rightarrow \text{Hom}_R(U_{i+1}[s], U_{i+1}[s]) \rightarrow \text{Hom}_R(U_i[s+1], U_{i+1}[s]) \rightarrow \text{Hom}_R(U_i, U_{i+1}[s]) = 0$$

$$0 = \text{Hom}_R(U_i[s+1], U_i) \rightarrow \text{Hom}_R(U_i[s+1], U_i[s+1]) \rightarrow \text{Hom}_R(U_i[s+1], U_{i+1}[s]) \rightarrow 0$$

Hence $\dim_k \text{End}_R(U_i) = \dim_k \text{End}_R(U_i[s+1]) = \dim_k \text{End}_R(U_{i+1}[s])$. Since we have the injective morphisms of rings $\text{End}_R(U_i[s+1]) \rightarrow \text{End}(U_i)$, $f \mapsto f|_{U_i}$, and $\text{End}_R(U_i[s+1]) \rightarrow \text{End}(U_{i+1}[s])$, $f \mapsto \bar{f}$, it turns out that $\text{End}_R(U_i)$, $\text{End}_k(U_i[s+1])$ and $\text{End}_R(U_{i+1}[s])$ are isomorphic k -algebras for any $1 \leq s < r$. The result now follows because $i \in \{1, \dots, r\}$ is arbitrary. \square

Proposition 1.10. Let $\mathcal{U} \subseteq \mathbb{U}$ be a set of quasi-simple modules. Set $\alpha = \dim_{\text{End}_R G} G$ and $\alpha_U = \dim_{\text{End}_R U} \text{Ext}_R^1(U, R)$ for each $U \in \mathcal{U}$. The following statements hold true.

- (1) If \mathcal{U} is a union of cliques, then

$$R_{\mathcal{U}}/R \cong \bigoplus_{U \in \mathcal{U}} U[\infty]^{(\alpha_U)}.$$

In particular, if $\mathcal{U} = \mathbb{U}$, then

$$T_{\mathcal{U}} = G^{(\alpha)} \oplus \left(\bigoplus_{U \in \mathbb{U}} U[\infty]^{(\alpha_U)} \right).$$

- (2) Let \mathfrak{t}_λ be a tube of rank $r > 1$, let $\mathcal{U} = \{U_1, \dots, U_m\} \subseteq \mathbb{U}$ be a set of $m < r$ quasi-simples in \mathfrak{t}_λ where $U_{i+1} = \tau^- U_i$ for all $1 \leq i < m$. Then $R_{\mathcal{U}}/R$ is a direct sum of modules on the coray ending at U_m . More precisely,

$$R_{\mathcal{U}}/R \cong U_1[m]^{(\alpha_{U_1})} \oplus U_2[m-1]^{(\alpha_{U_2})} \oplus \dots \oplus U_m^{(\alpha_{U_m})}.$$

Proof: First of all, by Proposition 1.7(6), we can write $R_{\mathcal{U}}/R$ as a directed union $\bigcup N_i = \varinjlim N_i$ with the N_i 's finite extensions of elements in \mathcal{U} .

(1) Suppose that \mathcal{U} is a union of cliques. Then $R_{\mathcal{U}}/R$ is divisible. Indeed, if V is a quasi-simple not in \mathcal{U} , then $\text{Ext}_R^1(V, R_{\mathcal{U}}/R) = \varinjlim \text{Ext}_R^1(V, N_i) = 0$. On the other hand, if $U \in \mathcal{U}$, then $\text{Ext}_R^1(U, R_{\mathcal{U}}/R) = 0$ because $T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting module with tilting class \mathcal{U}^\perp . So $R_{\mathcal{U}}/R$ is a divisible torsion regular module, hence a direct sum of Prüfer modules by [32, 4.5, Lemma 3].

Observe that for $U, V \in \mathbb{U}$ we have

$$\text{Ext}_R^1(U[\infty], \tau V) \cong D \text{Hom}_R(V, U[\infty]) = 0 \text{ iff } V \neq U.$$

So, if $V \in \mathbb{U} \setminus \mathcal{U}$, then as \mathcal{U} is a union of cliques, $\tau V \in \mathcal{U}^\perp = (R_{\mathcal{U}}/R)^\perp$, which implies that the Prüfer module $V[\infty]$ cannot occur as a direct summand of $R_{\mathcal{U}}/R$. Similarly, if $U \in \mathcal{U}$, then clearly $\tau U \notin \mathcal{U}^\perp = (R_{\mathcal{U}}/R)^\perp$, thus the Prüfer module $U[\infty]$ must be a direct summand of $R_{\mathcal{U}}/R$. Therefore

$$R_{\mathcal{U}}/R \cong \bigoplus_{U \in \mathcal{U}} U[\infty]^{(\beta_U)}$$

for some cardinals β_U . Recall now that $\text{End}_R(U)$ is a division ring for $U \in \mathbb{U}$. Furthermore, $\dim_{\text{End}_R(U)} \text{Hom}_R(U, U[n]) = 1$ for all $n \geq 1$, and thus $\dim_{\text{End}_R(U)} \text{Hom}_R(U, U[\infty]) = 1$. Then, for a fixed $U \in \mathcal{U}$, the number of direct summands of $R_{\mathcal{U}}/R$ isomorphic to $U[\infty]$ coincides with $\dim_{\text{End}_R(U)} \text{Hom}_R(U, R_{\mathcal{U}}/R)$, which by Proposition 1.8(3) equals α_U . Therefore $\beta_U = \alpha_U$ for all $U \in \mathcal{U}$, as desired.

The statement for $\mathcal{U} = \mathbb{U}$ follows from Proposition 1.8(2).

(2) Suppose now that \mathcal{U} is defined as in (2). Note that the modules N_i above are finite direct sums of elements in the wing $\mathcal{W}_{U_1[m]}$ of \mathcal{U} , see Lemma 1.1. Set $Y = U_1 \oplus \dots \oplus U_m$. By Proposition 1.8(3) we get that $\text{Hom}_R(Y, R_{\mathcal{U}}/R) \cong \text{Ext}_R^1(Y, R)$, which implies that $\dim_k \text{Hom}_R(Y, R_{\mathcal{U}}/R) < \infty$. Therefore the directed union $R_{\mathcal{U}}/R = \bigcup N_i$ is finite, which means that $R_{\mathcal{U}}/R$ is a finite direct sum of elements in $\mathcal{W}_{U_1[m]}$. In particular $R_{\mathcal{U}}/R$ and $R_{\mathcal{U}}$ are finite dimensional over k (this is well known, see [14, Theorem 4.2] and [35, Theorem 13] or [18, 10.1]).

Since the number of direct summands of $R_{\mathcal{U}}/R$ isomorphic to some module in the ray determined by U_i equals $\dim_{\text{End}_R(U_i)} \text{Hom}_R(U_i, R_{\mathcal{U}}/R)$, the total number of indecomposable direct summands of $R_{\mathcal{U}}/R$ equals $\alpha_{U_1} + \dots + \alpha_{U_m}$ by Proposition 1.8(3).

Let us consider the modules in the wing $\mathcal{W}_{U_1[m]}$ that lie on the coray ending at $U_m \in \mathcal{U}$. These are $\{U_m, U_{m-1}[2], \dots, U_2[m-1], U_1[m]\} = \{U_{m-i+1}[i] \mid i = 1, \dots, m\}$.

For a fixed $1 \leq i \leq m$, the number of direct summands of $R_{\mathcal{U}}/R$ admitting a non-zero morphism from $U_{m-i+1}[i]$ equals $\dim_{\text{End}_R(U_{m-i+1}[i])} \text{Hom}_R(U_{m-i+1}[i], R_{\mathcal{U}}/R)$. If $i = 1$, this number agrees with α_{U_m} by Proposition 1.8(3). This shows that $R_{\mathcal{U}}/R$ has α_{U_m} summands isomorphic to U_m .

For $i \geq 2$, we observe that any morphism from $U_{m-i+1}[i]$ to $R_{\mathcal{U}}/R$ which is not injective factors through $U_{m-i+2}[i-1]$. Thus the number of direct summands of $R_{\mathcal{U}}/R$ which are isomorphic to

$U_{m-i+1}[i]$ equals

$$\dim_{\text{End}_R(U_{m-i+1}[i])} \text{Hom}_R(U_{m-i+1}[i], R_{\mathcal{U}}/R) - \dim_{\text{End}_R(U_{m-i+2}[i-1])} \text{Hom}_R(U_{m-i+2}[i-1], R_{\mathcal{U}}/R).$$

We want to show that this number agrees with $\alpha_{U_{m-i+1}}$. To this end, we claim that

$$\dim_{\text{End}_R(U_{m-i+1}[i])} \text{Hom}_R(U_{m-i+1}[i], R_{\mathcal{U}}/R) = \alpha_{U_m} + \cdots + \alpha_{U_{m-i+1}}$$

for $i = 1, \dots, m$. This is clear for $i = 1$. We proceed by recurrence and suppose our claim holds true for i . From the exact sequence $0 \rightarrow U_{m-i} \rightarrow U_{m-i}[i+1] \rightarrow U_{m-i+1}[i] \rightarrow 0$ we obtain the exact sequence

$$0 \rightarrow \text{Hom}_R(U_{m-i+1}[i], R_{\mathcal{U}}/R) \rightarrow \text{Hom}_R(U_{m-i}[i+1], R_{\mathcal{U}}/R) \rightarrow \text{Hom}_R(U_{m-i}, R_{\mathcal{U}}/R) \rightarrow 0,$$

hence

$$\dim_k \text{Hom}_R(U_{m-i}[i+1], R_{\mathcal{U}}/R) = \dim_k \text{Hom}_R(U_{m-i+1}[i], R_{\mathcal{U}}/R) + \dim_k \text{Hom}_R(U_{m-i}, R_{\mathcal{U}}/R).$$

But for every indecomposable module $X \in \mathcal{W}_{U_1[m]}$, we can compute $\dim_k \text{Hom}_R(X, R_{\mathcal{U}}/R) = \gamma \cdot \dim_{\text{End}_R(X)} \text{Hom}_R(X, R_{\mathcal{U}}/R)$ where $\gamma = \dim_k \text{End}_R(X)$ does not depend on X by Lemma 1.9. Therefore, dividing by γ , and using that $\dim_{\text{End}_R(U_{m-i})} \text{Hom}_R(U_{m-i}, R_{\mathcal{U}}/R) = \alpha_{U_{m-i}}$, we obtain the claim for $i+1$.

So $R_{\mathcal{U}}/R$ has $\alpha_{U_{m-i+1}}$ summands isomorphic to $U_{m-i+1}[i]$ for any $1 \leq i \leq m$, and (2) is proven. \square

Here are some further results on universal localization that will be needed in Sections 5 and 6.

Proposition 1.11. [14, 35, 18]

- (1) Let \mathcal{Y} be a set of quasi-simple modules, and set $\tau\mathcal{Y} = \{\tau V \mid V \in \mathcal{Y}\}$.
 - (a) If $S \in \mathbb{U} \setminus (\mathcal{Y} \cup \tau\mathcal{Y})$, then $S \otimes_R R_{\mathcal{Y}} \cong S$.
 - (b) If $S \in \tau\mathcal{Y} \setminus \mathcal{Y}$, then S belongs to a tube \mathfrak{t}_{λ} of rank $r > 1$, and there exists $2 \leq m \leq r$ such that $S \otimes_R R_{\mathcal{Y}} \cong S[m]$.
- (2) Assume that $\mathcal{U} \subset \mathbb{U}$ is a set of quasi-simple R -modules that contains no complete cliques. Then:
 - (a) The universal localization $R_{\mathcal{U}}$ is a tame hereditary k -algebra with $\text{rk } K_0(R_{\mathcal{U}}) = \text{rk } K_0(R) - |\mathcal{U}|$.
 - (b) The set $\{S \otimes_R R_{\mathcal{U}} \mid S \in \mathbb{U} \setminus \mathcal{U}\}$ is a complete irredundant set of quasi-simple $R_{\mathcal{U}}$ -modules.
 - (c) The set $\mathfrak{t}_{\mathcal{U}} = \{V \otimes_R R_{\mathcal{U}} \mid V \in \mathfrak{t} \text{ with } \text{Hom}_R(V, U) = \text{Hom}_R(U, V) = 0 \text{ for all } U \in \mathcal{U}\}$ is a complete irredundant set of finite dimensional indecomposable regular $R_{\mathcal{U}}$ -modules.
 - (d) In particular, if \mathfrak{t}_{λ} is a tube of rank $r > 1$ with quasi-simples $U_1, U_2 = \tau^{-1}U_1, \dots, U_r = \tau^{-1}U_{r-1}$, and $\mathcal{U} = \{U_2, \dots, U_{m+1}\}$ for some $m < r$, then the tube $\mathfrak{t}_{\lambda} \otimes_R R_{\mathcal{U}}$ in the Auslander-Reiten quiver of $R_{\mathcal{U}}$ is given by the quasi-simple $R_{\mathcal{U}}$ -modules

$$U_1 \otimes_R R_{\mathcal{U}}, \tau^{-1}(U_1 \otimes_R R_{\mathcal{U}}) = U_{m+2} \otimes_R R_{\mathcal{U}}, \dots, \tau^{-1}(U_{r-1} \otimes_R R_{\mathcal{U}}) = U_r \otimes_R R_{\mathcal{U}}.$$

- (e) The set $\{S[\infty] \mid S \in \mathbb{U} \setminus \mathcal{U}\}$, is a complete irredundant set of Prüfer $R_{\mathcal{U}}$ -modules. We have $(S \otimes_R R_{\mathcal{U}})[\infty] = S[\infty]$ for each $S \in \mathbb{U} \setminus \mathcal{U}$.
- (3) Assume that $\mathcal{V} \subset \mathbb{U}$ is a set of quasi-simple R -modules that contains a complete clique. Then $R_{\mathcal{V}}$ is a hereditary order. Moreover, $\{S \otimes_R R_{\mathcal{V}} \mid S \in \mathbb{U} \setminus \mathcal{V}\}$ is a complete irredundant set of simple $R_{\mathcal{V}}$ -modules, and $\{S[\infty] \mid S \in \mathbb{U} \setminus \mathcal{V}\}$, is a complete irredundant set of injective envelopes of simple $R_{\mathcal{V}}$ -modules. We have injective envelopes $E(S \otimes_R R_{\mathcal{V}}) = S[\infty]$ for each $S \in \mathbb{U} \setminus \mathcal{V}$.

- (4) Assume that $\mathcal{U} \subset \mathbb{U}$ and $\mathcal{V} \subseteq \mathbb{U} \setminus \mathcal{U}$. Then $R_{\mathcal{U} \cup \mathcal{V}} = (R_{\mathcal{U}})_{\mathcal{V}'}$ where $\mathcal{V}' = \{V \otimes_R R_{\mathcal{U}} \mid V \in \mathcal{V}\}$. In particular, there is an injective ring epimorphism $R_{\mathcal{U}} \rightarrow R_{\mathbb{U}}$.

Proof: (1) (a) If $S \in \mathbb{U} \setminus (\mathcal{Y} \cup \tau\mathcal{Y})$, then $\text{Hom}_R(V, S) = \text{Ext}_R^1(V, S) = \text{Hom}_R(S, \tau V) = 0$ for all $V \in \mathcal{Y}$. That is, S is an $R_{\mathcal{Y}}$ -module by Proposition 1.7(4), and therefore $S \otimes_R R_{\mathcal{Y}} = S$.

(b) Let $S \in \tau\mathcal{Y} \setminus \mathcal{Y}$. Then S belongs to a tube \mathfrak{t}_λ of rank $r > 1$. Choose the numbering $S = U_1, U_2 = \tau^{-1}U_1, \dots, U_r = \tau^{-1}U_{r-1}$ for the quasi-simples in \mathfrak{t}_λ . Since, by assumption, $S \notin \mathcal{Y}$, there is m with $2 \leq m \leq r$ such that $U_2, \dots, U_m \in \mathcal{Y}$ and $\tau^{-1}U_m \notin \mathcal{Y}$.

For each $p = 1, \dots, m-1$, the exact sequence $0 \rightarrow S[p] \rightarrow S[p+1] \rightarrow U_{p+1} \rightarrow 0$ induces

$$\cdots \rightarrow \text{Tor}_1^R(U_{p+1}, R_{\mathcal{Y}}) \rightarrow S[p] \otimes_R R_{\mathcal{Y}} \rightarrow S[p+1] \otimes_R R_{\mathcal{Y}} \rightarrow U_{p+1} \otimes_R R_{\mathcal{Y}} \rightarrow 0.$$

Clearly $\text{Tor}_1^R(U_{p+1}, R_{\mathcal{Y}}) = U_{p+1} \otimes_R R_{\mathcal{Y}} = 0$ as $U_{p+1} \in \mathcal{Y}$. Hence we obtain that $S \otimes_R R_{\mathcal{Y}} \cong S[2] \otimes_R R_{\mathcal{Y}} \cong \dots \cong S[m] \otimes_R R_{\mathcal{Y}}$. Note that $S[m]$ is an $R_{\mathcal{Y}}$ -module because $\text{Hom}_R(\mathcal{Y}, S[m]) = 0$ and $\text{Ext}_R^1(\mathcal{Y}, S[m]) \cong D \text{Hom}_R(S[m], \tau\mathcal{Y}) = 0$. Thus $S[m] \otimes_R R_{\mathcal{Y}} \cong S[m]$ as desired.

(2) Statement (a) is [14, Theorem 4.2(1)]. The shape of the quasi-simple and the regular $R_{\mathcal{U}}$ -modules follows from [35, Theorem 10] (cf. [37, Theorem 3.5]), as noted in [14, 2.3, 2.4, Section 4]. See also [18, 10.1]. The statement on $\mathfrak{t}_\lambda \otimes R_{\mathcal{U}}$ is shown in [14, Section 4.2].

It remains to prove (e). Let S be a quasi-simple R -module not in \mathcal{U} . Since Prüfer modules are divisible and $\text{Hom}_R(\mathcal{U}, S[\infty]) = 0$, it follows that $S[\infty] \in \mathcal{U}^\wedge$ is a right $R_{\mathcal{U}}$ -module by Proposition 1.7(4). Further, if S belongs to the R -tube \mathfrak{t}_λ , $S[\infty]$ is filtered by the quasi-simple $R_{\mathcal{U}}$ -modules $\{S \otimes_R R_{\mathcal{U}} \mid S \in \mathfrak{t}_\lambda \setminus \mathcal{U}\}$ by (1). Then $\{(S \otimes_R R_{\mathcal{U}})[n] \mid n \in \mathbb{N}\}$ is a ray on the $R_{\mathcal{U}}$ -tube $\mathfrak{t}_\lambda \otimes R_{\mathcal{U}}$, and $S[\infty] = \varinjlim_n (S \otimes_R R_{\mathcal{U}})[n]$.

(3) By [14, 4.2], $R_{\mathcal{V}}$ is a hereditary order, and by [14, Section 3] (or [5, 6.5]) and [35, Theorem 10], the set $\{S \otimes_R R_{\mathcal{V}} \mid S \in \mathbb{U} \setminus \mathcal{V}\}$ is an irredundant set of simple $R_{\mathcal{V}}$ -modules.

Let $S \in \mathbb{U} \setminus \mathcal{V}$, and suppose that S belongs to the R -tube \mathfrak{t}_λ . By (1), $S[\infty]$ is filtered by the simple $R_{\mathcal{V}}$ -modules $U \otimes_R R_{\mathcal{V}}$, where U runs through the quasi-simple modules in $\mathfrak{t}_\lambda \setminus \mathcal{V}$. By [35, Theorem 10], there exists an equivalence of categories from the category of bound R -modules M such that

$$\text{Hom}_R(M, V) = \text{Hom}_R(V, M) = 0 \text{ for all } V \in \mathcal{V} \tag{1}$$

to the category of bound $R_{\mathcal{V}}$ -modules that restricts to an equivalence from the category of regular R -modules satisfying (1) to the category of torsion $R_{\mathcal{V}}$ -modules. Thus $S[\infty]$ is a uniserial $R_{\mathcal{V}}$ -module that contains $S \otimes_R R_{\mathcal{V}}$, and the injective envelope $E(S \otimes_R R_{\mathcal{V}})$ of $S \otimes_R R_{\mathcal{V}}$ has to contain $S[\infty]$. But by [20, Theorem 19(c)], $E(S \otimes_R R_{\mathcal{V}})$ is also uniserial and has the same filtration as $S[\infty]$, so they must coincide.

(4) is shown in [14, 2.4] as a consequence of [34] and [35]. \square

2. PARAMETRIZING TILTING MODULES.

Tilting classes are in one-to-one-correspondence with certain subcategories of $\text{mod } R$. Recall that a subcategory $\mathcal{S} \subset \text{mod } R$ is said to be *resolving* provided \mathcal{S} is closed under direct summands, extensions, and kernels of epimorphisms, and R belongs to \mathcal{S} . Observe that, since R is hereditary,

a subcategory $\mathcal{S} \subset \text{mod } R$ is resolving whenever it is closed under direct summands and extensions and contains R , see [4, 1.1].

Bazzoni and Herbera proved in [8] that every tilting class $\mathcal{B} = T^\perp$ is determined by a class of finitely presented modules. More precisely, $\mathcal{B} = \mathcal{S}^\perp$ where $\mathcal{S} = {}^\perp \mathcal{B} \cap \text{mod } R$. Combining this with [4, Theorem 2.2] and [39, Theorem 4.14] we obtain

Theorem 2.1. (1) The tilting classes in $\text{Mod } R$ correspond bijectively to the resolving subcategories of $\text{mod } R$. The correspondence is given by the mutually inverse assignments

$$\alpha : \mathcal{B} \mapsto {}^\perp \mathcal{B} \cap \text{mod } R \quad \text{and} \quad \beta : \mathcal{S} \mapsto \mathcal{S}^\perp$$

(2) The cotilting classes in $R\text{Mod}$ correspond bijectively to the resolving subcategories of $\text{mod } R$. The correspondence is given by the mutually inverse assignments

$$\gamma : \mathcal{C} \mapsto {}^\top \mathcal{C} \cap \text{mod } R \quad \text{and} \quad \delta : \mathcal{S} \mapsto \mathcal{S}^\top = {}^\perp(\mathcal{S}^*)$$

(3) The above correspondences yield a one-to-one-correspondence between tilting classes in $\text{Mod } R$ and cotilting classes in $R\text{Mod}$.

Remark 2.2. (1) $\alpha, \beta, \gamma, \delta$ are order-reversing: If $\mathcal{B}_1, \mathcal{B}_2$ are two tilting classes with $\mathcal{B}_1 \subset \mathcal{B}_2$, then $\alpha(\mathcal{B}_2) \subset \alpha(\mathcal{B}_1)$, and the analogous property holds for the remaining assignments.

(2) Any resolving subcategory of $\text{mod } R$ is closed under submodules, since it occurs as ${}^\perp \mathcal{B} \cap \text{mod } R$ for some class $\mathcal{B} \subset \text{Mod } R$ and all modules in $\text{Mod } R$ have injective dimension at most one.

(3) Let \mathcal{S} be a subcategory of $\text{mod } R$ containing R , and assume that \mathcal{S} is closed under predecessors, that is, if $X \in \text{mod } R$ is an indecomposable module with a nonzero map $X \rightarrow S$ to a module $S \in \mathcal{S}$, then $X \in \mathcal{S}$. Then it is easy to see that \mathcal{S} is resolving.

In particular we have the following examples:

Example 2.3. The category $\text{add } \mathbf{p}$ is a resolving subcategory of $\text{mod } R$ with $\beta(\text{add } \mathbf{p}) = \mathbf{p}^\perp = \text{Gen } \mathbf{L}$, and $\delta(\text{add } \mathbf{p}) = {}^\perp_R \mathbf{q} = \text{Cogen}_R \mathbf{W}$ where \mathbf{L} and \mathbf{W} are the Lukas and the Reiten-Ringel tilting modules respectively. \square

Example 2.4. Let \mathbf{t}' be a nonempty union of tubes, and let \mathcal{U} be the set of quasi-simple modules in \mathbf{t}' . Then the category $\text{add}(\mathbf{p} \cup \mathbf{t}')$ is a resolving subcategory of $\text{mod } R$ with $\beta(\text{add}(\mathbf{p} \cup \mathbf{t}')) = \mathbf{t}'^\perp = \text{Gen } T_{\mathcal{U}}$ where $T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$.

In fact, if $Z \in \mathbf{p}$ and S is quasi-simple, then there is a nonzero map from Z to the ray $\{S[n] \mid n \in \mathbb{N}\}$ defined by S , cf. [38, XII, 3.6]. So, by the Auslander-Reiten formula we deduce that the modules in \mathbf{t}'^\perp cannot have direct summands in \mathbf{p} , and therefore $\mathbf{t}'^\perp \subset \mathbf{p}^\perp$ and $(\text{add}(\mathbf{p} \cup \mathbf{t}'))^\perp = \mathbf{t}'^\perp$. This implies $\beta(\text{add}(\mathbf{p} \cup \mathbf{t}')) = \mathcal{U}^\perp = \text{Gen } T_{\mathcal{U}}$ by Example 1.6.

In particular, $\beta(\text{add}(\mathbf{p} \cup \mathbf{t})) = \mathbf{t}^\perp = \text{Gen } \mathbf{W}$. Moreover, with dual arguments one proves that $\delta(\text{add}(\mathbf{p} \cup \mathbf{t})) = {}^\perp_R \mathbf{t} = {}_R \mathbf{t}^\circ$ is the category of all torsion-free left R -modules. \square

The examples above give a complete list of large tilting modules over the Kronecker algebra, as we are going to see in Corollary 2.8, as a consequence of the general Theorem 2.7.

Lemma 2.5. Let T be a tilting R -module with tilting class $\mathcal{B} = T^\perp$, and $\mathcal{S} = {}^\perp\mathcal{B} \cap \text{mod } R$. Then $D(T)$ is a cotilting module with cotilting class ${}^\perp(\mathcal{S}^*) = \{{}_R X \mid D(X) \in \mathcal{B}\}$.

Proof: The well-known Ext-Tor relations yield ${}^\perp(\mathcal{S}^*) = \mathcal{S}^\tau$ and ${}^\perp D(T) = T^\tau = \{{}_R X \mid D(X) \in \mathcal{B}\}$. Now, if $D(X) \in \mathcal{B}$, and $\mathcal{A} = {}^\perp\mathcal{B}$, then $\text{Ext}_R^1(\mathcal{A}, D(X)) = 0$, hence $\text{Tor}_1^R(\mathcal{A}, X) = 0$, and in particular $X \in \mathcal{S}^\tau$. Conversely, since T is a direct limit of modules from \mathcal{S} by [39, 4.4], we have $\mathcal{S}^\tau \subset T^\tau$. So, we have shown ${}^\perp D(T) = {}^\perp(\mathcal{S}^*)$.

We now deduce that $D(T)$ is a cotilting module. In fact, the conditions (T1) and (T3) for T yield the dual conditions (C1) and (C3) for $D(T)$. Moreover, applying the Ext-Tor relations we obtain that $D(T) \in \mathcal{S}^\tau$ since $T \in \mathcal{S}^\perp$. So, $D(T) \in {}^\perp D(T)$, and since ${}^\perp D(T) = \mathcal{S}^\tau$ is closed under products, we infer $\text{Ext}_R^1(D(T)^\kappa, D(T)) = 0$ for any cardinal κ , that is, the dual condition (C2) is also satisfied. \square

Lemma 2.6. The following statements are equivalent for a tilting R -module T .

- (1) T is equivalent to a finitely generated tilting module.
- (2) $D(T)$ is equivalent to a finitely generated cotilting module.
- (3) All indecomposable direct summands of $D(T)$ are finitely generated.

Proof: We will freely use the results on endofinite modules collected in Lemma 1.2.

(1) \Rightarrow (3): Let T' be a finite-dimensional tilting module equivalent to T . Clearly, $\text{Add } T = \text{Add } T'$ implies $\text{Prod } D(T) = \text{Prod } D(T')$. Then the indecomposable direct summands of $D(T)$ belong to $\text{Prod } D(T') = \text{Add } D(T')$, and are therefore isomorphic to indecomposable direct summands of $D(T')$.

(3) \Rightarrow (2): By a well-known result of Bongartz [9], the number of isoclasses of indecomposable direct summands of $D(T)$ is bounded by the number of isoclasses of simple R -modules, and $D(T)$ is equivalent to a finitely generated cotilting module.

(2) \Rightarrow (1): Let ${}_R C$ be a finite-dimensional cotilting module equivalent to $D(T)$. Then $D(T)$ belongs to $\text{Prod } C = \text{Add } C$, and is thus isomorphic to a direct sum of copies of a finite number of indecomposable finitely generated modules. In particular, this implies that $D(T)$ is endofinite. But then T is a pure submodule of the endofinite module $D^2(T)$ and is therefore a direct summand of $D^2(T)$ by [16, 4.3]. In particular, also T is isomorphic to a direct sum of copies of a finite number of indecomposable finitely generated modules, which proves (1). \square

Theorem 2.7. Let T be a tilting R -module with tilting class $\mathcal{B} = T^\perp$, and $\mathcal{S} = {}^\perp\mathcal{B} \cap \text{mod } R$. Assume that T is not equivalent to a finitely generated tilting module. Then the following hold true.

- (1) T is a regular module and $\text{Gen } \mathbf{W} \subset \text{Gen } T \subset \text{Gen } \mathbf{L}$.
- (2) There is a subset $\mathbf{t}' \subset \mathbf{t}$ such that $\mathcal{S} = \text{add } (\mathbf{p} \cup \mathbf{t}')$.
- (3) If $\mathbf{t}' = \emptyset$, then T is equivalent to the Lukas tilting module \mathbf{L} .
- (4) If \mathbf{t}' is a non-empty union of tubes, and \mathcal{U} is the set of quasi-simple modules in \mathbf{t}' , then T is equivalent to $T_{\mathcal{U}}$.

Proof: By assumption and Lemma 2.6, the module ${}_R D(T)$ has an indecomposable direct summand M which is infinite dimensional. Observe that M is pure-injective as it is a summand of a dual

module. From Lemma 1.2(6) and Lemma 2.5, we infer that \mathcal{S}^* cannot contain modules from ${}_R\mathbf{p}$, hence \mathcal{S} cannot contain modules from \mathbf{q} . Similarly, ${}^\perp D(T)$ cannot contain modules from ${}_R\mathbf{q}$, hence $\mathcal{B} = \mathcal{S}^\perp$ cannot contain modules from \mathbf{p} . But then $\mathcal{B} \subset \mathbf{p}^\perp = \text{Gen } \mathbf{L}$, thus $\mathbf{p} \subset {}^\perp \mathcal{B} \cap \text{mod } R = \mathcal{S}$. So, T is a regular module, and we have verified (1) and (2). Now (3) and (4) follow immediately from Examples 2.3 and 2.4. \square

Corollary 2.8. Over the Kronecker-algebra, every tilting module is either equivalent to a finitely generated tilting module, or to precisely one of the modules in the following list:

- the Lukas tilting module \mathbf{L} ,
- the tilting modules of the form $T_{\mathcal{U}}$ for a non-empty set of quasi-simples \mathcal{U} .

In other words, there is a one-one-correspondence between the subsets of \mathfrak{T} and the equivalence classes of large tilting modules.

Proof: Assume that T is not equivalent to a finitely generated tilting module. With the notation of Theorem 2.7, we note that \mathbf{t}' can only contain modules from homogeneous tubes. Then, with any regular module $M \in \mathbf{t}'$, the resolving subcategory \mathcal{S} contains also its regular socle S by Remark 2.2(2), and so it contains the whole (homogeneous) tube S belongs to. This shows that \mathbf{t}' is a union of tubes, so the claim follows from Theorem 2.7. In particular, the large tilting modules are parametrized, up to equivalence, by the subsets of \mathfrak{T} ; hereby, the empty set corresponds to the equivalence class of \mathbf{L} . \square

3. FINITE DIMENSIONAL DIRECT SUMMANDS

In this section we describe the finite dimensional direct summands of a large tilting module T . They are regular modules whose indecomposable summands belong to non-homogeneous tubes. We show that these summands are arranged in disjoint wings, and that the number of summands from each wing equals the number of quasi-simple modules in that wing. Moreover, the summands contributed by each tube \mathbf{t}_λ are determined by the intersection $\mathbf{t}_\lambda \cap \mathcal{S}$ of the tube with the resolving subcategory \mathcal{S} corresponding to T . Special attention will be devoted to the case when \mathcal{S} contains a complete ray from \mathbf{t}_λ .

Lemma 3.1. If T is a large tilting R -module, then every finitely generated indecomposable module $X \in \text{Add } T$ is a regular module from a non-homogeneous tube, and its regular length $m < r$ is bounded by the rank r of the tube. Thus there are at most finitely many non-isomorphic finitely generated indecomposable modules that can occur as direct summands of large tilting modules.

Proof: Suppose that T has tilting class $\mathcal{B} = T^\perp$ and set $\mathcal{S} = {}^\perp \mathcal{B} \cap \text{mod } R$.

Notice that X is isomorphic to a direct summand of T (cf. Lemma 1.2), so it follows from Theorem 2.7(1) that X is a regular module, and there exist a tube \mathbf{t}_λ and a quasi-simple module $S \in \mathbf{t}_\lambda$ such that $X = S[m]$. Now $0 = \text{Ext}_R^1(X, X) \cong D \text{Hom}(S[m], \tau S[m])$ implies that the tube \mathbf{t}_λ has rank $r > 1$.

Choose the numbering $S = U_1, U_2 = \tau^- U_1, \dots, U_r = \tau^- U_{r-1}$ for the quasi-simples in \mathbf{t}_λ . Recall that $\text{Hom}_R(S[m], U_i[m-i+1]) \neq 0$ for all $1 \leq i \leq m$, where we suppose that $U_i = U_j$ whenever $i \equiv j$

mod r . Now, if $m \geq r$, we consider the module $S[m-r+1]$. Since it is a submodule of $X \in \mathcal{S}$ and \mathcal{S} is closed under submodules, we have $S[m-r+1] \in \mathcal{S}$. On the other hand, $\text{Ext}_R^1(S[m-r+1], S[m]) = D \text{Hom}_R(S[m], U_r[m-r+1]) \neq 0$, contradicting the fact that $S[m] \in \text{Add } T \subseteq \mathcal{S}^\perp$. So, we conclude that $m < r$.

Since there are at most finitely many (at most three) non-homogeneous tubes, the foregoing shows that there are at most finitely many non-isomorphic finitely generated indecomposable modules that can occur as direct summands of large tilting modules. \square

From now on in this section, we fix a tilting R -module T with tilting class $\mathcal{B} = T^\perp$. We work in a more general setting which is needed for the proof of our main result Theorem 5.6: we assume that $\mathcal{S} = {}^\perp \mathcal{B} \cap \text{mod } R$ does not contain any non-zero preinjective module, thus $\mathcal{S} = \text{add}(\mathbf{p}' \cup \mathbf{t}')$ where $\mathbf{p}' \subseteq \mathbf{p}$ and $\mathbf{t}' \subseteq \mathbf{t}$. Of course, every large tilting module satisfies this assumption by Theorem 2.7(2).

Remark 3.2. If $X \in \text{Add } T \cap \text{mod } R$, then $X \in \mathcal{S}$. Indeed, $\text{Add } T \cap \text{mod } R = \mathcal{B} \cap {}^\perp \mathcal{B} \cap \text{mod } R = \mathcal{B} \cap \mathcal{S}$.

Lemma 3.3. Let \mathbf{t}_λ be a tube of rank $r > 1$, and let S be a quasi-simple module in \mathbf{t}_λ . Choose the numbering $S = U_1, U_2 = \tau^- U_1, \dots, U_r = \tau^- U_{r-1}$ for the quasi-simples in \mathbf{t}_λ .

- (1) If \mathcal{S} contains some, but not all modules from the ray $\{S[n] \mid n \in \mathbb{N}\}$, then there is $m < r$ such that $S[m] \in \text{Add } T$. More precisely, if $S[m]$ is the module of maximal regular length in $\mathcal{S} \cap \{S[n] \mid n \in \mathbb{N}\}$, then $S[m] \in \text{Add } T$.
- (2) If $S[m] \in \mathbf{t}_\lambda \cap \text{Add } T$, then the rays starting at U_2, \dots, U_{m+1} are not completely contained in \mathcal{S} . In fact, $U_2[m], U_3[m-1], \dots, U_{m+1} \notin \mathcal{S}$.
- (3) If $S[m] \in \mathbf{t}_\lambda \cap \text{Add } T$, then $\mathcal{W}_{S[m]} \cap \text{Add } T$ contains precisely m modules which are uniquely determined by $\mathcal{S} \cap \mathcal{W}_{S[m]}$.

Proof: (1) Assume that $S[r] \in \mathcal{S}$. We claim that $S[n] \in \mathcal{S}$ for all $n \geq 1$. For \mathcal{S} is closed under submodules, thus $S[l] \in \mathcal{S}$ for all $1 \leq l \leq r$. If $n > r$, write $n = kr + l$ with $r \leq kr < n$ and $1 \leq l \leq r$ and consider the exact sequence $0 \rightarrow S[kr] \rightarrow S[n] \rightarrow S[l] \rightarrow 0$. Now the claim follows by induction on n since \mathcal{S} is closed under extensions.

Thus there exists $m < r$ such that $S[m] \in \mathcal{S}$ and $S[m+1] \notin \mathcal{S}$. We prove that $S[m] \in \text{Add } T$. By Remark 3.2, we have to show that $S[m] \in \mathcal{S}^\perp$. Take a module $Z \in \mathcal{S} = \text{add}(\mathbf{p}' \cup \mathbf{t}')$, w.l.o.g. assume that Z is indecomposable. If $Z \in \mathbf{p}'$, then $\text{Ext}_R^1(Z, S[m]) = D \text{Hom}_R(\tau^- S[m], Z) = 0$. If $Z \in \mathbf{t}'$, we can assume w.l.o.g. that Z belongs to \mathbf{t}_λ . If $\text{Ext}_R^1(Z, S[m]) = D \text{Hom}_R(S[m], \tau Z) \neq 0$, we would have $Z = U_{i+1}[m-i+1+l]$ for $1 \leq i \leq m$ and $0 \leq l$. But then the exact sequence $0 \rightarrow U_1[i] \rightarrow U_1[m+1+l] \rightarrow U_{i+1}[m-i+1+l] \rightarrow 0$ together with the fact that \mathcal{S} is closed under extensions would imply that $S[m+1+l] \in \mathcal{S}$, contradicting the choice of m . We conclude that $S[m]$ belongs to \mathcal{S}^\perp and thus to $\text{Add } T$.

(2) All modules of regular length at most m on the coray ending at U_m are quotients of $S[m]$ and therefore belong to the tilting class \mathcal{B} . Hence the modules of regular length at most m on the coray ending at U_{m+1} cannot be in \mathcal{S} by the AR-formula. This yields the claim, because these modules lie on the rays starting at U_2, \dots, U_{m+1} .

(3) We show by induction on m that $\mathcal{W}_{S[m]} \cap \text{Add } T$ contains m modules. Our proof will show how the m modules are determined by $\mathcal{S} \cap \mathcal{W}_{S[m]}$. The result clearly holds for $m = 1$. Let $m > 1$. First

of all, note that the modules $U_2[m-1], U_3[m-2], \dots, U_m$ on the coray ending at U_m are in \mathcal{S}^\perp because they are epimorphic images of $S[m]$.

Suppose that none of the modules $U_2[m-1], U_3[m-2], \dots, U_m$ belongs to \mathcal{S} . Then no regular module containing any of these modules can belong to \mathcal{S} . On the other hand, for $X \in \mathbf{p} \cup \mathbf{t}$ we have $\text{Ext}_R^1(X, S[m-1]) = D \text{Hom}_R(U_2[m-1], X) \neq 0$ if and only if X is a regular module that contains one of the modules $U_2[m-1], U_3[m-2], \dots, U_m$. Hence $S[m-1]$ belongs to \mathcal{S}^\perp , and as a submodule of $S[m] \in \mathcal{S}$ it also belongs to \mathcal{S} , therefore $S[m-1] \in \text{Add } T$. So $\text{Add } T$ contains precisely m modules in $\mathcal{W}_{S[m]}$: these are $S[m]$ and the $m-1$ modules in $\mathcal{W}_{S[m-1]}$ given by the induction hypothesis.

Suppose now that one of $U_2[m-1], U_3[m-2], \dots, U_m$ belongs to \mathcal{S} . Choose $U_{i+1}[m-i] \in \mathcal{S}$ of maximal regular length. Then $U_{i+1}[m-i] \in \text{Add } T$, and the induction hypothesis implies that $\text{Add } T$ contains precisely $m-i$ modules in $\mathcal{W}_{U_{i+1}[m-i]}$.

Since $U_{i+1}[m-i]$ (and its submodules on the ray starting at U_{i+1}) are in \mathcal{S} , no module of regular length at most $m-i$ on the ray starting at U_i can belong to \mathcal{S}^\perp . This shows that $S[i], S[i+1], \dots, S[m-1] \notin \mathcal{S}^\perp$. We claim that $S[i-1] \in \text{Add } T$. To this end, we note that for $X \in \mathbf{p} \cup \mathbf{t}$ we have $\text{Ext}_R^1(X, S[i-1]) = D \text{Hom}_R(U_2[i-1], X) \neq 0$ if and only if X is regular and contains one of the modules $U_2[i-1], U_3[i-2], \dots, U_i$ as a submodule. But none of $U_2[i-1], U_3[i-2], \dots, U_i$ can belong to \mathcal{S} . Indeed, this follows from the choice of $U_{i+1}[m-i]$, by using that each of the modules $U_2[m-1], U_3[m-2], \dots, U_i[m-i+1] \notin \mathcal{S}$ can be written as an extension of one of the modules $U_2[i-1], U_3[i-2], \dots, U_i$ by the module $U_{i+1}[m-i] \in \mathcal{S}$.

So, we infer that $S[i-1] \in \text{Add } T$, and the induction hypothesis implies that $\text{Add } T$ contains precisely $i-1$ modules in $\mathcal{W}_{S[i-1]}$. We conclude that $\text{Add } T$ contains precisely m modules in $\mathcal{W}_{S[m]}$: these are the $m-i$ modules in $\mathcal{W}_{U_{i+1}[m-i]}$, the $i-1$ modules in $\mathcal{W}_{S[i-1]}$, and $S[m]$. \square

The following result shows that the indecomposable summands of T from a tube \mathbf{t}_λ are arranged in disjoint wings, and that the union of such wings does not contain all quasi-simples from \mathbf{t}_λ .

Corollary 3.4. Let X, X' be two finitely generated indecomposable modules in $\text{Add } T$, and let $\mathcal{W}_X, \mathcal{W}_{X'}$ be the corresponding wings. Then either $\mathcal{W}_X \subset \mathcal{W}_{X'}$ or $\mathcal{W}_{X'} \subset \mathcal{W}_X$ or $\mathcal{W}_X \cap \mathcal{W}_{X'} = \emptyset$. Moreover, given a tube \mathbf{t}_λ of rank $r > 1$, the quasi-simple modules in the union of all wings \mathcal{W}_X with $X \in \mathbf{t}_\lambda \cap \text{Add } T$ do not form a complete clique, and there are at most $r-1$ isomorphism classes of modules in $\mathbf{t}_\lambda \cap \text{Add } T$.

Proof: We can assume w.l.o.g. that X, X' belong to the same tube \mathbf{t}_λ . Let S, S' be quasi-simples in \mathbf{t}_λ such that $X = S[m]$ and $X' = S'[m']$. Assume that $m \leq m'$, and suppose that $\mathcal{W}_X \not\subseteq \mathcal{W}_{X'}$ and $\mathcal{W}_X \cap \mathcal{W}_{X'} \neq \emptyset$.

We have to consider two cases. In the first case, the coray \mathbf{c}' that contains $S'[m']$ meets the ray \mathbf{r} determined by S in a module $S[l] \in \mathbf{c}' \cap \mathbf{r}$ with $1 \leq l \leq m$. We even have $l < m$ since otherwise $\mathcal{W}_X \subseteq \mathcal{W}_{X'}$. Then $S[l+1] \in \mathcal{S}$ and $\tau S[l+1] \in \mathcal{S}^\perp$ because \mathcal{S} and \mathcal{S}^\perp are closed under submodules and images respectively. But $\text{Ext}_R^1(S[l+1], \tau S[l+1]) = \text{Hom}_R(\tau S[l+1], \tau S[l+1]) \neq 0$, a contradiction. In the second case, the coray \mathbf{c} that contains $S[m]$ meets the ray \mathbf{r}' determined by S' in a module $S'[l] \in \mathbf{c} \cap \mathbf{r}'$, where again $l < m$ (otherwise $\mathcal{W}_X \subseteq \mathcal{W}_{X'}$). Then $S'[l+1] \in \mathcal{S}$ and $\tau S'[l+1] \in \mathcal{S}^\perp$. But $\text{Ext}_R^1(S'[l+1], \tau S'[l+1]) = \text{Hom}_R(S'[l+1], S'[l+1]) \neq 0$, again a contradiction.

For the proof of the second statement, let $U_1 = S, U_2 = \tau^-U_1, \dots, U_m$ be the quasi-simple modules in \mathcal{W}_X . Then it follows from Lemma 3.1 that $m < r$ and from Lemma 3.3(2) that $\tau^-U_m \notin \mathcal{S}$ cannot be a submodule of a module $X' \in \text{Add } T$. Thus it cannot belong to any wing $\mathcal{W}_{X'}$ with $X' \in \mathfrak{t}_\lambda \cap \text{Add } T$.

Finally, by Lemma 3.3(3) the number of isomorphism classes of modules in $\mathfrak{t}_\lambda \cap \text{Add } T$ equals the number of quasi-simple modules in the union of all wings involved, hence it is at most $r - 1$. \square

Let us now deal with the case that \mathcal{S} contains a complete ray from \mathfrak{t}_λ .

Lemma 3.5. Let \mathfrak{t}_λ be a tube of rank $r > 1$, and let S be a quasi-simple module in \mathfrak{t}_λ . Suppose that the ray $\{S[n] \mid n \geq 1\}$ starting at S is completely contained in \mathcal{S} . Choose the numbering $S = U_1, U_2 = \tau^-U_1, \dots, U_r = \tau^-U_{r-1}$ for the quasi-simples in \mathfrak{t}_λ . The following assertions hold true.

- (1) If the ray $\{U_2[n] \mid n \geq 1\}$ starting at U_2 is completely contained in \mathcal{S} , then $S[n] \notin \text{Add } T$ for all $n \geq 1$.
- (2) If $2 < i \leq r$ is the least number such that the ray $\{U_i[n] \mid n \geq 1\}$ starting at U_i is completely contained in \mathcal{S} , then $S[i-2]$ is the module of maximal regular length in $\{S[n] \mid n \geq 1\} \cap \text{Add } T$.
- (3) If $\{S[n] \mid n \geq 1\}$ is the only ray of \mathfrak{t}_λ which is completely contained in \mathcal{S} , then $S[r-1] \in \text{Add } T$.

Proof: (1) Clearly $\text{Ext}_R^1(U_2[n], S[n]) = D \text{Hom}_R(S[n], S[n]) \neq 0$.

(2) We have to verify $S[i-2] \in \mathcal{S}^\perp$. Observe that, since \mathcal{S} is closed under submodules, $\text{Ext}_R^1(Z, S[i-2]) = D \text{Hom}_R(U_2[i-2], Z) = 0$ for all $Z \in \mathcal{S}$ if and only if $U_2[i-2], U_3[i-3], \dots, U_{i-1} \notin \mathcal{S}$. So, assume that one of the modules $U_2[i-2], U_3[i-3], \dots, U_{i-1}$ belongs to \mathcal{S} , say $U_j[i-j] \in \mathcal{S}$ with $2 \leq j \leq i-1$. Since the rays starting at U_2, \dots, U_{i-1} , are not completely contained in \mathcal{S} , it follows from Lemma 3.3(1) that $U_j[l] \in \text{Add } T$ for some $l \geq i-j$. As \mathcal{S}^\perp is closed under epimorphic images, there exists a module in $U_{i-1}[t] \in \mathcal{S}^\perp$ on the ray starting at U_{i-1} . But this is a contradiction because $\text{Ext}_R^1(U_i[t], U_{i-1}[t]) = D \text{Hom}_R(U_{i-1}[t], U_{i-1}[t]) \neq 0$.

Hence $S[i-2] \in \text{Add } T$. Moreover, no module of the form $S[i-2+t]$ with $t > 0$ is in $\text{Add } T$, because otherwise its epimorphic image $U_{i-1}[t]$ would be a module in \mathcal{S}^\perp on the ray starting at U_{i-1} .

(3) Proceed as in (2) and show that $U_2[r-2], \dots, U_{r-1} \notin \mathcal{S}$. \square

If \mathcal{S} contains some, but not all rays from a tube \mathfrak{t}_λ , then it certainly contains the rays with modules of maximal regular length in $\mathfrak{t}_\lambda \cap \text{Add } T$, as we are going to see next.

Lemma 3.6. Let \mathfrak{t}_λ be a nonhomogeneous tube. Suppose that \mathcal{S} contains a complete ray from \mathfrak{t}_λ . For every module $X \in \mathfrak{t}_\lambda \cap \text{Add } T$ there is a module $S[m] \in \mathfrak{t}_\lambda \cap \text{Add } T$ lying on a ray $\{S[n] \mid n \geq 1\}$ which is completely contained in \mathcal{S} such that X belongs to the wing $\mathcal{W}_{S[m]}$. More precisely, $S[m]$ can be chosen to be either $S[i-2]$ as in Lemma 3.5(2) or $S[r-1]$ as in Lemma 3.5(3).

Proof: Let $S' \in \mathfrak{t}_\lambda$ be a quasi-simple such that $S'[m'] \in \mathfrak{t}_\lambda \cap \text{Add } T$ for some $m' \geq 1$. Choose the numbering $U_1, U_2 = \tau^-U_1, \dots, U_r = \tau^-U_{r-1}$ for the quasi-simples in \mathfrak{t}_λ where the ray starting at U_1 is completely contained in \mathcal{S} , $S' = U_j$ for some $j \in \{1, \dots, r-1\}$, but no ray starting at U_l is completely contained in \mathcal{S} for $2 \leq l \leq j$. Note that also the ray starting at $\tau^-S' = U_{j+1}$ is not completely contained in \mathcal{S} by Lemma 3.3(2).

Set $S = U_1$. If there is no other $i \in \{1, \dots, r\}$ such that the ray starting at U_i is completely contained in \mathcal{S} , then $S[r-1] \in \text{Add } T$ by Lemma 3.5(3). The result then holds by Lemma 3.1 and Lemma 3.3(2).

If $i \in \{j+2, \dots, r\}$ is the first number such that the ray $\{U_i[n] \mid n \geq 1\}$ is completely contained in \mathcal{S} , then $S[i-2] \in \text{Add } T$ by Lemma 3.5(2). Since $U_j[m'] = S'[m'] \in \text{Add } T$, we know that the rays starting at $U_{j+1}, \dots, U_{j+m'}$ are not completely contained in \mathcal{S} by Lemma 3.3(2). Hence $i \geq j+m'+1$. Thus $i-2 \geq j$ and $i-2 \geq m'$. The first inequality implies that $U_j = S' \in \mathcal{W}_{S'[m']} \cap \mathcal{W}_{S[i-2]}$. By Corollary 3.4, the second inequality implies that $\mathcal{W}_{S'[m']} \subseteq \mathcal{W}_{S[i-2]}$. Therefore $S'[m'] \in \mathcal{W}_{S[i-2]}$. \square

Let us summarize our discussion on $\mathbf{t}_\lambda \cap \text{Add } T$.

Proposition 3.7. Let \mathbf{t}_λ be a tube of rank r . Then $\mathbf{t}_\lambda \cap \mathcal{S}$ determines $\mathbf{t}_\lambda \cap \text{Add } T$. More precisely:

- (1) If $\mathbf{t}_\lambda \cap \mathcal{S} = \emptyset$, then $\mathbf{t}_\lambda \cap \text{Add } T = \emptyset$.
- (2) If $\mathbf{t}_\lambda \subseteq \mathcal{S}$, then $\mathbf{t}_\lambda \cap \text{Add } T = \emptyset$.
- (3) If $\emptyset \neq \mathbf{t}_\lambda \cap \mathcal{S} \subsetneq \mathbf{t}_\lambda$, then $\mathbf{t}_\lambda \cap \mathcal{S}$ determines unique quasi-simples $S_1, \dots, S_l \in \mathbf{t}_\lambda$ and unique $m_1, \dots, m_l \in \mathbb{N}$ such that
 - (a) $S_j[m_j] \in \mathbf{t}_\lambda \cap \text{Add } T$ for $j = 1, \dots, l$.
 - (b) $\mathcal{W}_{S_{j_1}[m_{j_1}]} \cap \mathcal{W}_{S_{j_2}[m_{j_2}]} = \emptyset$ if $j_1 \neq j_2$.
 - (c) $\mathbf{t}_\lambda \cap \text{Add } T \subseteq \bigcup_{j=1}^l \mathcal{W}_{S_j[m_j]}$.

For each $j \in \{1, \dots, l\}$, there are exactly m_j modules from $\mathcal{W}_{S_j[m_j]}$ in $\mathbf{t}_\lambda \cap \text{Add } T$ and they are uniquely determined by $\mathcal{S} \cap \mathcal{W}_{S_j[m_j]}$. Therefore there are exactly $m_1 + \dots + m_l < r$ modules in $\mathbf{t}_\lambda \cap \text{Add } T$.

Proof: By Remark 3.2, every finite dimensional indecomposable module in $\text{Add } T$ belongs to \mathcal{S} . Thus (1) follows.

(2) holds by Lemma 3.5(1).

(3) If $\mathbf{t}_\lambda \cap \mathcal{S} \neq \emptyset$ contains no complete ray, then there exist unique quasi-simples S_1, \dots, S_l and $m_1, \dots, m_l \in \mathbb{N}$ verifying (a), (b) and (c) by Lemma 3.3(1) and Corollary 3.4.

If $\mathbf{t}_\lambda \cap \mathcal{S}$ contains a complete ray, then there exist unique quasi-simples S_1, \dots, S_l and $m_1, \dots, m_l \in \mathbb{N}$ verifying (a), (b) and (c) by Lemma 3.6 and Corollary 3.4.

In both cases Lemma 3.3(3) implies that $\mathbf{t}_\lambda \cap \text{Add } T$ contains exactly m_j modules from each $\mathcal{W}_{S_j[m_j]}$ and that these m_j modules are uniquely determined by $\mathcal{S} \cap \mathcal{W}_{S_j[m_j]}$.

Altogether, $\mathbf{t}_\lambda \cap \text{Add } T$ consists of $m_1 + \dots + m_l$ modules, and $m_1 + \dots + m_l < r$ by Corollary 3.4. \square

Definition 3.8. Let \mathbf{t}_λ be a tube. The modules $S_1[m_1], \dots, S_l[m_l]$ satisfying (a), (b) and (c) in Proposition 3.7 will be called the *vertices* of T in \mathbf{t}_λ .

We now want to describe the regular modules that can occur as the finite dimensional part of T .

Definition 3.9. Recall that a module Y is said to be *exceptional* if $\text{Ext}_R^1(Y, Y) = 0$. Inspired by [33, 4.4], we will say that a finite dimensional regular multiplicity free exceptional R -module Y is a *branch module* if it satisfies the following condition:

- (B) For each quasi-simple module S and $m \in \mathbb{N}$ such that $S[m]$ is a direct summand of Y , there exist precisely m direct summands of Y that belong to $\mathcal{W}_{S[m]}$.

Let T be a tilting module with tilting class $\mathcal{B} = T^\perp$ such that $\mathcal{S} = {}^\perp\mathcal{B} \cap \text{mod } R$ does not contain any non-zero preinjective module. By Lemma 3.3(3), the direct sum Y of a complete irredundant set of finitely generated indecomposable direct summands of T is a branch module. The following result shows that there do not exist any other branch modules. We will even see in Theorem 5.6 that every branch module does occur as a direct summand of a large tilting module.

Lemma 3.10. Every finite dimensional regular multiplicity free exceptional module Z is a direct summand of a finite dimensional tilting R -module $H = H_0 \oplus Y$ satisfying the following properties:

- (a) $H_0 \neq 0$ is a preprojective module.
- (b) Y is a branch module with the same quasi-simple composition factors as Z .
- (c) $H^\perp = Z^\perp = Y^\perp$.
- (d) $\mathcal{S}_H = {}^\perp(H^\perp) \cap \text{mod } R$ does not contain any non-zero preinjective module.

In particular, if Z is a branch module, then $H = H_0 \oplus Z$.

Proof: The module Z is a partial tilting module, so by a well known construction due to Bongartz, taking a universal extension $0 \rightarrow R \xrightarrow{\iota} R_0 \xrightarrow{\pi} Z^{(c)} \rightarrow 0$ where $c = \dim_k \text{Ext}_R^1(Z, R)$, we obtain a finitely generated tilting R -module $H = R_0 \oplus Z$ with $H^\perp = Z^\perp$. Hence $\mathfrak{q} \subseteq H^\perp$ and therefore $\mathcal{S}_H = {}^\perp(H^\perp) \cap \text{mod } R$ does not contain any non-zero preinjective module. So $R_0 = H_0 \oplus Y_0$ with H_0 preprojective and Y_0 regular, and $H_0 \neq 0$ since there are no finite dimensional regular tilting modules over R (indeed, if $H_0 = 0$, then H is a direct sum of modules from non-homogeneous tubes, and the number of isomorphism classes from each such tube is smaller than the rank of the tube by Proposition 3.7, so the number of pairwise non-isomorphic indecomposable summands of H is strictly smaller than the number of isomorphism classes of simples, see the table in [32, p.335]). Observe that the regular module $\ker \pi|_{Y_0}$ is contained in the preprojective module $\text{im } \iota$. Thus $\ker \pi|_{Y_0} = 0$ and $Y_0 \subseteq Z^{(c)}$.

We can suppose that $H = H_0 \oplus Y$ where $Y = Y' \oplus Z$ is a direct sum of a complete irredundant set of the indecomposable direct summands of $Y_0 \oplus Z$. Then Y is a branch module by Lemma 3.3(3), and it has the same quasi-simple composition factors as Z since $Y_0 \subseteq Z^{(c)}$.

Finally, note that any exceptional module which is a direct sum of modules from a wing $\mathcal{W}_{S[m]}$ can have at most m non-isomorphic indecomposable summands. So, if Z is a branch module, then the fact that $Y_0 \subseteq Z^{(c)}$ implies $Y_0 \in \text{add } Z$ and therefore $Y = Z$. \square

Remark 3.11. Let Z be a branch module. It can be proved that the tilting module H above is equivalent to $R_{\mathcal{U}} \oplus Z$ where $R_{\mathcal{U}}$ is the universal localization of R at the set \mathcal{U} of quasi-simple composition factors of Z .

4. DECOMPOSING TILTING MODULES

Throughout this section, we fix a tilting R -module T with tilting class $\mathcal{B} = T^\perp$ and $\mathcal{S} = {}^\perp\mathcal{B} \cap \text{mod } R$. We assume that T is not equivalent to a finite dimensional tilting module. We prove a structure result for the modules in \mathcal{B} , from which we derive a canonical decomposition for T .

We are going to use two torsion pairs first studied by Ringel in [32]. The first is the split torsion pair $(\mathcal{D}, \mathcal{D}^\circ)$ whose torsion class is the class \mathcal{D} of the divisible modules. We call a module *reduced*

if it belongs to the corresponding torsion-free class \mathcal{D}^o . The second is the non-split torsion pair $(\text{Gen } \mathbf{t}, \mathcal{F})$ with torsion class $\text{Gen } \mathbf{t}$. Here $(\text{Gen } \mathbf{t})^o = \mathcal{F}$ is the class of all torsion-free modules, cf. 1.3 and 1.4.

We will further need the following canonical decomposition of the regular modules in $\text{Gen } \mathbf{t}$. Write

$$\mathbf{t} = \bigcup_{\lambda \in \mathfrak{T}} \mathbf{t}_\lambda$$

where the \mathbf{t}_λ are the tubes in the Auslander-Reiten quiver of R , and set $\mathcal{T}_\lambda = \varinjlim \text{add } \mathbf{t}_\lambda$. For $X \in \text{Mod } R$ denote by $\mathbf{t}_\lambda(X)$ the maximal submodule of X belonging to \mathcal{T}_λ . As shown in [32, 4.5], every regular module $X \in \text{Gen } \mathbf{t}$ has a unique decomposition

$$X = \bigoplus_{\lambda \in \mathfrak{T}} \mathbf{t}_\lambda(X).$$

We will say that a Prüfer module $S[\infty]$ (or an adic module $S[-\infty]$) *belongs to a tube* \mathbf{t}_λ if S is a quasi-simple module in (the mouth of) \mathbf{t}_λ .

Let us start by investigating the modules in the tilting class $\mathcal{B} = \text{Gen } T$. Since \mathcal{S} consists of finitely presented modules, the class \mathcal{B} is *definable*, i. e., it is closed under direct limits, direct products, and pure submodules.

Proposition 4.1. Let $X \in \mathcal{B} = \mathcal{S}^\perp$. Then

- (1) $X = X_{\mathcal{D}} \oplus X_{\text{red}}$ where $X_{\mathcal{D}} \in \mathcal{D}$ is divisible, and X_{red} is reduced.
- (2) There is a pure-exact sequence $0 \rightarrow X' \rightarrow X_{\text{red}} \rightarrow \overline{X} \rightarrow 0$ where $\overline{X} \in \mathcal{B}$ is torsion-free, and $X' \in \text{Gen } \mathbf{t}$.
- (3) $X' = \bigoplus_{\lambda \in \mathfrak{T}} X_\lambda$, and for each λ there is a pure-exact sequence $0 \rightarrow A_\lambda \rightarrow X_\lambda \rightarrow Z_\lambda \rightarrow 0$ where A_λ is a direct sum of modules in $\mathbf{t}_\lambda \cap \mathcal{B}$, and $Z_\lambda \in \mathcal{B}$ is a direct sum of Prüfer modules belonging to the tube \mathbf{t}_λ .

Proof: For (1) and (2), we refer to [32, 4.7 and 4.1].

(3) Note that the torsion-free class of reduced modules is closed under submodules, and the tilting class $\mathcal{B} = \mathcal{S}^\perp$ is definable, hence closed under pure-submodules. So, we infer from (1) and (2) that X' is a reduced module in \mathcal{B} . Since preinjective modules are divisible, it follows that X' has no indecomposable summands from \mathbf{q} . Moreover, X' has no indecomposable summands from \mathbf{p} because $X' \in \text{Gen } \mathbf{t}$. Thus X' is a regular module in $\text{Gen } \mathbf{t}$ and has a decomposition $X' = \bigoplus_{\lambda \in \mathfrak{T}} \mathbf{t}_\lambda(X')$ as above by [32, 4.5]. We set $X_\lambda = \mathbf{t}_\lambda(X')$. From [32, Theorem G and 4.8] we know that there is a pure-exact sequence $0 \rightarrow A_\lambda \rightarrow X_\lambda \rightarrow Z_\lambda \rightarrow 0$ where A_λ is a direct sum of indecomposable modules of finite length, and Z_λ has no indecomposable direct summand of finite length. Thus Z_λ is regular, and is therefore a direct sum of Prüfer modules. Again, we see that A_λ is a regular module in \mathcal{B} , and since $\text{Hom}_R(\mathbf{t}_\nu, \mathcal{T}_\lambda) = \varinjlim \text{Hom}_R(\mathbf{t}_\nu, \mathbf{t}_\lambda) = 0$ for $\nu \neq \lambda$, we infer that A_λ is a direct sum of modules in $\mathbf{t}_\lambda \cap \mathcal{B}$. Similarly, we see that $Z_\lambda \in \mathcal{B}$ and that the Prüfer modules occurring as direct summands of Z_λ admit non-zero maps from \mathbf{t}_λ and therefore belong to the tube \mathbf{t}_λ . \square

We can now refine the structure result of Proposition 4.1 to the modules in $\text{Add } T$. By Lemma 3.1, there are at most finitely many non-isomorphic finitely generated indecomposable modules in $\text{Add } T$. We denote by Y the direct sum of a complete irredundant set of such modules, which is a branch

module by Lemma 3.3(3). Of course, $Y = \bigoplus_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda(Y)$ where $\mathfrak{t}_\lambda(Y)$ is the direct sum of a complete irredundant set of modules in $\mathfrak{t}_\lambda \cap \text{Add } T$.

Proposition 4.2. Every module $X \in \text{Add } T$ has a unique direct sum decomposition

$$X = \bigoplus_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda(X) \oplus \overline{X}$$

where \overline{X} is torsion-free and each $\mathfrak{t}_\lambda(X)$ has a decomposition in torsion modules with local endomorphism ring. The indecomposable summands of $\mathfrak{t}_\lambda(X)$ are isomorphic to direct summands of $\mathfrak{t}_\lambda(T)$ and are either modules from \mathfrak{t}_λ or Prüfer modules belonging to \mathfrak{t}_λ . Moreover, every torsion (respectively, torsion-free) direct summand of X is a direct summand of the torsion part $\bigoplus_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda(X)$ (respectively, of the torsion-free part \overline{X}).

Proof: Let $X \in \text{Add } T$. We know from [32, 4.1] that there is a pure-exact sequence $0 \rightarrow X' \rightarrow X \rightarrow \overline{X} \rightarrow 0$ where $\overline{X} \in \mathcal{B}$ is torsion-free, and $X' \in \text{Gen } \mathfrak{t}$. Note that $X \in \text{Add } T = {}^\perp \mathcal{B} \cap \mathcal{B}$, and ${}^\perp \mathcal{B}$ is closed under submodules, while \mathcal{B} is closed under pure submodules because it is a definable class. So, we infer that $X' \in \text{Add } T$ is a regular module in $\text{Gen } \mathfrak{t}$, which by [32, 4.5] has a decomposition $X' = \bigoplus_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda(X')$. As in the proof of Proposition 4.1, we deduce from [32, Theorem G and 4.8] that for each λ there is a pure-exact sequence $0 \rightarrow A_\lambda \rightarrow \mathfrak{t}_\lambda(X') \rightarrow Z_\lambda \rightarrow 0$ where A_λ is a direct sum of modules in $\mathfrak{t}_\lambda \cap \mathcal{B}$, and Z_λ is a direct sum of Prüfer modules belonging to the tube \mathfrak{t}_λ . Again, we see that $A_\lambda \in \text{Add } T$, which implies by Lemma 3.1 that A_λ has only finitely many non-isomorphic indecomposable direct summands. In particular, this shows that A_λ is endofinite, thus pure-injective (cf. Lemma 1.2), so the pure-exact sequence $0 \rightarrow A_\lambda \rightarrow \mathfrak{t}_\lambda(X') \rightarrow Z_\lambda \rightarrow 0$ splits, and $\mathfrak{t}_\lambda(X')$ is a direct sum of modules in $\mathfrak{t}_\lambda \cap \text{Add } T$ and Prüfer modules belonging to \mathfrak{t}_λ . In particular, $\mathfrak{t}_\lambda(X')$ has a decomposition in modules with local endomorphism ring.

We infer that $X' = \bigoplus_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda(X')$, being a direct sum of modules isomorphic to indecomposable direct summands of Y or to Prüfer modules, belongs to $\text{Add}(Y \oplus \mathbf{W})$. Now Y is finite dimensional and therefore Σ -pure-injective (that is, every direct sum of copies of Y is pure-injective), and \mathbf{W} is Σ -pure-injective because $\text{Add } \mathbf{W} = \text{Prod } \mathbf{W}$, see [31, 10.1]. Thus X' is pure-injective, and the pure-exact sequence $0 \rightarrow X' \rightarrow X \rightarrow \overline{X} \rightarrow 0$ splits, that is, $X = \bigoplus_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda(X) \oplus \overline{X}$ has the stated decomposition. The uniqueness of \overline{X} and the $\mathfrak{t}_\lambda(X)$ follows directly from torsion theory.

Let A be a direct summand of $X = X' \oplus \overline{X}$. Then there are morphisms $\iota = (\iota', \overline{\iota}) : A \rightarrow X$ and $\pi = (\pi', \overline{\pi}) : X \rightarrow A$ such that $1_A = \pi \iota = \pi' \iota' + \overline{\pi} \overline{\iota}$. If A is torsion, then $\overline{\iota} = 0$, so A is a direct summand of X' . Similarly, if A is torsion-free, then $\pi' = 0$ and A is a direct summand of \overline{X} . In particular, each summand A of $\mathfrak{t}_\lambda(X)$ belongs to $\text{Add } \mathfrak{t}_\lambda(T)$. As $\mathfrak{t}_\lambda(T)$ has a decomposition in modules with local endomorphism ring, we deduce from the Theorem of Krull-Remak-Schmidt-Azumaya that A is isomorphic to an indecomposable direct summand of $\mathfrak{t}_\lambda(T)$, see e.g. [23, 7.3.4]. \square

The following result will be useful when dealing with the torsion-free part \overline{X} in the structure results from Propositions 4.1 and 4.2.

Lemma 4.3. Let \mathfrak{t}_λ be a tube.

- (1) \mathcal{S} contains a complete ray $\{S[n] \mid n \geq 1\}$ from \mathfrak{t}_λ if and only if \mathcal{B} does not contain any adic module belonging to \mathfrak{t}_λ .

- (2) Suppose that \mathbf{t}_λ is a tube of rank $r > 1$ such that \mathcal{S} contains no complete ray from \mathbf{t}_λ . Let \mathcal{U} denote the set of quasi-simple modules in the union of all wings determined by the vertices of T in \mathbf{t}_λ . Then for a quasi-simple module $S \in \mathbf{t}_\lambda$, the adic module $S[-\infty]$ belongs to \mathcal{B} if and only if $S \notin \tau\mathcal{U} = \{\tau U \mid U \in \mathcal{U}\}$. Thus \mathcal{B} contains precisely $r - |\mathcal{U}|$ pairwise non-isomorphic adic modules belonging to \mathbf{t}_λ .
- (3) Let \mathcal{U} be a set of quasi-simple modules in \mathbf{t}_λ . Every torsion-free module in \mathcal{B} is contained in \mathcal{U}^\perp if and only if all adic modules in \mathcal{B} belonging to \mathbf{t}_λ are contained in \mathcal{U}^\perp (equivalently, every torsion-free module in \mathcal{B} is an $R_{\mathcal{U}}$ -module if and only if all the adic modules in \mathcal{B} belonging to \mathbf{t}_λ are $R_{\mathcal{U}}$ -modules.)

Proof: We start by proving the only-if part of (1). Suppose that \mathcal{S} contains the complete ray $\{S[n] \mid n \geq 1\}$. Choose the numbering $S = U_1, U_2 = \tau^-U_1, \dots, U_r = \tau^-U_{r-1}, r \geq 1$, for the quasi-simples in \mathbf{t}_λ . Consider $U_i[-\infty]$ for some $1 \leq i \leq r$. Then $\text{Ext}_R^1(S[i+1], U_i[-\infty]) \cong D \text{Hom}_R(U_i[-\infty], \tau S[i+1]) = D \text{Hom}_R(U_i[-\infty], U_r[i+1]) \neq 0$. Therefore $U_i[-\infty] \notin \mathcal{S}^\perp$ and hence $U_i[-\infty] \notin \mathcal{B}$.

Next, we prove (2) and the if-part of (1). First of all, observe that for any quasi-simple $S \in \mathbf{t}_\lambda$ we have $\text{Ext}_R^1(\mathbf{p}, S[-\infty]) = 0$ because $S[-\infty]$ has no non-zero preprojective summands. Also $\text{Ext}_R^1(\mathbf{t}_\mu, S[-\infty]) = 0$ for all $\mu \neq \lambda$.

So, if $\mathbf{t}_\lambda \cap \mathcal{S} = \emptyset$, then all adic modules belonging to \mathbf{t}_λ are in \mathcal{B} (and indeed, this is the case $\mathcal{U} = \tau\mathcal{U} = \emptyset$ in (2)).

Assume now $\mathbf{t}_\lambda \cap \mathcal{S} \neq \emptyset$, and suppose that \mathcal{S} does not contain a complete ray from \mathbf{t}_λ . We know from Lemma 3.3 that $\mathbf{t}_\lambda \cap \mathcal{S}$ is contained in the extension closure \mathcal{W} of \mathcal{U} . By Proposition 3.7, \mathcal{U} does not contain a complete clique. So, there are quasi-simple modules $S \in \mathbf{t}_\lambda \setminus \tau\mathcal{U}$. For such S we have $\text{Ext}_R^1(U, S[-\infty]) = D \text{Hom}_R(S[-\infty], \tau U) = 0$ for all $U \in \mathcal{U}$, so $S[-\infty] \in \mathcal{U}^\perp = \mathcal{W}^\perp$ by Lemma 1.1, and combining this with our first observation, we conclude that $S[-\infty] \in \mathcal{S}^\perp = \mathcal{B}$. On the other hand, if $S \in \tau\mathcal{U}$, it is easy to see that $\text{Ext}_R^1(X_j, S[-\infty]) = D \text{Hom}_R(S[-\infty], \tau X_j) \neq 0$ for X_j a vertex of T in \mathbf{t}_λ , which shows $S[-\infty] \notin \mathcal{B}$ and completes the proof of (1) and (2).

(3) First of all, we note that the class of all torsion-free modules $\mathcal{F} = \mathbf{t}^\circ$, as well as the classes \mathcal{B} and \mathcal{U}^\perp , are definable classes. Indeed, \mathcal{F} is clearly closed under direct products and submodules, and it is closed under direct limits since \mathbf{t} consists of finitely presented modules. As for $\mathcal{B} = \mathcal{S}^\perp$ and \mathcal{U}^\perp , closure under direct products is clear, and closure under direct limits and pure submodules follows from the fact that \mathcal{S} and \mathcal{U} consist of finitely presented modules.

We are now ready to consider a torsion-free module $X \in \mathcal{B}$. Take the pure-injective envelope I of X , which is again a torsion-free module in \mathcal{B} as definable classes are closed under pure-injective envelopes, see [19, 3.1.10]. Moreover, I is the pure-injective envelope of $\bigoplus_{l \in L} I_l$, where $\{I_l \mid l \in L\}$ is a complete irredundant set of indecomposable summands of I , cf. [22, Chapter 8]. Now the I_l are indecomposable pure-injective torsion-free modules in \mathcal{B} , and they are in $\text{Gen } \mathbf{L} = \mathbf{p}^\perp$ by Theorem 2.7(1). We infer from the classification of the indecomposable pure-injective modules reviewed in Lemma 1.2 that I_l is either the generic module G or an adic module. Notice that G is divisible and thus belongs to \mathcal{U}^\perp . Moreover, using the Auslander-Reiten formula, it is easy to see that $I_l \in \mathcal{U}^\perp$ if I_l is an adic module belonging to a tube \mathbf{t}_μ with $\mu \neq \lambda$. So, if we assume that all adic modules I_l belonging to \mathbf{t}_λ are contained in \mathcal{U}^\perp , then also $\bigoplus_{l \in L} I_l$ and its pure-injective envelope I are in \mathcal{U}^\perp , and therefore also the pure submodule X .

Conversely, recall that any adic module is torsion-free. \square

Let us determine the branch module Y when the tilting class $\mathcal{B} = \mathcal{S}^\perp$ is the class of modules that are Ext-orthogonal to a ray in a non-homogeneous tube, or in other words, $\mathcal{S} = \text{add}(\mathbf{p} \cup \mathbf{t}')$ with \mathbf{t}' being a ray. This is a special case that will play an important role in the sequel.

Example 4.4. Let S be a quasi-simple module, and assume $\mathcal{S} = \text{add}(\mathbf{p} \cup \{S[n] \mid n \in \mathbb{N}\})$. Then

$$\mathcal{B} = S[\infty]^\perp, \text{ and } S[\infty] \in \text{Add } T.$$

Moreover, if S belongs to a tube \mathbf{t}_λ of rank $r > 1$, then

$$Y = S \oplus S[2] \oplus \cdots \oplus S[r-1]$$

(and $T \sim S \oplus S[2] \oplus \cdots \oplus S[r-1] \oplus S[\infty] \oplus R\mathbf{t}_\lambda$, as we will see later in 5.9).

Indeed, we show as in Example 2.4 that $\mathcal{S}^\perp = \bigcap_{n \geq 1} S[n]^\perp$. Moreover, as $S[n] \subset S[\infty]$, we have $S[\infty]^\perp \subseteq S[n]^\perp$ for all $n \geq 1$, hence $S[\infty]^\perp \subseteq \mathcal{B}$. For the reverse inclusion, note that $S[\infty]$ is filtered by $S[r]$, thus every $X \in \mathcal{B} \subset S[r]^\perp$ also belongs to $S[\infty]^\perp$ by [19, 3.1.2].

Now we deduce $S[\infty] \in {}^\perp(S[\infty]^\perp) = {}^\perp\mathcal{B}$. Moreover, $S[\infty]$ also belongs to \mathcal{B} as it is divisible and thus satisfies $\text{Ext}_R^1(S[n], S[\infty]) = 0$ for all $n \in \mathbb{N}$. So, we conclude $S[\infty] \in \text{Add } T$.

For the last claim, observe that every finitely generated indecomposable module in $\text{Add } T$ must belong to \mathcal{B} , and also to ${}^\perp\mathcal{B} \cap \text{mod } R = \mathcal{S}$ and thus to the ray $\{S[n] \mid n \in \mathbb{N}\}$. So, we deduce from $\mathbf{t}_\lambda \cap \mathcal{B} = \mathcal{W}_{S[r-1]}$ that $S, S[2], \dots, S[r-1]$ is a complete irredundant set of the finitely generated indecomposable modules in $\text{Add } T$. \square

We are now ready for the main result of this section.

Theorem 4.5. There is a unique direct sum decomposition

$$T = \bigoplus_{\lambda \in \mathfrak{X}} \mathbf{t}_\lambda(T) \oplus \overline{T}$$

where \overline{T} is torsion-free, and $\mathbf{t}_\lambda(T)$ is a direct sum of copies of the indecomposable direct summands of $\mathbf{t}_\lambda(Y)$ and of Prüfer modules belonging to \mathbf{t}_λ . More precisely, for each tube \mathbf{t}_λ of rank r , the summand $\mathbf{t}_\lambda(T)$ is given as follows:

- (i) if \mathcal{S} contains some modules from \mathbf{t}_λ , but no complete ray, then $\mathbf{t}_\lambda(T)$ is a direct sum of at most $r-1$ pairwise non-isomorphic modules from \mathbf{t}_λ that are arranged in the disjoint wings determined by the vertices of T in \mathbf{t}_λ , and the number of non-isomorphic summands from each wing equals the number of quasi-simple modules in that wing;
- (ii) if \mathcal{S} contains some rays from \mathbf{t}_λ , then $\mathbf{t}_\lambda(T)$ has precisely r pairwise non-isomorphic indecomposable summands: these are the s Prüfer modules corresponding to the $s \leq r$ rays from \mathbf{t}_λ contained in \mathcal{S} , and $r-s$ modules from \mathbf{t}_λ which are arranged in the disjoint wings determined by the vertices of T in \mathbf{t}_λ ;
- (iii) $\mathbf{t}_\lambda(T) = 0$ whenever $\mathbf{t}_\lambda \cap \mathcal{S} = \emptyset$.

Proof: The existence of the decomposition follows from Proposition 4.2. Observe that every indecomposable direct summand of $\mathbf{t}_\lambda(Y)$ lies in $\mathbf{t}_\lambda \cap \text{Add } T$ and therefore occurs as a direct summand in $\mathbf{t}_\lambda(T)$ by Lemma 1.2(1).

We now turn to the additional statements. Note first that the finite dimensional direct summands of $\mathbf{t}_\lambda(T)$ are contained in $\mathbf{t}_\lambda \cap \mathcal{S}$, and further, recall that ${}^\perp\mathcal{B}$ is closed under submodules, so with

every Prüfer module $S[\infty]$ it contains also the corresponding ray $\{S[n] \mid n \in \mathbb{N}\}$. This proves (iii) and shows that $\mathfrak{t}_\lambda(T)$ cannot contain infinite dimensional direct summands if $\mathfrak{t}_\lambda \cap \mathcal{S}$ does not contain complete rays. Thus, in case (i), the non-isomorphic indecomposable summands of $\mathfrak{t}_\lambda(T)$ are precisely the indecomposable direct summands of $\mathfrak{t}_\lambda(Y)$, and they have the stated properties by Proposition 3.7.

It remains to prove (ii). Assume that $\mathfrak{t}_\lambda \cap \mathcal{S}$ contains a complete ray $\{S[n] \mid n \in \mathbb{N}\}$. Then $\mathcal{B} \subset \bigcap_{n \geq 1} S[n]^\perp = S[\infty]^\perp$ by Example 4.4, so $S[\infty] \in {}^\perp\mathcal{B}$. Further, using that $\mathcal{S} = \text{add}(\mathfrak{p} \cup \mathfrak{t}')$ for some $\emptyset \neq \mathfrak{t}' \subseteq \mathfrak{t}$, we see that $S[\infty]$ lies in \mathcal{B} , hence in $\text{Add } T$. We then infer from Proposition 4.2 that $S[\infty]$ is a direct summand in $\mathfrak{t}_\lambda(T)$.

Now we determine the remaining indecomposable summands of $\mathfrak{t}_\lambda(T)$. If \mathcal{S} contains the whole tube \mathfrak{t}_λ , then it follows from Lemma 3.5(1) that $\mathfrak{t}_\lambda(T)$ has no finite dimensional summands, and therefore it is a direct sum of all Prüfer modules belonging to \mathfrak{t}_λ . If \mathfrak{t}_λ has rank $r > 1$, and \mathcal{S} contains $1 \leq s < r$ complete rays from \mathfrak{t}_λ , we get from Lemma 3.6 and Proposition 3.7 that there are exactly $r - s$ finite dimensional indecomposable summands in $\mathfrak{t}_\lambda(T)$. \square

Remark 4.6. There seems to be an asymmetry between case (i) and (ii) in Theorem 4.5 above: the number s of pairwise non-isomorphic indecomposable summands of $\mathfrak{t}_\lambda(T)$ equals the rank r of \mathfrak{t}_λ when \mathcal{S} contains some rays from \mathfrak{t}_λ , but is smaller than r otherwise. Note however that in the latter case s coincides with the number of quasi-simple modules in the union of the wings determined by the vertices of T in \mathfrak{t}_λ . So, the “missing” summands are somehow “replaced” by the $r - s$ adic modules in \mathcal{B} established by Lemma 4.3(2). This aspect will become more clear in Remark 6.2 of the Appendix.

5. CLASSIFYING TILTING MODULES

Let again T be a tilting R -module with tilting class $\mathcal{B} = T^\perp$, and $\mathcal{S} = {}^\perp\mathcal{B} \cap \text{mod } R$. We assume that T is not equivalent to a finitely generated tilting module. Then we know from Theorem 2.7 that there is a subset $\mathfrak{t}' \subset \mathfrak{t}$ such that $\mathcal{S} = \text{add}(\mathfrak{p} \cup \mathfrak{t}')$. We have seen in Theorem 2.7 that T is equivalent either to the Lukas tilting modules if \mathfrak{t}' is empty, or to a tilting module arising from universal localization in case \mathfrak{t}' is a non-empty union of tubes. We now discuss the general case.

Recall that we denote by Y the branch module defined as the direct sum of a complete irredundant set of the finitely generated indecomposable modules in $\text{Add } T$. Thus Y is a finite dimensional direct summand of T by Lemma 3.1 and Lemma 1.2(1).

Our aim is to reduce the classification problem to the situation considered in Theorem 2.7. To this end, we will show that T is equivalent to a tilting module of the form $Y \oplus M$, where M has no finite dimensional indecomposable direct summands and is a tilting module over a suitable universal localization of R . We will prove this step by step, by considering the finitely many non-homogeneous tubes \mathfrak{t}_λ where $\mathfrak{t}_\lambda \cap \text{Add } T \neq \emptyset$.

We first give a general criterion for constructing a tilting module of the desired form.

Lemma 5.1. Let $Y' \in \text{Add } T$, and let M be a module satisfying condition (T2), i.e. $\text{Ext}_R^1(M, M^{(\kappa)}) = 0$ for any cardinal κ . Then $Y' \oplus M$ is a tilting module equivalent to T if and only if the following hold true.

- (a) $\mathcal{B} \subset M^\perp$,
- (b) $M \in \mathcal{B}$,
- (c) $T \in \text{Add}(Y' \oplus M)$.

Proof: For the only-if-part, note that $\text{Add}(Y' \oplus M) = \text{Add} T \subset \mathcal{B} = (Y' \oplus M)^\perp$, which immediately yields (a), (b), (c).

For the if-part, we show that $Y' \oplus M$ is tilting. Condition (T1) is trivially verified. In order to check (T2), let α be a cardinal. Then $\text{Ext}_R^1((Y' \oplus M), (Y' \oplus M)^{(\alpha)}) \cong \text{Ext}_R^1(Y', Y'^{(\alpha)}) \oplus \text{Ext}_R^1(M, M^{(\alpha)}) \oplus \text{Ext}_R^1(M, Y'^{(\alpha)}) \oplus \text{Ext}_R^1(Y', M^{(\alpha)})$. Now the first term vanishes since $Y' \in \text{Add} T$, the second by assumption on M , the third term vanishes by (a) since \mathcal{B} is closed under direct sums and therefore $Y'^{(\alpha)} \in \mathcal{B}$, and the last term vanishes because $M^{(\alpha)} \in \mathcal{B} = T^\perp \subset Y'^\perp$ by property (b). Finally, condition (T3) is satisfied by property (c).

So, $Y' \oplus M$ is a tilting module with $\text{Add} T \subset \text{Add}(Y' \oplus M)$, thus $(Y' \oplus M)^\perp \subset T^\perp$. Conversely, $T^\perp = \mathcal{B} \subset Y'^\perp \cap M^\perp = (Y' \oplus M)^\perp$, showing that $Y' \oplus M$ is equivalent to T . \square

Now we proceed with our reduction. Given a non-homogeneous tube \mathfrak{t}_λ where $\mathfrak{t}_\lambda \cap \text{Add} T \neq \emptyset$, we want to replace T by an equivalent tilting module of the form $\mathfrak{t}_\lambda(Y) \oplus M$ where M is a tilting module over a suitable universal localization $R_{\mathcal{U}}$ of R . To this end, we replace the resolving subcategory \mathcal{S} by its localization

$$\mathcal{S}_{\mathcal{U}} = \{A \otimes_R R_{\mathcal{U}} \mid A \in \mathcal{S}\}$$

and choose M to be a tilting $R_{\mathcal{U}}$ -module with tilting class $\mathcal{B}_{\mathcal{U}} = \mathcal{S}_{\mathcal{U}}^\perp$. The existence of M is guaranteed by [19, 5.2.2]. We formulate criteria that will allow to perform the replacement.

Proposition 5.2. Let \mathfrak{t}_λ be a tube of rank $r > 1$, and let \mathcal{U} be a set of $m < r$ quasi-simples in \mathfrak{t}_λ with extension closure \mathcal{W} . Assume that M is an $R_{\mathcal{U}}$ -tilting module with tilting class

$$\mathcal{B}_{\mathcal{U}} = \{X \in \text{Mod } R_{\mathcal{U}} \mid \text{Ext}_{R_{\mathcal{U}}}^1(A \otimes_R R_{\mathcal{U}}, X) = 0 \text{ for all } A \in \mathcal{S}\}$$

such that

- (i) $\mathcal{W} \cup \mathcal{W}^\circ$ contains the subset $\mathfrak{t}' \subset \mathfrak{t}$ with $\mathcal{S} = \text{add}(\mathfrak{p} \cup \mathfrak{t}')$,
- (ii) $\text{Add}(\mathfrak{t}_\lambda \cap \mathcal{B}) \subset M^\perp = \{X \in \text{Mod } R \mid \text{Ext}_R^1(M, X) = 0\}$,
- (iii) every adic module in \mathcal{B} belonging to \mathfrak{t}_λ is contained in \mathcal{U}^\perp .

Then $\mathfrak{t}_\lambda(Y) \oplus M$ is a tilting R -module equivalent to T .

Proof: As $\mathfrak{t}_\lambda(Y) \in \text{Add} T$ and M satisfies (T2), we only have to verify the conditions in Lemma 5.1.

(a) We prove $\mathcal{B} \subset M^\perp$ in two steps.

Step 1: We show $\mathcal{B} \cap \mathcal{U}^\perp \subset M^\perp$. Take $X \in \mathcal{B} \cap \mathcal{U}^\perp$. We claim $M \in {}^\perp X$. Since the $R_{\mathcal{U}}$ -tilting module M is filtered by the modules in $\mathcal{S}_{\mathcal{U}} = \{A \otimes_R R_{\mathcal{U}} \mid A \in \mathcal{S}\}$ by [39, Lemma 4.5], it suffices to show that $\mathcal{S}_{\mathcal{U}} \subset {}^\perp X$. So, let $A \in \mathcal{S}$, w.l.o.g. A indecomposable. Then $A \in \mathfrak{p} \subset \mathcal{W}^\circ$ or $A \in \mathfrak{t}' \subset \mathcal{W} \cup \mathcal{W}^\circ$ by (i). If $A \in \mathcal{W}$, then $A \otimes_R R_{\mathcal{U}} = 0$ by Proposition 1.7(5), so we can assume w.l.o.g. $A \in \mathcal{W}^\circ$. Then we know from Proposition 1.7(7) that there is a short exact sequence $0 \rightarrow A \rightarrow A \otimes_R R_{\mathcal{U}} \rightarrow A \otimes_R R_{\mathcal{U}}/R \rightarrow 0$ where the two outer terms $A \in \mathcal{S}$ and $A \otimes_R R_{\mathcal{U}}/R \in {}^\perp(\mathcal{U}^\perp)$ belong to ${}^\perp X$, so we infer that the middle-term $A \otimes_R R_{\mathcal{U}}$ belongs to ${}^\perp X$ as well.

Step 2: We now consider an arbitrary $X \in \mathcal{B}$ and apply the structure result in 4.1. Since the divisible module $X_{\mathcal{D}}$ belongs to $\mathcal{B} \cap \mathcal{U}^\perp \subset M^\perp$, and M^\perp is closed under extensions, it is enough to show

that X' and \overline{X} are in M^\perp . Observe first that X' and \overline{X} are in \mathcal{B} since \mathcal{B} is closed under pure submodules and epimorphic images. Furthermore, we know from Lemma 4.3 and condition (iii) that the torsion-free module $\overline{X} \in \mathcal{B}$ is contained in \mathcal{U}^\perp . So, we conclude from Step 1 that $\overline{X} \in M^\perp$.

Now let us turn to $X' = \bigoplus_{\mu \in \mathfrak{S}} X_\mu$. Since $\mathcal{B} \cap \mathcal{U}^\perp$ is closed under direct sums, we have $\bigoplus_{\mu \neq \lambda} X_\mu \in \mathcal{B} \cap \mathcal{U}^\perp \subset M^\perp$, so we only have to consider X_λ . Recall that there is a pure-exact sequence $0 \rightarrow A_\lambda \rightarrow X_\lambda \rightarrow Z_\lambda \rightarrow 0$ where A_λ is a direct sum of modules in $\mathfrak{t}_\lambda \cap \mathcal{B}$, and $Z_\lambda \in \mathcal{B}$ is a direct sum of Prüfer modules belonging to the tube \mathfrak{t}_λ . Then Z_λ is divisible and therefore in M^\perp by Step 1, and $A_\lambda \in M^\perp$ by (ii), thus also $X_\lambda \in M^\perp$.

(b) We now prove $M \in \mathcal{B}$. Let $A \in \mathcal{S}$, and assume w.l.o.g. that A is indecomposable. As in (a) we infer from (i) that $A \in \mathcal{W} \cup \mathcal{W}^\circ$. If $A \in \mathcal{W}$, then $\text{Ext}_R^1(A, M) = 0$ because M is an $R_{\mathcal{U}}$ -module and thus belongs to \mathcal{W}^\wedge by Proposition 1.7(4). If $A \in \mathcal{W}^\circ$, then we know from Proposition 1.7(7) that A embeds in $A \otimes_R R_{\mathcal{U}} \in {}^\perp M$, hence $A \in {}^\perp M$, and the claim is verified.

(c) Finally, we check that $T \in \text{Add}(\mathfrak{t}_\lambda(Y) \oplus M)$. By Theorem 4.5 there is a decomposition $T = \bigoplus_{\mu \in \mathfrak{S}} \mathfrak{t}_\mu(T) \oplus \overline{T}$ where \overline{T} is torsion-free, and each $\mathfrak{t}_\mu(T)$ is a direct sum of copies of $\mathfrak{t}_\mu(Y)$ and Prüfer modules belonging to \mathfrak{t}_μ . Moreover, a Prüfer module $S[\infty]$ occurs as a direct summand in $\mathfrak{t}_\mu(T)$ if and only if $\mathfrak{t}_\mu \cap \mathcal{S}$ contains the complete ray $\{S[n] \mid n \geq 1\}$, again by Theorem 4.5. Observe that complete rays in \mathcal{S} are contained in \mathcal{W}° by (i). So, we deduce that the Prüfer modules occurring as direct summands in $\mathfrak{t}_\lambda(T)$ do not belong to quasi-simples in \mathcal{U} and are therefore contained in \mathcal{U}° , and even in \mathcal{U}^\wedge as they are divisible modules. Thus $\mathfrak{t}_\lambda(T)$ is the direct sum of a module in $\text{Add } \mathfrak{t}_\lambda(Y)$ with a module in \mathcal{U}^\wedge . Of course, also the $\mathfrak{t}_\mu(T)$ with $\mu \neq \lambda$ belong to \mathcal{U}^\wedge . Finally, the torsion-free module \overline{T} is contained in \mathcal{U}° , and even in \mathcal{U}^\wedge by Lemma 4.3 and condition (iii). So, our claim will be proven once we show that $\text{Add } T \cap \mathcal{U}^\wedge \subset \text{Add } M$.

Let us thus consider $X \in \text{Add } T \cap \mathcal{U}^\wedge$. First of all, $X \in \mathcal{B} \subset M^\perp$ by (a). Moreover, X is an $R_{\mathcal{U}}$ -module, hence $\text{Ext}_{R_{\mathcal{U}}}^1(M, X) = \text{Ext}_R^1(M, X) = 0$. Therefore X belongs to the $R_{\mathcal{U}}$ -tilting class $\mathcal{B}_{\mathcal{U}}$, and there is an exact sequence $0 \rightarrow M_1 \rightarrow M_0 \xrightarrow{f} X \rightarrow 0$ with $M_0, M_1 \in \text{Add } M$ by [19, 5.1.8(d)]. Note that $\text{Add } M \subseteq X^\perp$ because $M \in \mathcal{B} = T^\perp$ and $X \in \text{Add } T$. Hence the exact sequence splits and $X \in \text{Add } M$.

Now the proof of the Proposition is complete. \square

In order to specify the set \mathcal{U} at which we will localize, we have to distinguish two cases, depending on whether $\mathfrak{t}_\lambda \cap \mathcal{S}$ contains a complete ray or not.

Definition 5.3. Let \mathfrak{t}_λ be a tube of rank $r > 1$, and let $S_1[m_1], \dots, S_l[m_l]$ be the vertices of T in \mathfrak{t}_λ . We define a set \mathcal{U} of quasi-simple modules as follows. If $\mathfrak{t}_\lambda \cap \mathcal{S}$ does not contain a complete ray, then \mathcal{U} consists of the quasi-simple modules in the union of the wings $\bigcup_{j=1}^l \mathcal{W}_{S_j[m_j]}$. If $\mathfrak{t}_\lambda \cap \mathcal{S}$ contains a complete ray, then \mathcal{U} consists of the quasi-simples in \mathfrak{t}_λ whose ray is not completely contained in \mathcal{S} .

We remark that the set \mathcal{U} consists of exactly $m_1 + \dots + m_l < r$ quasi-simple modules. Indeed, this is clear in the first case by Corollary 3.4. In the second case, the rays that are not completely contained in \mathcal{S} correspond to the $m_1 + \dots + m_l$ quasi-simples in $\bigcup_{j=1}^l \mathcal{W}_{\tau - S_j[m_j]}$ by Lemma 3.5.

Proposition 5.4. Let \mathfrak{t}_λ be a tube of rank $r > 1$ such that $\mathfrak{t}_\lambda \cap \mathcal{S} \neq \emptyset$ does not contain complete rays. Let \mathcal{U} be defined as in Definition 5.3, and let M be an $R_{\mathcal{U}}$ -tilting module with tilting class

$\mathcal{B}_{\mathcal{U}} = \{X \in \text{Mod } R_{\mathcal{U}} \mid \text{Ext}_{R_{\mathcal{U}}}^1(A \otimes_R R_{\mathcal{U}}, X) = 0 \text{ for all } A \in \mathcal{S}\}$. Then $\mathbf{t}_{\lambda}(Y) \oplus M$ is a tilting R -module equivalent to T such that neither \mathbf{t}_{λ} nor the $R_{\mathcal{U}}$ -tube $\mathbf{t}_{\lambda} \otimes R_{\mathcal{U}}$ have modules from $\text{Add } M$.

Proof: Let $S_1[m_1], \dots, S_l[m_l]$ be the vertices of T in \mathbf{t}_{λ} . For each $j \in \{1, \dots, l\}$, let \mathcal{U}_j consist of the m_j quasi-simples in $\mathcal{W}_{S_j[m_j]}$. By definition, $\mathcal{U} = \bigcup_{j=1}^l \mathcal{U}_j$. We denote by \mathcal{W} the extension closure of \mathcal{U} and recall from Lemma 1.1 that \mathcal{W} consists of all finite direct sums of modules in $\mathcal{W}' = \bigcup_{i=1}^l \mathcal{W}_{S_j[m_j]}$.

We verify conditions (i)-(iii) in Proposition 5.2.

(i) We claim $\mathbf{t}' \subset \mathcal{W} \cup \mathcal{W}^{\circ}$. Indeed, if $A \in \mathbf{t}' \cap \mathbf{t}_{\nu}$ with $\nu \neq \lambda$, then clearly $A \in \mathcal{U}^{\circ}$, which coincides with \mathcal{W}° by Lemma 1.7(3). If $A \in \mathbf{t}' \cap \mathbf{t}_{\lambda}$, then, by the assumption, A lies on a ray $\{S'[n] \mid n \in \mathbb{N}\}$ which is not completely contained in \mathcal{S} . Then, by Lemma 3.3(1), A is a submodule of $S'[m] \in \text{Add } T$. By the definition of vertices, $\mathcal{W}_{S'[m]} \subseteq \mathcal{W}$, and therefore $A \in \mathcal{W}$.

(ii) Let us now verify $\text{Add}(\mathbf{t}_{\lambda} \cap \mathcal{B}) \subset M^{\perp}$. Choose $A \in \text{Add}(\mathbf{t}_{\lambda} \cap \mathcal{B})$. By [40] there is an indecomposable decomposition of the form

$$A = \bigoplus_{p \in P} W_p^{(\alpha_p)} \oplus \bigoplus_{q \in Q} X_q^{(\alpha_q)}$$

where $\{W_p \mid p \in P\}$ is a complete irredundant set of modules in \mathcal{W}' , and $\{X_q \mid q \in Q\}$ is a complete irredundant set of modules in $(\mathbf{t}_{\lambda} \cap \mathcal{B}) \setminus \mathcal{W}'$. Note that the index set P is finite.

First of all, we prove that $\mathcal{U} \subset M^{\perp}$. We fix a $j \in \{1, \dots, l\}$, and choose the numbering $U_1 = S_j, U_2 = \tau^{-1}U_1, \dots, U_{m_j} = \tau^{-1}U_{m_j-1}$ for the quasi-simples in \mathcal{U}_j . For $1 \leq i < m_j$ we have $\text{Ext}_R^1(M, U_i) \cong \text{DHom}_R(U_{i+1}, M) = 0$ since M is an $R_{\mathcal{U}}$ -module. Moreover, $U_{m_j} \in \mathcal{B}$ because it is a quotient of $S_j[m_j] \in \text{Add } T$, and $U_{m_j} \in \mathcal{U}^{\perp}$ because $\text{Ext}_R^1(U, U_{m_j}) \cong \text{DHom}_R(\tau^{-1}U_{m_j}, U) = 0$ for all $U \in \mathcal{U}$ as $\tau^{-1}U_{m_j} \notin \mathcal{U}$. Now we infer as in Step 1 of the proof of Proposition 5.2 that $\mathcal{B} \cap \mathcal{U}^{\perp} \subset M^{\perp}$, so $U_{m_j} \in M^{\perp}$ as well. Hence $\mathcal{U} \subset M^{\perp}$, thus also $\mathcal{W}' \subset M^{\perp}$, yielding by Lemma 1.2(2) and (3) that $\bigoplus_{p \in P} W_p^{(\alpha_p)} \in M^{\perp}$.

Next, we consider $X \in (\mathbf{t}_{\lambda} \cap \mathcal{B}) \setminus \mathcal{W}'$. Note that $S_j[m_j] \in \text{Add } T$ implies $S_j[t] \in \mathcal{S}$ for all $t \leq m_j$, thus $0 = \text{Ext}_R^1(S_j[t], X) \cong \text{DHom}_R(X, \tau S_j[t])$. Then X cannot lie on a coray ending at $\tau U_1, U_1, \dots, U_{m_j-1}$. Hence $\text{Ext}_R^1(U_t, X) \cong \text{DHom}_R(X, \tau U_t) = 0$ for all $1 \leq t \leq m_j$, which shows that $X \in \mathcal{U}^{\perp}$. Therefore $\bigoplus_{q \in Q} X_q^{(\alpha_q)} \in \mathcal{B} \cap \mathcal{U}^{\perp}$ as \mathcal{B} and \mathcal{U}^{\perp} are closed under direct sums. Now we

infer as in Step 1 of the proof of Proposition 5.2 that $\mathcal{B} \cap \mathcal{U}^{\perp} \subset M^{\perp}$, and we conclude that $A \in M^{\perp}$ as desired.

(iii) Finally, we check that every adic module in \mathcal{B} belonging to \mathbf{t}_{λ} is contained in \mathcal{U}^{\perp} . So suppose that $I = U[-\infty] \in \mathcal{B}$ for some quasi-simple $U \in \mathbf{t}_{\lambda}$. As in (ii), we fix a $j \in \{1, \dots, l\}$, and we see that $0 = \text{Ext}_R^1(S_j[t], I) \cong \text{DHom}_R(I, \tau S_j[t])$ for all $1 \leq t \leq m_j$, hence $U \notin \{\tau U_1, U_1, \dots, U_{m_j-1}\}$. This implies $\text{Ext}_R^1(U_t, I) \cong \text{DHom}_R(I, \tau U_t) = 0$ for all $1 \leq t \leq m_j$, $j = 1, \dots, l$, that is, $I \in \mathcal{U}^{\perp}$.

Therefore $\mathbf{t}_{\lambda}(Y) \oplus M$ is a tilting R -module equivalent to T . Now we prove the remaining assertions. By Proposition 1.11(1) and (2), the R -tube \mathbf{t}_{λ} contains the quasi-simple modules and therefore all modules in the $R_{\mathcal{U}}$ -tube $\mathbf{t}_{\lambda} \otimes R_{\mathcal{U}}$. Moreover, since $\text{Mod } R_{\mathcal{U}}$ is a full subcategory of $\text{Mod } R$ closed under direct sums and direct summands, $\text{Add}_{R_{\mathcal{U}}} M = \text{Add}_R M$. So, it is enough to show that \mathbf{t}_{λ} has no submodules from $\text{Add } M$.

Assume that $Z \in \mathbf{t}_{\lambda} \cap \text{Add } M$. Then Z is an $R_{\mathcal{U}}$ -module, because $\text{Mod } R_{\mathcal{U}}$ is a full subcategory of $\text{Mod } R$ closed under direct sums and direct summands. On the other hand, as $\text{Add}(\mathbf{t}_{\lambda}(Y) \oplus M) =$

Add T , we deduce that Z belongs to $\mathfrak{t}_\lambda \cap \text{Add } T$, thus to $\mathcal{W}_{S_j[m_j]}$ for some $j \in \{1, \dots, l\}$. But then it follows from Proposition 1.7(5) that $Z \otimes_R R_{\mathcal{U}} = 0$, a contradiction. \square

Proposition 5.5. Let \mathfrak{t}_λ be a tube of rank $r > 1$ such that $\mathfrak{t}_\lambda \cap \mathcal{S} \neq \mathfrak{t}_\lambda$ contains a complete ray. Let \mathcal{U} be as in Definition 5.3, and let M be an $R_{\mathcal{U}}$ -tilting module with tilting class $\mathcal{B}_{\mathcal{U}} = \{X \in \text{Mod } R_{\mathcal{U}} \mid \text{Ext}_{R_{\mathcal{U}}}^1(A \otimes_R R_{\mathcal{U}}), X) = 0 \text{ for all } A \in \mathcal{S}\}$. Then $\mathfrak{t}_\lambda(Y) \oplus M$ is a tilting R -module equivalent to T such that neither \mathfrak{t}_λ nor the $R_{\mathcal{U}}$ -tube $\mathfrak{t}_\lambda \otimes R_{\mathcal{U}}$ have modules from $\text{Add } M$.

Proof: Let $S_1[m_1], \dots, S_l[m_l]$ be the vertices of T in \mathfrak{t}_λ . For each $j \in \{1, \dots, l\}$, let \mathcal{U}_j consist of the m_j quasi-simples lying in $\mathcal{W}_{\tau^{-1}S_j[m_j]}$ and choose the numbering $U_{j1} = \tau^{-1}S_j, U_{j2} = \tau^{-1}U_{j1}, \dots, U_{jm_j} = \tau^{-1}U_{j(m_j-1)}$ for these quasi-simples. By definition, $\mathcal{U} = \bigcup_{j=1}^l \mathcal{U}_j$. We denote by \mathcal{W} the extension closure of \mathcal{U} and recall from Lemma 1.1 that \mathcal{W} consists of all finite direct sums of modules in $\mathcal{W}' = \bigcup_{i=1}^l \mathcal{W}_{\tau^{-1}S_j[m_j]}$.

Let us verify conditions (i)-(iii) in Proposition 5.2.

(i) We claim $\mathfrak{t}' \subset \mathcal{W} \cup \mathcal{W}^\circ$. If $A \in \mathfrak{t}' \cap \mathfrak{t}_\mu$ with $\mu \neq \lambda$, then clearly $A \in \mathcal{U}^\circ$ which coincides with \mathcal{W}° by Proposition 1.7. If $A \in \mathfrak{t}' \cap \mathfrak{t}_\lambda$ and A lies on a ray which is completely contained in \mathfrak{t}' , then $A \in \mathcal{U}^\circ$ because \mathcal{U} consists of the quasi-simples in \mathfrak{t}_λ whose ray is not completely contained in \mathcal{S} , cf. Definition 5.3. Then, as before, $A \in \mathcal{W}^\circ$. Assume now that $A \in \mathfrak{t}' \cap \mathfrak{t}_\lambda$ lies on a ray $\{S'[n] \mid n \in \mathbb{N}\}$ which is not completely contained in \mathcal{S} . There exists $t \in \mathbb{N}$ such that $S'[t] \in \text{Add } T$ and $A = S'[v]$ with $v \leq t$ by Lemma 3.3(1). Note that $S' \neq S_j$ because the ray starting at S_j is completely contained in \mathcal{S} , and also that $S' \neq U_{jm_j}$ because $U_{jm_j} \notin \mathcal{S}$ by Lemma 3.3(2). But, by Proposition 3.7, $S'[t] \in \mathcal{W}_{S_j[m_j]}$ for some j . Hence $S'[v] \in \mathcal{W}_{U_{j1}[m_j-1]} \subseteq \mathcal{W}_{\tau^{-1}S_j[m_j]}$.

(ii) In order to verify $\text{Add}(\mathfrak{t}_\lambda \cap \mathcal{B}) \subset M^\perp$, we first observe that no module $\tau S_j[n]$ on the ray starting at τS_j can belong to \mathcal{B} . Consider now a module $X \in \mathfrak{t}_\lambda \cap \mathcal{B}$, and assume $X \notin \mathcal{U}^\perp$. Then there are $j \in \{1, \dots, l\}$ and $i \in \{1, \dots, m_j\}$ such that $0 \neq \text{Ext}_R^1(U_{ji}, X) \cong \text{DHom}_R(X, \tau U_{ji})$, hence X lies on one of the corays ending at $S_j = \tau U_{j1}, U_{j1}, \dots, U_{jm_j-1}$. But then X must belong to the wing $\mathcal{W}_{S_j[m_j]}$, because otherwise there is an epimorphism from X to a module $\tau S_j[n]$ with $2 \leq n \leq m_j + 1$, which would imply $\tau S_j[n] \in \mathcal{B}$. Now recall that the $R_{\mathcal{U}}$ -module M belongs to \mathcal{U}^\perp , hence $\text{Ext}_R^1(M, \tau U_{ji}) \cong \text{DHom}_R(U_{ji}, M) = 0$ for all $1 \leq i \leq m_j, 1 \leq j \leq l$, which proves that $\{S_j, U_{j1}, \dots, U_{jm_j-1}\} \subset M^\perp$ and therefore $X \in \mathcal{W}_{S_j[m_j]} \subset M^\perp$.

As in the proof of Proposition 5.2, step 2, we have $\mathcal{B} \cap \mathcal{U}^\perp \subset M^\perp$, so the claim follows.

(iii) is trivially satisfied, because the assumption that $\mathfrak{t}_\lambda \cap \mathcal{S}$ contains a complete ray implies by Lemma 4.3 that \mathcal{B} does not contain any adic module belonging to \mathfrak{t}_λ .

Hence $\mathfrak{t}_\lambda(Y) \oplus M$ is a tilting R -module equivalent to T . As in Proposition 5.4, we observe that there are no modules in $\mathfrak{t}_\lambda \cap \text{Add } M$. Finally, since \mathcal{U} consists of the quasi-simples whose ray is not completely contained in \mathcal{S} , we infer from Proposition 1.11 that the $R_{\mathcal{U}}$ -tube $\mathfrak{t}_\lambda \otimes R_{\mathcal{U}}$ is completely contained in $\mathcal{S}_{\mathcal{U}} = \{A \otimes_R R_{\mathcal{U}} \mid A \in \mathcal{S}\} \subset {}^\perp \mathcal{B}_{\mathcal{U}} \cap \text{mod } R_{\mathcal{U}}$. But then the tilting $R_{\mathcal{U}}$ -module M cannot contain direct summands from this tube by Proposition 3.7(2). \square

Now we are in a position to prove our main result.

Theorem 5.6. Let R be a tame hereditary algebra with $\mathfrak{t} = \bigcup_{\lambda \in \mathfrak{T}} \mathfrak{t}_\lambda$. Every tilting R -module is either equivalent to a finitely generated tilting module, or to precisely one module $T_{(Y, \Lambda)}$ in the following list:

- (1) $T_{(Y, \emptyset)} = Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$ where Y is a branch module, and \mathcal{U} is the set of quasi-simple composition factors of Y .
- (2) $T_{(Y, \Lambda)} = Y \oplus R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$ where Y is a branch module, $\emptyset \neq \Lambda \subseteq \mathfrak{T}$, and \mathcal{U}, \mathcal{V} are defined as follows:
 - (i) If $\lambda \in \Lambda$, then $\mathbf{t}_{\lambda} \cap \mathcal{V}$ is the complete clique in \mathbf{t}_{λ} , and $\mathbf{t}_{\lambda} \cap \mathcal{U}$ is the set of all the quasi-simples in \mathbf{t}_{λ} that appear in a regular composition series of $\tau^{-}Y$.
 - (ii) If $\lambda \notin \Lambda$, then $\mathbf{t}_{\lambda} \cap \mathcal{V} = \mathbf{t}_{\lambda} \cap \mathcal{U}$ consists of all the quasi-simples in \mathbf{t}_{λ} that appear in a regular composition series of Y .

Moreover, the large tilting modules are parametrized, up to equivalence, by the elements of $\mathcal{Y} \times \mathcal{P}(\mathfrak{T})$, where $\mathcal{P}(\mathfrak{T})$ denotes the power set of \mathfrak{T} , and $\mathcal{Y} = \{Y_1, \dots, Y_t\}$ is a complete irredundant set of branch modules over R .

Proof: Let T be a tilting R -module with tilting class $\mathcal{B} = T^{\perp}$ and $\mathcal{S} = {}^{\perp}\mathcal{B} \cap \text{mod } R$. Assume that T is not equivalent to a finitely generated tilting module. Thus there exists $\mathbf{t}' \subset \mathbf{t}$ such that $\mathcal{S} = \text{add}(\mathbf{p} \cup \mathbf{t}')$ by Theorem 2.7. By Lemma 3.1, there are at most finitely many non-isomorphic finitely generated indecomposable modules in $\text{Add } T$ and all of them are regular modules from some non-homogeneous tube. Let us denote by Y the direct sum of a complete irredundant set of such modules. By Lemma 3.3(3), Y is a branch module. We want to show that T is equivalent to $T_{(Y, \Lambda)}$ where $\Lambda = \{\lambda \in \mathfrak{T} \mid \mathbf{t}_{\lambda} \cap \mathcal{S} \text{ contains a complete ray}\}$.

Step 1: Assume that $\text{Add } T$ does not contain finitely generated modules. Then \mathbf{t}' is empty or a union of tubes by Proposition 3.7. In the first case, $\Lambda = \emptyset$, $Y = 0$ and $\mathcal{U} = \emptyset$, hence $T_{(Y, \Lambda)} = \mathbf{L}$, which is equivalent to T by Theorem 2.7. If \mathbf{t}' is a union of tubes, then $\Lambda = \{\lambda \mid \mathbf{t}_{\lambda} \subseteq \mathbf{t}'\}$, $Y = 0$ and $\mathcal{U} = \emptyset$, and \mathcal{V} consists of the quasi-simples in \mathbf{t}' . Hence $T_{(Y, \Lambda)} = R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R$, which is equivalent to T by Theorem 2.7, as desired.

Step 2: Assume now that $\text{Add } T$ contains some finitely generated indecomposable module. So, let us consider a tube \mathbf{t}_{λ} of rank $r > 1$ and such that $\mathbf{t}_{\lambda} \cap \text{Add } T \neq \emptyset$. Let \mathcal{U}_{λ} be as in Definition 5.3. Set $\mathcal{B}_{\mathcal{U}_{\lambda}} = \{X \in \text{Mod } R_{\mathcal{U}_{\lambda}} \mid \text{Ext}_R^1(A \otimes_R R_{\mathcal{U}_{\lambda}}, X) = 0 \text{ for all } A \in \mathcal{S}\}$ and $\mathcal{S}_{\mathcal{U}_{\lambda}} = {}^{\perp}\mathcal{B}_{\mathcal{U}_{\lambda}} \cap \text{mod } R_{\mathcal{U}_{\lambda}}$.

We have to distinguish two cases depending on whether $\mathbf{t}_{\lambda} \cap \mathcal{S}$ contains a complete ray or not.

Suppose first that $\mathbf{t}_{\lambda} \cap \mathcal{S}$ does not contain a complete ray, that is, $\lambda \notin \Lambda$. Then \mathcal{U}_{λ} consists of the quasi-simples that appear in a regular composition series of $\mathbf{t}_{\lambda}(Y)$, so $\mathcal{U}_{\lambda} = \mathbf{t}_{\lambda} \cap \mathcal{U} = \mathbf{t}_{\lambda} \cap \mathcal{V}$.

Suppose now that $\mathbf{t}_{\lambda} \cap \mathcal{S}$ contains a complete ray, that is, $\lambda \in \Lambda$. Here \mathcal{U}_{λ} consists of the quasi-simples in \mathbf{t}_{λ} whose ray is not completely contained in \mathcal{S} , which coincide with the regular composition factors of $\tau^{-}(\mathbf{t}_{\lambda}(Y))$, or in other words, with the quasi-simples in \mathbf{t}_{λ} that appear in the regular series of $\tau^{-}Y$. Thus $\mathcal{U}_{\lambda} = \mathbf{t}_{\lambda} \cap \mathcal{U}$.

Let M_{λ} be a tilting module over the tame hereditary algebra $R_{\mathcal{U}_{\lambda}}$ with tilting class $\mathcal{B}_{\mathcal{U}_{\lambda}}$. It follows from Propositions 5.4 and 5.5 that T is equivalent to $\mathbf{t}_{\lambda}(Y) \oplus M_{\lambda}$ over R , and $\text{Add } M_{\lambda}$ has no modules from \mathbf{t}_{λ} . Over $R_{\mathcal{U}_{\lambda}}$, we know that $\text{Add } M_{\lambda}$ has no module from the $R_{\mathcal{U}_{\lambda}}$ -tube $\mathbf{t}_{\lambda} \otimes R_{\mathcal{U}_{\lambda}}$, and further, the modules from the other tubes that belong to $\mathcal{S}_{\mathcal{U}_{\lambda}}$ are the same as before. Indeed, if $\mu \neq \lambda$, then $\mathbf{t}_{\mu} \otimes R_{\mathcal{U}_{\lambda}} = \mathbf{t}_{\mu}$ because every element in \mathbf{t}_{μ} is already an $R_{\mathcal{U}_{\lambda}}$ -module. Hence $(\mathbf{t}_{\mu} \otimes R_{\mathcal{U}_{\lambda}}) \cap \mathcal{S}_{\mathcal{U}_{\lambda}} = \mathbf{t}_{\mu} \cap \mathcal{S}$.

Step 3: Now we apply Propositions 5.4 and 5.5 as in Step 2 repeatedly (at most twice more) until we obtain that T is equivalent to $Y \oplus M$ where M is a tilting module over a universal localization $R_{\mathcal{U}}$ at the set \mathcal{U} from Definition 5.3, and $\text{Add}_{R_{\mathcal{U}}} M$ does not contain finitely generated $R_{\mathcal{U}}$ -modules. Note

that \mathcal{U} is a union of quasi-simples from different tubes, and it does not contain a complete clique by Proposition 3.7. Thus $R_{\mathcal{U}}$ is a tame hereditary algebra, and Step 1 yields that M is equivalent either to the Lukas tilting module over $R_{\mathcal{U}}$, or to a tilting module of the form $(R_{\mathcal{U}})_{\mathcal{V}'} \oplus (R_{\mathcal{U}})_{\mathcal{V}'} / R_{\mathcal{U}}$ for a set \mathcal{V}' of quasi-simple $R_{\mathcal{U}}$ -modules which is a union of cliques over $R_{\mathcal{U}}$.

In the first case we know from [3, Theorem 6] that M is equivalent to $\mathbf{L} \otimes_R R_{\mathcal{U}}$. Observe that, by construction, this first case holds if and only if \mathcal{S} does not contain a complete ray, and that \mathcal{U} is the set of quasi-simples that appear in the regular composition series of Y . Therefore T is equivalent to $T_{(Y, \emptyset)}$.

In the second case we apply Proposition 1.11. By construction, $\mathcal{V}' = \{V \otimes R_{\mathcal{U}} \mid V \in \mathcal{R}\}$ where \mathcal{R} is a set of quasi-simple R -modules defined as follows: if $\lambda \in \Lambda$, then $\mathbf{t}_{\lambda} \cap \mathcal{R}$ is the complement of $\mathbf{t}_{\lambda} \cap \mathcal{U}$, and $\mathbf{t}_{\lambda} \cap \mathcal{R} = \emptyset$ otherwise. Then $\mathcal{V} = \mathcal{U} \cup \mathcal{R}$ and $(R_{\mathcal{U}})_{\mathcal{V}'} \cong R_{\mathcal{V}}$. Thus T is equivalent to $T_{(Y, \Lambda)}$, as desired.

Step 4: Conversely, we show that for any branch module Y and any subset $\Lambda \subseteq \mathfrak{T}$, there exists a tilting R -module of the form $T_{(Y, \Lambda)}$ as above.

First of all, by Lemma 3.10, there exists a finitely generated tilting R -module $H = H_0 \oplus Y$ with $H_0 \neq 0$ preprojective and $\mathcal{S}_H = {}^{\perp}(H^{\perp}) \cap \text{mod } R = \text{add}(\mathbf{p}' \cup \mathbf{t}')$ where $\mathbf{p}' \subseteq \mathbf{p}$ and $\mathbf{t}' \subseteq \mathbf{t}$.

We claim that \mathbf{t}' does not contain any complete ray. Indeed, if \mathbf{t}' contains a ray, then we infer as in Example 2.4 that the modules in \mathbf{t}'^{\perp} cannot have direct summands in \mathbf{p} . But $H^{\perp} = \mathcal{S}_H^{\perp} \subset \mathbf{t}'^{\perp}$ contains the preprojective module $H_0 \neq 0$, a contradiction. Therefore the claim holds true.

Suppose that $\Lambda = \emptyset$. Consider $\mathcal{S} = \text{add}(\mathbf{p} \cup \mathbf{t}')$. Then \mathcal{S} is a resolving subcategory of $\text{mod } R$ because so is \mathcal{S}_H . Hence there exists a tilting R -module T with $\mathcal{S} = {}^{\perp}(T^{\perp}) \cap \text{mod } R$ by Theorem 2.1(1). By Remark 3.2, T has neither preinjective nor preprojective direct summands. Since there are no finite dimensional regular tilting R -modules (cf. the proof of Lemma 3.10), we infer that T is a large tilting R -module. By Steps 1-3 above, T is then equivalent to a tilting module of the form $T_{(Y', \emptyset)} = Y' \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$ where \mathcal{U} is the set of quasi-simple modules that appear in a regular composition series of Y' . But we know from Proposition 3.7 that $\text{Add } T \cap \mathbf{t}_{\lambda}$ is determined by $\mathcal{S} \cap \mathbf{t}_{\lambda}$, which coincides with $\mathcal{S}_H \cap \mathbf{t}_{\lambda}$ for all $\lambda \in \mathfrak{T}$. Hence $Y \cong Y'$, and $T_{(Y, \emptyset)}$ is a tilting module equivalent to T .

Suppose now that $\Lambda \neq \emptyset$. By Lemma 3.3(1), the set \mathbf{t}' is contained in the union \mathcal{W}' of the wings determined by the vertices $S_1[m_1], \dots, S_l[m_l]$ of H . We now want to enlarge \mathbf{t}' by inserting some rays from the tubes \mathbf{t}_{λ} with $\lambda \in \Lambda$, namely, the rays corresponding to the set \mathcal{R} of all quasi-simples in $\bigcup_{\lambda \in \Lambda} \mathbf{t}_{\lambda}$ that do not appear in the regular composition series of $\tau^{-1}Y$. So, let $\mathbf{t}'' \subseteq \mathbf{t}$ be obtained from \mathbf{t}' by adding these rays, that is, $\mathbf{t}'' = \mathbf{t}' \cup \{S[n] \mid S \in \mathcal{R}, n \in \mathbb{N}\}$.

We claim that $\text{add}(\mathbf{p} \cup \mathbf{t}'')$ is a resolving subcategory of $\text{mod } R$. To this end, we start by observing that $\text{add}(\mathbf{t}'')$ is closed under regular submodules by construction, since so is $\text{add}(\mathbf{t}')$.

Next, we prove that $\text{add}(\mathbf{t}'')$ is closed under extensions. Consider an extension

$$0 \longrightarrow R'_1 \oplus R''_1 \xrightarrow{\iota} X \xrightarrow{\pi} R'_2 \oplus R''_2 \longrightarrow 0$$

of two modules in $\text{add}(\mathbf{t}'')$, where we suppose that R''_i is a direct sum of indecomposables from the inserted rays $\{S[n] \mid S \in \mathcal{R}, n \in \mathbb{N}\}$, while R'_i is a direct sum of indecomposables lying on the remaining rays of \mathbf{t}' . Let Z be an indecomposable regular direct summand of X . The module $\ker \pi|_Z$ is a regular submodule of Z , hence indecomposable. Further, $\ker \pi|_Z$ is a submodule of $R'_1 \oplus R''_1$, hence a submodule of an indecomposable summand of $R'_1 \oplus R''_1$. If it is a submodule of an indecomposable

summand of R_1'' , we are done, because Z then belongs to $\{S[n] \mid S \in \mathcal{R}, n \in \mathbb{N}\} \subset \mathbf{t}''$. So suppose that $\ker \pi|_Z$ is a submodule of R_1' . Thus Z is a module lying on a ray starting at a quasi-simple $S \in \mathbf{t}' \setminus \mathcal{R}$. By construction, the kernel of a non-zero map from Z to a module in $\{S[n] \mid S \in \mathcal{R}, n \in \mathbb{N}\}$ cannot belong to the union of the wings \mathcal{W}' . On the other hand, recall that $\ker \pi|_Z \in \mathbf{t}' \subset \mathcal{W}'$. This shows that $\pi(Z)$ is a submodule of R_2' and belongs to \mathbf{t}' . Now Z is an extension of two elements in \mathbf{t}' , and therefore lies in $\mathbf{t}' \subseteq \text{add } \mathbf{t}''$ as desired.

Finally, we deduce that $\text{add}(\mathbf{p} \cup \mathbf{t}'')$ is closed under extensions (and is therefore resolving). Consider an extension

$$0 \longrightarrow P_1 \oplus R_1 \xrightarrow{\iota} X \xrightarrow{\pi} P_2 \oplus R_2 \longrightarrow 0$$

where $P_i \in \text{add } \mathbf{p}$ and $R_i \in \text{add } \mathbf{t}''$ for each i . Firstly, X has no preinjective direct summand. Hence $X = P \oplus R$ with $P \in \text{add } \mathbf{p}$ and $R \in \text{add } \mathbf{t}$. We have to prove that R belongs to $\text{add } \mathbf{t}''$. Observe that $\pi(R) = R_2'$ is a regular submodule of R_2 . Thus $R_2' \in \text{add } \mathbf{t}''$. Now $\ker \pi|_R$ is a regular module because $\text{add } \mathbf{t}$ is closed under kernels. Hence it is a submodule of R_1 and thus $\ker \pi|_R \in \text{add } \mathbf{t}''$. Therefore $R \in \text{add } \mathbf{t}''$ because it is the extension of two modules in $\text{add } \mathbf{t}''$.

So, $\mathcal{S} = \text{add}(\mathbf{p} \cup \mathbf{t}'')$ is a resolving subcategory of $\text{mod } R$. By Theorem 2.1(1), there exists a tilting R -module T with ${}^\perp(T^\perp) \cap \text{mod } R = \mathcal{S}$, and by the discussion above, T is equivalent to a tilting R -module $T_{(Y', \Lambda)}$ as in (2). By construction, the vertices of T are exactly the vertices $S_1[m_1], \dots, S_l[m_l]$ of H (the only difference being that the vertices in the tubes $\mathbf{t}_\lambda, \lambda \in \Lambda$, now lie on rays that are completely contained in \mathcal{S} , while they are not completely contained in \mathcal{S}_H). Moreover, we know from Proposition 3.7 that $\mathbf{t}_\lambda \cap \text{Add } T$ is determined by the intersections with the corresponding wings $\mathcal{S} \cap \mathcal{W}_{S_j[m_j]}$, which coincide with $\mathcal{S}_H \cap \mathcal{W}_{S_j[m_j]}$ for all $1 \leq j \leq l$. Hence $Y \cong Y'$, and $T_{(Y, \Lambda)}$ is a tilting module equivalent to T , as desired.

Step 5: Finally, we establish the parametrization. Observe first that \mathcal{Y} is indeed a finite set by Lemma 3.1. Furthermore, we have just seen that the assignment $(Y, \Lambda) \rightarrow T_{(Y, \Lambda)}$ is a well-defined surjective map from $\mathcal{Y} \times \mathcal{P}(\mathfrak{T})$ to the set of equivalence classes of tilting modules. It remains to verify the injectivity. Suppose that $T_{(Y, \Lambda)}$ and $T_{(Y', \Lambda')}$ are equivalent tilting modules, where $Y, Y' \in \mathcal{Y}$ and Λ, Λ' are subsets of \mathfrak{T} . Proposition 4.2 then implies that the torsion parts of $T_{(Y, \Lambda)}$ and $T_{(Y', \Lambda')}$, which are direct sums of modules with local endomorphism ring, must coincide up to multiplicity of the summands. We give a precise description of these summands in Remark 5.7 below: for $T_{(Y, \Lambda)}$, they are the indecomposable summands of Y and the Prüfer modules $S[\infty], S \in \mathcal{R}$, where \mathcal{R} is a set of quasi-simples with $\mathcal{R} \cap \mathbf{t}_\lambda \neq \emptyset$ if and only if $\lambda \in \Lambda$, and correspondingly for $T_{(Y', \Lambda')}$. We conclude that the torsion parts determine Y, Λ and Y', Λ' , respectively, and we infer that $Y = Y'$ and $\Lambda = \Lambda'$. This concludes the proof of the Theorem. \square

Remark 5.7. (1) Let Y be a branch R -module and Λ a subset of \mathfrak{T} . The tilting R -module $T_{(Y, \Lambda)}$ from Theorem 5.6 is equivalent to a tilting R -module of the form

$$L_{(Y, \Lambda)} = \bigoplus_{S \in \mathcal{R}} S[\infty] \oplus Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{V}}),$$

where \mathcal{V} consists of all quasi-simples in $\bigcup_{\lambda \in \Lambda} \mathbf{t}_\lambda$ and all the regular composition factors of Y , and \mathcal{R} is the set of quasi-simples in $\bigcup_{\lambda \in \Lambda} \mathbf{t}_\lambda$ that are not regular composition factors of $\tau^- Y$. In particular, $\mathcal{R} \cap \mathbf{t}_\lambda \neq \emptyset$ if and only if $\lambda \in \Lambda$.

(2) Let Z be a finitely generated multiplicity-free regular exceptional module, and let Δ be a set of quasi-simple modules. Set

$$E = \bigoplus_{S \in \Delta} S[\infty] \oplus Z.$$

Then E is a direct summand of a large tilting R -module T if and only if no element of Δ is a regular composition factor of τ^-Z . In this event, T is equivalent to $T_{(Y,\Lambda)}$ where Y is a branch module having Z as a direct summand, and $\{\lambda \in \mathfrak{T} \mid \mathbf{t}_\lambda \cap \Delta \neq \emptyset\} \subseteq \Lambda$

Proof: (1) If $\Lambda = \emptyset$, then $\mathcal{R} = \emptyset$, and the result holds by Theorem 5.6(1).

Suppose that $\Lambda \neq \emptyset$. Let \mathcal{U}, \mathcal{V} be defined as in Theorem 5.6(2). Then \mathcal{V} is as stated above, and $\mathcal{V} \setminus \mathcal{U} = \mathcal{R}$. Moreover, we know from Propositions 1.11(2) and 1.10(1) that $R_{\mathcal{V}}/R_{\mathcal{U}}$ is the direct sum of all Prüfer $R_{\mathcal{U}}$ -modules corresponding to the tubes $\mathbf{t}_\lambda \otimes R_{\mathcal{U}}, \lambda \in \Lambda$, which are precisely the Prüfer R -modules corresponding to the quasi-simples in $\bigcup_{\lambda \in \Lambda} \mathbf{t}_\lambda \setminus \mathcal{U}$, that is, to the quasi-simples from \mathcal{R} . Hence $\text{Add}(R_{\mathcal{V}}/R_{\mathcal{U}}) = \text{Add}(\bigoplus_{S \in \mathcal{R}} S[\infty])$. Furthermore, as remarked in Definition 5.3, the cardinality of $\mathcal{U} \cap \mathbf{t}_\lambda$ is always strictly smaller than the rank of \mathbf{t}_λ , so $\mathcal{R} \cap \mathbf{t}_\lambda \neq \emptyset$ if and only if $\lambda \in \Lambda$. Let \mathbf{L} be the Lukas tilting R -module. By (T3), there exists a short exact sequence

$$0 \longrightarrow R \longrightarrow L_0 \longrightarrow L_1 \longrightarrow 0 \quad (2)$$

with $L_0, L_1 \in \text{Add } \mathbf{L}$. By [3, Lemma 4(iii)], $\text{Tor}_1^R(\mathbf{L}, R_{\mathcal{V}}) = 0$. So, applying $- \otimes_R R_{\mathcal{V}}$ to (2), we obtain the short exact sequence

$$0 \longrightarrow R_{\mathcal{V}} \longrightarrow L_0 \otimes_R R_{\mathcal{V}} \longrightarrow L_1 \otimes_R R_{\mathcal{V}} \longrightarrow 0. \quad (3)$$

By [3, Theorem 5], $\mathbf{L} \otimes_R R_{\mathcal{V}}$ is a projective $R_{\mathcal{V}}$ -module, and therefore $L_0 \otimes_R R_{\mathcal{V}}$ and $L_1 \otimes_R R_{\mathcal{V}}$ are projective $R_{\mathcal{V}}$ -modules. Thus (3) splits, and $\text{Add}_R R_{\mathcal{V}} = \text{Add}_R(\mathbf{L} \otimes_R R_{\mathcal{V}})$. Hence $\text{Add}_R T_{(Y,\Lambda)} = \text{Add}_R L_{(Y,\Lambda)}$, and $L_{(Y,\Lambda)}$ is a tilting R -module equivalent to $T_{(Y,\Lambda)}$.

(2) Suppose that E is a direct summand of a large tilting R -module T . Let Y be a branch module and $\Lambda \subseteq \mathfrak{T}$ be such that $L_{(Y,\Lambda)} = \bigoplus_{S \in \mathcal{R}} S[\infty] \oplus Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{V}})$ is equivalent to T , where \mathcal{R} is defined as in (1). Set $\Lambda' = \{\lambda \in \mathfrak{T} \mid \mathbf{t}_\lambda \cap \Delta \neq \emptyset\}$. By Proposition 4.2, Z is a direct summand of Y , $\Lambda' \subseteq \Lambda$ and $\Delta \subseteq \mathcal{R}$, so no element of Δ is a regular composition factor of τ^-Z .

Conversely, suppose that no element of Δ is a regular composition factor of τ^-Z . By Lemma 3.10, there exists at least one branch module Y such that Z is a direct summand of Y and no element of Δ is a regular composition factor of τ^-Y . For all such Y , and for Λ containing Λ' , we get that $L_{(Y,\Lambda)}$ is a tilting module with $\Delta \subseteq \mathcal{R}$. Therefore E is a direct summand of $L_{(Y,\Lambda)}$. \square

Corollary 5.8. Let R be a tame hereditary algebra with $\mathbf{t} = \bigcup_{\lambda \in \mathfrak{T}} \mathbf{t}_\lambda$. Let T be a large tilting R -module, and let

$$T = \bigoplus_{\lambda \in \mathfrak{T}} \mathbf{t}_\lambda(T) \oplus \bar{T}$$

be a decomposition as in Theorem 4.5. Set $\mathcal{B} = T^\perp$ and $\mathcal{S} = {}^\perp \mathcal{B} \cap \text{mod } R$. There are two cases.

(1) If \mathcal{S} contains no complete rays, then T is equivalent to $T_{(Y,\emptyset)} = Y \oplus (\mathbf{L} \otimes R_{\mathcal{U}})$ where Y is a branch module, and \mathcal{U} is the set of quasi-simple composition factors of Y . Thus \mathcal{U} is a set of quasi-simple modules that contains no complete cliques. Moreover,

(a) $\mathcal{F} \cap \mathcal{B}$ consists of the torsion-free $R_{\mathcal{U}}$ -modules with no direct summand from $\mathbf{p} \otimes_R R_{\mathcal{U}}$.

- (b) \overline{T} is equivalent to the Lukas tilting module over $R_{\mathcal{U}}$.
- (2) If \mathcal{S} contains some rays, then T is equivalent to $T_{(Y,\Lambda)} = Y \oplus R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$ where Y is a branch module, and $\mathcal{U} \subset \mathcal{V}$ are sets of quasi-simple modules as in Theorem 5.6(2). Thus \mathcal{U} contains no complete clique and \mathcal{V} contains complete cliques. Moreover,
- (a) $\mathcal{F} \cap \mathcal{B}$ consists of the torsion-free $R_{\mathcal{V}}$ -modules.
- (b) \overline{T} is a projective generator for $R_{\mathcal{V}}$.

Proof: According to Theorem 5.6, we see that \mathcal{S} contains no complete rays (respectively, does contain some ray) if and only if T is equivalent to a tilting module as in (1) (respectively, (2)). Observe further that, given a subset $\mathcal{Y} \subset \mathbb{U}$, an $R_{\mathcal{Y}}$ -module X , and a quasi-simple S , we have

$$(*) \quad \text{Hom}_R(S, X) \cong \text{Hom}_{R_{\mathcal{Y}}}(S \otimes_R R_{\mathcal{Y}}, X).$$

In case (1), $R_{\mathcal{U}}$ is a tame hereditary algebra with preprojective component $\mathbf{p} \otimes_R R_{\mathcal{U}}$, and $\{S \otimes_R R_{\mathcal{U}} \mid S \in \mathbb{U} \setminus \mathcal{U}\}$ is a complete irredundant set of quasi-simple $R_{\mathcal{U}}$ -modules, cf. Proposition 1.11(2) and [18, 10.1]. So, (*) shows that an $R_{\mathcal{U}}$ -module is torsion-free over $R_{\mathcal{U}}$ if and only if it is torsion-free over R .

Now assume that $X \in \mathcal{F} \cap \mathcal{B}$. Then X is generated by $\mathbf{L} \otimes_R R_{\mathcal{U}}$, thus $X \in \mathcal{U}^\perp$ because the same holds true for the $R_{\mathcal{U}}$ -module $\mathbf{L} \otimes_R R_{\mathcal{U}}$. Hence X is an $R_{\mathcal{U}}$ -module which is generated by $\mathbf{L} \otimes_R R_{\mathcal{U}}$. Since $\mathbf{L} \otimes_R R_{\mathcal{U}}$ is equivalent to the Lukas tilting module over $R_{\mathcal{U}}$ [3, Theorem 6], it follows that X has no direct summand in $\mathbf{p} \otimes_R R_{\mathcal{U}}$. Conversely, if X is a torsion-free $R_{\mathcal{U}}$ -module which has no direct summand from $\mathbf{p} \otimes_R R_{\mathcal{U}}$, then it is generated by the Lukas tilting module over $R_{\mathcal{U}}$, whence $X \in \text{Gen}(\mathbf{L} \otimes_R R_{\mathcal{U}}) \subset \mathcal{B}$.

For assertion (b), first note that \overline{T} is an $R_{\mathcal{U}}$ -module by (a), and $\text{Ext}_{R_{\mathcal{U}}}^1(\overline{T}, \overline{T}^{(I)}) = \text{Ext}_R^1(\overline{T}, \overline{T}^{(I)}) = 0$ for any set I . Next observe that $\text{Add } \overline{T} = \text{Add}(\mathbf{L} \otimes_R R_{\mathcal{U}})$ by Proposition 4.2. Since $\text{Mod } R_{\mathcal{U}}$ is a full subcategory of $\text{Mod } R$ closed under direct sums and direct summands, it follows that $\text{Add}_{R_{\mathcal{U}}} \overline{T} = \text{Add}_{R_{\mathcal{U}}}(\mathbf{L} \otimes_R R_{\mathcal{U}})$, and therefore \overline{T} is a tilting $R_{\mathcal{U}}$ -module equivalent to $\mathbf{L} \otimes_R R_{\mathcal{U}}$.

We now turn to case (2). Here $R_{\mathcal{V}}$ is a hereditary order in $R_{\mathbb{U}}$ by [14, 4.2], and $\{S \otimes_R R_{\mathcal{V}} \mid S \in \mathbb{U} \setminus \mathcal{V}\}$ is a complete irredundant set of simple $R_{\mathcal{V}}$ -modules, cf. Proposition 1.11(3). Moreover, by definition an $R_{\mathcal{V}}$ -module X is torsion-free if its torsion submodule $\{x \in X \mid xs = 0 \text{ for some regular element } s \in R_{\mathcal{V}}\}$ is zero, or equivalently, if the canonical map $X \rightarrow X \otimes_R R_{\mathbb{U}}$ is an embedding.

If $X \in \mathcal{F} \cap \mathcal{B}$, then $X \in \text{Gen } R_{\mathcal{V}}$. Thus $X \in \mathcal{V}^\perp$ because the same holds true for $R_{\mathcal{V}}$. Hence X is an $R_{\mathcal{V}}$ -module, which is torsion-free as $X \hookrightarrow X \otimes_R R_{\mathbb{U}}$ by Proposition 1.7. For the converse, having $\text{Gen } R_{\mathcal{V}} \subset \mathcal{B}$, it is enough to show that any torsion-free $R_{\mathcal{V}}$ -module is also torsion-free over R . This is clear in case $\mathcal{V} = \mathbb{U}$, so we can assume w.l.o.g. that \mathcal{V} is properly contained in \mathbb{U} . Then, as is well known, all simple $R_{\mathcal{V}}$ -modules are torsion, so (*) yields the claim, and (a) is verified.

For assertion (b), we show as in case (1) that \overline{T} is an $R_{\mathcal{V}}$ -module such that $\text{Add } \overline{T} = \text{Add } R_{\mathcal{V}}$. \square

We remark that the projective $R_{\mathcal{V}}$ -modules are well understood, see for example [27] and [37, §4].

Example 5.9. Let \mathbf{t}_λ be a tube of rank $r > 1$, and let S be a quasi-simple module in \mathbf{t}_λ . If T is a tilting module with $\mathcal{S} = \text{add}(\mathbf{p} \cup \{S[n] \mid n \in \mathbb{N}\})$, then

$$T \sim S \oplus S[2] \oplus \cdots \oplus S[r-1] \oplus S[\infty] \oplus R_{\mathbf{t}_\lambda}.$$

Indeed, we have already computed $Y = \mathbf{t}_\lambda(T) = S \oplus S[2] \oplus \cdots \oplus S[r-1]$ in Example 4.4. Choose the numbering $S = U_1, U_2 = \tau^{-1}U_1, \dots, U_r = \tau^{-1}U_{r-1}$ for the quasi-simples in \mathbf{t}_λ , and set $\mathcal{U} = \{U_2, \dots, U_r\}$. Consider the universal localization $R_{\mathcal{U}}$. Following the proof of Theorem 5.6, $T \sim S \oplus S[2] \oplus \cdots \oplus S[r-1] \oplus M$ where M is a tilting $R_{\mathcal{U}}$ -module whose tilting class $\mathcal{B}_{\mathcal{U}}$ is given by $\mathcal{B}_{\mathcal{U}} = \{X \in \text{Mod } R_{\mathcal{U}} \mid \text{Ext}_{R_{\mathcal{U}}}^1(A \otimes R_{\mathcal{U}}, X) = 0 \text{ for all } A \in \mathcal{S}\}$. Note that $\{P \otimes R_{\mathcal{U}} \mid P \in \mathbf{p}\}$ is the preprojective component of $R_{\mathcal{U}}$ by [18, 10.1], and $\{S[n] \otimes R_{\mathcal{U}} \mid n \in \mathbb{N}\}$ is a homogeneous $R_{\mathcal{U}}$ -tube with mouth $\mathcal{V}' = \{S \otimes R_{\mathcal{U}}\}$. Hence $M \cong (R_{\mathcal{U}})_{\mathcal{V}'} \oplus (R_{\mathcal{U}})_{\mathcal{V}'}/R_{\mathcal{U}} \cong R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$ where $\mathcal{V} = \mathcal{U} \cup \{S\} = \{U_1, \dots, U_r\}$. We conclude that $R_{\mathcal{V}} = R_{\mathbf{t}_\lambda}$. Moreover, we deduce as in Remark 5.7 that $R_{\mathcal{V}}/R_{\mathcal{U}}$ is a direct sum of copies of $S[\infty]$. This proves the claim. \square

We now turn to the tilting modules arising from ring epimorphisms studied in [5].

Corollary 5.10. Let T be a tilting R -module which is not equivalent to a finite dimensional tilting module. Set $\mathcal{B} = T^\perp$ and $\mathcal{S} = {}^\perp\mathcal{B} \cap \text{mod } R$. The following statements are equivalent.

- (1) There exists an injective ring epimorphism $\lambda: R \rightarrow R'$ such that $\text{Tor}_1^R(R', R') = 0$ and $R' \oplus R'/R$ is a tilting R -module equivalent to T .
- (2) T is equivalent to a tilting module $T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ with $\mathcal{U} \subseteq \mathcal{U}$.

Moreover, under these conditions, \mathcal{S} must contain some rays.

Proof: In [25, Theorem 6.1] it is proved that λ as in (1) can be chosen as a universal localization of R . We will give a different proof for that and also show that T is equivalent to $T_{\mathcal{U}}$ as stated.

By Proposition 4.2, both modules $R', R'/R \in \text{Add } T$ are direct sums of their torsion part and their torsion-free part. We denote by $\overline{R'}$ the torsion-free part of R' and observe that

$$\mathcal{B} = \text{Gen } T = \text{Gen } R'_R = (R'/R)^\perp, \text{ and in particular, } \mathcal{F} \cap \mathcal{B} \subseteq \text{Gen } \overline{R'}.$$

Suppose that \mathcal{S} contains no complete ray. Then T is equivalent to a tilting R -module of the form $Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$ as in Corollary 5.8(1). Since $\mathbf{L} \otimes_R R_{\mathcal{U}} \in \mathcal{F} \cap \mathcal{B}$, we have $\overline{R'} \neq 0$. Moreover, it follows from Proposition 4.2 that any torsion-free module in $\text{Add } T$ belongs to $\text{Add}(\mathbf{L} \otimes_R R_{\mathcal{U}})$, and any torsion module in $\text{Add } T$ belongs to $\text{Add } Y$. Now $\mathbf{L} \otimes_R R_{\mathcal{U}}$ is equivalent to the Lukas tilting module over the tame hereditary algebra $R_{\mathcal{U}}$ by [3, Theorem 6], and we know from [28, Lemma 3.3(a)] that $\text{Hom}_{R_{\mathcal{U}}}(A, B) \neq 0$ for any two nonzero $A, B \in \text{Add}(\mathbf{L} \otimes_R R_{\mathcal{U}})$. This shows that any torsion-free module $0 \neq A \in \text{Add } T$ satisfies $\text{Hom}_R(A, \overline{R'}) \neq 0$. Note that $\text{Hom}_R(R'/R, R') = 0$, see for example [5, 2.6]. So, we infer that R'/R is a torsion module, hence $R'/R \in \text{Add } Y$. In particular, it follows that $(R'/R)^\perp = Y^\perp$. By Lemma 3.10, the branch module Y can be completed to a finite dimensional tilting module H with tilting class $\text{Gen } H = Y^\perp$. But then $\text{Gen } T = (R'/R)^\perp = \text{Gen } H$, contradicting the assumption that T is not equivalent to a finite-dimensional tilting R -module.

So, \mathcal{S} contains some rays, and T must be equivalent to a tilting module of the form $Y \oplus R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$ as in Corollary 5.8(2). Since $R_{\mathcal{V}} \in \mathcal{F} \cap \mathcal{B}$, we have $\overline{R'} \neq 0$. Moreover, it follows from Proposition 4.2 that any torsion-free module $A \in \text{Add } T$ belongs to $\text{Add } R_{\mathcal{V}}$ and is therefore a projective $R_{\mathcal{V}}$ -module. So $0 \neq A \in \mathcal{F} \cap \mathcal{B} \subseteq \text{Gen } \overline{R'}$ implies $A \in \text{Add } \overline{R'}$ and $\text{Hom}_R(A, \overline{R'}) \neq 0$. Again, from $\text{Hom}_R(R'/R, R') = 0$ we infer that R'/R is a torsion module, hence a direct sum of Prüfer modules and finite-dimensional torsion modules. Observe that if $S[\infty] \in \text{Add } T$ belongs to a tube of rank $r > 1$, then it is filtered by $S[r]$, which belongs to $\{S[n] \mid n \geq 1\} \subseteq \mathcal{S}$ by (the proof of) Theorem 4.5.

Thus R'/R is filtered by non-projectives in \mathcal{S} , and we can assume that λ is a universal localization by [1, Corollary 3.5].

We then know from [37, 2.3] that T is equivalent to $R_{\mathbf{E}} \oplus R_{\mathbf{E}}/R$ for some full exact abelian subcategory \mathbf{E} of $\text{mod } R$ which is closed under extensions. By [37, 2.6] and [5, 4.12 and 4.13] we have $T^\perp = \mathbf{E}^\perp$, hence $\mathcal{S} = {}^\perp(\mathbf{E}^\perp) \cap \text{mod } R$ contains \mathbf{E} , and from the bijection between resolving subcategories of $\text{mod } R$ and tilting classes given in Theorem 2.1 we infer that \mathcal{S} is the resolving closure of \mathbf{E} . In particular, it follows that $\mathcal{S} = \text{add}(\mathbf{p} \cup \mathbf{E})$. So, the set $\mathbf{t}' = \mathcal{S} \cap \mathbf{t}$ coincides with $\mathbf{E} \cap \mathbf{t}$ and is therefore closed under cokernels.

We claim that T is equivalent to $T_{\mathcal{U}'}$ where \mathcal{U}' is the set of quasi-simple modules in \mathbf{t}' . Indeed, $\text{Gen } T = \mathbf{t}'^\perp$ as \mathbf{t}' contains a complete ray (cf. Example 2.4), and by Example 1.6 it remains to show $\mathbf{t}'^\perp = (\mathcal{U}')^\perp$. Take $S[m] \in \mathbf{t}'$. Since \mathcal{S} is closed under submodules, all $S[n]$ with $n \leq m$ are in \mathbf{t}' as well, and so are the cokernels of the inclusions $S[n] \hookrightarrow S[n']$ for $n < n' \leq m$. Thus \mathbf{t}' contains the wing $\mathcal{W}_{S[m]}$, and \mathcal{U}' contains the quasi-simples from that wing. But then $(\mathcal{U}')^\perp \subset S[m]^\perp$, and the proof is complete. \square

We know from [3, Corollary 9] that the Lukas tilting module \mathbf{L} is noetherian over its endomorphism ring. The following result generalizes this.

Corollary 5.11. Let T be a tilting R -module which is not equivalent to a finite dimensional tilting module. Set $\mathcal{B} = T^\perp$ and $\mathcal{S} = {}^\perp\mathcal{B} \cap \text{mod } R$. The following statements are equivalent.

- (1) T is noetherian over its endomorphism ring.
- (2) T is equivalent to a tilting module $T_{(Y, \emptyset)} = Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$ as in Corollary 5.8(1).
- (3) \mathcal{S} contains no complete rays.

Proof: We know from [2, 9.9] that T is noetherian over its endomorphism ring if and only if $D(T)$ is Σ -pure-injective. For (2) \Rightarrow (1), we proceed as in the proof of [3, Corollary 9]. Suppose that T is equivalent to $Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$. Then $\text{Add } T = \text{Add}(Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}}))$, and $D(T) \in \text{Prod}(D(Y) \oplus D(\mathbf{L} \otimes_R R_{\mathcal{U}}))$. Since Y is finite dimensional, Y and $D(Y)$ are (right and left, respectively) endofinite modules. Moreover, by Lemma 2.5, the dual of the Lukas tilting $R_{\mathcal{U}}$ -module $D(\mathbf{L} \otimes_R R_{\mathcal{U}})$ is a cotilting $R_{\mathcal{U}}$ -module whose cotilting class is the class of $R_{\mathcal{U}}$ -modules without preinjective summands. Then $D(\mathbf{L} \otimes_R R_{\mathcal{U}})$ is equivalent to the Reiten-Ringel cotilting module over $R_{\mathcal{U}}$ and therefore it is a Σ -pure-injective $R_{\mathcal{U}}$ -module. By [17, 1.36], $D(\mathbf{L} \otimes_R R_{\mathcal{U}})$ is also Σ -pure-injective over R . Hence $D(T)$ is Σ -pure-injective, and the claim is proven.

For the remaining implications, we show that T is not noetherian over its endomorphism ring whenever it is equivalent to a tilting module $Y \oplus R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$ as in Corollary 5.8(2). Indeed, in the latter case, the indecomposable direct summands of the torsion part of $Y \oplus R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$ are direct summands of T , and we see as in Remark 5.7 that $R_{\mathcal{V}}/R_{\mathcal{U}}$ is a non-trivial direct sum of Prüfer modules. Hence $D(T)$ has an adic module as a direct summand. Since adic modules are not Σ -pure-injective modules, $D(T)$ is not. \square

Let us now describe the dual property.

Corollary 5.12. Let T be a tilting R -module which is not equivalent to a finite dimensional tilting module. Set $\mathcal{B} = T^\perp$ and $\mathcal{S} = {}^\perp\mathcal{B} \cap \text{mod } R$. The following statements are equivalent.

- (1) T is (Σ) -pure-injective.
- (2) $G \in \text{Add } T$.
- (3) \mathcal{S} contains a complete ray from each tube.
- (4) T is equivalent to a tilting module $T_{(Y, \mathfrak{T})} = Y \oplus R_{\mathbb{U}} \oplus R_{\mathbb{U}}/R_{\mathcal{U}}$ where Y is a branch module and \mathcal{U} consists of the quasi-simples that appear in a regular composition series of $\tau^{-1}Y$.
- (5) T is a cotilting right R -module with ${}^{\perp}T = {}^{\perp}\mathcal{B}$
- (6) There exists a cotilting left R -module C such that $D(C)$ is a tilting module equivalent to T .

Proof: Recall that $T = Y \oplus Z \oplus \overline{T}$ where Y is a direct sum of copies of finitely many finite-dimensional modules, Z is a direct sum of Prüfer modules, and \overline{T} is a non-zero torsion-free module. Now Y is endofinite, hence Σ -pure-injective, and $Z \in \text{Add } \mathbf{W}$ is (Σ) -pure-injective by [31, 10.1]. So, we have that T is (Σ) -pure-injective if and only if so is \overline{T} .

(1) \Rightarrow (2) and (3): By Corollary 5.8, either \overline{T} is equivalent to the Lukas tilting module over the tame hereditary algebra $R_{\mathcal{U}}$, where \mathcal{U} is a set of quasi-simple modules that contains no complete cliques, or \overline{T} is a projective generator for $R_{\mathcal{V}}$, where \mathcal{V} is a set of quasi-simple modules that contains complete cliques.

In the first case, we know from [3, Proposition 7 and Example 8] that $\text{Add } \overline{T}$ does not contain indecomposable pure-injective $R_{\mathcal{U}}$ -modules, and therefore \overline{T} is not a pure-injective $R_{\mathcal{U}}$ -module. By [22, 8.62], an $R_{\mathcal{U}}$ -module is pure-injective over $R_{\mathcal{U}}$ if and only if it is pure-injective over R . So, we conclude that T is not pure-injective.

Let us consider the second case. If \mathcal{V} is properly contained in \mathbb{U} , then $R_{\mathcal{V}}$ is a hereditary order in $R_{\mathbb{U}}$ which is not simple artinian, and from the classification of the indecomposable pure-injective $R_{\mathcal{V}}$ -modules in [30, 3.3] we know that no projective $R_{\mathcal{V}}$ -module can be pure-injective. So, \overline{T} can only be pure-injective if $\mathcal{V} = \mathbb{U}$ and $\overline{T} \in \text{Add } G$.

In particular, we see that $\text{Add } T$ contains G , but does not contain adic modules. On the other hand, the class \mathcal{F} of all torsion-free modules coincides with ${}^{\perp}(G^{\perp})$ by [3, Proposition 7] and is therefore contained in ${}^{\perp}(T^{\perp})$. We infer that all adic modules are in ${}^{\perp}(T^{\perp}) \setminus \mathcal{B}$. By Lemma 4.3 it follows that \mathcal{S} contains a complete ray from each tube.

(3) \Rightarrow (4) by Theorem 5.6.

(4) \Rightarrow (2): $R_{\mathbb{U}}$ is a direct sum of copies of G by Proposition 1.8(2).

(2) \Rightarrow (1): If $G \in \text{Add } T$, then $\overline{T} \in G^{\perp}$ is Σ -pure-injective by [3, Proposition 7].

(1) \Rightarrow (5): Since T is Σ -pure-injective, any module in $\text{Add } T$ is also Σ -pure-injective, and thus every pure embedding into a module in $\text{Add } T = \mathcal{B} \cap {}^{\perp}\mathcal{B}$ splits. Hence [6, Corollary 2.3] implies that ${}^{\perp}\mathcal{B}$ is closed under direct limits. Now (5) follows from [6, Corollary 3.3].

(5) \Rightarrow (1): Every cotilting module over an arbitrary ring is pure-injective by [11].

(3) \Rightarrow (6): By all the foregoing, we can suppose that T is equivalent to $X = Y \oplus G \oplus \bigoplus_{S \in \mathcal{Y}} S[\infty]$ where Y is a finite-dimensional module and $\mathcal{Y} \subseteq \mathbb{U}$. Note that, for each $\lambda \in \mathfrak{T}$, $\mathbf{t}_{\lambda}(X)$ has precisely r_{λ} , the rank of \mathbf{t}_{λ} , pairwise non-isomorphic direct summands. Observe that DX is a cotilting module. By [10, 3.9], (the r_{λ}) nonisomorphic direct summands of DX that belong to ${}_{R}\mathbf{t}_{\lambda}$ are precisely the duals of the nonisomorphic direct summands of $\mathbf{t}_{\lambda}(X)$. Hence, again by [10, 3.9], DX is equivalent to $C = DY \oplus G \oplus \bigoplus_{S \in \mathcal{Y}} (DS)[- \infty]$.

Condition (5) implies that $\text{Add } T = \text{Prod } T$ by [6, Corollary 3.3]. Therefore DC is a tilting module equivalent to T .

(6) \Rightarrow (1): The dual of any left R -module is a pure-injective right R -module. \square

6. APPENDIX: THE CLASSIFICATION OF COTILTING MODULES

Combining work of Buan and Krause [10, 3.9] with some combinatorial arguments from [11] and with Bazzoni's result [7] stating that every cotilting module over an arbitrary ring is pure-injective, one obtains a classification of cotilting modules over tame hereditary algebras which we recall below. We now recover this classification by an elementary proof that only uses the results from Sections 2–4.

Theorem 6.1. Let R be a tame hereditary algebra with $\mathbf{t} = \bigcup_{\lambda \in \mathfrak{T}} \mathbf{t}_\lambda$. Let C be a cotilting left R -module with an indecomposable direct summand which is not finitely generated. The following hold true:

- (I) Each indecomposable direct summand of C is either generic or of the form $S[n]$ for some quasi-simple left R -module S and some $n \in \mathbb{N} \cup \{\infty, -\infty\}$.
- (II) For each tube ${}_{R}\mathbf{t}_\lambda$, $\lambda \in \mathfrak{T}$, let \mathcal{I}_λ be the set of non-isomorphic indecomposable direct summands of C which are of the form $S[n]$ for some $n \in \mathbb{N} \cup \{\infty, -\infty\}$ and quasi-simple $S \in {}_{R}\mathbf{t}_\lambda$. Then the number of elements in \mathcal{I}_λ equals the rank of ${}_{R}\mathbf{t}_\lambda$.

Proof: Let us fix a cotilting left R -module C having an indecomposable direct summand which is not finitely generated. We know from Theorem 2.1 that the cotilting class ${}^\perp C$ is of the form ${}^\perp(\mathcal{S}^*)$ where

$$\mathcal{S} = \tau({}^\perp C) \cap \text{mod } R$$

is a resolving subcategory of $\text{mod } R$. Furthermore, if T is a tilting module with tilting class \mathcal{S}^\perp , then we know from Lemma 2.5 that $D(T)$ is a cotilting module equivalent to C . More precisely, denoting as before $\mathcal{B} = T^\perp$, we have

$${}^\perp C = {}^\perp(\mathcal{S}^*) = \{{}_R X \mid D(X) \in \mathcal{B}\}.$$

Moreover,

$$({}^\perp C)^\perp \cap R\text{mod} = \mathcal{S}^*$$

because every finitely generated left R -module X is of the form $X = D(W)$ for some $W \in \text{mod } R$, and the condition $D(W) \in ({}^\perp C)^\perp$ means by the Ext-Tor-relations that $W \in \tau({}^\perp C) \cap \text{mod } R = \mathcal{S}$.

Recall that the modules in $\text{Prod } C = \text{Prod } D(T)$ are pure-injective. In particular, this implies that ${}^\perp C$ is closed under direct limits. Since ${}^\perp C$ is also closed under submodules, it follows that ${}^\perp C = \varinjlim ({}^\perp C \cap R\text{mod})$, see [10, 1.1]. If I is a pure-injective left R -module, we thus have

$$(\#) \quad I \in ({}^\perp C)^\perp \text{ if and only if } \text{Ext}_R^1(A, I) = 0 \text{ for all } A \in {}^\perp C \cap R\text{mod}.$$

Step 1: We compute the indecomposable modules in $\text{Prod } C$. First of all, we have

$$(0) \quad \text{Prod } C \cap R\text{mod} = \{D(W) \mid W \in \text{Add } T \cap \text{mod } R\}.$$

In fact, if X is a finitely generated left R -module of the form $X = D(W)$ with $W \in \text{mod } R$, then by the observations above, the condition $X \in \text{Prod } C = {}^\perp C \cap ({}^\perp C)^\perp$ means that $W \in \mathcal{S} \cap \mathcal{B} = \text{Add } T \cap \text{mod } R$, so (0) is verified.

Recall that there are at most finitely many non-isomorphic finitely generated indecomposable right R -modules in $\text{Add } T$. As before, we denote by Y the direct sum of a complete irredundant set of

such modules. Then $D(Y)$ is the direct sum of a complete irredundant set of finitely generated indecomposable left R -modules in $\text{Prod } C$.

Next, we compute the adics and the Prüfer modules in $\text{Prod } C$. Observe that adic and Prüfer modules are dual to each other. So, Lemma 4.3, Theorem 4.5 and Remark 4.6 yield

$$D(T) \cong \prod_{\lambda \in \mathfrak{T}} D(\mathfrak{t}_\lambda(T)) \oplus D(\overline{T})$$

where $D(\overline{T})$ is divisible without finite dimensional direct summands, hence a direct sum of Prüfer modules and copies of G , and $D(\mathfrak{t}_\lambda(T))$ is a direct product of copies of the indecomposable direct summands of $D(\mathfrak{t}_\lambda(Y))$ and of adic modules belonging to the corresponding tube ${}_{R}\mathfrak{t}_\lambda$ in $R\text{mod}$. More precisely, the following statements hold true for a tube ${}_{R}\mathfrak{t}_\lambda$ of rank r :

- (1) if \mathcal{S}^* contains some modules from ${}_{R}\mathfrak{t}_\lambda$, but no complete coray, then $D(\mathfrak{t}_\lambda(T))$ is a direct sum of copies of s pairwise non-isomorphic modules from ${}_{R}\mathfrak{t}_\lambda$, and ${}^\perp C$ contains precisely $r - s$ pairwise non-isomorphic Prüfer modules belonging to ${}_{R}\mathfrak{t}_\lambda$;
- (2) if \mathcal{S}^* contains some corays from ${}_{R}\mathfrak{t}_\lambda$, then ${}^\perp C$ does not contain any Prüfer module belonging to ${}_{R}\mathfrak{t}_\lambda$, and $D(\mathfrak{t}_\lambda(T))$ has precisely r pairwise non-isomorphic indecomposable summands: these are the s adic modules corresponding to the $s \leq r$ corays from ${}_{R}\mathfrak{t}_\lambda$ contained in \mathcal{S}^* , and $r - s$ modules from ${}_{R}\mathfrak{t}_\lambda$;
- (3) $D(\mathfrak{t}_\lambda(T)) = 0$ whenever ${}_{R}\mathfrak{t}_\lambda \cap \mathcal{S}^* = \emptyset$.

It remains to show that an indecomposable module belongs to $\text{Prod } C$ if and only if it is isomorphic to a module in the following list:

- the indecomposable summands of $D(\mathfrak{t}_\lambda(T))$, $\lambda \in \mathfrak{T}$,
- the Prüfer modules in ${}^\perp C$,
- the generic left R -module ${}_R G$.

For the if-part, we verify that all these modules belong to $\text{Prod } C$. This is clear for the indecomposable summands of $D(\mathfrak{t}_\lambda(T))$, $\lambda \in \mathfrak{T}$. For the other modules, recall first from Theorem 2.7 that $\mathcal{B} \subset \mathfrak{p}^\perp$, and $\mathcal{S} = \text{add}(\mathfrak{p} \cup \mathfrak{t}')$ for some subset $\mathfrak{t}' \subset \mathfrak{t}$. Then ${}^\perp C \subset {}^\perp {}_R \mathfrak{q}$, and $\mathcal{S}^* = \text{add}({}_R \mathfrak{q} \cup \mathfrak{t}'')$ for some subset $\mathfrak{t}'' \subset {}_R \mathfrak{t}$. Thus every $A \in {}^\perp C \cap R\text{mod}$ belongs to ${}_R \mathfrak{p} \cup {}_R \mathfrak{t}$, and $\text{Ext}_R^1(A, I) = 0$ for any divisible module I without indecomposable preprojective summands. In particular, we deduce from (#) that all Prüfer modules and the generic module ${}_R G$ belong to $({}^\perp C)^\perp$. Furthermore, since ${}_R G$ is a torsion-free module without indecomposable preinjective summands, we also have ${}_R G \in {}^\perp(\mathcal{S}^*) = {}^\perp C$. This shows that all modules in our list belong to $\text{Prod } C$.

For the only-if-part, let X be an indecomposable module in $\text{Prod } C$. Then X is pure-injective, and we can assume w.l.o.g. that X is neither a Prüfer module nor generic. If X is finite dimensional, then by (0) it is isomorphic to a finite dimensional indecomposable summand of $D(\mathfrak{t}_\lambda(T))$ for some $\lambda \in \mathfrak{T}$. If $X = S[-\infty]$ is an adic module, then the class $({}^\perp C)^\perp$, being closed under epimorphic images, must contain the whole coray ending at S . So X is the adic module corresponding to a coray in \mathcal{S}^* , and by (2) it is isomorphic to an indecomposable summand of $D(\mathfrak{t}_\lambda(T))$ for some $\lambda \in \mathfrak{T}$. This completes the proof of the claim.

Step 2: Now statement (I) in the Theorem follows immediately from Step 1. For statement (II), we fix a tube ${}_{R}\mathfrak{t}_\lambda$ of rank r and let \mathcal{I}_λ be the set of non-isomorphic indecomposable direct summands of

C which are adic, Prüfer or finite dimensional modules belonging to ${}_{R}\mathbf{t}_\lambda$. We have to verify that \mathcal{I}_λ has precisely r elements. By (2) and (3) in Step 1, we need only to show that all Prüfer modules in $\text{Prod } C$ belonging to ${}_{R}\mathbf{t}_\lambda$ and all indecomposable summands of $D(\mathbf{t}_\lambda(T))$ occur as direct summands of C and therefore are elements of \mathcal{I}_λ . Note that this is clear for the finite dimensional summands, as it is well known that a finite dimensional indecomposable module arises as a direct summand of a product of modules $\prod C_j$ if and only if it is a direct summand of one of the factors C_j . For the other modules, we distinguish two cases.

(i) Suppose first that \mathcal{S}^* contains $s > 0$ corays from ${}_{R}\mathbf{t}_\lambda$. By (2) we have only to consider the s adic modules corresponding to these corays. Let X be one of these adic modules. Then there is a quasi-simple right R -module $S \in \mathbf{t}_\lambda$ such that $X = D(S[\infty])$ and \mathcal{S} contains the complete ray starting at S . Choose the numbering $S = U_1$, $U_2 = \tau^-U_1$, \dots , $U_r = \tau^-[U_{r-1}]$ for the quasi-simple modules in \mathbf{t}_λ . Then we have the numbering $D(U_r) = \tau[D(U_{r-1})]$, \dots , $D(U_2) = \tau[D(U_1)]$, $D(U_1) = D(S)$ for the quasi-simple modules in ${}_{R}\mathbf{t}_\lambda$.

Let m be the greatest positive integer such that $S[m] \in \text{Add } T$, or $m = 0$ if $S[m] \notin \text{Add } T$ for all $m \geq 1$. Consider $A = D(U_r[m+1]) \in {}_{R}\mathbf{t}_\lambda$. Since $S = \tau^-U_r$, we have $\text{Ext}_R^1(A, X) = \text{Ext}_R^1(S[\infty], U_r[m+1]) = D\text{Hom}_R(S[m+1], S[\infty]) \neq 0$. But this means that $A \notin {}^\perp C$ as $X \in \text{Prod } C$. Now ${}^\perp C = \bigcap_N {}^\perp N$ where N runs through all indecomposable direct summands of C by [10, 2.2]. Thus there must be an indecomposable direct summand N of C with $\text{Ext}_R^1(A, N) \neq 0$, and of course, N cannot be divisible, nor can it belong to a tube ${}_{R}\mathbf{t}_\mu$ with $\mu \neq \lambda$, so N is a finite dimensional or an adic module belonging to ${}_{R}\mathbf{t}_\lambda$.

Note that $D\text{Hom}_R(N, \tau A) \cong \text{Ext}_R^1(A, N) \neq 0$, and $\tau A \cong D(S[m+1])$ lies on the coray ending at $D(S)$. Moreover, $U_2[m], U_3[m-1], \dots, U_{m+1} \notin \mathcal{S}$ by Lemma 3.3(2), hence $D(U_2[m]), D(U_3[m-1]), \dots, D(U_{m+1}) \notin \mathcal{S}^*$. Since the finite dimensional quotients of N lie in $({}^\perp C)^\perp \cap R\text{mod} = \mathcal{S}^*$, we deduce that N does neither lie on one of the corays ending at $D(U_2), D(U_3), \dots, D(U_{m+1})$ nor it is an adic module determined by one of these corays. Further, N does not lie on the coray ending at $D(S)$ by choice of m . It follows that $X = D(S)[- \infty] = N$ is the desired direct summand of C .

(ii) Suppose now that \mathcal{S}^* contains no complete coray from ${}_{R}\mathbf{t}_\lambda$. By (3) we have only to consider the $r - s$ Prüfer modules in $\text{Prod } C$ belonging to ${}_{R}\mathbf{t}_\lambda$. Let $X = S[\infty]$ be one of these Prüfer left R -modules. Take the greatest positive integer m such that $S[m] \in \text{Prod } C$, or $m = 0$ if $S[m] \notin \text{Prod } C$ for all $m \geq 1$. Then $A = S[m+1] \in {}_{R}\mathbf{t}_\lambda$ is cogenerated by C , so there must be an indecomposable direct summand N of C with $\text{Hom}_R(A, N) \neq 0$. Of course, N cannot be torsion-free, nor can it belong to a tube ${}_{R}\mathbf{t}_\mu$ with $\mu \neq \lambda$, so N is a finite dimensional or a Prüfer module belonging to ${}_{R}\mathbf{t}_\lambda$. Choose the numbering $S = U_1$, $U_2 = \tau^-U_1$, \dots , $U_r = \tau^-[U_{r-1}]$ for the quasi-simple modules in ${}_{R}\mathbf{t}_\lambda$. As in Lemma 3.3(2), we show that $U_2[m], U_3[m-1], \dots, U_{m+1} \notin {}^\perp C$. Since the finite dimensional submodules of N lie in ${}^\perp C$, we deduce that N does neither lie on one of the rays starting at U_2, U_3, \dots, U_{m+1} nor it is a Prüfer module determined by one these rays. Further, N does not lie on the ray starting at S by choice of m . It follows that $X = S[\infty] = N$ is the desired direct summand of C . \square

Remark 6.2. Assume that ${}_{R}\mathbf{t}_\lambda$ is a tube of rank r having no complete coray in \mathcal{S}^* and having precisely $s \geq 0$ non-isomorphic indecomposable modules in $\text{Prod } C$. As we have seen above, the set \mathcal{I}_λ contains $r - s$ Prüfer modules. They arise as duals of the $r - s$ adic modules in \mathcal{B} established by

Lemma 4.3(2), see also Remark 4.6. We will now give an alternative explanation for the occurrence of these Prüfer modules by using Proposition 1.11.

Let the notation be as above. According to Theorem 5.6 and Corollary 5.8, we distinguish two cases.

(1) \mathcal{S}^* contains no complete corays. Then, up to equivalence, $T = Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$ as in Corollary 5.8(1). By [3, Theorem 6], $\mathbf{L} \otimes_R R_{\mathcal{U}}$ is equivalent to the Lukas tilting module over $R_{\mathcal{U}}$, so $D(\mathbf{L} \otimes_R R_{\mathcal{U}})$ is a cotilting $R_{\mathcal{U}}$ -module equivalent to the Reiten-Ringel tilting $R_{\mathcal{U}}$ -module $\mathbf{W}_{\mathcal{U}}$. Hence $\text{Prod}_{R_{\mathcal{U}}} D(\mathbf{L} \otimes_R R_{\mathcal{U}}) = \text{Prod}_{R_{\mathcal{U}}} \mathbf{W}_{\mathcal{U}} = \text{Add}_{R_{\mathcal{U}}} \mathbf{W}_{\mathcal{U}}$, and $\text{Prod} D(T) = \text{Add}(D(Y) \oplus \mathbf{W}_{\mathcal{U}})$. Therefore any module in $\text{Prod} C$ is a direct sum of indecomposable direct summands of $D(Y)$ and of Prüfer $R_{\mathcal{U}}$ -modules.

By assumption, there are precisely $s \geq 0$ non-isomorphic indecomposable modules in $\mathfrak{t}_{\lambda} \cap \text{Add} T$ (in fact, in $\text{Add} \mathfrak{t}_{\lambda}(Y)$), whose duals give the indecomposables in ${}_R \mathfrak{t}_{\lambda} \cap \text{Prod} C$. By construction, $\mathcal{U} \cap \mathfrak{t}_{\lambda}$ has s elements. Hence the $R_{\mathcal{U}}$ -tube $\mathfrak{t}_{\lambda} \otimes R_{\mathcal{U}}$ has $r - s$ quasi-simples, and $\text{Prod} C$ has precisely $r - s$ Prüfer left $R_{\mathcal{U}}$ -modules belonging to this tube.

(2) \mathcal{S}^* contains some corays. Then, up to equivalence, $T = Y \oplus R_{\mathcal{V}}/R_{\mathcal{U}} \oplus R_{\mathcal{V}}$ as in Corollary 5.8(2). Thus $\text{Prod} C = \text{Prod}(D(Y) \oplus D(R_{\mathcal{V}}/R_{\mathcal{U}}) \oplus D(R_{\mathcal{V}}))$, and the Prüfer modules in $\text{Prod} C$ are all in $\text{Prod} D(R_{\mathcal{V}})$ because there are no nonzero morphism from a Prüfer module neither to a torsion-free module nor to a regular module.

By assumption, $\mathfrak{t}_{\lambda} \cap \mathcal{S}$ does not contain a complete ray, and according to the construction, \mathcal{V} cannot contain all quasi-simple R -modules in \mathfrak{t}_{λ} . More precisely, $\mathfrak{t}_{\lambda}(T)$ has s pairwise non-isomorphic indecomposable direct summands, whose duals give the indecomposables in ${}_R \mathfrak{t}_{\lambda} \cap \text{Prod} C$. They are arranged in disjoint wings from \mathfrak{t}_{λ} , and the quasi-simple modules in $\mathcal{V} \cap \mathfrak{t}_{\lambda}$ are precisely the quasi-simples in these wings. So, there are exactly s quasi-simple modules in $\mathcal{V} \cap \mathfrak{t}_{\lambda}$. Each of the remaining $r - s$ quasi-simple modules $S \in \mathfrak{t}_{\lambda} \setminus \mathcal{V}$ gives rise to a simple $R_{\mathcal{V}}$ -module $S \otimes_R R_{\mathcal{V}}$ with projective presentation $0 \rightarrow \mathfrak{m} \rightarrow R_{\mathcal{V}} \rightarrow S \otimes_R R_{\mathcal{V}} \rightarrow 0$ for some maximal right ideal \mathfrak{m} . Applying D , we obtain the exact sequence $0 \rightarrow D(S \otimes_R R_{\mathcal{V}}) \rightarrow D(R_{\mathcal{V}}) \rightarrow D(\mathfrak{m}) \rightarrow 0$.

Observe that $D(R_{\mathcal{V}})$ is an injective left $R_{\mathcal{V}}$ -module [26, Corollary 3.6C] that contains the simple left $R_{\mathcal{V}}$ -module $D(S \otimes_R R_{\mathcal{V}})$. Thus the injective envelope ${}_R S[\infty]$ of $D(S \otimes_R R_{\mathcal{V}})$ is a direct summand of $D(R_{\mathcal{V}})$. We conclude that $\text{Prod} C$ has precisely $r - s$ Prüfer left $R_{\mathcal{V}}$ -modules belonging to this tube.

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