Stability of spherically symmetric steady states in galactic dynamics against general perturbations

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Abstract

Certain steady states of the Vlasov-Poisson system can be characterized as minimizers of an energy-Casimir functional, and this fact implies a nonlinear stability property of such steady states. In previous investigations by Y. Guo and the author stability was obtained only with respect to spherically symmetric perturbations. In the present investigation we show how to remove this unphysical restriction.

1 Introduction

In astrophysics the Vlasov-Poisson system

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \tag{1.1}$$

$$\triangle U = 4\pi\rho, \ \lim_{|x|\to\infty} U(t,x) = 0, \tag{1.2}$$

$$\rho(t,x) = \int f(t,x,v)dv, \tag{1.3}$$

is used to model the time evolution of large stellar systems such as galaxies or a globular clusters. Here $f = f(t,x,v) \geq 0$ denotes the density of the stars in phase space, $t \in \mathbb{R}$ denotes time, $x, v \in \mathbb{R}^3$ denote position and velocity respectively, ρ is the spatial mass density, and U the gravitational potential. The model neglects relativistic effects or collisions among the stars.

In a series of papers Y. Guo and the author have developed a variational technique to construct stable steady states of this system, cf. [6, 7, 8, 14]: Under some assumptions on the function Q the energy-Casimir functional

$$\mathcal{H}_{\mathcal{C}}(f) := \frac{1}{2}\iint |v|^2 f(x,v)\,dv\,dx - \frac{1}{8\pi}\int |\nabla U_f(x)|^2 dx + \iint Q(f(x,v))\,dv\,dx \tag{1.4}$$

has a minimizer f_0 in some function set, this minimizer is easily seen to be a steady state of the Vlasov-Poisson system, and its minimizing property implies nonlinear stability. These investigations were restricted to the case of spherical symmetry (or "flat" axial symmetry in the case of [14]), and the aim of the present paper is to remove this restriction. A physically realistic perturbation, say by the gravitational pull of some distant galaxy, is hardly spherically symmetric.

The variational equation for the minimizer f_0 shows that f_0 depends only on the particle energy

$$E = \frac{1}{2}|v|^2 + U_0(x), \tag{1.5}$$

which is a conserved quantity along characteristics; U_0 is the potential induced by f_0. In the case of spherical symmetry steady states may also depend on a further conserved quantity, the modulus of angular momentum squared:

$$L := |x|^2|v|^2 - (x\cdot v)^2. \tag{1.6}$$

To obtain such steady states, the function Q in (1.4) must also depend on L, and [6, 7, 8] actually treated this more general case. However, the present investigation does not cover the case of L-dependent steady states, the reason being as follows: For the minimizing property of f_0 to imply stability $\mathcal{H}_{\mathcal{C}}$ should be conserved along solutions of the Vlasov-Poisson system. This is the case for not necessarily symmetric solutions if $\mathcal{H}_{\mathcal{C}}$ is as in (1.4), but it requires spherical symmetry if Q depends on L.

To put the present investigation into perspective we compare the variational approach sketched above with other approaches. To this end we recall the well known class of polytropic steady states where

$$f_0(x,v) = (E_0 - E)_+^k L^l. \tag{1.7}$$

Here $(\cdot)_+$ denotes the positive part, $E_0 \in \mathbb{R}$ is a constant, and $k > -1$, $l > -1$, $k + l + 1/2 > 0$, $k < 3l + 7/2$; only for this range of exponents do these steady states have compact support and finite mass. The first nonlinear stability result for the Vlasov-Poisson system in the present stellar dynamics case is due to G. Wolansky [19]. It is restricted to spherically symmetric perturbations of the polytropes with exponents $l > -1$, $0 < k < l + 3/2$ with $k \neq -l - 1/2$ and uses a variational approach for a reduced functional which is not defined on a set of phase space densities f but on a set of mass functions $M(r) := \int_{|y| \leq r} \rho(y)\, dy$, $r \geq 0$ the radial coordinate. In particular, is does not yield a stability estimate directly for the phase space distribution f. In a recent paper Y.-H. Wan proves stability by a careful investigation of the quadratic and and higher order parts in a Taylor expansion of $\mathcal{H}_\mathcal{C}$ about a steady state. He has to assume the existence of the steady state, requires a strong condition on f_0 which is satisfied by the polytropes only for $k = 1$ and $l = 0$, but his arguments do not require spherical symmetry of the admissible perturbations, cf. [18]. Finally, the approach in [6, 7, 8] gives the existence of the steady states (and actually provides new ones), covers the polytropes for $l > -1$ and $0 < k < l + 3/2$, and, as we believe, has the simplest proof of the three approaches. With the present investigation we remove the only restriction this approach had so far when compared with [18], namely spherical symmetry of the admissible perturbations. We also mention [1] where stability for the limiting case $k = 7/2$ and $l = 0$ of polytropes which have finite mass but infinite support is treated by a variational technique.

The paper proceeds as follows: In the next section we establish some preliminary estimates which in particular show that $\mathcal{H}_\mathcal{C}$ is bounded from below and the positive terms in $\mathcal{H}_\mathcal{C}$ are bounded along minimizing sequences. In Section 3 the existence of a minimizer of $\mathcal{H}_\mathcal{C}$ is established. Most of the technical steps can be taken over from [7, 8], since in these papers spherical symmetry was only used to prevent mass from running off to spatial infinity along a minimizing sequence. To control this in the nonsymmetric case we use a concentration-compactness lemma due to P.-L. Lions. In Section 4 we show that such minimizers are spherically symmetric steady states of the Vlasov-Poisson system with finite mass and compact support. The stability properties of the steady states are then discussed in the last section. Here we point out one problem: If f_0 is a steady state then $f_0(x + Vt, v + V)$ for any given velocity $V \in \mathbb{R}^3$ is a solution of the Vlasov-Poisson system which for V small starts close to f_0, but travels away from f_0 at a linear rate in t. This trivial "instability", which cannot be present for spherically symmetric perturbations, has to be dealt with, and incidentally, both [18] and the present paper handle this by comparing f_0 with an appropriate shift in x-space of the time dependent perturbed solution $f(t)$. In our case this shift arises from the application of the concentration-compactness lemma.

We conclude the introduction with some further references. Global classical solutions to the initial value problem for the Vlasov-Poisson system were first established in [12], cf. also [16]. Many references to discussions of the stability problem in the astrophysics literature can be found in the monograph [4]. A rigorous investigation of linearized stability is given in [2]. For the plasma physics case, where the sign in the Poisson equation (1.2) is reversed, the stability problem is much easier and better understood. We refer to [3, 9, 10, 13]. Finally, a very general condition which guarantees finite mass and compact support of steady states, but not their stability, is established in [15].

2 Preliminaries

For a measurable function $f = f(x,v)$ we define

$$\rho_f(x) := \int f(x,v)\,dv,\ x \in \mathbb{R}^3,$$

and

$$U_f := -\rho_f * \frac{1}{|\cdot|}.$$

As to the existence of this convolution see Lemma 1 below. Next we define

$$\begin{aligned} E_{\mathrm{kin}}(f) &:= \frac{1}{2}\int\!\!\int |v|^2 f(x,v)\,dv\,dx, \\ E_{\mathrm{pot}}(f) &:= -\frac{1}{8\pi}\int |\nabla U_f(x)|^2 dx = -\frac{1}{2}\int\!\!\int \frac{\rho_f(x)\rho_f(y)}{|x-y|}dx\,dy, \\ \mathcal{C}(f) &:= \int\!\!\int Q(f(x,v))\,dv\,dx, \end{aligned}$$

and

$$\mathcal{H}_{\mathcal{C}}(f) := \mathcal{C}(f) + E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f),\ \mathcal{P}(f) := \mathcal{C}(f) + E_{\mathrm{kin}}(f),$$

where Q is a given function satisfying certain assumptions specified below. Note that $\mathcal{P}$ is the positive part of the energy-Casimir functional $\mathcal{H}_\mathcal{C}$. We will minimize $\mathcal{H}_\mathcal{C}$ over the set

$$\mathcal{F}_M := \Big\{ f \in L^1(\mathbb{R}^6) \mid f \geq 0,\ \iint f\,dv\,dx = M,\ \mathcal{P}(f) < \infty \Big\}, \tag{2.1}$$

where $M>0$ is prescribed. The function Q which determines the Casimir functional has to satisfy the following

Assumptions on Q: $Q \in C^1([0,\infty[) \cap C^2(]0,\infty[)$, $Q \geq 0$, and there exist constants $C_1, C_2 > 0$, $F_0 > 0$, and $0 < k_1, k_2, k_3 < 3/2$ such that:

(Q1) $Q(f) \geq C_1 f^{1+1/k_1},\ f \geq F_0$.

(Q2) $Q(f) \leq C_2 f^{1+1/k_2},\ 0 \leq f \leq F_0$.

(Q3) $Q(\lambda f) \geq \lambda^{1+1/k_3} Q(f),\ f \geq 0,\ 0 \leq \lambda \leq 1$.

(Q4) $Q''(f) > 0,\ f > 0$, and $Q'(0) = 0$.

On their support the steady states obtained later will be of the form

$$f_0(x,v) = (Q')^{-1}(E_0 - E)$$

with some $E_0 < 0$ and E as defined in (1.5); under the assumptions above Q' is strictly increasing with range $[0,\infty[$. A typical example of a function Q satisfying the assumptions is

$$Q(f) = f^{1+1/k},\ f \geq 0, \tag{2.2}$$

with $0 < k < 3/2$ which leads to a steady state of polytropic form (1.7).

We collect some estimates for ρ_f and U_f induced by an element $f \in \mathcal{F}_M$. As in the rest of the paper constants denoted by C are positive, may depend on M and Q, and may change their value from line to line.

Lemma 1 *Let $n_1 := k_1 + 3/2$ so that $1 + 1/n_1 > 4/3$. Then for any $f \in \mathcal{F}_M$ the following holds:*

(a) $f \in L^{1+1/k_1}(\mathbb{R}^6)$ *with*

$$\iint f^{1+1/k_1} dv\,dx \leq C(1 + \mathcal{P}(f)).$$

(b) $\rho_f \in L^{1+1/n_1}(\mathbb{R}^3)$ *with*

$$\int \rho_f^{1+1/n_1} dx \leq C(1+\mathcal{P}(f)).$$

(c) $U_f \in L^{12}(\mathbb{R}^3)$ *with* $\nabla U_f \in L^2(\mathbb{R}^3)$, *and*

$$\int |\nabla U_f|^2 dx \leq C\|\rho_f\|_{6/5}^2 \leq C\|\rho_f\|_{1+1/n_1}^{(n_1+1)/3}.$$

The two representations of $E_{\text{pot}}(f)$ *stated above are indeed equal.*

Proof. As to (a) and (b) we refer to [7, Lemma 1] or [8, Lemma 1]. The estimates for U_f follow from the generalized Young's inequality, and the equality of the two representations for $E_{\text{pot}}(f)$ follows by regularizing ρ_f, integrating by parts, and then passing to the limit. □

An an immediate corollary of the lemma above one can show that on $\mathcal{F}_M$ the functional $\mathcal{H}_\mathcal{C}$ is bounded from below in such a way that $\mathcal{P}$—and thus certain norms of f and ρ_f—remain bounded along minimizing sequences—note that $n_1 < 3$:

Lemma 2 *For every* $M > 0$ *there exists a constant* $C > 0$ *such that*

$$\mathcal{H}_\mathcal{C}(f) \geq \mathcal{P}(f) - C(1+\mathcal{P}(f))^{n_1/3},\ f \in \mathcal{F}_M,$$

in particular,

$$h_M := \inf_{\mathcal{F}_M} \mathcal{H}_\mathcal{C} > -\infty.$$

3 Existence of minimizers

The behaviour of $\mathcal{H}_\mathcal{C}$ and M under scaling transformations can be used to show that h_M is negative and to relate the h_M's for different values of M:

Lemma 3 (a) *Let* $M > 0$. *Then* $-\infty < h_M < 0$.

(b) *There exists* $\alpha > 0$ *such that for all* $0 < M_1 \leq M_2$,

$$h_{M_1} \geq \left(\frac{M_1}{M_2}\right)^{1+\alpha} h_{M_2}.$$

For the proof we refer to [7, Lemma 4] or [8, Lemma 4]; spherical symmetry was not used in those proofs, cf. also [14, Lemma 4] where we explicitly kept track of where symmetry was used. A simple consequence of part (b) of the lemma above is that

$$h_M < h_{M-m} + h_m,\ 0 < m < M,$$

which is condition (S.2) in [11, Theorem II.1], but we prefer to work with (b).

For easier reference we state the concentration-compactness lemma which replaces the splitting estimates used in [7, 8]. This is Lemma I. 1 in [11], cf. also [17, 4.3]. The fact that we have functions of two variables x and v but consider the various balls

$$B_R := \{x \in \mathbb{R}^3 \mid |x| \le R\}$$

only in x-space requires no changes in the proof.

Lemma 4 *Let $(f_n) \subset L^1(\mathbb{R}^6)$ with $f_n \ge 0$ and $\int f_n = M$, $n \in \mathbb{N}$. Then there exists a subsequence (f_{n_k}) such that one of the following assertions holds:*

(i) $\exists (a_k) \subset \mathbb{R}^3\ \forall \epsilon > 0\ \exists R > 0,\ k_0 \in \mathbb{N}$*:*

$$\int_{a_k + B_R} \int f_{n_k} dv\, dx \ge M - \epsilon,\ k \ge k_0.$$

(ii) $\forall R > 0$*:*

$$\lim_{k\to\infty} \sup_{a \in \mathbb{R}^3} \int_{a + B_R} \int f_{n_k} dv\, dx = 0.$$

(iii) $\exists m \in]0, M[\ \forall \epsilon > 0\ \exists k_0 \in \mathbb{N},\ (f_k^1),\ (f_k^2) \subset L^1(\mathbb{R}^6)$*:*

$$\|f_{n_k} - (f_k^1 + f_k^2)\|_1 \le \epsilon,\ \left|\int f_k^1 - m\right| \le \epsilon,\ \left|\int f_k^2 - (M-m)\right| \le \epsilon,\ k \ge k_0,$$

$$\mathrm{dist}\left(\mathrm{supp}\, f_k^1, \mathrm{supp}\, f_k^2\right) \to \infty,\ k \to \infty,$$

and

$$0 \le f_k^1, f_k^2, g_k \le f_{n_k},\ f_k^1 f_k^2 = f_k^1 g_k = f_k^2 g_k = 0\ \text{a. e.},\ k \ge k_0$$

where $g_k := f_{n_k} - f_k^1 - f_k^2$.

In the case of a minimizing sequence for $\mathcal{H}_\mathcal{C}$ Lemma 3 can be used to exclude possibilities (ii) and (iii) in the previous lemma:

Lemma 5 *Let $(f_n)\subset\mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}_\mathcal{C}$. Then in Lemma 4 only (i) holds.*

Proof. For $R>1$ define

$$K_R(x):=\begin{cases} 1/|x| & , 1/R\leq |x|\leq R,\\ R & , |x|<1/R,\\ 0 & , |x|>R,\end{cases}$$

and

$$F_R(x):=\frac{1}{|x|}1_{\{|x|>R\}}(x),\ G_R(x):=\left(\frac{1}{|x|}-R\right)1_{\{|x|<1/R\}}(x)$$

so that

$$\frac{1}{|x|}=K_R(x)+F_R(x)+G_R(x),\ x\in\mathbb{R}^3. \tag{3.1}$$

Here 1_A denotes the indicator function of the set A. Assume (ii) holds and split

$$\frac{1}{4\pi}\int|\nabla U_n|^2dx=\int\int\frac{\rho_n(x)\rho_n(y)}{|x-y|}dy\,dx=I_1+I_2+I_3$$

according to (3.1). Since (ρ_n) is bounded in $L^{4/3}(\mathbb{R}^3)$ and $\|\rho_n\|_1=M$, $n\in\mathbb{N}$, we find

$$\begin{aligned}|I_1| &\leq R\int\int_{|x-y|<R}\rho_n(x)\,\rho_n(y)\,dx\,dy\leq RM\sup_{y\in\mathbb{R}^3}\int_{y+B_r}\rho_n(x)\,dx,\\ |I_2| &\leq \frac{1}{R}\int\int\rho_n(x)\,\rho_n(y)\,dx\,dy=M^2R^{-1},\\ |I_3| &\leq \|\rho_n\|_{4/3}\|\rho_n*G_R\|_4\leq C\|\rho_n\|_{4/3}^2\|G_R\|_2\leq CR^{-1/2};\end{aligned}$$

for the last term we used Hölder's and Young's inequality. Since this holds for any $R>1$, we conclude by (ii) that $E_{\rm pot}(f_{n_k})\to 0$ along the subsequence obtained in Lemma 4. Hence $h_M\geq 0$, a contradiction to Lemma 3 (a).

Assume that (iii) holds. We denote the subsequence obtained in Lemma 4 by (f_n). Let $m\in]0,M[$ be according to (iii) and $\epsilon>0$ arbitrary. With $m_n:=\int\rho_n^1$, $M_n:=\int\rho_n^2$, obvious definitions for ρ_n^i and $\sigma_n:=\int g_n dv$ we have

$$|m_n-m|\leq\epsilon,\ |M_n-(M-m)|\leq\epsilon$$

and

$$\mathcal{P}(f_n) = \mathcal{P}(f_n^1 + f_n^2 + g_n) = \mathcal{P}(f_n^1) + \mathcal{P}(f_n^2) + \mathcal{P}(g_n) \geq \mathcal{P}(f_n^1) + \mathcal{P}(f_n^2).$$

Moreover

$$E_{\rm pot}(\rho_n) = E_{\rm pot}(\rho_n^1) + E_{\rm pot}(\rho_n^2) - I_1 - I_2 + I_3$$

where

$$\begin{aligned} I_1 &:= \int\!\!\int \frac{\rho_n^1(x)\rho_n^2(y)}{|x-y|}\,dx\,dy, \\ I_2 &:= \int\!\!\int \frac{\rho_n(x)\sigma_n(y)}{|x-y|}\,dx\,dy, \\ I_3 &:= \frac{1}{2}\int\!\!\int \frac{\sigma_n(x)\sigma_n(y)}{|x-y|}\,dx\,dy. \end{aligned}$$

To estimate I_1 observe that for n sufficiently large, $\mathrm{dist}\,(\mathrm{supp}\,\rho_n^1, \mathrm{supp}\,\rho_n^2) > 1/\epsilon$ so that

$$|I_1| \leq M^2\epsilon.$$

To estimate I_2 use the generalized Young's inequality and interpolation to find

$$|I_2| \leq C\|\rho_n\|_{6/5}\|\sigma_n\|_{6/5} \leq C\|\sigma_n\|_1^{1/3} \leq C\epsilon^{1/3}.$$

As to I_3 it suffices to observe that this term is nonnegative. Thus for any $\epsilon < 1$ and all sufficiently large n we find, using Lemma 3 (b),

$$\begin{aligned} h_M &\geq \mathcal{P}(f_n) + E_{\rm pot}(f_n) - \epsilon \\ &\geq \mathcal{P}(f_n^1) + \mathcal{P}(f_n^2) + E_{\rm pot}(f_n^1) + E_{\rm pot}(f_n^2) - C\epsilon^{1/3} \\ &\geq h_{m_n} + h_{M_n} - C\epsilon^{1/3} \\ &\geq \left[\left(\frac{m_n}{M}\right)^{1+\alpha} + \left(\frac{M_n}{M}\right)^{1+\alpha}\right] h_M - C\epsilon^{1/3}; \end{aligned}$$

clearly $0 < m_n, M_n < M$ for $\epsilon > 0$ sufficiently small. To continue we define

$$C_\alpha := -\inf_{x\in]0,1[} \frac{(1-x)^{1+\alpha} + x^{1+\alpha} - 1}{(1-x)x} > 0.$$

Since $h_M < 0$ it follows that

$$\begin{aligned} 1 &\leq \left(\frac{m_n}{M}\right)^{1+\alpha} + \left(\frac{M_n}{M}\right)^{1+\alpha} + C\epsilon^{1/3} \\ &= \left(\frac{m}{M}\right)^{1+\alpha} + \left(\frac{M-m}{M}\right)^{1+\alpha} + C\epsilon^{1/3} \\ &\quad + \left(\frac{m_n}{M}\right)^{1+\alpha} - \left(\frac{m}{M}\right)^{1+\alpha} + \left(\frac{M_n}{M}\right)^{1+\alpha} - \left(\frac{M-m}{M}\right)^{1+\alpha} \\ &\leq 1 - C_\alpha \left(1 - \frac{m}{M}\right)\frac{m}{M} + C\epsilon^{1/3} + 2\frac{1+\alpha}{M}\epsilon,\ \epsilon \in]0,1[, \end{aligned}$$

and this is a contradiction. Thus only assertion (i) can hold. □

Theorem 1 *Let $M > 0$. Let $(f_n) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}_\mathcal{C}$. Then there is a minimizer $f_0 \in \mathcal{F}_M$, a subsequence (f_{n_k}), and a sequence $(a_k) \subset \mathbb{R}^3$ such that*

$$\mathcal{H}_\mathcal{C}(f_0) = \inf_{\mathcal{F}_M} \mathcal{H}_\mathcal{C} =: h_M$$

and $f_{n_k}^{a_k} \rightharpoonup f_0$ weakly in $L^{1+1/k_1}(\mathbb{R}^6)$. For the induced potentials we have $\nabla U_{n_k}^{a_k} \to \nabla U_0$ strongly in $L^2(\mathbb{R}^3)$. Here $f^a(x,v) := f(x+a,v)$.

Proof. Let (f_n) be a minimizing sequence. Use Lemma 5 to choose a subsequence, denoted by (f_n) again and a sequence $(a_n) \subset \mathbb{R}^3$ such that (i) in Lemma 4 holds. Let $\bar{f}_n(x,v) := f_n(x+a_n,v)$. This is again a minimizing sequence, because $\mathcal{H}_\mathcal{C}$ is translation invariant. By Lemma 2, $(\mathcal{P}(\bar{f}_n))$ is bounded and thus $(\bar{f}_n)$ is bounded in $L^{1+1/k_1}(\mathbb{R}^6)$. Thus there exists a weakly convergent subsequence, denoted by $(\bar{f}_n)$ again:

$$\bar{f}_n \rightharpoonup f_0 \text{ weakly in } L^{1+1/k_1}(\mathbb{R}^6).$$

Clearly, $f_0 \geq 0$ a. e. By Lemma 1 $(\bar{\rho}_n) = (\rho_{\bar{f}_n})$ is bounded in $L^{1+1/n_1}(\mathbb{R}^3)$. After extracting a further subsequence

$$\bar{\rho}_n \rightharpoonup \rho_0 := \rho_{f_0} \text{ weakly in } L^{1+1/n_1}(\mathbb{R}^3).$$

Also by weak convergence $E_{\mathrm{kin}}(f_0) \leq \liminf_{n\to\infty} E_{\mathrm{kin}}(\bar{f}_n)$. By (Q4) the functional $\mathcal{C}$ is convex. Thus by Mazur's Lemma and Fatou's Lemma

$$\mathcal{C}(f_0) \leq \limsup_{n\to\infty} \mathcal{C}(\bar{f}_n),$$

and

$$\mathcal{P}(f_0) \le \limsup_{n\to\infty} \mathcal{P}(\bar f_n).$$

We show that $f_0 \in \mathcal{F}_M$. Let $\epsilon > 0$. By (i) in Lemma 4 and the boundedness of $E_{\text{kin}}(\bar f_n)$ there exists $R > 0$ such that

$$M \ge \int_{B_R} \int f_0 dv\, dx \ge M - \epsilon$$

which implies that $\int\int f_0 = M$ and $f_0 \in \mathcal{F}_M$. It remains to deal with the potential energy:

$$\begin{aligned}\frac{1}{8\pi}\int \left|\nabla U_{\bar f_n} - \nabla U_0\right|^2 dx &= \frac{1}{2}\int\int \frac{(\bar\rho_n(x)-\rho_0(x))(\bar\rho_n(y)-\rho_0(y))}{|x-y|} dx\, dy \\ &= I_1 + I_2 + I_3\end{aligned}$$

where the latter integral is split according to (3.1).

Estimate of I_1: Define

$$U_n^R := -\bar\rho_n * K_R,\ U_0^R := -\rho_0 * K_R,\ R > 0.$$

Since $K_R \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ weak convergence of $\bar\rho_n$ implies that for any $R > 0$,

$$U_n^R \to U_0^R,\ n \to \infty, \text{ pointwise on } \mathbb{R}^3.$$

Since $\int U_n^R = \int U_0^R$ we find $U_n^R \to U_0^R$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ for any $R > 0$. Now Hölder's inequality, an interpolation argument, and Young's inequality together with the boundedness of the ρ's in $L^{4/3}(\mathbb{R}^3)$ imply that

$$\begin{aligned}|I_1| &\le \|\bar\rho_n - \rho_0\|_{4/3} \|U_n^R - U_0^R\|_4 \le C \|U_n^R - U_0^R\|_{12}^{9/11} \|U_n^R - U_0^R\|_1^{2/11} \\ &\le C \|\bar\rho_n - \rho_0\|_{4/3}^{9/11} \|U_n^R - U_0^R\|_1^{2/11} \to 0,\ n \to \infty,\ R > 0.\end{aligned}$$

Obviously

$$|I_2| \le \frac{4M^2}{R},$$

and again by Hölder's and Young's inequality

$$|I_3| \le \|\bar\rho_n - \rho_0\|_{4/3} \|(\bar\rho_n - \rho_0) * G_R\|_4 \le C \|G_R\|_2 \le C R^{-1/2}.$$

Thus $\nabla \bar U_{\bar f_n} \to \nabla U_0$ in $L^2(\mathbb{R}^3)$ for $n \to \infty$, and the proof is complete. □

4 Properties of minimizers

Theorem 2 *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}_\mathcal{C}$. Then*

$$f_0(x,v) = \begin{cases} (Q')^{-1}(E_0 - E) & , E_0 - E > 0, \\ 0 & , E_0 - E \leq 0 \end{cases}$$

where

$$E := \frac{1}{2}|v|^2 + U_0(x),$$

$$E_0 := \frac{1}{M} \int\int (Q'(f_0) + E)\, f_0 \, dv\, dx,$$

and U_0 is the potential induced by f_0. In particular, f_0 is a steady state of the Vlasov-Poisson system.

For the proof we refer to [7, Thm. 2] where a somewhat stronger condition (Q4) was used or [8, Thm. 2] where (Q4) is as in the present paper; if anything, the fact that we do not require spherical symmetry of the functions in $\mathcal{F}_M$ makes the proof of this theorem easier.

There now arise a couple of questions which are all interrelated: Firstly, in which sense does f_0 satisfy the stationary Vlasov-Poisson system? Up to now, the Poisson equation holds in the sense of distributions, and the Vlasov equation in the sense that f_0 is constant along characteristics, but ∇U_0 is not sufficiently regular to define classical characteristics to begin with. Secondly, we know that if we minimize $\mathcal{H}_\mathcal{C}$ over the space of spherically symmetric functions in $\mathcal{F}_M$ we obtain a spherically symmetric minimizer with compact support. Are the minimizers that we obtain in the present, more general context still spherically symmetric and compactly supported? Are they unique? These questions are considered next; C^k_c and C^k_b denote the space of C^k functions with compact support and with bounded derivatives up to order k, respectively:

Theorem 3 *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}_\mathcal{C}$ so that by Theorem 2 $f_0(x,v) = \phi(E)$ with ϕ determined by Q.*

(a) *Assume that*

$$\phi(E) \leq C'_1 (E_0 - E)^{k_1},\ E \to -\infty$$

and

$$\phi(E) \geq C_2'(E_0 - E)^{k_2},\ E \to E_0-$$

for positive constants C_1', C_2' *(as is illustrated by the polytropes these assumptions are compatible with the general assumptions on* Q*). Then* $E_0 < 0$, $\rho_0 \in C_c^1(\mathrm{I\!R}^3)$, $U_0 \in C_b^2(\mathrm{I\!R}^3)$ *with* $\lim_{|x|\to\infty} U_0(x) = 0$, *and the steady state is spherically symmetric with respect to some point in* $\mathrm{I\!R}^3$.

(b) *If in particular* $Q(f) = f^{1+1/k}$, $f \geq 0$, *with* $0 < k < 3/2$ *then up to a shift in* x*-space the minimizer is unique in* $\mathcal{F}_M$.

Proof. To prove part (a) the basic idea is to use Sobolev embedding to obtain the desired regularity, establish the appropriate behaviour of U_0 at infinity, and then apply a result by Gidas, Ni and Nirenberg to conclude the spherical symmetry, cf. [5, Thm. 4]. First we show that

$$-U_0(x) \geq \frac{M}{3|x|},\ |x| \to \infty. \tag{4.1}$$

To see this, choose $R > 0$ such that

$$\int_{|y|\leq R} \rho_0(y)\, dy > \frac{M}{2}.$$

Since

$$\left| \frac{1}{|x-y|} - \frac{1}{|x|} \right| \leq \frac{R}{(|x|-R)^2},\ |x| \geq 2R,\ |y| \leq R$$

we obtain (4.1) by restricting the convolution integral defining U_0 to the ball $\{|y| \leq R\}$ and expanding the kernel as indicated.

Next we claim that

$$\rho_0 \in L^4(\mathrm{I\!R}^3). \tag{4.2}$$

To see this, we observe that f_0 depends only on the particle energy E via the function ϕ, and thus

$$\rho_0(x) = h_\phi(U_0(x)),\ x \in \mathrm{I\!R}^3 \tag{4.3}$$

where

$$h_\phi(u) := 4\pi\sqrt{2} \int_u^\infty \phi(E) \sqrt{E-u}\, dE,\ u \in \mathrm{I\!R}; \tag{4.4}$$

note that $h_\phi(u) = 0$ for $u \geq E_0$. The general assumptions on Q and the additional assumption in the theorem imply that

$$h_\phi(u) \leq C\left(1 + (E_0 - u)^{k_1+3/2}\right),\ u \leq E_0.$$

If we use this estimate on the set where ρ_0 is large—this set has finite measure—and the integrability of ρ_0 on the complement we find that

$$\int \rho_0(x)^p\, dx \leq C + \int (-U_0(x))^{(k_1+3/2)p} dx,$$

and since by Lemma 1 (c) $U_0 \in L^{12}(\mathbb{R}^3)$ this is finite for $p = 12/(k_1 + 3/2) > 4$.

The next step is to show that

$$U_0 \in L^\infty(\mathbb{R}^3),\ U_0(x) \to 0,\ |x| \to \infty. \tag{4.5}$$

To see this we split the potential in the following way:

$$\begin{aligned} -U_0(x) &= \int_{|x-y|<1/R} \frac{\rho_0(y)}{|x-y|} dy + \int_{1/R \leq |x-y| < R} \cdots + \int_{|x-y| \geq R} \cdots \\ &\leq C\|\rho_0\|_4 \left(\int_0^{1/R} r^{2-4/3} dr\right)^{3/4} + R \int_{|y| \geq |x|-R} \rho_0(y)\, dy + \frac{M}{R},\ |x| \geq R, \end{aligned}$$

and since $R > 1$ is arbitrary and $\rho_0 \in L^1(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ assertion (4.5) follows.

We are now in the position to show that

$$E_0 < 0 \text{ and } \operatorname{supp} \rho_0 \text{ is compact.} \tag{4.6}$$

By (4.3) and (4.5) the second assertion follows from the first. Assuming that $E_0 > 0$ immediately implies that $\rho_0(x) \geq h_\phi(E_0/2) > 0$ for all sufficiently large $|x|$ which contradicts the integrability of ρ_0. Assume that $E_0 = 0$. Then the estimate for ϕ from below implies

$$h_\phi(u) \geq C(-u)^{k_2+3/2},\ u \to 0-.$$

But then (4.1) implies that

$$\rho_0(x) \geq C|x|^{-k_2-3/2}$$

for all sufficiently large $|x|$, and since $k_2 + 3/2 < 3$ this again contradicts the integrability of ρ_0. Thus only the alternative $E_0 < 0$ remains, and (4.6) is established.

Next we establish the desired regularity of the steady state. Since $U_0 \in L^\infty(\mathbb{R}^3)$ this is also true for ρ_0, cf. (4.3). This in turn implies that the first order derivatives of U_0 are bounded, i. e., $U_0 \in W^{1,\infty}(\mathbb{R}^3)$. By Sobolev embedding $U_0 \in C_b(\mathbb{R}^3)$, thus $\rho_0 \in C_c(\mathbb{R}^3)$. This in turn implies that $U_0 \in C^1_b(\mathbb{R}^3)$, thus $\rho_0 \in C^1_c(\mathbb{R}^3)$, and thus $U_0 \in C^2_b(\mathbb{R}^3)$. Observe that the function h_ϕ defined in (4.4) is continuously differentiable.

If we define $V := -U_0 > 0$ and expand $1/|x-y|$ in powers of y to third order for $y \in \operatorname{supp} \rho_0$ and $|x|$ large we find that the assumptions in [5, Thm. 4] hold. Thus U_0 is spherically symmetric about some point in $\mathbb{R}^3$, and the proof of part (a) is complete.

As to part (b) we first observe that up to some shift U_0 as a function of the radial variable $r := |x|$ solves the equation

$$\frac{1}{r^2}(r^2 U_0')' = c_k (E_0 - U_0)_+^{k+3/2},\ r > 0, \tag{4.7}$$

with some appropriately defined constant c_k. Here $'$ denotes the derivative with respect to r. The function $E_0 - U_0$ is a solution of the singular ordinary differential equation

$$\frac{1}{r^2}(r^2 z')' = -c_k z_+^{k+3/2},\ r > 0. \tag{4.8}$$

Now observe that solutions $z \in C([0,\infty[) \cap C^2(]0,\infty[)$ of (4.8) with z' bounded near $r=0$ are uniquely determined by $z(0)$. This is due to the fact that for such a solution the equation implies that $z'(0)$ exists and is zero; clearly, $U_0'(0) = 0$. Moreover, if z is such a solution then so is

$$z_\alpha(r) := \alpha z(\alpha^\gamma r),\ r \geq 0$$

for any $\alpha > 0$ where $\gamma := (k+1/2)/2$, and $z_\alpha(0) = \alpha z(0)$. Now assume there exists another minimizer in $\mathcal{F}_M$, i. e., up to a shift another solution U_1 of (4.7) with cut-off energy $E_1 < 0$. Uniqueness for (4.8) yields some $\alpha > 0$ such that

$$E_1 - U_1(r) = \alpha E_0 - \alpha U_0(\alpha^\gamma r),\ r \geq 0.$$

However, both steady states have the same total mass M, so that

$$\begin{aligned} M &= c_k \int_0^\infty r^2 (E_1 - U_1(r))_+^{k+3/2} dr \\ &= \alpha^{k+3/2-3\gamma} c_k \int_0^\infty r^2 (E_0 - U_0(r))_+^{k+3/2} dr = \alpha^{k+3/2-3\gamma} M. \end{aligned}$$

Since the exponent of α is not zero, this implies that $\alpha = 1$, and considering limits at spatial infinity we conclude that $E_0 = E_1$ and $U_0 = U_1$. $\Box$

5 Dynamical stability

To investigate the dynamical stability of f_0 we note that

$$\mathcal{H}_\mathcal{C}(f) - \mathcal{H}_\mathcal{C}(f_0) = d(f, f_0) - \frac{1}{8\pi}\|\nabla U_f - \nabla U_0\|_2^2,\ f \in \mathcal{F}_M \tag{5.1}$$

where

$$d(f, f_0) := \int\int \Big[Q(f) - Q(f_0) + (E - E_0)(f - f_0)\Big]\, dv\, dx.$$

Moreover, a simple Taylor expansion shows that

$$d(f, f_0) \geq 0,\ f \in \mathcal{F}_M$$

always, and under further restrictions on Q, say for $Q(f) = f^{1+1/k}$ with $1 \leq k < 3/2$, we even have

$$d(f, f_0) \geq C\|f - f_0\|_2^2,\ f \in \mathcal{F}_M.$$

Theorem 4 *Assume that the minimizer f_0 is unique in $\mathcal{F}_M$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that for any solution $t \mapsto f(t)$ of the Vlasov-Poisson system with $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$,*

$$d(f(0), f_0) + \frac{1}{8\pi}\|\nabla U_{f(0)} - \nabla U_0\|_2^2 < \delta$$

implies that for every $t \geq 0$ there exists $a \in \mathbb{R}^3$ such that

$$d(f^a(t), f_0) + \frac{1}{8\pi}\|\nabla U_{f^a(t)} - \nabla U_0\|_2^2 < \epsilon.$$

Proof. For $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$ there exists a unique classical solution to the corresponding initial value problem, $f(t) \in \mathcal{F}_M$, $t \geq 0$, and $\mathcal{H}_\mathcal{C}$ is constant along $f(t)$.

Now assume the assertion of the theorem were false. Then there exist $\epsilon > 0$, $t_n > 0$, and $f_n(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$ such that

$$d(f_n(0), f_0) + \frac{1}{8\pi} \|\nabla U_{f_n(0)} - \nabla U_0\|_2^2 \to 0, \; n \to \infty,$$

but

$$\inf_{a \in \mathbb{R}^3} \left(d(f_n^a(t_n), f_0) + \frac{1}{8\pi} \|\nabla U_{f_n^a(t_n)} - \nabla U_0\|_2^2 \right) \geq \epsilon, \; n \in \mathbb{N}. \tag{5.2}$$

Since $\mathcal{H}_\mathcal{C}$ is conserved, (5.1) implies that

$$\lim_{n \to \infty} \mathcal{H}_\mathcal{C}(f_n(t_n)) = \lim_{n \to \infty} \mathcal{H}_\mathcal{C}(f_n(0)) = h_M,$$

i. e., $(f_n(t_n)) \subset \mathcal{F}_M$ is a minimizing sequence of $\mathcal{H}_\mathcal{C}$. By Theorem 1 , we deduce that—up to a subsequence—$\|\nabla U_{f_n^{a_n}(t_n)} - \nabla U_0\|_2^2 \to 0$. Since $(f_n^{a_n}(t_n))$ is a minimizing sequence as well, (5.1) implies that $d(f_n^{a_n}(t_n), f_0) \to 0$, a contradiction to (5.2). □

Final remarks

(a) We do in general have no control over the shift vectors a. One might think that taking only initial data with

$$\iint v f(x,v)\, dv\, dx = \iint x f(x,v)\, dv\, dx = 0 \tag{5.3}$$

might avoid the necessity of the shifts, since this condition propagates and eliminates the trivial instability due to perturbations of the form $f_0(x + tV, v + V)$ with $V \in \mathbb{R}^3$ fixed. However, as pointed out in [18], it is conceivable that an appropriate, small perturbation causes a small fraction of the total mass distribution to move off in one direction and the bulk of the distribution in the other direction in such a way that (5.3) holds, but one still has to shift the reference frame with the bulk of the distribution to save the stability estimate.

(b) If we restrict the set $\mathcal{F}_M$ to spherically symmetric functions then clearly all shift vectors $a = 0$, and we recover the results in [7, 8] for the L-independent case.

(c) The question whether steady states which depend on angular momentum L are stable against nonsymmetric perturbations remains open, since it is then no longer true that $\mathcal{H}_\mathcal{C}$ is conserved along nonsymmetric solutions.

(d) Another open problem is the uniqueness of the minimizers if Q is not of the polytropic form (2.2). We have found no substitute for the scaling argument used to analyse solutions of the equation (4.8) in the general case. However, should the minimizer not be unique (not even locally) then one still obtains a stability result in the sense that the whole set of minimizers is stable, cf. [7].

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