Biorthogonality in $\mathcal{A}$-Pairings and Hyperbolic Decomposition Theorem for $\mathcal{A}$-Modules

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Abstract

In this paper, as part of a project initiated by A. Mallios consisting of exploring new horizons for *Abstract Differential Geometry* (à la Mallios), [8, 9, 10, 11], such as those related to the *classical symplectic geometry*, we show that results pertaining to biorthogonality in pairings of vector spaces do hold for biorthogonality in pairings of $\mathcal{A}$-modules. However, for the *dimension formula* the algebra sheaf $\mathcal{A}$ is assumed to be a PID. The dimension formula relates the rank of an $\mathcal{A}$-morphism and the dimension of the kernel (sheaf) of the same $\mathcal{A}$-morphism with the dimension of the source free $\mathcal{A}$-module of the $\mathcal{A}$-morphism concerned. Also, in order to obtain an analog of the Witt's hyperbolic decomposition theorem, $\mathcal{A}$ is assumed to be a PID while topological spaces on which $\mathcal{A}$-modules are defined are assumed *connected*.

Key Words: convenient $\mathcal{A}$-modules, quotient $\mathcal{A}$-modules, locally free $\mathcal{A}$-module of varying finite rank, orthogonally convenient $\mathcal{A}$-pairing, locally orthogonally convenient $\mathcal{A}$-pairing, hyperbolic decomposition theorem.

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1 Introduction

In this paper, we discuss *biorthogonality in pairings of $\mathcal{A}$-modules* and the *Witt's hyperbolic decomposition theorem for $\mathcal{A}$-modules,* where the *algebra sheaf* $\mathcal{A}$ in the second part of the work is considered to be a *PID,* that is for every open $U \subseteq X$, the algebra $\mathcal{A}(U)$ is a PID-algebra; in other words, given a free $\mathcal{A}$-module $\mathcal{E}$ and a sub-$\mathcal{A}$-module $\mathcal{F} \subseteq \mathcal{E}$, one has that $\mathcal{F}$ is *section-wise free.* See [16].

All the $\mathcal{A}$-modules in the paper are defined on a fixed topological space X. For the purpose of the Witt's hyperbolic decomposition theorem, the space X is assumed to be *connected.* The connectedness of X leads to some useful simplification, such as: *Locally free $\mathcal{A}$-modules of varying finite rank (Definition 3.1) with $\mathcal{A}$ a PID algebra sheaf are locally free $\mathcal{A}$-modules of finite rank, i.e. vector sheaves.*

For the sake of easy referencing, we recall some notions, which may be found in our recent papers: [12], [14], [15] and [16]. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules and $\phi : \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ an $\mathcal{A}$-bilinear morphism (or $\mathcal{A}$-bilinear form). The, we say that the triple $(\mathcal{F}, \mathcal{E}; \phi) \equiv (\mathcal{F}, \mathcal{E}; \mathcal{A}) \equiv ((\mathcal{F}, \mathcal{E}; \phi); \mathcal{A})$ constitutes a *pairing of $\mathcal{A}$-modules* or an *$\mathcal{A}$-pairing,* in short. Note the places of $\mathcal{F}$ in both the direct sum $\mathcal{E} \oplus \mathcal{F}$ and the triple $(\mathcal{F}, \mathcal{E}; \phi)$. This placing is in keeping with the classical case as is in Artin's book [2]. The sub-$\mathcal{A}$-module $\mathcal{F}^{\top_\phi}$ of $\mathcal{E}$ such that, for every open subset U of X, $\mathcal{F}^{\top_\phi}(U)$ consists of all $s \in \mathcal{E}(U)$ with $\phi_V(s|_V, \mathcal{F}(V)) = 0$ for any open $V \subseteq U$, is called the *left kernel* of $(\mathcal{F}, \mathcal{E}; \phi)$. In a similar way, one defines the *right kernel* of $(\mathcal{F}, \mathcal{E}; \phi)$ to be the sub-$\mathcal{A}$-module $\mathcal{E}^{\perp_\phi}$ of $\mathcal{F}$ such that, for any open subset U of X, $\mathcal{E}^{\perp_\phi}(U)$ consists of all sections $s\mathcal{F}(U)$ such that $\phi_V(\mathcal{E}(V), s|_V) = 0$ for every open $V \subseteq U$. If $(\mathcal{F}, \mathcal{E}; \phi)$ is a *pairing of free $\mathcal{A}$-modules* (or a *free $\mathcal{A}$-pairing* for short), for every open subset U of X,

$$\mathcal{F}^{\top_\phi}(U) = \mathcal{F}(U)^{\top_\phi} := \{s \in \mathcal{E}(U) : \ \phi_U(s, \mathcal{F}(U)) = 0\};$$

similarly,

$$\mathcal{E}^{\perp_\phi}(U) = \mathcal{E}(U)^{\perp_\phi} := \{s \in \mathcal{F}(U) : \ \phi_U(\mathcal{E}(U), s) = 0\}.$$

Now, let $(\mathcal{E}, \mathcal{E}; \phi) \equiv (\mathcal{E}, \phi)$ be a self $\mathcal{A}$-pairing such that if $r, s \in \mathcal{E}(U)$, where U is an open subset of X, then $\phi_U(r,s)=0$ if and only if $\phi_U(s,r)=0$. The left kernel $\mathcal{E}^{\top_\phi}$ is the same as the right kernel $\mathcal{E}^{\perp_\phi}$. In this case, we say that the $\mathcal{A}$-bilinear form ϕ is *orthosymmetric* and call $\mathcal{E}^{\perp_\phi}(=\mathcal{E}^{\top_\phi})$ the *radical sheaf* of $\mathcal{E}$, and denote it by $\text{rad}_\phi\mathcal{E}\equiv \text{rad}\,\mathcal{E}$. A self $\mathcal{A}$-pairing such that $(\mathcal{E},\phi)$ such that rad $\mathcal{E}\neq 0$ (resp. rad $\mathcal{E}=0$) is called *isotropic* (resp. *non-isotropic*); $\mathcal{E}$ is *totally isotropic* if ϕ is identically zero, i.e., $\phi_U(r,s)=0$ for all sections $r,s\in\mathcal{E}(U)$, with U any open subset of X. For any open $U\subseteq X$, a *non-zero section* $s\in\mathcal{E}(U)$ is called *isotropic* if $\phi_U(s,s)=0$. The $\mathcal{A}$-*radical* of a sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is defined as rad $\mathcal{F}:=\mathcal{F}\cap\mathcal{F}^{\perp_\phi}=\mathcal{F}\cap\mathcal{F}^{\top_\phi}$. If $(\mathcal{F},\mathcal{E};\phi)$ is a free $\mathcal{A}$-pairing, then for every open subset $U\subseteq X$,

$$(\text{rad}\ \mathcal{E})(U)=\text{rad}\ \mathcal{E}(U)\quad\text{and}\quad(\text{rad}\ \mathcal{F})(U)=\text{rad}\ \mathcal{F}(U),$$

where rad $\mathcal{E}(U)=\mathcal{E}(U)\cap\mathcal{E}(U)^{\perp_\phi}$ and rad $\mathcal{F}(U)=\mathcal{F}(U)\cap\mathcal{F}(U)^{\top_\phi}$. Given an $\mathcal{A}$-pairing $(\mathcal{E},\mathcal{E};\phi)$ with ϕ a symmetric or antisymmetric $\mathcal{A}$-bilinear morphism, sub-$\mathcal{A}$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be *mutually orthogonal* if for every open subset U of X, $\phi_U(r,s)=0$, for all $r\in\mathcal{E}_1(U)$ and $s\in\mathcal{E}_2(U)$. If $\mathcal{E}=\oplus_{i\in I}\mathcal{E}_i$, where the $\mathcal{E}_i$ are pairwise orthogonal sub-$\mathcal{A}$-modules of $\mathcal{E}$, we say that $\mathcal{E}$ is the direct orthogonal sum of the $\mathcal{E}_i$, and write $\mathcal{E}:=\mathcal{E}_1\perp\cdots\perp\mathcal{E}_i\perp\cdots$.

This paper grew out of our earlier efforts to understand the conditions defining *convenient* $\mathcal{A}$-*modules* (cf. [13] and [16]). For the sake of self-containedness, we recall a convenient $\mathcal{A}$-module is a self $\mathcal{A}$-pairing $\mathcal{E},\phi)$, where $\mathcal{E}$ is a free $\mathcal{A}$-module of finite rank and ϕ an orthosymmetric $\mathcal{A}$-bilinear morphism, such that the following conditions are satisfied: (1) *If $\mathcal{F}$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, then $\mathcal{F}^\perp\equiv\mathcal{F}^{\perp_\phi}=\mathcal{F}^{\top_\phi}\equiv\mathcal{F}^\top$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$;* (2) *Every free sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is orthogonally reflexive, i.e., $\mathcal{F}^{\perp\top}=\mathcal{F}$;* (3) *The intersection of any two free sub-$\mathcal{A}$-modules of $\mathcal{E}$ is a free sub-$\mathcal{A}$-module.* While symplectic vector spaces satisfy the above conditions, we have not come up yet with an example other than vector spaces. However, we show in the present account that if the space X is connected and the algebra sheaf $\mathcal{A}$ a PID, then every *symplectic orthogonally convenient $\mathcal{A}$-pairing* satisfies the conditions above. Every free $\mathcal{A}$-module $\mathcal{E}$ has a *naturally associated orthogonally convenient $\mathcal{A}$-pairing*: the $\mathcal{A}$-pairing $(\mathcal{E}^*,\mathcal{E};\nu)$, where

$$\nu_U(s,\psi):=\psi_U(s)\in\mathcal{A}(U)$$

for every open subset $U\subseteq X$ and sections $s\in\mathcal{E}(U)$, $\psi\equiv(\psi_V)_{U\supseteq V,\ open}\in\mathcal{E}^*(U)$. The $\mathcal{A}$-pairing $(\mathcal{E}^*,\mathcal{E};\nu)$ is called the *canonical $\mathcal{A}$-pairing of $\mathcal{E}$ and*

$\mathcal{E}^*$.

Condition (2) of convenient $\mathcal{A}$-modules has prompted us to study extensively biorthogonality in pairings of $\mathcal{A}$-modules. One may refer to [2], [4], [6], [5], [7] and [17] for biorthogonality in pairings of vector spaces.

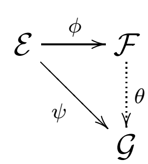
Notation. We assume throughout the paper, unless otherwise mentioned, that the pair $(X, \mathcal{A})$ is an *algebraized space* ([9, p. 96]), where $\mathcal{A}$ is a unital $\mathbb{C}$-algebra sheaf such that *every nowhere-zero section of $\mathcal{A}$ is invertible.* Furthermore, all free $\mathcal{A}$-modules are considered to be *torsion-free,* that is, for eny open subset $U \subseteq X$ and nowhere-zero section $s \in \mathcal{E}(U)$, if $as = 0$, where $a \in \mathcal{A}(U)$, then $a = 0$. Next, in the course of the paper, the notation $s \in \mathcal{E}(U)$ signifies that s is a section of an $\mathcal{A}$-module $\mathcal{E}$ over an open subset $U \subseteq X$. Finally, left and right kernels in a canonical $\mathcal{A}$-pairing $\mathcal{E}^*, \mathcal{E}; \nu)$ are simply denoted using superscripts $\perp$ and $\top$ instead of the more formal ones $\perp_\nu$ and $\top^\nu$.

2 Universal property of quotient $\mathcal{A}$-modules

This section contains proofs of the basic results on biorthogonality in canonical parings of $\mathcal{A}$-modules, namely Proposition 2.1 and Theorems 2.4 and 2.5.

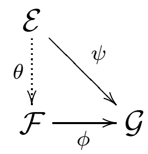
Theorem 2.1 *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{A}$-modules.*

1. *Let $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ be a surjective $\mathcal{A}$-morphism. Then, if $\psi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ such that $\ker \phi \subseteq \ker \psi$, there exists a unique $\theta \in \mathrm{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ such that the diagram*



commutes. In other words, the mapping $\theta \mapsto \theta \circ \phi$ is an $\mathcal{A}$-isomorphism *from* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ *onto the sub-$\mathcal{A}$-module of* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ *consisting of $\mathcal{A}$-morphisms whose kernel contains* $\ker\phi$.

2. *Let* $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ *be an injective $\mathcal{A}$-morphism. Then, if* $\psi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ *such that* $\mathrm{Im}\psi \subseteq \mathrm{Im}\phi$, *there exists a unique* $\theta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ *making the diagram*



commute. More precisely, the mapping $\theta \mapsto \phi \circ \theta$ *is an $\mathcal{A}$-isomorphism from* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ *onto the sub-$\mathcal{A}$-module of* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ *consisting of $\mathcal{A}$-morphism whose image is contained in* $\mathrm{Im}\phi$.

Proof. *Assertion* 1. **Uniqueness.** Let θ_1, $\theta_2 \in Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ be such that $\psi = \theta_1 \circ \phi$ and $\psi = \theta_2 \circ \phi$. Fix an open subset U in X; since ϕ_U is surjective, the equation $\theta_{1,U} \circ \phi_U = \theta_{2,U} \circ \phi_U$ implies that $\theta_{1,U} = \theta_{2,U}$. Thus, $\theta_1 = \theta_2$.

Existence. Fix an open subset U in X and consider an element (section) $t \in \mathcal{F}(U)$. Since ϕ_U is surjective, there exists an element $s \in \mathcal{E}(U)$ such that $t = \phi_U(s)$. Now, suppose there exists a $r \in \mathcal{F}(U)$ with $u \in \ker\psi_U$ and $v \notin \ker\psi_U$ as its pre-images by ϕ_U, i.e.

$$\phi_U(v) = r = \phi_U(u)$$

with $u \in \ker\psi_U$ and $v \notin \ker\psi_U$. Since ϕ_U is linear, $\phi_U(v - u) = 0$; so $v - u \in \ker\phi_U \subseteq \ker\psi_U$. But $u \in \ker\psi_U$, so $v \in \ker\psi_U$, which yields a *contradiction.* We conclude that such a situation cannot occur. Furthermore, the element $\psi_U(s)$ does only depend on t. Let θ_U be the $\mathcal{A}(U)$-morphism sending $\mathcal{F}(U)$ into $\mathcal{G}(U)$ and such that

$$\theta_U(t) = \psi_U(s);$$

that

$$\psi_U = \theta_U \circ \phi_U$$

is clear.

Next, let us consider the *complete presheaves of sections* of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, respectively, viz.

$$\Gamma(\mathcal{E}) \equiv (\Gamma(U,\mathcal{E}), \alpha_V^U), \quad \Gamma(\mathcal{F}) \equiv (\Gamma(U,\mathcal{F}), \beta_V^U), \quad \Gamma(\mathcal{G}) \equiv (\Gamma(U,\mathcal{G}), \delta_V^U).$$

Given open subsets U and V of X such that $V \subseteq U$, since $\psi \in Hom_{\mathcal{A}}(\mathcal{E},\mathcal{G})$, one has

$$\psi_V \circ \alpha_V^U = \delta_V^U \circ \psi_U. \tag{1}$$

But $\psi_U = \theta_U \circ \phi_U$ and $\psi_V = \theta_V \circ \phi_V$, therefore, (1) becomes

$$\theta_V \circ \phi_V \circ \alpha_V^U = \delta_V^U \circ \theta_U \circ \phi_U$$

or

$$\theta_V \circ \beta_V^U \circ \phi_U = \delta_V^U \circ \theta_U \circ \phi_U. \tag{2}$$

Since ϕ_U is surjective, it follows from (2) that

$$\theta_V \circ \beta_V^U = \delta_V^U \circ \theta_U,$$

which means that $\theta \equiv (\theta_U)_{X\supseteq U,\ open}$ is an $\mathcal{A}$-morphism of $\mathcal{F}$ into $\mathcal{G}$ such that

$$\psi = \theta \circ \phi,$$

as required.

Assertion 2. **Uniqueness.** Let $\theta_1,\ \theta_2 \in Hom_{\mathcal{A}}(\mathcal{E},\mathcal{F})$ be such that $\psi = \phi \circ \theta_1$ and $\psi = \phi \circ \theta_2$. As ϕ is injective, the relation

$$\phi \circ \theta_1 = \phi \circ \theta_2$$

implies that $\theta_1 = \theta_2$, so uniqueness is obtained.

Existence. Fix an open subset U in X and consider an element $s \in \mathcal{E}(U)$; since $\mathrm{Im}\psi \subseteq \mathrm{Im}\phi$, there exists a $t \in \mathcal{F}(U)$ such that

$$\phi_U(t) = \psi_U(s). \tag{3}$$

But ϕ_U is injective, therefore such an element t is unique. Now, let θ_U be the mapping of $\mathcal{E}(U)$ into $\mathcal{F}(U)$ sending an element $s \in \mathcal{E}(U)$ to an element $t \in \mathcal{F}(U)$ such that (3) is satisfied. It is immediate that θ_U is $\mathcal{A}(U)$-linear, and one has

$$\psi_U = \phi_U \circ \theta_U.$$

Finally, let $\Gamma(\mathcal{E}) \equiv (\Gamma(U,\mathcal{E}),\alpha_V^U)$, $\Gamma(\mathcal{F}) \equiv (\Gamma(U,\mathcal{F}),\beta_V^U)$, $\Gamma(\mathcal{G}) \equiv (\Gamma(U,\mathcal{G}),\delta_V^U)$ be as above the complete presheaves of sections of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, respectively. Given open subsets U and V of X such that $V \subseteq U$, since $\psi \in Hom_{\mathcal{A}}(\mathcal{E},\mathcal{G})$, one has

$$\psi_V \circ \alpha_V^U = \delta_V^U \circ \psi_U. \tag{4}$$

But $\psi_U = \phi_U \circ \theta_U$ and $\psi_V = \phi_V \circ \theta_V$, therefore, we deduce from (4) that

$$\phi_V \circ \theta_V \circ \alpha_V^U = \delta_V^U \circ \phi_U \circ \theta_U$$

or

$$\phi_V \circ \theta_V \circ \alpha_V^U = \phi_V \circ \beta_V^U \circ \theta_U. \tag{5}$$

Since ϕ_V is injective, it is clear from (5) that

$$\theta_V \circ \alpha_V^U = \beta_V^U \circ \theta_U,$$

which is to say that $\theta \equiv (\theta_U)_{X\supseteq U,\ open}$ is an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{F}$ such that

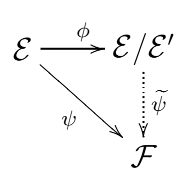
$$\psi = \phi \circ \theta,$$

and the proof is complete. ■

The *universal property of quotient $\mathcal{A}$-modules* is then obtained as a corollary of Theorem 2.1. More precisely, one has

Corollary 2.1 (***Universal property of quotient $\mathcal{A}$-modules***) *Let $\mathcal{E}$ be an $\mathcal{A}$-module, $\mathcal{E}'$ a sub-$\mathcal{A}$-module of $\mathcal{E}$, and ϕ the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto $\mathcal{E}/\mathcal{E}'$. The pair $(\mathcal{E}/\mathcal{E}',\phi)$ satisfies the following universal property:*

Given any pair $(\mathcal{F},\psi)$ consisting of an $\mathcal{A}$-module $\mathcal{F}$ and an $\mathcal{A}$-morphism $\psi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{F})$ such that $\mathcal{E}' \subseteq \ker\psi$, there exists a unique $\mathcal{A}$-morphism $\widetilde{\psi} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}/\mathcal{E}',\mathcal{F})$ such that the diagram



commutes, i.e.

$$\psi = \widetilde{\psi} \circ \phi.$$

The kernel of $\widetilde{\psi}$ *equals the image by* ϕ *of the kernel of* ψ*, and the image of* $\widetilde{\psi}$ *equals the image of* ψ*.*

The mapping

$$\theta \mapsto \theta \circ \phi$$

is an $\mathcal{A}$*-isomorphism of the* $\mathcal{A}$*-module* $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}/\mathcal{E}', \mathcal{F})$ *onto the sub-*$\mathcal{A}$*-module of* $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ *consisting of* $\mathcal{A}$*-morphisms of* $\mathcal{E}$ *into* $\mathcal{F}$ *whose kernel contains* $\mathcal{E}'$*.*

Proof. Apply assertion 1 of Theorem 2.1. ■

Similarly to the classical case (cf. [4, p. 15, Corollary 1]), we also have the following corollary, the proof of which is an easy exercise and is, for that reason, omitted.

Corollary 2.2 *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be* $\mathcal{A}$*-modules and* $\phi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$*. Then,*

(1) $\mathcal{E}/\ker\phi = \mathrm{Im}\phi$ *within an* $\mathcal{A}$*-isomorphism.*

(2) *Given a sub-*$\mathcal{A}$*-module* $\mathcal{F}'$ *of* $\mathcal{F}$*,* $\mathcal{E}' \equiv \phi^{-1}(\mathcal{F}')$ *is a sub-*$\mathcal{A}$*-module of* $\mathcal{E}$ *containing* $\ker\phi$*; moreover,* $\mathcal{F}' = \phi(\mathcal{E}')$ *if* ϕ *is surjective.*

(3) *Conversely, if* $\mathcal{E}'$ *is a sub-*$\mathcal{A}$*-module of* $\mathcal{E}$ *containing* $\ker\phi$*, then* $\mathcal{F}' \equiv \mathrm{Im}\mathcal{E}'$ *is a sub-*$\mathcal{A}$*-module of* $\mathcal{F}$ *such that* $\mathcal{E}' = \phi^{-1}(\mathcal{F}')$*.*

As a further application of the universal property of quotient $\mathcal{A}$-modules, we have

Corollary 2.3 *Let* $\mathcal{E}$ *be a free* $\mathcal{A}$*-module, and* $\mathcal{E}_1$ *a free sub-*$\mathcal{A}$*-module of* $\mathcal{E}$*. Then, the* $\mathcal{A}$*-morphism* $\phi \equiv (\phi_U)_{X\supseteq U,\ open} \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}^*, \mathcal{E}_1^*)$ *such that every* ϕ_U *maps an element* $(\psi_V)_{U\supseteq V,\ open}$ *of* $\Gamma(\mathcal{E}^*)(U) \equiv \mathcal{E}^*(U)$ *onto its restriction*

$(\psi_V|_{\mathcal{E}_1(V)})_{U\supseteq V,\ open} \in \mathcal{E}_1^*(U)$ *is surjective, and has* $\mathcal{E}_1^\perp \subseteq \mathcal{E}^*$ *as its kernel. Moreover,*

$$\mathcal{E}^*/\mathcal{E}_1^\perp = \mathcal{E}_1^*$$

within an $\mathcal{A}$*-isomorphism.*

Proof. That $\ker\phi = \mathcal{E}_1^\perp$ is clear. Now, let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ complementing $\mathcal{E}_1$. It follows (cf. [9, p. 137, relation (6.21)] that

$$\mathcal{E}^* = \mathcal{E}_1^* \oplus \mathcal{E}_2^*,$$

so that if U is open in X and

$$\psi \equiv (\psi_V)_{U\supseteq V,\ open} \in \mathcal{E}_1^*(U) \quad \text{and} \quad \theta \equiv (\theta_V)_{U\supseteq V,\ open} \in \mathcal{E}_2^*(U),$$

then

$$\Omega \equiv \psi + \theta \in \mathcal{E}^*(U).$$

If V is open in U and $s \in \mathcal{E}(V)$, so s is uniquely written as $s = r + t$ where $r \in \mathcal{E}_1(V)$ and $t \in \mathcal{E}_2(V)$, then

$$\Omega_V(s) = \psi_V(s) + \theta_V(t).$$

Consequently,

$$\phi_U(\Omega) = (\Omega_V|_{\mathcal{E}_1(V)})_{U\supseteq V,\ open} = \psi;$$

thus ϕ_U is surjective. Hence, applying Corollary 2.2 (1), we obtain an $\mathcal{A}$-isomorphism

$$\mathcal{E}^*/\mathcal{E}_1^\perp \simeq \mathcal{E}_1^*.$$

■

Now, let us introduce the notion of $\mathcal{A}$-*projection.*

Definition 2.1 Let $\mathcal{E}$ be an $\mathcal{A}$-module, $\mathcal{F}$ and $\mathcal{G}$ two *supplementary sub-*$\mathcal{A}$*-modules* of $\mathcal{E}$. (Thus, for every open subset $U \subseteq X$, every section $s \in \mathcal{E}(U)$ can be uniquely written as $s = r + t$, where $r \in \mathcal{F}(U)$ and $t \in \mathcal{G}(U)$.) The $\mathcal{A}$-*endomorphism*

$$\pi^{\mathcal{F}} \equiv (\pi_U^{\mathcal{F}})_{X\supseteq U,\ open} \in Hom_{\mathcal{A}}(\mathcal{E},\mathcal{E}) := \mathcal{E}nd_{\mathcal{A}}(\mathcal{E})$$

such that, for any section $s \in \mathcal{E}(U) \equiv \Gamma(\mathcal{E})(U) := \Gamma(U, \mathcal{E})$,

$$\pi_U^{\mathcal{F}}(s) \equiv \pi_U^{\mathcal{F}}(r+t) := r,$$

where $s = r + t$ with $r \in \mathcal{F}(U)$ and $t \in \mathcal{G}(U)$, is called the **$\mathcal{A}$-projection onto $\mathcal{F}$ (parallel to $\mathcal{G}$)**. In a similar way, one define the **$\mathcal{A}$-projection onto $\mathcal{G}$ (parallel to $\mathcal{F}$)**.

Proposition 2.1 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, $\mathcal{E}_1$ and $\mathcal{E}_2$ two free sub-$\mathcal{A}$-modules of $\mathcal{E}$ the direct sum of which is $\mathcal{A}$-isomorphic to $\mathcal{E}$, $\pi_1 \equiv \pi^{\mathcal{E}_1}$, $\pi_2 \equiv \pi^{\mathcal{E}_2}$ the corresponding $\mathcal{A}$-projections. Then,*

$$\mathcal{E}^* = \mathcal{E}_1^{\perp} \oplus \mathcal{E}_2^{\perp},$$

and the $\mathcal{A}$-projections $\pi_1' \equiv \pi^{\mathcal{E}_1^{\perp}}$, $\pi_2' \equiv \pi^{\mathcal{E}_2^{\perp}}$ associated with this direct decomposition are given by setting

$$\pi'_{1,U}(\alpha) := (\alpha_V \circ \pi_{2,V})_{U \supseteq V,\ open} \quad and \quad \pi'_{2,U}(\alpha) := (\alpha_V \circ \pi_{1,V})_{U \supseteq V,\ open}$$

for any $\alpha \equiv (\alpha_V)_{U \supseteq V,\ open} \in \mathcal{E}^*(U)$.

The proof of Proposition 2.1 requires some part of [12, p. 404, Theorem 2.2], which we restate here for easy referencing.

Theorem 2.2 *Let $(\mathcal{E}^*, \mathcal{E}; \mathcal{A})$ be the canonical free $\mathcal{A}$-pairing determined by $\mathcal{E}$. Then, for any open subset $U \subseteq X$,*

$$\dim \mathcal{E}^*(U) = \dim \mathcal{E}(U).$$

If $\phi \in \mathcal{E}^*(U)$ and $\phi_U(s) = 0$ for all $s \in \mathcal{E}(U)$, then $\phi = 0$; on the other hand, if $\phi(s) = 0$ for all $\phi \in \mathcal{E}^*(U)$, then $s = 0$.

Now, let us get to the proof of Proposition 2.1.

Proof. (**Proposition 2.1**) Fix an open set U in X. That $(\alpha_V \circ \pi_{2,V})_{U \supseteq V,\ open}$ and $(\alpha_V \circ \pi_{1,V})_{U \supseteq V,\ open}$ belong to $\mathcal{E}_1^{\perp}(U)$ and $\mathcal{E}_2^{\perp}(U)$, respectively, is obvious. For any open $V \subseteq U$, the relation

$$\alpha_V = \alpha_V \circ \pi_{1,V} + \alpha_V \circ \pi_{2,V}$$

shows that

$$\mathcal{E}^*(U) = \mathcal{E}_1^\perp(U) + \mathcal{E}_2^\perp(U).$$

Finally, suppose that there exists $\beta \equiv (\beta_V)_{U\supseteq V,\ open}$ in $\mathcal{E}_1^\perp(U) \cap \mathcal{E}_2^\perp(U)$; since $\beta_V(s) = 0$ for any open $V \subseteq U$ and any $s \in \mathcal{E}(V) = \mathcal{E}_1(V) \oplus \mathcal{E}_2(V)$, it follows that $\beta = 0$ (cf. Theorem 2.2). Thus,

$$\mathcal{E}^*(U) = \mathcal{E}_1^\perp(U) \oplus \mathcal{E}_2^\perp(U)$$

and hence

$$\mathcal{E}^* = \mathcal{E}_1^\perp \oplus \mathcal{E}_2^\perp$$

as claimed. ■

From [12], we also recall the following result a particular case of which will be needed below.

Theorem 2.3 *Let $(\mathcal{F}, \mathcal{E}; \mathcal{A})$ be an $\mathcal{A}$-pairing such that the right $\mathcal{A}$-kernel, i.e. $\mathcal{E}^\perp$, is identically 0. Moreover, let $\mathcal{E}_0$ and $\mathcal{F}_0$ be sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then, there exist natural $\mathcal{A}$-isomorphisms* ***into****:*

$$\mathcal{E}/\mathcal{F}_0^\perp \longrightarrow \mathcal{F}_0^* \quad \textit{and} \quad \mathcal{E}_0^\perp \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*.$$

An interesting result may be derived from Theorem 2.3, viz.:

Theorem 2.4 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, $\mathcal{E}_1$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and ϕ the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto (the free sub-$\mathcal{A}$-module) $\mathcal{E}/\mathcal{E}_1$. The $\mathcal{A}$-morphism*

$$\Lambda \equiv (\Lambda_U)_{X\supseteq U,\ open} : (\mathcal{E}/\mathcal{E}_1)^* \longrightarrow \mathcal{E}^*$$

such that, given any open subset $U \subseteq X$ and a section $\psi \equiv (\psi_V)_{U\supseteq V,\ open} \in (\mathcal{E}/\mathcal{E}_1)^*(U) := \mathrm{H}om_{\mathcal{A}|_U}((\mathcal{E}/\mathcal{E}_1)|_U, \mathcal{A}|_U)$,

$$\Lambda_U(\psi) := (\psi_V \circ \phi_V)_{U\supseteq V,\ open}$$

is an $\mathcal{A}$-isomorphism of $(\mathcal{E}/\mathcal{E}_1)^*$ onto $\mathcal{E}_1^\perp$, where $\mathcal{E}_1^\perp$ is the $\mathcal{A}$-orthogonal of $\mathcal{E}_1$ in the canonical $\mathcal{A}$-pairing $(\mathcal{E}^*, \mathcal{E}; \mathcal{A})$.

Proof. It is clear that Λ is indeed an $\mathcal{A}$-morphism. Now, let us fix an open set U in X and let us consider a section $\psi \equiv (\psi_V)_{U\supseteq V,\ open} \in (\mathcal{E}/\mathcal{E}_1)^*(U)$. Then, $\Lambda_U(\psi) = 0$ if for any open V in U and $s \in \mathcal{E}(V)$,

$$\Lambda_U(\psi)(s) = 0.$$

But

$$\Lambda_U(\psi)(s) = (\psi_V \circ \phi_V)(s) = \psi_V(\phi_V(s)) = 0,$$

therefore, by Theorem 2.2,

$$\psi_V = 0.$$

It follows that

$$\ker \Lambda_U = 0,$$

and consequently

$$\ker \Lambda = 0;$$

in other words, Λ is injective.

Next, for every $\psi \equiv (\psi_V)_{U\supseteq V,\ open} \in (\mathcal{E}/\mathcal{E}_1)^*(U)$, where the open set U is fixed in X,

$$\Lambda_U(\psi)(s) = (\psi_V \circ \phi_V)(s) = 0,$$

where s is any element in $\mathcal{E}_1(V)$; that is

$$\Lambda_U(\psi) \in \mathcal{E}_1^{\perp}(U),$$

from which we deduce that

$$\text{Im}\Lambda \subseteq \mathcal{E}_1^{\perp}.$$

Finally, still under the assumption that U is an open set fixed in X, let us consider, for every open $V \subseteq U$, the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}(V) & \xrightarrow{\phi_V} & (\mathcal{E}/\mathcal{E}_1)(V) \\ & \searrow^{\psi_V\circ\phi_V} & \downarrow{\scriptstyle \psi_V} \\ & & \mathcal{A}(V) \end{array}.$$

The *universal property of quotient $\mathcal{A}$-modules* (cf. Corollary 2.1) shows that, given an element $\sigma_V \in \mathcal{H}om_{\mathcal{A}(V)}(\mathcal{E}(V), \mathcal{A}(V))$ such that $\ker \phi_V \subseteq \ker \sigma_V$, i.e., $\sigma_V(\mathcal{E}_1(V)) = 0$, there is a unique $\psi_V \in \mathcal{H}om_{\mathcal{A}(V)}((\mathcal{E}/\mathcal{E}_1)(V), \mathcal{A}(V))$ such that

$$\sigma_V = \psi_V \circ \phi_V.$$

It is clear that the family $\sigma \equiv (\sigma_V)_{U\supseteq V,\ open}$ is an $\mathcal{A}$-morphism $\mathcal{E}|_U \longrightarrow \mathcal{A}|_U$ satisfying the property that:

$$\sigma = \psi \circ \phi.$$

Thus, Λ is surjective and the proof is finished. ■

As a result, based essentially on everything above, we have

Theorem 2.5 *Let $(\mathcal{E}^*, \mathcal{E}; \mathcal{A})$ be the canonical free $\mathcal{A}$-pairing and $\mathcal{E}_1$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$. Then,*

(1) $(\mathcal{E}_1^\perp)^\top = \mathcal{E}_1$ *within an $\mathcal{A}$-isomorphism.*

(2) $\mathcal{E}_1$ *has finite dimension if and only if $\mathcal{E}_1^\perp$ has finite codimension in $\mathcal{E}^*$, and then one has*
$$\dim \mathcal{E}_1 = \mathrm{codim}_{\mathcal{E}^*} \mathcal{E}_1^\perp.$$

(3) $\mathcal{E}_1$ *has finite codimension in $\mathcal{E}$ if and only if $\mathcal{E}_1^\perp$ has finite dimension, and*
$$\mathrm{codim}_{\mathcal{E}} \mathcal{E}_1 = \dim \mathcal{E}_1^\perp.$$

Proof. *Assertion* (1). Let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$, complementing $\mathcal{E}_1$. By Proposition 2.1,
$$\mathcal{E}^* \simeq \mathcal{E}_1^\perp \oplus \mathcal{E}_2^\perp.$$
We already know that $\mathcal{E}_1 \subseteq (\mathcal{E}_1^\perp)^\top$. Now, consider a section $s \in (\mathcal{E}_1^\perp)^\top(U)$; there exist $r \in \mathcal{E}_1(U)$ and $t \in \mathcal{E}_2(U)$ such that $s = r + t$. The section t is orthogonal to $\mathcal{E}_2^\perp(U)$, and since r and s are orthogonal to $\mathcal{E}_1^\perp(U)$, we then have that t is orthogonal to $\mathcal{E}_1^\perp(U) \oplus \mathcal{E}_2^\perp(U) \simeq \mathcal{E}^*(U)$. It follows from Theorem 2.2 that $t = 0$; thus $(\mathcal{E}_1^\perp)^\top(U) \subseteq \mathcal{E}_1(U)$, and hence $(\mathcal{E}_1^\perp)^\top \subseteq \mathcal{E}_1$.

Assertion (2). Since $\mathcal{E}_1$ is free, it follows that $\mathcal{E}_1^* \simeq \mathcal{E}_1$ (cf. [9, p. 298, (5.2)]). Thus, $\mathcal{E}_1$ has finite dimension if and only if $\mathcal{E}_1^*$ has finite dimension, and
$$\dim \mathcal{E}_1^* = \dim \mathcal{E}_1.$$

But, by Corollary 2.3, $\mathcal{E}^*/\mathcal{E}_1^\perp$ is $\mathcal{A}$-isomorphic to $\mathcal{E}_1^*$, therefore

$$\dim \mathcal{E}_1 = \operatorname{codim}_{\mathcal{E}^*}\mathcal{E}_1^\perp.$$

Assertion (3). Let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ complementing $\mathcal{E}_1$, that is $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. But $\mathcal{E}/\mathcal{E}_1$ is $\mathcal{A}$-isomorphic to $\mathcal{E}_2$ (cf. [15]), therefore $\mathcal{E}/\mathcal{E}_1$ is free; consequently $\mathcal{E}/\mathcal{E}_1$ has finite dimension if and only if $(\mathcal{E}/\mathcal{E}_1)^*$ has finite dimension, and one has

$$(\mathcal{E}/\mathcal{E}_1)^* \simeq \mathcal{E}/\mathcal{E}_1$$

so that

$$\operatorname{codim}_{\mathcal{E}}\mathcal{E}_1 = \dim \mathcal{E}/\mathcal{E}_1 = \dim(\mathcal{E}/\mathcal{E}_1)^*.$$

But, by Theorem 2.4, $(\mathcal{E}/\mathcal{E}_1)^* \simeq \mathcal{E}_1^\perp$ within an $\mathcal{A}$-isomorphism, so the assertion is corroborated. ■

3 Biorthogonality in dual free $\mathcal{A}$-modules

Let us introduce a set of notions we will be concerned with in the sequel.

Definition 3.1 *An $\mathcal{A}$-module $\mathcal{E}$ is called a* ***locally free $\mathcal{A}$-module of varying finite rank*** *if there exist an open covering $\mathcal{U} \equiv (U_\alpha)_{\alpha \in I}$ of X and numbers $n(\alpha) \in \mathbb{N}$ for every open set U_α such that*

$$\mathcal{E}|_{U_\alpha} = \mathcal{A}^{n(\alpha)}|_{U_\alpha}.$$

The open covering $\mathcal{U}$ is called a ***local frame.***

It is clear that if the topological space X is *connected* the *local* (*constant*) *ranks* are *equal,* and thus locally free $\mathcal{A}$-modules of varying finite rank on a connected topological space are *vector sheaves.* See [9, p. 127, Definition 4.3] for vector sheaves. In particular, if the algebra sheaf $\mathcal{A}$ is a PID, we have the following result.

Corollary 3.1 *Let X be a connected topological space, $\mathcal{A}$ a PID algebra sheaf on X and $\mathcal{E}$ an $\mathcal{A}$-module on X. Then, every sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is a locally free sub-$\mathcal{A}$-module of finite rank, i.e. a vector sheaf. Consequently, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are sub-$\mathcal{A}$-modules of $\mathcal{E}$, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is a vector sheaf.*

Proof. Obvious. ■

Example 3.1 Consider a free $\mathcal{A}$-module $\mathcal{E}$, where $\mathcal{A}$ is a PID-algebra sheaf. Then, every sub-$\mathcal{A}$-module of $\mathcal{E}$ is a locally free $\mathcal{A}$-module of varying finite rank.

Definition 3.2 *Let $(\mathcal{F}, \mathcal{E}; \phi)$ be a pairing of* free $\mathcal{A}$-modules *$\mathcal{E}$ and $\mathcal{F}$.*

(*i*) *$(\mathcal{F}, \mathcal{E}; \phi)$ is called an* ***orthogonally convenient $\mathcal{A}$-pairing*** *if for all free sub-$\mathcal{A}$-modules $\mathcal{E}_0$ and $\mathcal{F}_0$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, their orthogonal $\mathcal{E}_0^{\perp\phi}$ and $\mathcal{F}_0^{\top\phi}$ are free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively.*

(*ii*) *$(\mathcal{F}, \mathcal{E}; \phi)$ is called a* ***locally orthogonally convenient $\mathcal{A}$-pairing*** *if for all locally free sub-$\mathcal{A}$-modules of varying finite rank $\mathcal{E}_0$ and $\mathcal{F}_0$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, their orthogonal $\mathcal{E}_0^{\perp\phi}$ and $\mathcal{F}^{\top\phi}$ are locally free sub-$\mathcal{A}$-modules of varying finite rank of $\mathcal{F}$ and $\mathcal{E}$, respectively.*

Definition 3.3 *The $\mathcal{A}$-pairing $(\mathcal{E}^*, \mathcal{E}; \phi)$, where $\mathcal{E}$ is a free $\mathcal{A}$-module and such that for every open $U \subseteq X$,*

$$\phi_U(\psi, r) := \psi_U(r),$$

where $\psi \in \mathcal{E}^*(U) := Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$ and $r \in \mathcal{E}(U)$, is called the ***canonical $\mathcal{A}$-pairing*** *of $\mathcal{E}$ and $\mathcal{E}^*$.*

For every open $U \subseteq X$, let (ε_i^U) and (ε_i^{*U}) be the canonical bases (of sections) of $\mathcal{E}(U)$ and $\mathcal{E}^*(U)$, where $\mathcal{E}$ and $\mathcal{E}^*$ are *canonically paired into $\mathcal{A}$*. The family $\phi \equiv (\phi_U)_{X \supseteq U,\ open}$ such that

$$\phi_U(\varepsilon_i^U) := \varepsilon_i^{*U}$$

is an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{E}^*$. Furthermore, the kernel of ϕ is exactly the same as the left kernel of the canonical $\mathcal{A}$-pairing $(\mathcal{E}^*, \mathcal{E}; \mathcal{A})$. Indeed, $\ker \phi = 0 = \mathcal{E}^{*\top}$.

Theorem 3.1 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of finite dimension. The canonical pairing $((\mathcal{E}^*, \mathcal{E}; \nu); \mathcal{A})$ is orthogonally convenient.*

Proof. First, we notice by Theorem 2.2 that both kernels, i.e. $(\mathcal{E}^*)^\top$ and $\mathcal{E}^\perp$, are 0. Let $\mathcal{E}_0$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and consider the second map of Theorem 2.3: $\mathcal{E}_0^\perp \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*$. It is an $\mathcal{A}$-isomorphism *into*, and we shall show that *it is onto.* Fix an open set U in X, and let $\psi \in (\mathcal{E}/\mathcal{E}_0)^*(U) := \mathcal{H}om_{\mathcal{A}|_U}((\mathcal{E}/\mathcal{E}_0)|_U, \mathcal{A}|_U)$. Let us consider a family $\overline{\psi} \equiv (\overline{\psi}_V)_{U \supseteq V,\ open}$ such that

$$\overline{\psi}_V(r) := \psi_V(r + \mathcal{E}_0(V)), \quad r \in \mathcal{E}(V). \tag{6}$$

It is easy to see that $\overline{\psi}_V$ is $\mathcal{A}(V)$-linear for any open $V \subseteq U$. Now, let $\{\rho_W^V\}$, $\{\overline{\rho}_W^V\}$ and $\{\tau_W^V\}$ be the restriction maps for the (*complete*) *presheaves of sections* of $\mathcal{E}$, $\mathcal{E}/\mathcal{E}_0$ and $\mathcal{A}$, respectively. The restriction maps $\overline{\rho}_W^V$ are defined by setting

$$\overline{\rho}_W^V(r + \mathcal{E}_0(V)) := \rho_W^V(r) + \mathcal{E}_0(W), \quad r \in \mathcal{E}(V)\ .$$

It clearly follows that

$$\begin{aligned} (\tau_W^V \circ \overline{\psi}_V)(r) &= \tau_W^V(\psi_V(r + \mathcal{E}_0(V))) \\ &= \psi_W(\rho_W^V(r) + \mathcal{E}_0(W)) \\ &= \overline{\psi}_W(\rho_W^V(r)) \\ &= (\overline{\psi}_W \circ \rho_W^V)(r), \end{aligned}$$

from which we deduce that

$$\tau_W^V \circ \overline{\psi}_V = \overline{\psi}_W \circ \rho_W^V,$$

which implies that

$$\overline{\psi} \in \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U) =: \mathcal{E}^*(U).$$

Suppose $r \in \mathcal{E}_0(V)$, where V is open in U. Then

$$\overline{\psi}_V(r) = \psi_V(r + \mathcal{E}_0(V)) = \psi_V(\mathcal{E}_0(V)) = 0,$$

therefore

$$\nu_V(\overline{\psi}|_V, \mathcal{E}_0(V)) = \overline{\psi}_V(\mathcal{E}_0(V)) = 0,$$

i.e. $\overline{\psi} \in \mathcal{E}_0^{\perp}(U)$. We contend that $\overline{\psi}$ has the given ψ as image under the second map of Theorem 2.3, and this will show the ontoness of the map thereof and that $\mathcal{E}_0^{\perp}$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}^*$.

Let us find the image of $\overline{\psi}$. Consider the pairing $((\mathcal{E}/\mathcal{E}_0, \mathcal{E}_0^{\perp}; \Theta); \mathcal{A})$ such that for any open $V \subseteq X$, we have

$$\Theta_V(\alpha, r + \mathcal{E}_0(V)) := \nu_V(\alpha, r) = \alpha_V(r),$$

where $\alpha \in \mathcal{E}_0^{\perp}(V) \subseteq \mathcal{E}^*(V)$, $r \in \mathcal{E}(V)$. Clearly, the left kernel of this new pairing is 0. For $\alpha = \overline{\psi} \in \mathcal{E}_0^{\perp}(U) \subseteq \mathcal{E}^*(U)$, we have

$$\Theta_U(\overline{\psi}, r + \mathcal{E}_0(U)) = \overline{\psi}_U(r)$$

where $r \in \mathcal{E}(U)$, and the map

$$\overline{\Theta}_U : \mathcal{E}_0^{\perp}(U) \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*(U)$$

given by

$$\overline{\psi} \longmapsto \overline{\Theta}_{U,\overline{\psi}} \equiv \big((\overline{\Theta}_{U,\overline{\psi}})_V\big)_{U \supseteq V,\ open}$$

and such that for any $r \in \mathcal{E}(V)$

$$(\overline{\Theta}_{U,\overline{\psi}})_V(r + \mathcal{E}_0(V)) := \Theta_V(\overline{\psi}|_V, r + \mathcal{E}_0(V)) = \overline{\psi}_V(r) = \psi_V(r + \mathcal{E}_0(V))$$

is the image. Thus the image of $\overline{\psi}$ is ψ, hence the map $\mathcal{E}_0^{\perp}(U) \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*(U)$ is onto, and therefore an $\mathcal{A}(U)$-isomorphism. Since $\mathcal{E}/\mathcal{E}_0$ is free by Corollary 2.2, so are $(\mathcal{E}/\mathcal{E}_0)^*$ and $\mathcal{E}_0^{\perp}$ free.

Now, let $\mathcal{F}_0$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}^* \cong \mathcal{E}$ (cf. Mallios [9, p.298, (5.2)]); on considering $\mathcal{F}_0$ as a free sub-$\mathcal{A}$-module of $\mathcal{E}$, according to all that precedes above $\mathcal{F}_0^{\perp}$ is free in $\mathcal{E}^* \cong \mathcal{E}$, and so the proof is finished. ■

Definition 3.4 Let $\mathcal{E}$ and $\mathcal{F}$ be free $\mathcal{A}$-modules. An $\mathcal{A}$-morphism $\phi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is called **free** if Im ϕ is a free sub-$\mathcal{A}$-module of $\mathcal{F}$. The dimension of Im ϕ is called the **rank** of ϕ, and is denoted **rank** ϕ.

We may now state the counterpart of the *fundamental theorem* of the classical theory, see [4, p. 54, Théorème 6.4].

Theorem 3.2 *Let $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ be a free $\mathcal{A}$-morphism mapping a free $\mathcal{A}$-module $\mathcal{E}$ into a free $\mathcal{A}$-module $\mathcal{F}$. Then, the rank of ϕ is finite if and only if the kernel of ϕ has finite codimension in $\mathcal{E}$. Moreover, one has*

$$\mathrm{rank}\phi := \dim \mathrm{Im}\phi = \mathrm{codim}_{\mathcal{E}} \ker \phi.$$

Proof. Corollary 2.2(1) shows that the quotient free $\mathcal{A}$-module $\mathcal{E}/\ker\phi$ is $\mathcal{A}$-isomorphic to Im ϕ. ■

Corollary 3.2 *Let $\mathcal{A}$ be a PID algebra sheaf and $\mathcal{E}$, $\mathcal{F}$ free $\mathcal{A}$-modules. Then, if $\dim \mathcal{E}$ is finite, every free $\mathcal{A}$-morphism $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ has finite rank, and*

$$\mathrm{rank}(\phi) + \dim \ker(\phi) = \dim \mathcal{E}. \tag{7}$$

The formula above is called the ***dimension formula.***

Proof. Indeed, given that every $\mathcal{A}(U)$, where U is open in X, is a PID algebra, it follows that $\ker(\phi_U)$ is a free sub-$\mathcal{A}(U)$- module of the free $\mathcal{A}(U)$-module $\mathcal{E}(U)$. By elementary module theory (see, for instance, [1, p. 173, Proposition 8.8] or [3, p. 105, Corollary 2]), we have

$$\dim \ker(\phi_U) + \dim \mathrm{Im}(\phi_U) = \dim \mathcal{E}(U).$$

Since for any subsets U and V of X, $\dim \ker(\phi_U) = \dim \ker(\phi_V)$, it follows that $\ker(\phi)$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and therefore

$$\dim \ker(\phi) + \dim \mathrm{Im}(\phi) = \dim \mathcal{E},$$

or

$$\dim \ker(\phi) + \mathrm{rank}(\phi) = \dim \mathcal{E}.$$

■

Theorem 3.3 *Let $(\mathcal{E}^*, \mathcal{E}; \mathcal{A})$ be the canonical free $\mathcal{A}$-pairing, and $\mathcal{F}$ a free sub-$\mathcal{A}$-module of $\mathcal{E}^*$. $\mathcal{F}$ has finite dimension if and only if $\mathcal{F}^\top$ has finite codimension in $\mathcal{E}$; moreover, one has*

$$\dim \mathcal{F} = \mathrm{codim}_{\mathcal{E}} \mathcal{F}^\top; \quad (\mathcal{F}^\top)^\perp = \mathcal{F}.$$

Proof. The case $\mathcal{F} = 0$ is trivial.

Suppose that $\mathcal{F}$ has finite dimension; let U be an open subset of X, $(e_1^{U*}, \ldots, e_n^{U*})$ a canonical (local) gauge of $\mathcal{F}$ (cf. [9, p. 291, (3.11) along with p. 301, (5.17) and (5.18)]), and $\phi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{A}^n)$ be such that if $s \in \mathcal{E}(U)$,

$$\phi_U(s) := (e_1^{U*}(s), \ldots, e_n^{U*}(s)).$$

It is clear that ϕ is indeed an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{A}^n$ whose kernel is $\mathcal{F}^\top$, which is a free sub-$\mathcal{A}$-module of $\mathcal{E}$ for the simple reason that *canonical free $\mathcal{A}$-pairings are orthogonally convenient,* see Theorem 3.1. It is also clear that Im ϕ is $\mathcal{A}$-isomorphic to the free $\mathcal{A}$-module $\mathcal{A}^n$; thus, by Theorem 3.2, one has

$$\mathrm{rank}(\phi) := \mathrm{codim}_{\mathcal{E}} \mathcal{F}^\top = \dim \mathcal{F}. \tag{8}$$

According to Theorem 2.5(3), $(\mathcal{F}^\top)^\perp$ has finite dimension, and

$$\dim(\mathcal{F}^\top)^\perp = \mathrm{codim}_{\mathcal{E}} \mathcal{F}^\top. \tag{9}$$

Since $\mathcal{F}$ is contained in $(\mathcal{F}^\top)^\perp$, we deduce from (8) and (9) that

$$\mathcal{F} = (\mathcal{F}^\top)^\perp.$$

Conversely, suppose that $\mathcal{F}^\top$ has finite codimension in $\mathcal{E}$; then $(\mathcal{F}^\top)^\perp$ has finite dimension, and thus $\mathcal{F}$ as well, as $\mathcal{F}$ is contained in $(\mathcal{F}^\top)^\perp$. ■

4 Biorthogonality with respect to arbitrary $\mathcal{A}$-bilinear forms

In this section, we investigate the results of the previous sections in a more general setting, that is, $\mathcal{A}$-pairings defined by arbitrary $\mathcal{A}$-bilinear morphisms. The section ends with the Witt's hyperbolic decomposition theorem for $\mathcal{A}$-modules.

Definition 4.1 Let $((\mathcal{F}, \mathcal{E}; \phi); \mathcal{A})$ be an $\mathcal{A}$-pairing. The $\mathcal{A}$-bilinear morphism ϕ is said to be **non-degenerate** if $\mathcal{E}^{\perp_\phi} = \mathcal{F}^{\top^\phi} = 0$, and degenerate otherwise. For every open $U \subseteq X$,

$$\mathcal{E}^{\perp_\phi}(U) := \{t \in \mathcal{F}(U) : \ \phi_V(\mathcal{E}(V), t|_V) = 0, \text{ for any open } V \subseteq U\}.$$

Similarly,

$$\mathcal{F}^{\top^\phi}(U) := \{s \in \mathcal{E}(U) : \phi_V(s|_V, \mathcal{F}(V)) = 0, \text{ for any open } V \subseteq U\}.$$

Definition 4.2 Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules (not necessarily free) and $\phi : \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ and $\mathcal{A}$-bilinear morphism. The $\mathcal{A}$-morphism

$$\phi^R \in Hom_{\mathcal{A}}(\mathcal{F}, Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{A}))$$

such that, for any open subset $U \subseteq X$ and sections $t \in \mathcal{F}(U)$ and $s \in \mathcal{E}(V)$, where $V \subseteq U$ is open,

$$\phi^R_U(t)(s) \equiv (\phi^R)_U(t)(s) := \phi_V(s, t|_V)$$

is called the **right insertion $\mathcal{A}$-morphism** associated with ϕ. Similarly, for every open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$ and $t \in \mathcal{F}(V)$, where V is open in U,

$$\phi^L_U(s)(t) \equiv (\phi^L)_U(s)(t) := \phi_V(s|_V, t)$$

defines an $\mathcal{A}$-morphism, denoted ϕ^L, of $\mathcal{E}$ into $\mathcal{F}^*$, i.e.,

$$\phi^L \in Hom_{\mathcal{A}}(\mathcal{E}, Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{A})).$$

The $\mathcal{A}$-morphism ϕ^L is called the **left insertion $\mathcal{A}$-morphism** associated with ϕ.

It is clear in the light of Definition 4.1 that if the $\mathcal{A}$-bilinear morphism $\phi : \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ is non-degenerate, then both insertion $\mathcal{A}$-morphisms ϕ^R and

ϕ^L are injective. Moreover, if $\mathcal{E}$ and $\mathcal{F}$ are free $\mathcal{A}$-modules of finite dimension, then

$$\mathcal{E} = \mathcal{F}$$

within an $\mathcal{A}$-isomorphism.

In other to differentiate a canonical $\mathcal{A}$-pairing $(\mathcal{E}^*, \mathcal{E}; \nu)$ from an arbitrary $\mathcal{A}$-pairing $(\mathcal{F}, \mathcal{E}; \phi)$, in which $\mathcal{F}$ may still be the dual $\mathcal{A}$-module $\mathcal{E}^*$, we will adopt the following notation for the orthogonal sub-$\mathcal{A}$-module associated with a given sub-$\mathcal{A}$-module:

Definition 4.3 Let $(\mathcal{F}, \mathcal{E}; \phi)$ be an $\mathcal{A}$-pairing and $\mathcal{G}$ a sub-$\mathcal{A}$-module of $\mathcal{E}$. The sub-$\mathcal{A}$-module $\mathcal{G}^{\perp_\phi} \subseteq \mathcal{F}$ such that, for every open $U \subseteq X$,

$$\mathcal{G}^{\perp_\phi} := \{t \in \mathcal{F}(U) : \ \phi_V(\mathcal{G}(V), t|_V) = 0, \text{for any open } V \subseteq U\}$$

is called the **ϕ-orthogonal** (or just **orthogonal** if there is no confusion to fear) **of** $\mathcal{G}$ in $\mathcal{F}$. Similarly, one defines the ϕ-orthogonal sub-$\mathcal{A}$-module $\mathcal{H}^{\top^\phi}$ of a given sub-$\mathcal{A}$-module $\mathcal{H}$ of $\mathcal{F}$, viz

$$\mathcal{H}^{\top^\phi} := \{s \in \mathcal{E}(U) : \ \phi_V(s|_V, \mathcal{H}(V)) = 0, \text{for any open } V \subseteq U\}.$$

While the notion of orthogonality with respect to arbitrary $\mathcal{A}$-bilinear forms generalizes orthogonality in canonical $\mathcal{A}$-pairings, the former may relate with the latter through the following lemma.

Lemma 4.1 *Let $(\mathcal{F}, \mathcal{E}; \phi)$ be a free $\mathcal{A}$-pairing, $\mathcal{G}$ and $\mathcal{H}$ free sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then,*

$$\mathcal{G}^{\perp_\phi} \simeq (\phi^L(\mathcal{G}))^\top, \tag{10}$$

and

$$\mathcal{H}^{\top^\phi} \simeq (\phi^R(\mathcal{H}))^\top. \tag{11}$$

Proof. Let U be an open subset of X. Since $\mathcal{G}$ is free, it is clear that for a section $t \in \mathcal{F}(U)$ to be in $\mathcal{G}^{\perp_\phi}$ it is necessary and sufficient that

$$\phi_U(\mathcal{G}(U), t) = 0.$$

But

$$(\phi_U^L(\mathcal{G}(U)))^{\top} = \{t \in \mathcal{F}(U) : \ \phi_U^L(\mathcal{G}(U))(t) := \phi_U(\mathcal{G}(U), t) = 0\},$$

therefore (10) holds as required.

In a similar way, one shows (11). ■

The case where $((\mathcal{F}, \mathcal{E}; \phi); \mathcal{A})$ is an *orthogonally convenient pairing* and ϕ is *degenerate* is interesting, for it yields the following result.

Proposition 4.1 *Given an orthogonally convenient pairing* $((\mathcal{F}, \mathcal{E}; \phi); \mathcal{A})$, *where* $\mathcal{E}$ *and* $\mathcal{F}$ *are free* $\mathcal{A}$*-modules of finite dimension, the free quotient* $\mathcal{A}$*-modules* $\mathcal{E}/\mathcal{F}^{\top\phi}$ *and* $\mathcal{F}/\mathcal{E}^{\perp\phi}$ *have the same dimension, i.e.*

$$\mathcal{E}/\mathcal{F}^{\top\phi} = \mathcal{F}/\mathcal{E}^{\perp\phi}$$

within an $\mathcal{A}$*-isomorphism.*

Proof. Since $((\mathcal{F}, \mathcal{E}; \phi); \mathcal{A})$ is *orthogonally convenient,* kernels $\mathcal{E}^{\perp\phi}$ and $\mathcal{F}^{\top\phi}$ are free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively. By [15], it follows that the quotient $\mathcal{A}$-modules $\mathcal{E}/\mathcal{F}^{\top\phi}$ and $\mathcal{F}/\mathcal{E}^{\perp\phi}$ are free, and for any open subset U of X,

$$(\mathcal{E}/\mathcal{F}^{\top\phi})(U) = \mathcal{E}(U)/\mathcal{F}^{\top\phi}(U) = \mathcal{E}(U)/\mathcal{F}(U)^{\top\phi}$$

and

$$(\mathcal{F}/\mathcal{E}^{\perp\phi})(U) = \mathcal{F}(U)/\mathcal{E}^{\perp\phi}(U) = \mathcal{F}(U)/\mathcal{E}(U)^{\perp\phi}$$

within $\mathcal{A}(U)$-isomorphism. Clearly, for a fixed open $U \subseteq X$, if $s \in \mathcal{E}(U)$ and $t, t_1 \in \mathcal{F}(U)$ such that $t - t_1 \in \mathcal{E}^{\perp\phi}(U)$, then

$$\phi_U(s, t) = \phi_U(s, t_1).$$

In the same vein, if $s = s_1 \mod \mathcal{F}^{\top\phi}(U)$ and $t = t_1 \mod \mathcal{E}^{\perp\phi}(U)$, then

$$\phi_U(s, t) = \phi_U(s_1, t_1).$$

Now, let us consider the $\mathcal{A}$-bilinear morphism

$$\overline{\phi} \equiv (\overline{\phi}_U)_{X \supseteq U,\ open} \equiv ((\overline{\phi})_U)_{X \supseteq U,\ open} : \mathcal{E}/\mathcal{F}^{\top_\phi} \oplus \mathcal{F}/\mathcal{E}^{\perp_\phi} \longrightarrow \mathcal{A},$$

induced by the $\mathcal{A}$-bilinear morphism ϕ, which is such that, for any open $U \subseteq X$ and sections $\overline{s} := \mathrm{cl}(s) \mod \mathcal{F}^{\top_\phi}(U)$, $\overline{t} := \mathrm{cl}(t) \mod \mathcal{E}^{\perp_\phi}(U)$ ($\mathrm{cl}(s)$ stand for the *equivalence class containing* s), one has

$$\overline{\phi}_U(\overline{s}, \overline{t}) := \phi_U(s,t).$$

It is clear that $\overline{\phi}_U(\overline{s}, \overline{t}) = 0$ for any $\overline{s} \in (\mathcal{E}/\mathcal{F}^{\top_\phi})(U) = \mathcal{E}(U)/\mathcal{F}^{\top_\phi}(U)$ is equivalent to $\phi_U(s,t) = 0$ for any $s \in \mathcal{E}(U)$; therefore $t \in \mathcal{E}^{\perp_\phi}(U) = 0$ and hence $\overline{t} = 0$. This implies that $(\mathcal{E}/\mathcal{F}^{\top_\phi})^{\perp_\phi} = 0$. Similarly, that $\overline{\phi}_U(\overline{s}, \overline{t}) = 0$ for any $\overline{t} \in (\mathcal{F}/\mathcal{E}^{\perp_\phi})(U) = \mathcal{F}(U)/\mathcal{E}^{\perp_\phi}(U)$ is equivalent to $\overline{s} = 0$, from which we deduce that $(\mathcal{F}/\mathcal{E}^{\perp_\phi})^{\top_\phi} = 0$. Hence, $\overline{\phi}$ is non-degenerate; so

$$\mathcal{E}/\mathcal{F}^{\top_\phi} = \mathcal{F}/\mathcal{E}^{\perp_\phi}$$

within an $\mathcal{A}$-isomorphism. ■

Based on Proposition 4.1, we make the following definition.

Definition 4.4 Let $\mathcal{E}$, $\mathcal{F}$ be free $\mathcal{A}$-modules, and $((\mathcal{F}, \mathcal{E}; \phi); \mathcal{A})$ an orthogonally convenient $\mathcal{A}$-pairing. The dimension of the free $\mathcal{A}$-module $\mathcal{E}/\mathcal{F}^{\top_\phi} \cong \mathcal{F}/\mathcal{E}^{\perp_\phi}$ is called the **rank of** ϕ**.**

Theorem 4.1 *Let $\mathcal{A}$ be a PID algebra sheaf, $(\mathcal{F}, \mathcal{E}; \phi)$ an orthogonally convenient $\mathcal{A}$-pairing with $\dim \mathcal{E}$ and $\dim \mathcal{F}$ finite. Moreover, let ϕ^L and ϕ^R be the left and right insertion $\mathcal{A}$-morphisms associated with ϕ. Then,*

(1) *For every free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, one has*

1.1) $\phi^L(\mathcal{G}) \simeq (\mathcal{G}^{\perp_\phi})^\perp$ *and* $\phi^R(\mathcal{H}) \simeq (\mathcal{H}^{\top_\phi})^\perp$.

1.2) $\dim \phi^L(\mathcal{G}) = \mathrm{codim}_{\mathcal{F}} \mathcal{G}^{\perp_\phi}$ *and* $\dim \phi^R(\mathcal{H}) = \mathrm{codim}_{\mathcal{E}} \mathcal{H}^{\top_\phi}$.

(2) *$\mathcal{A}$-morphisms ϕ^L and ϕ^R have the same rank:*

$$\mathrm{rank}(\phi^L) = \mathrm{rank}(\phi^R), \tag{12}$$

which is the rank of ϕ.

Proof. *Assertion* (1). Since $(\mathcal{F}, \mathcal{E}; \phi)$ is orthogonally convenient, the sub-$\mathcal{A}$-module $\mathcal{G}^{\perp_\phi}$ is free, and thus

$$\mathcal{G}^{\perp_\phi}(U) \simeq \mathcal{G}(U)^{\perp_\phi}$$

for every open $U \subseteq X$. By Lemma 4.1,

$$\mathcal{G}^{\perp_\phi} = (\phi^L(\mathcal{G}))^{\top}$$

within an $\mathcal{A}$-isomorphism. Applying Theorem 3.3, and since $\dim \mathcal{F}$ is finite, we have

$$(\mathcal{G}^{\perp_\phi})^{\perp} = \phi^L \mathcal{G}$$

within an $\mathcal{A}$-isomorphism. By the same theorem along with Theorem 2.5, it follows that

$$\dim \mathcal{G}^{\perp_\phi} + \dim \phi^L \mathcal{G} = \dim \mathcal{F},$$

from which we deduce that

$$\dim \phi^L \mathcal{G} = \mathrm{codim}_{\mathcal{F}} \mathcal{G}^{\perp_\phi}.$$

In particular,

$$\mathrm{rank}(\phi^L) = \mathrm{codim}_{\mathcal{F}} \mathcal{E}^{\perp_\phi}. \tag{13}$$

In a similar way, one shows the claims related to the induced $\mathcal{A}$-morphism ϕ^R by using the fact that $\dim \mathcal{E}$ is finite. The analog of (13) is

$$\mathrm{rank}(\phi^L) = \mathrm{codim}_{\mathcal{E}} \mathcal{F}^{\top_\phi}. \tag{14}$$

Assertion (2). That

$$\ker(\phi^L) \simeq \mathcal{E}^{\perp_\phi} \quad \text{and} \quad \ker(\phi^R) \simeq \mathcal{F}^{\top_\phi}$$

is immediate. Applying the *dimension formula* (Corollary 3.2), we obtain

$$\operatorname{rank}(\phi^R) := \dim \phi^R(\mathcal{F}) = \dim \mathcal{F} - \dim \mathcal{E}^{\perp_\phi} = \operatorname{codim}_{\mathcal{F}} \mathcal{E}^{\perp_\phi}, \tag{15}$$

and

$$\operatorname{rank}(\phi^L) := \dim \phi^L(\mathcal{E}) = \dim \mathcal{E} - \dim \mathcal{F}^{\top^\phi} = \operatorname{codim}_{\mathcal{E}} \mathcal{F}^{\top^\phi}. \tag{16}$$

From (13), (14), (15) and (16), one gets (12). ■

Corollary 4.1 *Let $\mathcal{A}$ be a PID algebra sheaf and $(\mathcal{F}, \mathcal{E}; \phi)$ an orthogonally convenient $\mathcal{A}$-pairing with free $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ both of finite dimension.*

(1) *For every free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, one has*

1.1) $\dim \mathcal{G}^{\perp_\phi} \geq \dim \mathcal{F} - \dim \mathcal{G}$ *and* $\dim \mathcal{H}^{\top^\phi} \geq \dim \mathcal{E} - \dim \mathcal{H}$

1.2) $(\mathcal{G}^{\perp_\phi})^{\top^\phi} \supseteq \mathcal{G}$ *and* $(\mathcal{H}^{\top^\phi})^{\perp_\phi} \supseteq \mathcal{H}$.

(2) *If ϕ is nondegenerate, then*

2.1) $\dim \mathcal{G}^{\perp_\phi} + \dim \mathcal{G} = \dim \mathcal{F} = \dim \mathcal{E} = \dim \mathcal{H}^{\top^\phi} + \dim \mathcal{H}$

2.2) $(\mathcal{G}^{\perp_\phi})^{\top^\phi} \simeq \mathcal{G}$ *and* $(\mathcal{H}^{\top^\phi})^{\perp_\phi} \simeq \mathcal{H}$.

Proof. *Assertion* (1). Theorem 4.1 shows that

$$\dim \phi^L(\mathcal{G}) = \operatorname{codim}_{\mathcal{F}} \mathcal{G}^{\perp_\phi} = \dim \mathcal{F} - \dim \mathcal{G}^{\perp_\phi}.$$

On the other hand, by virtue of Corollary 3.2, one has

$$\dim \phi^L(\mathcal{G}) = \dim \mathcal{G} - \dim(\ker \phi^L \cap \mathcal{G}).$$

It follows, in particular, that

$$\dim \mathcal{G} \geq \dim \phi^L(\mathcal{G}),$$

from which we have

$$\dim \mathcal{G}^{\perp_\phi} \geq \dim \mathcal{F} - \dim \mathcal{G}.$$

Likewise, one shows the second inequality of 1.1).

Assertion (2). If ϕ is nondegenerate, $\dim \mathcal{E} = \dim \mathcal{F}$; therefore ϕ^L is an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{F}^*$. Thus, $\dim \phi^L(\mathcal{G}) = \dim \mathcal{G}$, and

$$\dim \mathcal{G}^{\perp_\phi} = \dim \mathcal{F} - \dim \mathcal{G}.$$

Likewise, one has

$$\dim \mathcal{H}^{\top^\phi} = \dim \mathcal{E} - \dim \mathcal{H}.$$

Applying relation 2.1) to the free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{G}^{\perp_\phi}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, we see that

$$\dim (\mathcal{G}^{\perp_\phi})^{\top^\phi} = \dim \mathcal{G}.$$

Since $\mathcal{G}$ is contained in $(\mathcal{G}^{\perp_\phi})^{\top^\phi}$, it follows that

$$(\mathcal{G}^{\perp_\phi})^{\top^\phi} = \mathcal{G}$$

within an $\mathcal{A}$-isomorphism. In a similar way, we show that $(\mathcal{H}^{\top^\phi})^{\perp_\phi} = \mathcal{H}$ within an $\mathcal{A}$-isomorphism. ■

We will soon turn to the hyperbolic decomposition theorem for $\mathcal{A}$-modules. However, the so-called theorem requires some preparations.

Lemma 4.2 *Let $(\mathcal{E}, \phi)$ be a symplectic $\mathcal{A}$-module, U an open subset of X and $(r_1, \ldots, r_n) \subseteq \mathcal{E}(U)$ an arbitrary (local) gauge of $\mathcal{E}$. For any $r \equiv r_i$, $1 \leq i \leq n$, there exists a nowhere-zero section $s \in \mathcal{E}(U)$ such that $\phi_U(r, s)$ is nowhere zero.*

Proof. Without loss of generality, assume that $r_1 = r$. On the other hand, since the induced $\mathcal{A}$-morphism $\widetilde{\phi} \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^*)$ is one-to-one and both $\mathcal{E}$ and $\mathcal{E}^*$ have the same finite rank, it follows that the matrix D representing ϕ_U (see also [1, p. 357, Theorem 2.21, along with p. 356, Definition 2.19] or [4, p. 343, Proposition 20.3]), with respect to the basis $(r_1, \ldots, r_n)$, has a *nowhere-zero determinant*; so since

$$\det D = \sum_{i=1}^{n} (-1)^{1+i} \phi(r_1, r_i) \det D_{1i} = \phi(r_1, \sum_{i=1}^{n} (-1)^{1+i} \det D_{1i} r_i),$$

where D_{1i} is the minor of the corresponding $\phi(r_1, r_i)$, and $\det D$ nowhere zero, we thus have a section $s := \sum_{i=1}^{n}(-1)^{1+i}\det D_{1i} r_i \in \mathcal{E}(U)$ such that $\phi(r, s)$ is nowhere zero. ■

For the purpose of Theorem 4.3 below, we require the following result, see [13].

Theorem 4.2 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module of finite rank, equipped with an $\mathcal{A}$-bilinear morphism $\phi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$. Then, every non-isotropic free sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ is a direct summand of $\mathcal{E}$; viz.*

$$\mathcal{E} = \mathcal{F} \perp \mathcal{F}^{\perp_\phi}.$$

So, we have

Theorem 4.3 (***Hyperbolic Decomposition Theorem***) *Let X be a connected topological space, $\mathcal{A}$ a PID algebra sheaf on X, $(\mathcal{E}, \mathcal{E}; \phi)$ an orthogonally convenient self-pairing, where $\mathcal{E}$ is a (free-) $\mathcal{A}$-module on X and $\phi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ a non-degenerate skew-symmetric $\mathcal{A}$-bilinear morphism. Then, if $\mathcal{F}$ is a totally isotropic (free) sub-$\mathcal{A}$-module of rank k, there is a non-isotropic sub-$\mathcal{A}$-module $\mathcal{H}$ of $\mathcal{E}$ of the form*

$$\mathcal{H} := \mathcal{H}_1 \perp \cdots \perp \mathcal{H}_k,$$

where if $(r_{1,U}, \ldots, r_{k,U})$ is a basis of $\mathcal{F}(U)$ (with U an open subset of X), then $r_{i,u} \in \mathcal{H}_i(U)$ for $1 \leq i \leq k$.

Proof. Suppose that $k = 1$, i.e. $\mathcal{F} = \mathcal{A}$, within an $\mathcal{A}$-isomorphism. If $\mathcal{F}(X) = [r_X]$ with $r_X \in \mathcal{E}(X)$ a nowhere-zero section, then for every open $U \subseteq X$, $r_U \equiv r_X|_U$ generates the $\mathcal{A}(U)$-module $\mathcal{F}(U)$. Since ϕ_X is non-degenerate, by Lemma 4.2, there exists a nowhere-zero section $s_X \in \mathcal{E}(X)$ such that $\phi_X(r_X, s_X)$ is nowhere zero. The correspondence

$$U \longmapsto \mathcal{H}(U) := [r_U, s_U] \equiv [r_X|_U, s_X|_U],$$

where U runs over the open sets in X, along with the obvious restriction maps yields a *complete presheaf of $\mathcal{A}(U)$-modules* on X. Clearly, the pair $(\mathcal{H}, \overline{\phi})$, where $\overline{\phi}$ is the $\mathcal{A}$-bilinear $\overline{\phi} : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{A}$ such that

$$(r, s) \longmapsto \overline{\phi}_U(r, s) := \phi_U(r, s),$$

where $r, s \in \mathcal{H}(U)$, is *non-isotropic.* Hence, the theorem holds for the case $k = 1$. Let us now proceed by induction to $k > 1$. To this end, put $\mathcal{F}_{k-1} \simeq \mathcal{A}^{k-1}$ and $\mathcal{F}_k := \mathcal{F} \simeq \mathcal{A}^k$. Then, $\mathcal{F}_{k-1} \subsetneqq \mathcal{F}_k$, so $\mathcal{F}_k^{\perp_\phi} \subsetneqq \mathcal{F}_{k-1}^{\perp_\phi}$. Since *orthogonal of free sub-$\mathcal{A}$-modules in an orthogonally convenient $\mathcal{A}$-module are free sub-$\mathcal{A}$-modules,* the inclusion $\mathcal{F}_k^{\perp_\phi} \subsetneqq \mathcal{F}_{k-1}^{\perp_\phi}$ implies that, if $\mathcal{F}_{k-1}^{\perp_\phi} \simeq \mathcal{A}^m$ and $\mathcal{F}_k^{\perp_\phi} \simeq \mathcal{A}^n$ with $n < m$, then there exists a sub-$\mathcal{A}$-module $\mathcal{G} \subseteq \mathcal{F}_{k-1}^{\perp}$ such that $\mathcal{G} \simeq \mathcal{A}^{m-n}$. For every open $U \subseteq X$, pick a nowhere-zero section $s_{k,U} \in \mathcal{G}(U)$, and put $\mathcal{H}_k(U) = [r_{k,U}, s_{k,U}]$. The correspondence

$$U \longmapsto \mathcal{H}_k(U),$$

where U is open in X, along with the obvious restriction maps, is a complete presheaf of $\mathcal{A}(U)$-modules. Since $\phi_U(r_{i,U}, s_{k,U}) = 0$ for $1 \leq i \leq k-1$, $\phi_U(r_{k,U}, s_{k,U})$ is nowhere zero. Hence, $\mathcal{H}_k(U)$ is a non-isotropic $\mathcal{A}(U)$-plane containing $r_{k,U}$. By Theorem 4.3 $\mathcal{E} = \mathcal{H}_k \perp \mathcal{H}_k^{\perp_\phi}$. Since $r_{k,U}, s_{k,U} \in \mathcal{F}_{k-1}^{\perp_\phi}(U)$, $\mathcal{H}_k(U) \subseteq \mathcal{F}_{k-1}^{\perp_\phi}(U)$ for every open $U \subseteq X$; so $\mathcal{H}_k \subseteq \mathcal{F}_{k-1}^{\perp_\phi}$, which in turn implies that $\mathcal{F}_{k-1} \subseteq \mathcal{H}_k^{\perp_\phi}$. Apply an inductive argument to $\mathcal{F}_{k-1}$ regarded as a sub-$\mathcal{A}$-module of the non-isotropic $\mathcal{A}$-module $\mathcal{H}_k^{\perp_\phi}$. ■

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