SAMPLING IN REPRODUCING KERNEL BANACH SPACES ON LIE GROUPS

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ABSTRACT. We present sampling theorems for reproducing kernel Banach spaces on Lie groups. Recent approaches to this problem rely on integrability of the kernel and its local oscillations. In this paper we replace the integrability conditions by requirements on the derivatives of the reproducing kernel. The results are then used to obtain frames and atomic decompositions for Banach spaces of distributions stemming from a cyclic representation, and it is shown that this is particularly easy, when the cyclic vector is a Gårding vector for a square integrable representation.

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1. INTRODUCTION

The classical sampling theorem for band-limited functions states that a function can be reproduced from its samples at equidistant points. At the core of this statement lies the fact that a bounded interval has an orthonormal basis of complex exponentials. Extensions of this theorem for irregular sampling points have been found using the smoothness of the functions involved [13, 11, 14]. The irregularity and density of the sampling points is connected to the theory of frames [5, 1]: A sequence of vectors ϕ_i in a Hilbert space H is called a frame, if there are constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_i |(f, \phi_i)|^2 \le B\|f\|^2$$

for all $f \in H$. A vector f can be reconstructed by inversion of the frame operator

$$Sf = \sum_i (f, \phi_i)\phi_i$$

A Banach (or Hilbert) space of functions on a set D for which point evaluation is continuous is called a reproducing kernel Banach (or Hilbert) space. Sampling at points x_i provide a frame on a reproducing kernel Hilbert space H if for all $f \in H$

$$A\|f\|^2 \le \sum_i |c_i f(x_i)|^2 \le B\|f\|^2 \tag{1}$$

where c_i are constants. If this frame inequality is satisfied we can reconstruct f from its samples $f(x_i)$. For reproducing kernel Banach spaces the existence of a reconstruction operator

is not evident from a frame type inequality. However in [15] it was proven that reconstruction is possible if for $1 \leq p < \infty$ there are constants $0 < A \leq B < \infty$ such that

$$A\|f\|^p \leq \sum_i |c_i f(x_i)|^p \leq B\|f\|^p \qquad (2)$$

for all $f \in B$. For other types of reproducing kernel Banach spaces more care has to be taken and a lot more machinery is needed. The article [17] is concerned with reconstruction in reproducing kernel subspaces of $L^p(\mathbb{R}^n)$ and [9, 12] deals with Banach spaces defined via representations of locally compact groups. Common for these approaches is that a reproducing kernel is given by an integral over a locally compact group

$$f(x) = \int_G f(y)K(x,y)\,dy$$

This kernel is assumed to be integrable, i.e. for every x

$$\int_G |K(x,y)|\,dy < \infty$$

It is also assumed that for a compact set U, the local oscillations

$$M_U K(x,y) = \sup_{u,v \in U} |K(xu, yv) - K(x,y)|$$

satisfy

$$\int_G |M_U K(x,y)|\,dy < \infty$$

These assumptions are not satisfied for band-limited functions, since the reproducing kernel is the non-integrable sinc-function. Other non-integrable kernels are known (see for example the sections about Bergman spaces in [3, 4]) and this calls for a sampling theory without integrability conditions. The main idea in this article is to estimate local oscillations via derivatives, and therefore we restrict our attention to reproducing kernel Banach spaces on Lie groups.

Reproducing kernel Banach spaces show up naturally in connection with square integrable representations, which was first noted in the construction of coorbit spaces (see [8, 9]). In [3, 4] this work was generalized and coorbit spaces were defined as Banach spaces of distributions stemming from cyclic representations. As an application of our sampling theorems we obtain frames and atomic decompositions for coorbit spaces arising from cyclic (and not necessarily integrable) representations of Lie groups.

2. Examples with reproducing kernel Hilbert spaces

In this section we will cover sampling theorems for two cases of reproducing kernel Hilber spaces on groups. The two groups are $\mathbb{R}$ and the $(ax + b)$-group.

2.1. **Sampling of band-limited functions.** The Fourier transform is the extension to $L^2(\mathbb{R}^n)$ of the operator $\mathcal{F}$

$$\mathcal{F}f(w) = (2\pi)^{-n/2} \int f(x)e^{-iw\cdot x}\,dx$$

defined for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We will often denote the Fourier transform $\mathcal{F}f$ by $\widehat{f}$. A function in $L^2(\mathbb{R})$ is called Ω-band-limited if $\text{supp}(\widehat{f}) \subseteq [-\Omega, \Omega]$. The space L^2_Ω of Ω-band-limited functions is a reproducing kernel Hilbert space and satisfies

$$f(x) = \int f(y)\text{sinc}(x - y)\, dy$$

where

$$\text{sinc}(x) = \frac{\sin x}{x}$$

Therefore we need only provide a frame inequality like (1) in order to reconstruct Ω-band-limited functions. In [14] the following irregular sampling theorem for band-limited functions was used to provide sampling theorems for the wavelet and short time Fourier transforms.

Theorem 2.1. *Suppose that $f \in L^2(\mathbb{R})$ and $\text{supp}(\widehat{f}) \subseteq [-\Omega, \Omega]$. If $\{x_k\}_{k \in \mathbb{Z}}$ is any increasing sequence such that the maximal gap length δ satisfies*

$$\delta := \sup_{k \in \mathbb{Z}} (x_{k+1} - x_k) < \frac{\pi}{\Omega}$$

then

$$(1 - \delta\Omega/\pi)^2 \|f\|_2^2 \leq \sum_k \frac{x_{k+1} - x_{k-1}}{2} |f(x_k)|^2 \leq (1 + \delta\Omega/\pi)^2 \|f\|_2^2$$

To prove this it is first shown that for disjoint intervals $I_k \subseteq (x_k - \delta/2, x_k + \delta/2)$ with $\cup_k I_k = \mathbb{R}$ we have

$$\left\| f - \sum_k f(x_k) 1_{I_k} \right\|_{L^2} \leq \frac{\delta}{\pi} \|f'\|_{L^2} \tag{3}$$

This inequality follows from an application of Wirtinger's inequality. Then Bernstein's inequality $\|f'\|_{L^2} \leq \Omega \|f\|_{L^2}$ is utilized to obtain the frame inequality of the theorem above.

We now give an alternative approach to inequalities resembling (3). Note that this has already been presented as Lemma 4 in [13], however we include the calculations here to demnonstrate how they can be generalized. This is more straight forward than Wirtinger's inequality and uses the smoothness of band-limited functions and the fundamental theorem of calculus. Since many reproducing kernel spaces consist of differentiable functions this approach will carry over to such spaces. Define the local oscillation of a band-limited function f as

$$M^\delta f(x) = \sup_{|u| \leq \delta} |f(x + u) - f(x)|$$

Then an application of Hölder's inequality shows that

$$M^\delta f(x) = \sup_{|u| \leq \delta} |f(x + u) - f(x)|$$

$$= \sup_{|u| \leq \delta} \left| \int_0^u f'(x + t)\, dt \right|$$

$$\leq \sup_{|u| \leq \delta} \left(\int_0^{|u|} 1\, dt \right)^{1/2} \left(\int_0^u |f'(x + t)|^2\, dt \right)^{1/2}$$

$$\leq \delta \left(\int_{-\delta}^\delta |f'(x + t)|^2\, dt \right)^{1/2}$$

Applying Minkowski's inequality then gives the following oscillation estimate

$$\|M^\delta f\|_{L^2} \le \sqrt{2}\delta\|f'\|_{L^2} \tag{4}$$

From this follows

$$\left\| f - \sum_k f(x_k)1_{I_k} \right\|_{L^2} \le \|M^\delta f\|_{L^2} \le \sqrt{2}\delta\|f'\|_{L^2}$$

and we can again derive a frame inequality by use of Bernstein's inequality. Note that this estimate is not as sharp as (3), however it has the advantage that it can be generalized to other groups than $\mathbb{R}$.

In this paper we will derive oscillation estimates similar to (4) for (non-commutative) Lie groups in order to obtain sampling theorems. In the next subsection we work through the details for the non-commutative $(ax + b)$-group and show how this provides sampling theorems for the wavelet transform.

2.2. **Sampling of the wavelet transform.** In this section we present the ideas behind sampling for reproducing kernel Hilbert space related to the non-commutative $(ax + b)$-group. The approach will be generalized in section 3.

Let G be the $(ax + b)$-group which can be realized as a matrix group

$$G = \left\{ (a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a > 0, b \in \mathbb{R} \right\}$$

The left Haar measure on G is defined by

$$C_c(G) \ni f \to \int_0^\infty \int_{\mathbb{R}} f(a, b) \, \frac{da\,db}{a^2}$$

and we denote by $L^2(G)$ the space of square integrable functions with respect to this measure. For a function g let $g^\vee$ be the function

$$g^\vee(x) = g(x^{-1})$$

Convolution of two functions $f, g^\vee \in L^2(G)$ is given by

$$f * g(a, b) = \int_0^\infty \int_{\mathbb{R}} f(a_1, b_1)g((a_1, b_1)^{-1}(a, b)) \, \frac{da\,db}{a^2}$$

Assume that $\phi \in L^2(G)$ is a non-zero function for which $\phi^\vee \in L^2(G)$ and the mapping

$$L^2(G) \ni f \mapsto f * \phi \in L^2(G)$$

is continuous. Further assume that $\phi * \phi = \phi$, then the space $H_\phi = L^2(G) * \phi$ is a reproducing kernel Hilbert space and the reproducing kernel is given by convolution with ϕ. In order to obtain oscillation estimates we need some notation concerning differentiation. The Lie algebra of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

and the exponential function is the usual matrix exponential function

$$e^A = \sum_{k=0}^\infty A^k/k!$$

For $X \in \mathfrak{g}$ define the differential operator

$$Xf(x) = \left.\frac{d}{dt}\right|_{t=0} f(xe^{tX})$$

Denote by X_1, X_2 the basis for the Lie algebra $\mathfrak{g}$ of G for which

$$e^{tX_1} = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad e^{tX_2} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

For α a multi-index of length k with entries 1 or 2 we define the differential operators R^α

$$R^\alpha f = X_{\alpha(k)} X_{\alpha(k-1)} \cdots X_{\alpha(1)} f$$

When ϵ is a positive number we define the neighbourhood U_ϵ of the identity e by

$$U_\epsilon = \{ e^{t_1 X_1} e^{t_2 X_2} \mid -\epsilon < t_1, t_2 < \epsilon \}$$

Choose points $x_i \in G$ such that $G \subseteq \cup_i x_i U_\epsilon$. Let $U_i \subseteq x_i U_\epsilon$ be disjoint sets such that $G = \cup_i U_i$ and denote by 1_{U_i} the indicator function for U_i. The following lemma provides an estimate equivalent to (3) for the $(ax + b)$-group.

Lemma 2.2. *If $f \in L^2(G)$ is right differentiable up to order 2 and $R^\alpha f \in L^2(G)$ for $|\alpha| \le 2$, then*

$$\left\| f - \sum_i f(x_i) 1_{U_i} \right\|_{L^2} \le C_\epsilon (\|X_1 f\|_{L^2} + \|X_2 f\|_{L^2} + \|X_2 X_1 f\|_{L^2})$$

where $C_\epsilon \to 0$ as $\epsilon \to 0$.

Proof. For $x \in x_i U$ there are s_1 and s_2 in between $-\epsilon$ and ϵ such that $x_i = xe^{s_2 X_2} e^{s_1 X_1}$. Thus we get

$$
\begin{aligned}
|f(x) - f(x_i)| &= |f(x) - f(xe^{s_2 X_2} e^{s_1 X_1})| \\
&\le |f(x) - f(xe^{s_2 X_2})| + |f(xe^{s_2 X_2}) - f(xe^{s_2 X_2} e^{s_1 X_1})| \\
&= \left| \int_0^{s_2} \frac{d}{dt_2} f(xe^{t_2 X_2}) \, dt_2 \right| + \left| \int_0^{s_1} \frac{d}{dt_1} f(xe^{s_2 X_2} e^{t_1 X_1}) \, dt_1 \right| \\
&\le \int_{-\epsilon}^{\epsilon} \left| X_2 f(xe^{t_2 X_2}) \right| \, dt_2 + \int_{-\epsilon}^{\epsilon} \left| X_1 f(xe^{s_2 X_2} e^{t_1 X_1}) \right| \, dt_1
\end{aligned}
$$

Since

$$e^{t_2 X_2} e^{t_1 X_1} = e^{t_1 X_1} e^{t_2 e^{-t_1} X_2}$$

the term $|X_1 f(x e^{s_2 X_2} e^{t_1 X_1})|$ can be estimated by

$$
\begin{aligned}
|X_1 f(x e^{s_2 X_2} e^{t_1 X_1})| &= |X_1 f(x e^{t_1 X_1} e^{s_2 e^{-t_1} X_2})| \\
&\leq |X_1 f(x e^{t_1 X_1} e^{s_2 e^{-t_1} X_2}) - X_1 f(x e^{t_1 X_1})| + |X_1 f(x e^{t_1 X_1})| \\
&= \left| \int_0^{s_1} \frac{d}{dt_2} X_1 f(x e^{t_1 X_1} e^{t_2 e^{-t_1} X_2}) \, dt_2 \right| + |X_1 f(x e^{t_1 X_1})| \\
&= \left| \int_0^{s_1} e^{-t_1} X_2 X_1 f(x e^{t_1 X_1} e^{t_2 e^{-t_1} X_2}) \, dt_2 \right| + |X_1 f(x e^{t_1 X_1})| \\
&= \left| \int_0^{s_1} e^{-t_1} X_2 X_1 f(x e^{t_2 X_2} e^{t_1 X_1}) \, dt_2 \right| + |X_1 f(x e^{t_1 X_1})| \\
&\leq e^{\epsilon} \int_{-\epsilon}^{\epsilon} \left| X_2 X_1 f(x e^{t_2 X_2} e^{t_1 X_1}) \right| dt_2 + |X_1 f(x e^{t_1 X_1})|
\end{aligned}
$$

We therefore obtain the following estimate for $|f(x) - f(x_i)|$:

$$
\begin{aligned}
|f(x) - f(x_i)| &\leq \int_{-\epsilon}^{\epsilon} \left| X_2 f(x e^{t_2 X_2}) \right| dt_2 + \int_{-\epsilon}^{\epsilon} |X_1 f(x e^{t_1 X_1})| \, dt_1 \\
&\quad + e^{\epsilon} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \left| X_2 X_1 f(x e^{t_2 X_2} e^{t_1 X_1}) \right| dt_2 \, dt_1
\end{aligned}
$$

This expression no longer depends on i and using Fubini's theorem to change the order of integration (or Minkowski to move the L^2-norm inside the integrals over t_1 and t_2) we get

$$
\begin{aligned}
\left\| f - \sum_i f(x_i) 1_{U_i} \right\|_{L^2} &\leq \left\| \int_{-\epsilon}^{\epsilon} r_{e^{t_2 X_2}} |X_2 f| \, dt_2 \right\|_{L^2} \\
&\quad + \left\| \int_{-\epsilon}^{\epsilon} r_{e^{t_1 X_1}} |X_1 f| \, dt_1 \right\|_{L^2} \\
&\quad + e^{\epsilon} \left\| \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} r_{e^{t_2 X_2} e^{t_1 X_1}} |X_2 X_1 f| \, dt_2 \, dt_1 \right\|_{L^2} \\
&\leq \int_{-\epsilon}^{\epsilon} \| r_{e^{t_2 X_2}} X_2 f \|_{L^2} \, dt_2 \\
&\quad + \int_{-\epsilon}^{\epsilon} \| r_{e^{t_1 X_1}} X_1 f \|_{L^2} \, dt_1 \\
&\quad + e^{\epsilon} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \| r_{e^{t_2 X_2} e^{t_1 X_1}} X_2 X_1 f \|_{L^2} \, dt_2 \, dt_1 \\
&\leq C_{\epsilon} (\| X_1 f \|_{L^2} + \| X_2 f \|_{L^2} + \| X_2 X_1 f \|_{L^2})
\end{aligned}
$$

where we have also used that right translation inside $L^2(G)$ is continuous. Note that by the above calculations $C_{\epsilon} \to 0$ when $\epsilon \to 0$.

$\square$

We are now able to obtain the following sampling result for reproducing kernel subspaces of $L^2(G)$.

Theorem 2.3. *If right differentiation up to order 2 is continuous on $H_\phi = L^2(G) * \phi$, then we can choose points x_i such that the norms $\|f\|_{L^2}$ and $\|\{f(x_i)\}\|_{\ell^2}$ are equivalent. Thus $\ell_{x_i}\phi^\vee$ forms a frame for H_ϕ.*

Proof. First we note that if right differentiation is continuous then by Lemma 2.2 there is a constant C_ϵ such that

$$\left\| f - \sum_i f(x_i)1_{U_i} \right\|_{L^2} \leq C_\epsilon \|f\|_{L^2}$$

By [18] it is possible to choose ϵ and x_i such that the sets $x_i U_{\epsilon/4}$ are disjoint and $C_\epsilon < 1$. Further note that as shown in [9] both $\|\sum_i \lambda_i 1_{x_i U_{\epsilon/4}}\|_{L^2}$ and $\|\sum_i \lambda_i 1_{x_i U_\epsilon}\|_{L^2}$ define equivalent norms on ℓ^2. Since $1_{x_i U_{\epsilon/4}} \leq 1_{U_i}$ we get

$$\|\{f(x_i)\}\|_{\ell^2} \leq C\|\sum_i f(x_i)1_{x_i U_{\epsilon/4}}\|_{L^2}$$

$$\leq C\|\sum_i (f - f(x_i))1_{x_i U_{\epsilon/4}}\|_{L^2} + C\|\sum_i f 1_{x_i U_{\epsilon/4}}\|_{L^2}$$

$$\leq C\|f - \sum_i f(x_i)1_{U_i}\|_{L^2} + C\|\sum_i f 1_{U_i}\|_{L^2}$$

$$\leq C(1 + C_\epsilon)\|f\|_{L^2}$$

Similarly we have

$$\|f\|_{L^2} \leq \|\sum_i (f - f(x_i))1_{U_i}\|_{L^2} + \|\sum_i f(x_i)1_{U_i}\|_{L^2}$$

$$\leq C_\epsilon\|f\|_{L^2} + \|\sum_i f(x_i)1_{x_i U_\epsilon}\|_{L^2}$$

$$\leq C_\epsilon\|f\|_{L^2} + D\|\{f(x_i)\}\|_{\ell^2}$$

Since $C_\epsilon < 1$ the inequalities above can be combined into

$$\frac{1 - C_\epsilon}{D}\|f\|_{L^2} \leq \|\{f(x_i)\}\|_{\ell^2} \leq C(1 + C_\epsilon)\|f\|_{L^2}$$

which shows the equivalence of the norms.
   Since

$$f(x_i) = \int f(y)\phi^\vee(x_i^{-1}y)\,dy$$

the vectors $\ell_{x_i}\phi^\vee$ form a frame. $\qquad\square$

This leaves of course task of showing the continuity of left differentiation on H_ϕ. In the following example most statements are already proven in [16].

2.2.1. *Example: Wavelet transform.* Let π be the irreducible unitary representation of G on the space

$$L_+^2 = \{f \in L^2(\mathbb{R}) \mid \operatorname{supp}(\widehat{f}) \subseteq [0, \infty)\}$$

given by

$$\pi(a, b)f(x) = \frac{1}{\sqrt{a}}f\left(\frac{x - b}{a}\right)$$

Define the subspace

$$\mathcal{S}_+ = \{f \in \mathcal{S}(\mathbb{R}) \mid \operatorname{supp}(\widehat{f}) \subseteq [0, \infty)\} \subseteq L_+^2$$

then $\mathcal{S}_+$ is invariant under the differential operators

$$\pi(X)u = \lim_{t \to 0} \frac{\pi(e^{tX})u - u}{t}$$

for $X \in \mathfrak{g}$. Denote by $\pi(R^\alpha)$ the differential operator

$$\pi(R^\alpha)u = \pi(X_{\alpha(k)})\pi(X_{\alpha(k-1)}) \cdots \pi(X_{\alpha(1)})u$$

which also leaves $\mathcal{S}_+$ invariant. For a non-zero $u \in \mathcal{S}_+$ define the wavelet transform of $f \in L_+^2$ by

$$W_u(f)(a, b) = (f, \pi(a, b)u) = \int f(x) \frac{1}{\sqrt{a}} \overline{u\left(\frac{x - b}{a}\right)} \, dx$$

We can normalize u according to the Duflo-Moore theorem [6] such that for all $f \in L_+^2$

$$W_u(f) * W_u(u) = W_u(f)$$

Then the wavelet tranform W_u becomes an isometric isomorphism from L_+^2 into the reproducing kernel Hilbert space $L^2(G) * W_u(u)$. Also the functions $W_u(f)$ is smooth and for $X \in \mathfrak{g}$ we have

$$R^\alpha W_u(f) = W_{\pi(R^\alpha)u}(f)$$

The Duflo-Moore theorem again tells us that

$$W_{\pi(R^\alpha)u}(f) = W_u(f) * W_{\pi(R^\alpha)u}(u)$$

Since $\pi(R^\alpha)u \in \mathcal{S}_+$ we have $W_{\pi(R^\alpha)u}(u) \in L^1(G)$ as shown in [16] and therefore

$$\|W_{\pi(R^\alpha)u}(f)\|_{L^2} \le C_\alpha \|W_u(f)\|_{L^2}.$$

This shows that the continuities in Theorem 2.3 are satisfied and we can reconstruct $W_u(f)$ from it samples. In other words we can reconstruct $f \in L_+^2$ from sampling its wavelet coefficients.

3. Sampling in reproducing kernel Banach spaces on Lie groups

A Banach space of functions is called a reproducing kernel Banach space if point evaluation is continuous. We restrict our attention to reproducing kernel Banach spaces where the reproducing formula is given by a Lie group convolution. We derive local oscillation estimates for such spaces and use them to give a discrete characterization of the reproducing kernel space. In particular we obtain frame and atomic decompositions for reproducing kernel Banach spaces under certain smoothness conditions on the kernel.

3.1. **Reproducing kernel Banach spaces.** Let G be a locally compact group with left invariant Haar measure μ. Denote by ℓ_x and r_x the left and right translations given by

$$\ell_x f(y) = f(x^{-1}y) \qquad \text{and} \qquad r_x f(y) = f(yx)$$

A Banach space of functions is called left or right invariant if there is a constant C_x such that

$$\|\ell_x f\|_B \le C_x \|f\|_B \qquad \text{or} \qquad \|r_x f\|_B \le C_x \|f\|_B$$

respectively. We will always assume that for compact U there is a constant C_U such that for all $f \in B$

$$\sup_{y \in U} \|\ell_y f\|_B \leq C_U \|f\|_B \qquad \text{and} \qquad \sup_{y \in U} \|r_y f\|_B \leq C_U \|f\|_B \qquad (5)$$

For $1 \leq p < \infty$ the space $L^p(G)$ denotes the equivalence class of measurable functions for which

$$\|f\|_{L^p} = \left(\int |f(x)|^p \, d\mu(x) \right)^{1/p} < \infty$$

We will often write dx instead of $d\mu(x)$. The space $L^\infty(G)$ consists of equivalence classes of measurable functions for which

$$\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x \in G} |f(x)|$$

The spaces $L^p(G)$ are left and right invariant and satisfy (5) for $1 \leq p \leq \infty$, however the left and right translations are only continuous for $1 \leq p < \infty$.

When f, g are measurable functions on G for which the product $f(x)g(x^{-1}y)$ is integrable for all $y \in G$ we define the convolution $f * g$ as

$$f * g(y) = \int_G f(x)g(x^{-1}y) \, d\mu(x)$$

A Banach space of functions B is called solid if $|f| \leq |g|$ and $g \in B$ imply that $f \in B$. All spaces $L^p(G)$ are solid, but Sobolev subspaces are not.

In this article we will only work with reproducing kernel Banach spaces which can be constructed in the following manner. Let B be a solid left invariant Banach space of functions on G which satisfies (5). Assume that there is a non-zero $\phi \in B$ such that

$$\left| \int_G f(y)\phi(y^{-1}) \, dy \right| \leq C\|f\|_B$$

By the left invariance of B the convolution $f * \phi$ is well-defined by

$$f * \phi(x) = \int_G f(y)\phi(y^{-1}x) \, dy$$

Assume that ϕ satisfies the reproducing formula

$$\phi * \phi = \phi$$

and that convergence in B implies convergence (locally) in measure. Then the space

$$B_\phi = \{f \in B \mid f = f * \phi\}$$

is a non-zero reproducing kernel Banach subspace of B. Let us for completeness prove this statement (though it is already contained in [4] in disguise) by showing that B_ϕ is a closed subspace of B. Let $f_n \in B_\phi$ be a sequence converging to $f \in B$ then f_n converges locally in measure. Therefore there is a subsequence f_{n_k} converging to f almost everywhere. Thus for almost all $x \in G$ we have

$$|f(x) - f * \phi(x)| \leq |f(x) - f_{n_k}(x)| + |f_{n_k}(x) - f_{n_k} * \phi(x)| + |f_{n_k} * \phi(x) - f * \phi(x)|$$
$$\leq |f(x) - f_{n_k}(x)| + |f_{n_k}(x) - f_{n_k} * \phi(x)| + C\|f_{n_k} - f\|_B$$

The middle term is 0 and the two remaining terms can be made arbitrarily small so $f = f * \phi$ which shows that B_ϕ is closed in B. Point evaluation is continuous for $f \in B_\phi$ by the left invariance of B:

$$|f(x)| \leq C\|\ell_{x^{-1}}f\|_B \leq C_x\|f\|_B$$

The discretizations we will investigate on B_ϕ can be thought of as Riemann sums for the reproducing formula

$$f(x) = \int_G f(y)\phi(y^{-1}x)\,dy$$

which holds for all $f \in B_\phi$.

3.2. **Atomic decompositions and Banach frames.** We will derive atomic decompositions and frames for reproducing kernel Banach spaces, and here we introduce the two notions. Further we introduce sequence spaces and partitions of unity used to obtain the discrete characterizations.

Definition 3.1. Let B be a Banach space and $B^\#$ an associated Banach sequence space with index set I. If for $\lambda_i \in B^*$ and $\phi_i \in B$ we have

  (a) $\{\lambda_i(f)\}_{i \in I} \in B^\#$ for all $f \in B$
  (b) the norms $\|\lambda_i(f)\|_{B^\#}$ and $\|f\|_B$ are equivalent
  (c) f can be written $f = \sum_i \lambda_i(f)\phi_i$

then $\{(\lambda_i, \phi_i)\}$ is an atomic decomposition of B with respect to $B^\#$.

More generally a Banach frame for a Banach space can be defined as

Definition 3.2. Let B be a Banach space and $B^\#$ an associated Banach sequence space with index set I. If for $\lambda_i \in B^*$ we have

  (a) $\{\lambda_i(f)\}_{i \in I} \in B^\#$ for all $f \in B$
  (b) the norms $\|\lambda_i(f)\|_{B^\#}$ and $\|f\|_B$ are equivalent
  (c) there is a bounded reconstruction operator $S : B^\# \to B$ such that $S(\{\lambda_i(f)\}) = f$

then $\{\lambda_i\}$ is an Banach frame for B with respect to $B^\#$.

In Hilbert spaces the existence of the operator S is automatic given the equivalence of the norms $\|\lambda_i(f)\|_{B^\#}$ and $\|f\|_B$. Further, the operator S has been shown to exist for p-frames for reproducing kernel Banach spaces in [15]. For general Banach spaces this is not the case as is demonstrated in [2].

We will work with very particular Banach sequence spaces which are constructed from a solid Banach function space B. These spaces were introduced in [9]. For a compact neighbourhood U of the identity we call the sequence $\{x_i\}_{i \in I}$ U-relatively separated if $G \subseteq \cup_i x_i U$ and there is an N such that

$$\sup_i(\#\{j \mid x_i U \cap x_j U \neq \emptyset\}) \leq N$$

For a U-relatively separated sequence $X = \{x_i\}_{i \in I}$ define the space $B^\#(X)$ to be the collection of sequences $\{\lambda_i\}_{i \in I}$ for which

$$\sum_{i \in I} |\lambda_i| 1_{x_i U} \in B$$

when the sum is taken to be pointwise. If the compactly supported continuous functions are dense in B then this sum also converges in norm. Equipped with the norm

$$\|\{\lambda_i\}\|_{B^\#} = \|\sum_{i \in I} |\lambda_i| 1_{x_i U}\|_B$$

this is a solid Banach sequence space. In the case were $B = L^p(G)$ we get that $B^\#(X) = \ell^p(I)$. For fixed $X = \{x_i\}_{i \in I}$ the space $B^\#(X)$ only depends on the compact neighbourhood U up to norm equivalence. Further, if $X = \{x_i\}_{i \in I}$ and $Y = \{y_i\}_{i \in I}$ are two U-relatively separated sequences with same index set such that $x_i^{-1} y_i \in V$ for some compact set V, then $B^\#(X) = B^\#(Y)$ equivalent norms. For these properties consult Lemma 3.5 in [9].

Given a compact neighbourhood U of the identity the non-negative functions ψ_i are called a bounded uniform partition of unity subordinate to U (or U-BUPU), if there is a U-relatively separated sequence $\{x_i\}$, such that $\text{supp}(\psi_i) \subseteq x_i U$ and $\sum_i \psi_i = 1$. A partition of unity could consist of indicator functions, however on a Lie group G it is possible to find smooth U-BUPU's whenever U is contained in a ball of radius less than the injectivity radius of G (see for example [18, Lemma 2.1]). For the existence of U-BUPU's with elements contained in homogeneous Banach spaces see the paper [7].

3.3. **Local oscillation estimates on Lie groups.** Let G be Lie group with Lie algebra $\mathfrak{g}$ with exponential function $\exp : \mathfrak{g} \rightarrow G$. Then for $X \in \mathfrak{g}$ we define the right and left differential operators (if the limits exist)

$$R(X)f(x) = \lim_{t \to 0} \frac{f(x \exp(tX)) - f(x)}{t} \quad \text{and} \quad L(X)f(x) = \lim_{t \to 0} \frac{f(\exp(tX)x) - f(x)}{t}$$

Fix a basis $\{X_i\}_{i=1}^{\dim(G)}$ for $\mathfrak{g}$. For a multi index α of length $|\alpha| = k$ with entries between 1 and $\dim(G)$ we introduce the operator R^α of subsequent right differentiations

$$R^\alpha f = R(X_{\alpha(k)}) R(X_{\alpha(k-1)}) \cdots R(X_{\alpha(1)}) f$$

Similarly we introduce the operator L^α of subsequent left differentiations

$$L^\alpha f = L(X_{\alpha(k)}) L(X_{\alpha(k-1)}) \cdots L(X_{\alpha(1)}) f$$

We call f right (or left) differentiable of order n if for every x and all $|\alpha| \leq n$ the derivatives $R^\alpha f(x)$ (or $L^\alpha f(x)$) exist.

In the following we will often use this lemma

Lemma 3.3. *Let U be a compact set and fix a basis element $X_k \in \mathfrak{g}$. There is a constant C_U such that for any $y \in U$ and $|s| \leq \epsilon$*

(a) *If f is right differentiable of order 1, then*

$$|f(x e^{sX_k} y) - f(xy)| \leq C_U \int_{-\epsilon}^{\epsilon} \sum_{n=1}^{\dim(G)} |R(X_n) f(x e^{tX_k} y)| \, dt$$

(b) *If f is left differentiable of order 1, then*

$$|f(y e^{sX_k} x) - f(yx)| \leq C_U \int_{-\epsilon}^{\epsilon} \sum_{n=1}^{\dim(G)} |L(X_n) f(y e^{tX_k} x)| \, dt$$

The constant C_U depends only on U and $C_{U'} \leq C_U$ for $U' \subseteq U$.

Proof. First use the fundamental theorem of calculus to get

$$|f(xe^{sX_k}y) - f(xy)| = |f(xye^{sAd_{y^{-1}}(X_k)}) - f(xy)|$$

$$= \left| \int_0^s \frac{d}{dt} f(xye^{tAd_{y^{-1}}(X_k)}) \, dt \right|$$

$$\leq \int_0^s \left| \frac{d}{dt} f(xye^{tAd_{y^{-1}}(X_k)}) \right| \, dt$$

$$\leq \int_{-\epsilon}^{\epsilon} |R(Ad_{y^{-1}}(X_k)) f(xe^{tX_k}y)| \, dt$$

The adjoint representation can be written as

$$Ad_{y^{-1}}(X_k)) = c_1(y)X_1 + \cdots + c_n(y)X_n$$

where the coefficients c_i depend continuously on y (and also depend on X_k). So for smooth f we have the pointwise inequality

$$|R(Ad_{y^{-1}}(X_k))f| \leq |c_1(y)||R(X_1)f| + \cdots + |c_n(y)||R(X_n)f| \leq C_U(X_k) \sum_{n=1}^{\dim(G)} |R(X_n)f|$$

where $C_U(X_k)$ is $\max_{y \in U} |c_i(y)|$. Let $C_U = \max_k |C_U(X_k)|$, then we obtain

$$|f(xe^{sX_k}y) - f(xy)| \leq C_U \int_{-\epsilon}^{\epsilon} \sum_{n=1}^{\dim(G)} |R(X_n)f(xye^{tAd_{y^{-1}}(X_k)})| \, dt$$

$$= C_U \int_{-\epsilon}^{\epsilon} \sum_{n=1}^{\dim(G)} |R(X_n)f(xe^{tX_k}y)| \, dt$$

From the definition of C_U above it follows that $C_{U'} \leq C_U$ for $U' \subseteq U$. A similar argument works for left differentiations. $\square$

From now on we let U_ϵ denote the set

$$U_\epsilon = \left\{ \prod_{k=1}^{n} \exp(t_k X_k) \,\middle|\, -\epsilon \leq t_k \leq \epsilon \right\}.$$

We further define the right local oscillations

$$M_r^\epsilon f(x) = \sup_{u \in U_\epsilon} |f(xu^{-1}) - f(x)|$$

and the left local oscillations

$$M_l^\epsilon f(x) = \sup_{u \in U_\epsilon} |f(ux) - f(x)|$$

For formulating the next lemma we need some notation. By δ we denote an n-tuple $\delta = (\delta_1, \ldots, \delta_n)$ with $\delta_i \in \{0, 1\}$. The length $|\delta|$ of δ is the number of non-zero entries $|\delta| = \delta_1 + \cdots + \delta_n$. Further the function τ_δ is defined as

$$\tau_\delta(t_1, \ldots, t_n) = e^{\delta_1 t_1 X_1} \ldots e^{\delta_n t_n X_n}$$

Lemma 3.4. *If f is right differentiable of order $n = \dim(G)$ there is a constant C_ϵ such that*

$$M_r^\epsilon f(x) \leq C_\epsilon \sum_{1 \leq |\alpha| \leq n} \sum_{|\delta| = |\alpha|} \underbrace{\int_{-\epsilon}^\epsilon \cdots \int_{-\epsilon}^\epsilon}_{|\delta| \ integrals} |R^\alpha f(x \tau_\delta(t_1, \dots, t_n)^{-1})| (dt_1)^{\delta_1} \cdots (dt_n)^{\delta_n}$$

If f is left differentiable of order $n = \dim(G)$ there is a constant C_ϵ such that

$$M_l^\epsilon f(x) \leq C_\epsilon \sum_{1 \leq |\alpha| \leq n} \sum_{|\delta| = |\alpha|} \underbrace{\int_{-\epsilon}^\epsilon \cdots \int_{-\epsilon}^\epsilon}_{|\delta| \ integrals} |L^\alpha f(\tau_\delta(t_1, \dots, t_n)x)| (dt_1)^{\delta_1} \cdots (dt_n)^{\delta_n}$$

For $\epsilon' \leq \epsilon$ we have $C_{\epsilon'} \leq C_\epsilon$.

Proof. For any x there is an element $\sigma = e^{s_n X_n} \dots e^{s_1 X_1} \in U_\epsilon^{-1}$ such that

$$M_r^\epsilon f(x) = |f(x e^{s_n X_n} \dots e^{s_1 X_1}) - f(x)|$$

Denote by σ_k the element in U_ϵ^{-1} given by

$$\sigma_k = e^{s_n X_n} e^{s_{n-1} X_{n-1}} \dots e^{s_k X_k}$$

with the convention that $\sigma_{n+1} = e$. The elements σ_k depend on x, and we wish to estimate the function $M_r^\epsilon f(x) = |f(x\sigma_1) - f(x)|$ by an expression without any σ_k. To do so we make repeated use of the fundamental theorem of calculus in form of the previous lemma.

For any n-tuple δ of 0's and 1's and for a smooth function f we define

$$T_{\alpha, \delta} f(x) = \underbrace{\int_{-\epsilon}^\epsilon \cdots \int_{-\epsilon}^\epsilon}_{|\delta| \ integrals} |R^\alpha f(x \tau_\delta(t_1, \dots, t_n)^{-1})| (dt_1)^{\delta_1} (dt_2)^{\delta_2} \dots (dt_n)^{\delta_n}$$

We first show that if $\delta_m = 0$ for $m \geq k$, then

$$T_{\alpha, \delta} f(x \sigma_k) \leq T_{\alpha, \delta} f(x \sigma_{k+1}) + C_{U_\epsilon^{-1}} \sum_{|\alpha'| = |\delta'| = |\alpha| + 1} T_{\alpha', \delta'} f(x \sigma_{k+1})$$

where $\delta_m' = 0$ for $m \geq k + 1$. To show this note that

$$|R^\alpha f(x \sigma_k \tau_\delta(t_1, \dots, t_n)^{-1})|$$
$$\leq |R^\alpha f(x \sigma_{k+1} \tau_\delta(t_1, \dots, t_n)^{-1})|$$
$$\quad + |R^\alpha f(x \sigma_{k+1} e^{s_k X_k} \tau_\delta(t_1, \dots, t_n)^{-1}) - R^\alpha f(x \sigma_{k+1} \tau_\alpha(t_1, \dots, t_n)^{-1})|$$
$$\leq |R^\alpha f(x \sigma_{k+1} \tau_\delta(t_1, \dots, t_n)^{-1})|$$
$$\quad + C_{U_\epsilon^{-1}} \int_{-\epsilon}^\epsilon \sum_{n=1}^{\dim(G)} |R(X_n) R^\alpha f(x \sigma_{k+1} e^{t_k X_k} \tau_\delta(t_1, \dots, t_n)^{-1})| \, dt_k$$

The terms $R(X_n) R^\alpha f(x \sigma_{k+1} e^{t_k X_k} \tau_\delta(t_1, \dots, t_n)^{-1})$ are of the type

$$R^{\alpha'} f(x \sigma_{k+1} \tau_{\delta'}(t_1, \dots, t_n)^{-1})$$

with $|\alpha'| = |\alpha| + 1$ and $\delta'_m = 0$ for $m \geq k + 1$. Therefore

$$T_{\alpha,\delta} f(x\sigma_k)$$

$$= \int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon} |R^\alpha f(x\tau_\delta(t_1, \ldots, t_n)^{-1})| (dt_1)^{\delta_1} (dt_2)^{\delta_2} \ldots (dt_n)^{\delta_n}$$

$$\leq \int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon} |R^\alpha f(x\sigma_{k+1}\tau_\delta(t_1, \ldots, t_n)^{-1})| (dt_1)^{\delta_1} (dt_2)^{\delta_2} \ldots (dt_n)^{\delta_n}$$

$$+ C_{U_\epsilon^{-1}} \underbrace{\int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon}}_{|\alpha| + 1 \text{ integrals}} \sum_{m=1}^{n} |R(X_m) R^\alpha f(x\sigma_{k+1} e^{t_k X_k} \tau_\delta(t_1, \ldots, t_n)^{-1})| \, dt_k \, (dt_1)^{\delta_1} (dt_2)^{\delta_2} \ldots (dt_n)^{\delta_n}$$

$$\leq T_{\alpha,\beta} f(x\sigma_{k+1}) + C_{U_\epsilon^{-1}} \sum_{|\alpha'|=|\delta'|=|\alpha|+1} T_{\alpha',\delta'} f(x\sigma_{k+1})$$

The assumption that $\delta_m = 0$ for $m \geq k$ ensures that each $\tau_{\delta'}$ is in U_ϵ and thus the constant $C_{U_\epsilon^{-1}}$ shows up in the application of the previous lemma. As ϵ is chosen smaller this constant is thus bounded.

Estimating the right local oscillation we first obtain

$$M_r^\epsilon f(x) = |f(x\sigma_1) - f(x)|$$

$$\leq \sum_{l=1}^{n} |f(x\sigma_l) - f(x\sigma_{l+1})|$$

$$= \sum_{l=1}^{n} |f(x\sigma_{l+1} e^{s_l X_l}) - f(x\sigma_{l+1})|$$

$$\leq \sum_{l=1}^{n} \int_{-\epsilon}^{\epsilon} |R(X_l) f(x\sigma_{l+1} e^{t_l X_l})| \, dt_l$$

This is a finite sum of terms of the type $T_{\alpha,\delta} f(x\sigma_k)$ with $|\alpha| = |\delta| = 1$ and $2 \leq k \leq n+1$. Each of the terms with $2 \leq k \leq n$ can in turn be estimated by a sum of terms $T_{\alpha,\delta} f(x\sigma_k)$ with $1 \leq |\alpha| = |\delta| \leq 2$ for $3 \leq k \leq n+1$. Repeating these steps we find

$$M_r^\epsilon f(x) \leq C_\epsilon \sum_{1 \leq |\alpha|=|\beta| \leq n} T_{\alpha,\beta} f(x)$$

where C_ϵ is a constant for which $C_{\epsilon'} \leq C_\epsilon$ when $\epsilon' \leq \epsilon$.

The inequality for the left local oscillation is obtained analogously. $\qquad \square$

The following local oscillation estimate will be of great importance to our sampling results.

Theorem 3.5. *If $f \in B$ is right differentiable up to order $n = \dim(G)$ and the derivatives $R^\alpha f$ are in B for $1 \leq |\alpha| \leq n$, then*

$$\|M_r^\epsilon f\|_B \leq C_\epsilon \sum_{1 \leq |\alpha| \leq n} \|R^\alpha f\|_B$$

Here $C_\epsilon \to 0$ as $\epsilon \to 0$.

Proof. As in the proof of the previous lemma let

$$T_{\alpha,\delta}f(x) = \underbrace{\int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon}}_{|\delta| \text{ integrals}} |R^\alpha f(x\tau_\delta(t_1,\ldots,t_n)^{-1})| \, (dt_1)^{\delta_1}(dt_2)^{\delta_2}\ldots(dt_n)^{\delta_n}$$

We now show that there is a constant C (only depending on U and B) such that

$$\|T_{\alpha,\delta}f\|_B \leq C(2\epsilon)^{|\delta|}\|R^\alpha f\|_B$$

For this we use the Minkowski inequality to get

$$\|T_{\alpha,\delta}f\|_B = \left\| \int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon} |r_{\tau_\delta(t_1,\ldots,t_n)^{-1}}R^\alpha f| \, (dt_1)^{\delta_1}(dt_2)^{\delta_2}\ldots(dt_n)^{\delta_n} \right\|_B$$

$$\leq \int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon} \|r_{\tau_\delta(t_1,\ldots,t_n)^{-1}}R^\alpha f\|_B \, (dt_1)^{\delta_1}(dt_2)^{\delta_2}\ldots(dt_n)^{\delta_n}$$

According to (5) let C be the smallest constant such that for all $f \in B$ and for all $u \in U_\epsilon$ we have $\|r_{u^{-1}}f\|_B \leq C\|f\|_B$. Then

$$\|T_{\alpha,\delta}f\|_B \leq \int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon} C\|R^\alpha f\|_B \, (dt_1)^{\delta_1}(dt_2)^{\delta_2}\ldots(dt_n)^{\delta_n} \leq C(2\epsilon)^{|\delta|}\|R^\alpha f\|_B$$

Since $M_r^\epsilon f$ can be estimated by a finite sum of terms of the type $T_{\alpha,\delta}f$ with $|\delta| \geq 1$ the triangle inequality can be used to finish the proof. $\square$

Corollary 3.6. *If the functions in B_ϕ are smooth and the mappings*

$$B_\phi \ni f \mapsto R^\alpha f \in B$$

are continuous for $|\alpha| \leq \dim(G)$, then there is a C_ϵ such that

$$\|M_r^\epsilon f\|_B \leq C_\epsilon\|f\|_B$$

with $C_\epsilon \to 0$ as $\epsilon \to 0$.

It is typically not hard to show that if $f \in B_\phi$ then

$$R^\alpha f = f * R^\alpha \phi$$

Thus we need only check that convolution with $R^\alpha \phi$ is continuous. In the following case we will use differentiability of the kernel to obtain estimates of $M_r^\epsilon f$ for $f \in B_\phi$. This ties our results with [12], though we do not require the kernel to be integrable. The previous results are more general and are in particular very easy to verify for band-limited functions, whereas the following theorem is harder to apply in that case (the author is at present not aware of an application of this theorem to band-limited functions).

Theorem 3.7. *If the mappings*

$$B_\phi \ni f \mapsto |f| * |R^\alpha \phi| \in B$$

are continuous for $|\alpha| \leq \dim(G)$ then there is a C_ϵ such that

$$\|M_r^\epsilon f\|_B \leq C_\epsilon\|f\|_B$$

with $C_\epsilon \to 0$ as $\epsilon \to 0$.

Proof. For $f \in B_\phi$ we have that

$$M_r^\epsilon f(x) = \sup_{u \in U_\epsilon} |f(xu^{-1}) - f(x)|$$

$$= \sup_{u \in U_\epsilon} \left| \int f(y)[\phi(y^{-1}xu^{-1}) - \phi(y^{-1}x)] \, dy \right|$$

$$\leq \int |f(y)| M_r^\epsilon \phi(y^{-1}x) \, dy$$

Since ϕ is differentiable we know that $M_r^\epsilon \phi(y^{-1}x)$ can be estimated by

$$M_r^\epsilon \phi(x) \leq C_\epsilon \sum_{1 \leq |\alpha| \leq n} \sum_{|\delta|=|\alpha|} \underbrace{\int_{-\epsilon}^\epsilon \cdots \int_{-\epsilon}^\epsilon}_{|\delta| \text{ integrals}} |R^\alpha \phi(x\tau_\delta(t_1, \ldots, t_n)^{-1})| (dt_1)^{\delta_1} \cdots (dt_n)^{\delta_n}$$

Thus the assumption that the convolutions $|f| * |R^\alpha \phi|$ are continuous from B_ϕ to B and the right invariance of B can be used to finish the proof. $\qquad\square$

3.4. Atomic decompositions and frames for reproducing kernel Banach spaces.

In this section we will derive sampling theorems and atomic decompositions for the reproducing kernel Banach space B_ϕ. The results are similar to those in [8, 9, 10, 12] and more recently [19] and [17], but unlike these references we do not require integrability of the reproducing kernel.

The following sampling theorem can be utilized together with the result of Corollary 3.6 and Theorem 3.7.

Theorem 3.8. *Assume there is a C_ϵ such that $C_\epsilon \to 0$ for $\epsilon \to 0$ such that for all $f \in B_\phi$ we have $\|M_r^\epsilon f\|_B \leq C_\epsilon \|f\|_B$. We can choose ϵ small enough that for every U_ϵ-relatively separated set $\{x_i\}$ the norms $\|\{f(x_i)\}\|_{B^\#}$ and $\|f\|_B$ are equivalent.*

Remark 3.9. We would like to note that Theorem 3.8 is sufficient to prove that sampling provides a Banach frame in the case $B = L^p(G)$ according to [15, Theorem 3.1]. Thus we are able to obtain sampling theorems for cases where the convolution with the kernel is not a continuous projection.

Proof. Choose ϵ small enough that $C_\epsilon < 1$ and let $\{x_i\}$ be U_ϵ-relatively separated. Note that there is an N such that each $x_i U_\epsilon$ overlap with at most N other $x_j U_\epsilon$. The following calculation shows that $\{f(x_i)\} \in B^\#$.

$$\sum_i |f(x_i)| 1_{x_i U_\epsilon}(x) \leq \sum_i |f(x) - f(x_i)| 1_{x_i U_\epsilon}(x) + \sum_i |f(x)| 1_{x_i U_\epsilon}(x)$$

$$\leq \sum_i M_r^\epsilon f(x) 1_{x_i U_\epsilon}(x) + \sum_i |f(x)| 1_{x_i U_\epsilon}(x)$$

$$\leq N(M_r^\epsilon f(x) + |f(x)|)$$

Both the functions $M_r^\epsilon f$ and $|f|$ are in B by assumption and thus

$$\|\{f(x_i)\}\|_{B^\#} = \left\| \sum_i |f(x_i)| 1_{x_i U_\epsilon} \right\|_B \leq N(C_\epsilon + 1)\|f\|_B$$

We now show that $(1-C_\epsilon)\|f\|_B \leq \|\{f(x_i)\}\|_{B\#}$. Let ψ_i be a U_ϵ-uniform bounded partition of unity, i.e. $\operatorname{supp}(\psi_i) \subseteq x_i U_\epsilon$ and $\sum_i \psi_i = 1$ a.e.

$$|f(x)| = \sum_i |f(x)|\psi_i(x)$$

$$\leq \sum_i |f(x) - f(x_i)|\psi_i(x) + \sum_i |f(x_i)|\psi_i(x)$$

$$\leq M_r^\epsilon f(x) + \sum_i |f(x_i)| 1_{x_i U_\epsilon}(x)$$

Therefore

$$\|f\|_B \leq \|M_r^\epsilon f\|_B + \|\{f(x_i)\}\|_B^\# \leq C_\epsilon \|f\|_B + \|\{f(x_i)\}\|_B^\#$$

By assumption $C_\epsilon < 1$ so we obtain

$$(1 - C_\epsilon)\|f\|_B \leq \|\{f(x_i)\}\|_B^\#$$

This finishes the proof. $\square$

The following theorem provides a reconstruction operator in the case where convolution with ϕ is continuous on B.

Theorem 3.10. *Assume there is a C_ϵ such that $C_\epsilon \to 0$ for $\epsilon \to 0$ such that for all $f \in B_\phi$ we have $\|M_r^\epsilon f\|_B \leq C_\epsilon \|f\|_B$. If convolution with ϕ is continuous on B, then we can choose ϵ small enough that for any U_ϵ-BUPU $\{\psi_i\}$ with $\operatorname{supp}(\psi_i) \subseteq x_i U_\epsilon$ the operator $T_1 : B_\phi \to B_\phi$ given by*

$$T_1 f = \sum_i f(x_i)(\psi_i * \phi)$$

is invertible. The convergence of the sum is pointwise, and if $C_c(G)$ is dense in B then the convergence is also in norm.

Proof. We have that

$$\left| f(x) - \sum_i f(x_i)\psi_i(x) \right| \leq \sum_i |f(x) - f(x_i)|\psi_i(x) \leq \sum_i M_r^\epsilon f(x)\psi_i(x) = M_r^\epsilon f(x)$$

so the solidity of B ensures that

$$\left\| f - \sum f(x_i)\psi_i \right\|_B \leq C_\epsilon \|f\|_B$$

The continuity of convolution with the reproducing kernel gives

$$\left\| f - \left(\sum f(x_i)\psi_i \right) * \phi \right\|_{B_\phi} \leq C_\epsilon \|f\|_{B_\phi}$$

Lastly we show that the operator

$$\left(\sum f(x_i)\psi_i \right) * \phi$$

is indeed the operator T. To do so we use the dominated convergence theorem to swap the sum and the integral in

$$\int \left(\sum_i f(x_i)\psi_i(x) \right) \phi(x^{-1}y)\, dx$$

The sum $\sum_i f(x_i)\psi_i(x)$ is to be understood as the pointwise limit of partial sums. Any partial sum $F_P(x) = \sum_{i \in P} f(x_i)\psi_i(x)$ is dominated by

$$\begin{aligned} |F_P(x)| &\leq \sum_{i \in P} |f(x_i)||\psi_i(x) \\ &\leq \sum_{i \in P} |f(x_i) - f(x)|\psi_i(x) + \sum_{i \in P} |f(x)|\psi_i(x) \\ &\leq \sum_{i \in P} M_r^\epsilon f(x)\psi_i(x) + \sum_{i \in P} |f(x)|\psi_i(x) \\ &\leq M_r^\epsilon f(x) + |f(x)| \end{aligned}$$

Both $M_r^\epsilon f$ and $|f|$ are in B and therefore (by our assumptions on ϕ) the integrable function $(M_r^\epsilon f(x) + |f(x)|)|\phi(x^{-1}y)|$ dominates $|F_N(x)\phi(x^{-1}y)|$. This allows the sum and integral to be swapped to get

$$\left\| f - \sum f(x_i)(\psi_i * \phi) \right\|_{B_\phi} \leq C_\epsilon \|f\|_{B_\phi}$$

Choosing ϵ small enough that $C_\epsilon < 1$ the operator T can be inverted using its Neumann series.

In [19] it has been shown that if the compactly supported continuous functions are dense, then the sum converges in norm. We therefore skip that part of the proof. $\square$

The previous result in conjunction with Corollary 3.6 and Theorem 3.7 only requires continuity involving left differentiation. We will now state results that also involve right differentation.

Lemma 3.11. *Assume the convolutions $f \mapsto f * |L^\alpha \phi|$ are continuous $B \to B$ for $|\alpha| \leq \dim(G)$, then the mapping*

$$B^\# \ni \{\lambda_i\} \mapsto \sum_i \lambda_i \ell_{x_i} \phi \in B_\phi$$

is continuous.

Proof. Let $\{\lambda_i\} \in B^\#$ and define

$$f(x) = \sum_i \lambda_i \ell_{x_i} \phi(x)$$

with pointwise convergence of the sum. We will show that this defines a function in B. For every x we have

$$|f(x)| \leq \sum_i |\lambda_i||\phi(x_i^{-1}x)|$$

$$= \mu(U)^{-1} \int \sum_i |\lambda_i| 1_{x_iU}(y)|\phi(x_i^{-1}x)|\, dy$$

$$\leq \mu(U)^{-1} \left(\int \sum_i |\lambda_i| 1_{x_iU}(y)|\phi(y^{-1}x) - \phi(x_i^{-1}x)|\, dy + \int \sum_i |\lambda_i| 1_{x_iU}(y)|\phi(y^{-1}x)|\, dy \right)$$

$$\leq \mu(U)^{-1} \left(\int \sum_i |\lambda_i| 1_{x_iU}(y) M_\ell^\epsilon \phi(y^{-1}x)\, dy + \int \sum_i |\lambda_i| 1_{x_iU}(y)|\phi(y^{-1}x)|\, dy \right)$$

$$\leq \mu(U)^{-1} \left(\sum_i |\lambda_i| 1_{x_iU} \right) * (M_\ell^\epsilon \phi + |\phi|)(x)$$

The function $F = \sum_i |\lambda_i| 1_{x_iU}(y)$ is in B and our assumptions ensure that the functions $F * |\phi|$ and $F * M_\ell^\epsilon \phi$ are also in B. The solidity of B thus ensures that the function f is in B.

We will now show that f is reproduced by convolution with ϕ. Note that any partial sum $f_N = \sum_{i=1}^N \lambda_i \ell_{x_i}\phi$ is in B and us reproduced by convolution by ϕ. We have to show that $f(x) = \lim_{N\to\infty} f_N(x)$ is also reproduced by convolution with ϕ. The calculation above shows that any partial sum f_N is dominated by the function

$$G = \mu(U)^{-1} \left(\sum_i |\lambda_i| 1_{x_iU} \right) * (M_\ell^\epsilon \phi + |\phi|) \in B$$

Thus we have $|f_N(y)\phi(y^{-1}x)| \leq |G(y)\phi(y^{-1}x)|$ and the dominated convergence theorem gives

$$f * \phi(x) = (\lim_{N\to\infty} f_N) * \phi(x) = \lim_{N\to\infty} (f_N * \phi)(x) = \lim_{N\to\infty} f_N(x) = f(x).$$

The continuity of the mapping follows from the calculations above. $\qquad\square$

Theorem 3.12. *Assume that the convolutions $f \mapsto |f| * |L^\alpha \phi|$ are continuous $B \to B$ for $|\alpha| \leq \dim(G)$. We can choose ϵ and U_ϵ-relatively separated points $\{x_i\}$, such that for any U_ϵ-BUPU $\{\psi_i\}$ with $\mathrm{supp}(\psi_i) \subseteq x_iU_\epsilon$ the operator (we let $\lambda_i(f) = \int f(x)\psi_i(x)\, dx$)*

$$T_2 f = \sum_i \lambda_i(f)\ell_{x_i}\phi$$

is invertible on B_ϕ. The convergence of the sum is pointwise, and if $C_c(G)$ is dense in B then the convergence is also in norm. Further $\{\lambda_i(T_2^{-1}f), \ell_{x_i}\phi\}$ is an atomic decomposition for B_ϕ.

Proof. For $f \in B_\phi$ we have the following estimate

$$\sum_i |\lambda_i(f)| 1_{x_i U}(y) \le \sum_i |\lambda_i| 1_{x_i U}(y)$$

$$\le \sum_i \left| \int f(x) \psi_i(x) \, dx \right| 1_{x_i U}(y)$$

$$\le \sum_i \int |f(x)| 1_{x_i U}(x) \, dx 1_{x_i U}(y)$$

$$= \sum_i \int |f(x)| 1_{x_i U}(x) \, dx 1_{x_i U}(y)$$

For each y only N of the neighbourhoods $x_i U$ overlap, and also $1_{x_i U}(x) 1_{x_i U}(y) \le 1_{U^{-1}U}(x^{-1}y)$ so we get

$$\sum_i |\lambda_i| 1_{x_i U}(y) \le N \int |f(x)| 1_{U^{-1}U}(x^{-1}y) \, dx = N|f| * 1_{U^{-1}U}(y)$$

The function $|f| * 1_{U^{-1}U}$ is in B with norm bounded by $C\|f\|_B$ for some constant C (in the sense of Bochner integrals). Therefore the sequence $\lambda_i(f)$ is in $B^\#$ with norm estimated by

$$\|\{\lambda_i(f)\}\|_{B^\#} \le CN\|f\|_B$$

By Lemma 3.11 we see that $T_2 f \in B_\phi$ and

$$|f(x) - T_2 f(x)| = \left| f(x) - \sum_i \int f(y) \psi_i(y) \, dy \phi(x_i^{-1} x) \right|$$

$$= \left| \int f(y) \phi(y^{-1} x) \, dy - \sum_i \int f(y) \psi_i(y) \, dy \phi(x_i^{-1} x) \right|$$

$$= \left| \int \sum_i \psi_i(y) f(y) \phi(y^{-1} x) \, dy - \sum_i \int \psi_i(y) f(y) \psi_i(y) \, dy \phi(x_i^{-1} x) \right|$$

$$\le \sum_i \left| \int \psi_i(y) f(y) \phi(y^{-1} x) \, dy - \int \psi_i(y) f(y) \psi_i(y) \, dy \phi(x_i^{-1} x) \right|$$

$$= \sum_i \left| \int \psi_i(y) f(y) (\phi(y^{-1} x) - \phi(x_i^{-1} x)) \, dy \right|$$

$$\le \sum_i \int \psi_i(y) |f(y)| M_l^\epsilon \phi(y^{-1} x) \, dy$$

$$= \int |f(y)| M_l^\epsilon \phi(y^{-1} x) \, dy$$

$$= |f| * M_l^\epsilon \phi(x)$$

The continuity of the mappings

$$B \ni f \mapsto f * |L^\alpha \phi| \in B$$

and Lemma 3.4 ensure that $T_2 f \in B$ and T_2 is well-defined.

We will now show that $\{\ell_{x_i} \phi\}$ yields an atomic decomposition.

Since $T_2^{-1}f \in B_\phi$ we therefore have

$$\|\{\lambda_i(T_2^{-1}f)\}\|_{B\#} \leq C\|T_2^{-1}f\|_B \leq C\|f\|_B$$

Further the reconstruction formula $f = \sum_i \lambda(T_2^{-1}f)\ell_{x_i}\phi$ Lemma 3.11 tells us that

$$\|f\|_B \leq C\|\{\lambda_i(T_2^{-1}f)\}\|_{B\#}$$

thus showing that we have an atomic decomposition. $\qquad\square$

Theorem 3.13. *Assume the convolutions $f \mapsto f * |L^\alpha\phi|$ and $f \mapsto f * |R^\alpha\phi|$ are continuous $B \to B$ for $|\alpha| \leq \dim(G)$. Then we can choose ϵ and U_ϵ-relatively separated points $\{x_i\}$ such that for any U_ϵ-BUPU $\{\psi_i\}$ with $\operatorname{supp}(\psi_i) \subseteq x_iU_\epsilon$ the operator given by (we let $c_i = \int \psi_i(x)\, dx$)*

$$T_3f = \sum_i c_i f(x_i)\ell_{x_i}\phi$$

is invertible. The convergence of the sum is pointwise, and if $C_c(G)$ is dense in B then the convergence is also in norm. Further $\{c_iT_3^{-1}f(x_i), \ell_{x_i}\phi\}$ is an atomic decomposition for B_ϕ and $\{c_i\ell_{x_i}\phi\}$ is a frame.

Proof. Since

$$c_i = \int \psi_i(x)\, dx \leq \int 1_{x_iU}(x)\, dx = \mu(U)$$

we have

$$\left|\sum_i c_i f(x_i)1_{x_iU}(x)\right| \leq \mu(U)\sum_i [|f(x_i) - f(x)| + |f(x)|]1_{x_iU}(x)$$

$$\leq \mu(U)\sum_i [M_r^\epsilon f(x) + |f(x)|]1_{x_iU}(x)$$

$$\leq \mu(U)N[M_r^\epsilon f(x) + |f(x)|]$$

Therefore Theorem 3.7 gives

$$\|\{c_i f(x_i)\}\|_{B\#} \leq \mu(U)N(C_\epsilon + 1)\|f\|_B$$

Thus $T_3f \in B_\phi$ by Lemma 3.11 and

$$|f(x) - T_3f(x)| \leq \left|f(x) - \sum_i \left(\int f(y)\psi_i(y)\, dy\right)\ell_{x_i}\phi(x)\right|$$

$$+ \left|\sum_i \left(\int f(y)\psi_i(y)\, dy\right)\ell_{x_i}\phi(x) - \sum_i c_i f(x_i)\ell_{x_i}\phi(x)\right|$$

The first expression was estimated in the previous theorem, so we concentrate on

$$\left| \sum_i \left(\int f(y)\psi_i(y)\,dy \right) \ell_{x_i}\phi - \sum_i c_i f(x_i)\ell_{x_i}\phi \right|$$

$$\leq \sum_i \left(\int |f(y) - f(x_i)||\phi(x_i^{-1}x)|\psi_i(y)\,dy \right)$$

$$\leq \sum_i \int |f(y) - f(x_i)||\phi(y^{-1}x)|\psi_i(y)\,dy$$

$$+ \sum_i \int |f(y) - f(x_i)||\phi(y^{-1}x) - \phi(x_i^{-1}x)|\psi_i(y)\,dy$$

$$\leq \sum_i \left(\int Mf(y)|\phi(y^{-1}x)|\psi_i(y)\,dy \right)$$

$$+ \sum_i \int Mf(y)M_r^\epsilon \phi(y^{-1}x)\psi_i(y)\,dy$$

$$= M_l^\epsilon f * |\phi|(x) + M_l^\epsilon f * M_r^\epsilon \phi(x)$$

Now, by our assumptions the functions in the last expression are all in B, and their norms are dominated by the norms of convolution with of f with $|L^\alpha \phi|$ and $|R^\alpha \phi|$ for $|\alpha| \leq \text{Dim}(G)$. Therefore

$$\|f - T_3 f\|_B \leq C_\epsilon \|f\|_B$$

where $C_\epsilon \to 0$ as $\epsilon \to 0$.

Since $T_3^{-1} f \in B_\phi$ we thus have

$$\|\{c_i T_3^{-1} f(x_i)\}\|_{B\#} \leq C \|T_3^{-1} f\|_B \leq C \|f\|_B$$

Any $f \in B_\phi$ can be written

$$f = \sum_i c_i T_3^{-1} f(x_i)\ell_{x_i}\phi$$

and thus by Lemma 3.11 the norm of f satisfies

$$\|f\|_B \leq C \|\{c_i T_3^{-1} f(x_i)\}\|_{B\#}$$

which finishes the proof that $\{c_i T_3^{-1} f(x_i), \ell_{x_i}\phi\}$ forms an atomic decomposition of B_ϕ.   □

4. Coorbit Spaces on Lie groups

In this section we apply the sampling theorems from section 3 to a certain class of reproducing spaces. These spaces are images of Banach spaces of distributions (so-called coorbit spaces) under a wavelet transform. Thus we yield sampling theorems for a large class of Banach spaces including modulation spaces, Besov spaces, Bergman spaces and Hilbert spaces of band-limited functions. Similar sampling theorems are known for spaces related to irreducible and integrable representations [9, 10, 12]. We replace the integrability condition with smoothness arguments which also apply to non-integrable and non-irreducible cases.

4.1. **Construction of coorbit spaces.** Let S be a Fréchet space and let S^* be the conjugate linear dual equipped with the weak* topology. We assume that S is continuously imbedded and weakly dense in S^*. The conjugate dual pairing of elements $v \in S$ and $v' \in S^*$ will be denoted by $\langle v', v \rangle$. As usual define the contragradient representation (π^*, S^*) by

$$\langle \pi^*(x)v', v \rangle = \langle v', \pi(x^{-1})v \rangle.$$

Then π^* is a continuous representation of G on S^*. For a fixed vector $u \in S$ define the linear map $W_u : S^* \to C(G)$ by

$$W_u(v')(x) = \langle v', \pi(x)u \rangle.$$

The map W_u is called *the voice transform* or *the wavelet transform*.

In [4] we listed minimal conditions ensuring that spaces of the form

$$\mathrm{Co}_S^u B = \{ v' \in S^* | W_u(v') \in B \}$$

equipped with the norm $\|v'\| = \|W_u(v')\|_B$ are π^* invariant Banach spaces. The space $\mathrm{Co}_S^u B$ is called the coorbit space of B related to u and S.

Assumption 4.1. *Assume there is a non-zero cyclic vector $u \in S$ satisfying the following properties*

(R1) *the reproducing formula $W_u(v) * W_u(u) = W_u(v)$ is true for all $v \in S$*
(R2) *the mapping $Y \ni F \mapsto \int_G F(x) W_u(u)(x^{-1})\, dx \in \mathbb{C}$ is continuous*
(R3) *if $F = F * W_u(u) \in Y$ then the mapping $S \ni v \mapsto \int F(x)\langle \pi^*(x)u, v\rangle\, dx \in \mathbb{C}$ is in S^**
(R4) *the mapping $S^* \ni \phi \mapsto \int \langle \phi, \pi(x)u\rangle \langle \pi^*(x)u, u\rangle\, dx \in \mathbb{C}$ is weakly continuous*

A vector u satisfying Assumption 4.1 is called an *analyzing vector*.
The subspace B_u of B defined by

$$B_u = \{ F \in B | F = F * W_u(u) \},$$

is a reproducing kernel Banach space. By [4] it follows that $\mathrm{Co}_S^u B$ is $W_u : \mathrm{Co}_S^u B \to B_u$ is an isometric isomorphism intertwining π^* and left translation.

4.2. **Sampling of wavelet transform.** We now list conditions ensuring that we can obtain the frame inequality from Theorem 3.8. A vector $u \in S$ is called weakly differentiable in the direction $X \in \mathfrak{g}$ if there is a vector denoted $\pi(X)u \in S$ such that for all $v' \in S^*$

$$\langle v', \pi(X)u \rangle = \frac{d}{dt}\Big|_{t=0} \langle v', \pi(e^{tX})u \rangle$$

For the differential operators R^α we write $\pi(R^\alpha)u$ for a vector which satisfies

$$\langle v', \pi(R^\alpha)u \rangle = \langle v', \pi(X_{\alpha(k)})\pi(X_{\alpha(k-1)}) \cdots \pi(X_{\alpha(1)})u \rangle$$

We use the notation $\pi(R^\alpha)$ for the differential operators on S because they match the right differential operators R^α on B: if $f(x) = W_u(v)(x)$, then $R^\alpha f(x) = W_{\pi(R^\alpha)u}(v)(x)$.

Assumption 4.2. *Assume there is a non-zero cyclic vector $u \in S$ satisfying Assumption 4.1. Further assume that u is weakly differentiable up to order $\dim(G)$ and that*

(D1) *there are non-zero constants c_α such that $W_u(v) * W_{\pi(R^\alpha)u}(u) = c_\alpha W_{\pi(R^\alpha)u}(v)$ for all $v \in S$*
(D2) *the mapping $S^* \ni \phi \mapsto \int \langle \phi, \pi(x)u\rangle \langle \pi^*(x)u, \pi(R^\alpha)u\rangle\, dx \in \mathbb{C}$ is weakly continuous*
(D3) *the mappings $B_u \ni F \mapsto F * W_{\pi(R^\alpha)u}(u) \in B$ are continuous for all $|\alpha| \le \dim(G)$*

Remark 4.3. Notice that for $\alpha = 0$ the properties (D1) and (D2) correspond to (R1) and (R4) respectively. The condition (D2) is used to extend the convolution relation from (D1) to all $v \in S^*$.

Theorem 4.4. *If $u \in S$ satisfies Assumption 4.2 then we can choose ϵ small enough that for any U_ϵ-relatively separated set $\{x_i\}$ there are $0 < A_1 \leq A_2 < \infty$ such that*

$$A_1\|v'\|_{\mathrm{Co}_S^u B} \leq \|\{\langle v', \pi(x_i)u\rangle\}\|_{B^\#} \leq A_2\|v'\|_{\mathrm{Co}_S^u B}$$

If convolution with $W_u(u)$ is continuous on B, then $\pi(x_i)u$ is a frame for $\mathrm{Co}_S^u B$ with reconstruction operator

$$v' = W_u^{-1}T_1^{-1}\left(\sum_i W_u(v')(x_i)\psi_i * W_u(u)\right)$$

where $\{\psi_i\}$ is any U_ϵ-BUPU for which $\mathrm{supp}(\psi_i) \subseteq x_i U_\epsilon$.

Proof. Let us first show that (D1) and (D2) ensure that

$$W_u(v') * W_{\pi(R^\alpha)u}(u) = c_\alpha W_{\pi(R^\alpha)u}(v')$$

for $v' \in S^*$. Let v_β be a net in S converging to v'. Then

$$
\begin{aligned}
c_\alpha W_{\pi(R^\alpha)u}(v')(x) &= \lim_\beta c_\alpha W_{\pi(R^\alpha)u}(v_\beta)(x) \\
&= \lim_\beta W_u(v_\beta) * W_{\pi(R^\alpha)u}(u)(x) \\
&= \lim_\beta \int \langle v_\beta, \pi(xy)u\rangle\langle \pi^*(y)u, \pi(R^\alpha)u\rangle \, dy \\
&= \lim_\beta \int \langle \pi^*(x^{-1})v_\beta, \pi(y)u\rangle\langle \pi^*(y)u, \pi(R^\alpha)u\rangle \, dy \\
&= \int \langle \pi^*(x^{-1})v', \pi(y)u\rangle\langle \pi^*(y)u, \pi(R^\alpha)u\rangle \, dy \\
&= \int \langle v', \pi(y)u\rangle\langle u, \pi(y^{-1}x)\pi(R^\alpha)u\rangle \, dy \\
&= W_u(v') * W_{\pi(R^\alpha)u}(u)(x)
\end{aligned}
$$

Therefore, if $v' \in \mathrm{Co}_S^u B$ we have

$$W_{\pi(R^\alpha)u}(v') = \frac{1}{c_\alpha}W_u(v') * W_{\pi(R^\alpha)u}(u)$$

and the continuity requirement (D3) ensures that $W_{\pi(R^\alpha)u}(v') \in B$ and

$$\|W_{\pi(R^\alpha)u}(v')\|_B \leq C_\alpha\|W_u(v')\|_B$$

By Theorem 3.5 there is a constant C_ϵ such that

$$\|M_r^\epsilon W_u(v')\|_B \leq C_\epsilon\|W_u(v')\|_B$$

and $C_\epsilon \to 0$ as $\epsilon \to 0$. Theorem 3.8 shows that there are A_1, A_2 such that

$$A_1\|v'\|_{\mathrm{Co}_S^u B} \leq \|\{\langle v', \pi(x_i)u\rangle\}\|_{B^\#} \leq A_2\|v'\|_{\mathrm{Co}_S^u B}$$

which proves the norm equivalence. If convolution with $W_u(u)$ is continuous on B the reconstruction operator can be found using Theorem 3.10. □

Remark 4.5. For $B = L^p(G)$ the sequence space is $B^\# = \ell^p$ and in this case a reconstruction operator is automatic when the frame inequality is given (see [15]).

4.3. Gårding vectors and smooth representations.

In this section we will focus on square integrable group representations and its smooth vectors. In particular we will show that Gårding vectors are particularly nice to work with.

A unitary irreducible representation (π, H) is square integrable if there is a non-zero $u \in H$ such that the function $W_u(u)(x) = (u, \pi(x)u)$ is in $L^2(G)$. Any such vector u is called admissible. Duflo and Moore [6] proved the following

Theorem 4.6 (Duflo-Moore). *If $(\pi, \mathcal{H})$ is square integrable, then there is a positive densely defined operator C with domain $D(C)$ such that $W_u(u)$ is in $L^2(G)$ if and only if $u \in D(C)$. Furthermore, if $u_1, u_2 \in D(C)$ then*

$$\int_G (v_1, \pi(x)u_1)_H (\pi(x)u_2, v_2)_H \, dx = (Cu_2, Cu_1)_H (v_1, v_2)_H$$

By choosing u such that $\|Cu\|_H = 1$ we automatically obtain a reproducing formula

$$W_u(v) * W_u(u) = W_u(v)$$

for all $v \in H$.

A vector $v \in H$ is called smooth if the mapping

$$G \in x \mapsto \pi(x)v \in H$$

is smooth in the norm topology of H. The space of smooth vector is denoted H_π^∞ and is a Fréchet space when equipped with the seminorms

$$\|v\|_k = \sup_{|\alpha| \leq k} \|\pi(R^\alpha)v\|_H$$

For any $v \in H$ and any $f \in C_c^\infty(G)$ the vector $\pi(f)v$ defined by

$$\pi(f)v = \int f(x)\pi(x)v \, dx$$

is smooth and called a Gårding vector.

The following statement is an extension of a result found in [3] without proof.

Lemma 4.7. *If $u \in H_\pi^\infty$ is in the domain of the operator C from Theorem 4.6, then the map*

$$H_\pi^{-\infty} \ni \phi \mapsto \int \langle \phi, \pi(x)u \rangle \langle \pi(x)u, v \rangle \, dx \in \mathbb{C}$$

is continuous in the weak topology for $v \in H_\pi^\infty$. Thus both (R4) and (D2) are satisfied.

Proof. For vectors v in H_π^∞ and $w \in H$ the dual pairing $\langle w, v \rangle$ is the inner product (w, v) on H. For $v \in H_\pi^\infty$ we have

$$H \in w \mapsto \int (w, \pi(x)u)(\pi(x)u, v) \, dx = C_u(w, v)$$

and therefore the weakly defined vector

$$\pi(W_u(v)^\vee)u = \int (\pi(x)u, v)\pi(x)u \, dx = C_u v \in H_\pi^\infty$$

This proves the statement of the lemma. $\qquad\square$

Theorem 4.8. *Let (π, H) be a square integrable representation with smooth vectors $S = H_\pi^\infty$. Let B be a left and right invariant Banach function space and let $u \in S$ be such that Assumption 4.1 is satisfied and further the mapping*

$$B \ni F \mapsto F * W_u(u) \in B$$

is continuous. Then $\mathrm{Co}_S^u = \mathrm{Co}_S^{\widetilde{u}}$ for any (properly normalized) Gårding vector $\widetilde{u}$ and the vectors $\pi(x_i)\widetilde{u}$ form a Banach frame for both $\mathrm{Co}_S^{\widetilde{u}}$ and Co_S^u. Further $\pi(x_i)\widetilde{u}$ provide atomic decompositions for Co_S^u through Theorem 3.12 and Theorem 3.13.

Proof. Note, that if $f, g \in C_c^\infty(G)$ then admissible u and $v \in H$ we have

$$W_{\pi(f)u}(\pi(g)v) = g * W_u(u) * (f^\vee)$$

Since $g * L^2(G) * g^\vee \subseteq L^2(G)$ we thus see that any non-zero Gårding vector $\widetilde{u} = \pi(g)u$ is also admissible. Therefore we can normalize $\pi(g)u$ such that the reproducing formula

$$W_{\widetilde{u}}(v) * W_{\widetilde{u}}(\widetilde{u}) = W_{\widetilde{u}}(v)$$

is true. From now on let $\widetilde{u}$ be normalized accordingly. Further

$$\int_G F(x)W_{\pi(g)u}(v)(x^{-1})\, dx = \int_G F(x)(v, \pi(x^{-1})\pi(g)u)\, dx$$

$$= \int_G \int_G F(x)(v, \pi(x^{-1})\pi(y)u)\overline{g(y)}\, dy\, dx$$

$$= \int_G \int_G F(yx)(v, \pi(x^{-1})u)\overline{g(y)}\, dy\, dx$$

$$= \int_G \int_G F(y^{-1}x)(v, \pi(x^{-1})u)\overline{g^\vee(y)}\, dy\, dx$$

$$= \int_G g^* * F(x)W_u(v)(x)\, dx$$

where $g^*(x) = \overline{g(y^{-1})}$. Since $g^* * F \in Y$ and depends continuously on F (in the sense of Bochner integrals) the mapping

$$(F, v) \mapsto \int_G F(x)W_{\pi(g)u}(v)(x^{-1})\, dx$$

is continuous. This shows that $\mathrm{Co}_S^{\pi(g)u}B$ is a well-defined π^*-invariant Banach space.

We now show that the norms on $\mathrm{Co}_S^u B$ and $\mathrm{Co}_S^{\pi(g)u}B$ are equivalent. By the square integrability it follows that for any $v \in H$

$$W_u(v) * W_{\widetilde{u}}(u) = C_u W_{\widetilde{u}}(v)$$

and

$$W_{\widetilde{u}}(v) * W_u(\widetilde{u}) = C_{\widetilde{u}} W_u(v)$$

These two formulas can be extended to all $v \in S^* = H_\pi^\infty$ by Lemma 4.7. Since

$$F \mapsto F * W_u(\pi(g)u) = F * g * W_u(u)$$

$$F \mapsto F * W_{\widetilde{u}}(u) = F * W_u(u) * (g^\vee)$$

are both continuous mappings it follows that

$$\|W_u(v)\|_B = C\|W_{\widetilde{u}}(v) * W_u(\widetilde{u})\|_B \le C\|W_{\widetilde{u}}(v)\|_B$$

and

$$\|W_{\widetilde{u}}(v)\|_B = C\|W_u(v) * (g^\vee)\|_B \le C\|W_u(v)\|_B$$

Thus the norms on $\mathrm{Co}_S^u B$ and $\mathrm{Co}_S^{\pi(g)u} B$ are equivalent.

Finally we need to show that we can reconstruct $\phi \in \mathrm{Co}_S^{\pi(g)u} B$ from it samples. For this it suffices to show that

$$\|R^\alpha W_{\widetilde{u}}(\phi)\|_B \le C\|W_{\widetilde{u}}(\phi)\|_B$$

and apply Theorem 3.5 and for example Theorem 3.10. Note, that

$$R^\alpha W_{\widetilde{u}}(\phi) = W_u(\phi) * (R^\alpha g)^\vee$$

By the continuity of convolution with $R^\alpha g$ we thus see that $\|R^\alpha W_{\widetilde{u}}(\phi)\|_B \le C\|W_u(\phi)\|_B$ and the previously proven norm equivalence gives

$$\|R^\alpha W_{\widetilde{u}}(\phi)\|_B \le C\|W_{\widetilde{u}}(\phi)\|_B$$

to finish the proof that $\pi(x_i)\widetilde{u}$ is a frame. The statements about atomic decompositions follow similarly. $\qquad\square$

Remark 4.9. Note that we need not necessarily work with the smooth vectors H_π^∞. In the coorbit theory introduced by Feicthinger and Gröchenig [9] the space

$$S = H_w^1 = \{v \in H \mid W_u(v) \in L_w^1\}$$

is used in the construction of coorbit. Here w is a submultiplicative weight. In order to obtain sampling theorems they need to choose the analyzing vector in the space

$$B_w = \{u \in H \mid W_u(u), M_\epsilon^r W_u(u) \in L_w^1\}$$

For any u with $W_u(u) \in H_w^1$ it follows from our calculations that any Gårding vector $\pi(g)u$ is in B_w. Thus it is natural to use Gårding vectors in the discretization machinery of Feichtinger and Gröchenig.

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