# ON THE THIRD COHOMOLOGY GROUP OF COMMUTATIVE MONOIDS

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ABSTRACT. We interpret Grillet's symmetric third cohomology classes of commutative monoids in terms of strictly symmetric monoidal abelian groupoids. We state and prove a classification result that generalizes the well-known one for strictly commutative Picard categories by Deligne, Fröhlich and Wall, and Sinh.

## 1. INTRODUCTION AND SUMMARY

The category of commutative monoids is tripleable (monadic) over the category of sets [23], and so it is natural to specialize Barr-Beck cotriple cohomology [1] to define a cohomology theory for commutative monoids. This was done in the 1990s by Grillet, to whose papers [12, 13, 14] and book [15] we refer the readers interested in cohomology theory for commutative monoids (the topic of this paper). Although in Subsection 2.1 we review the basic facts about the resulting Grillet's cohomology, let us briefly recall here that, for each commutative monoid $M$, its cohomology groups in this theory, $H^n(M,\mathcal{A})$, take coefficients in abelian group valued functors $\mathcal{A}$ on the category $\mathbb{H}M$, whose objects are the elements of $M$ and whose morphisms are pairs $(a,b): a \to ab$, $a,b\in M$. Since these cohomology groups $H^n(M,\mathcal{A})$ can be computed, at less in low dimensions, by means of *symmetric cochains*, they are usually referred as the *symmetric cohomology groups* of the commutative monoid $M$.

For an arbitrary monoid $M$, that is, non necessarily commutative, and $\mathcal{A}:\mathbb{D}M\to\mathbf{Ab}$ any abelian group valued functor on the Leech category $\mathbb{D}M$ of morphisms $(a,b,c): b\to abc$, there are defined Leech's cohomology groups $H^n_{\mathrm{L}}(M,\mathcal{A})$ [20]. When the monoid $M$ is commutative, and $\mathcal{A}:\mathbb{H}M\to\mathbf{Ab}$ is any functor, then both cohomology groups $H^n(M,\mathcal{A})$ and $H^n_{\mathrm{L}}(M,\mathcal{A})$ are defined, where the coefficients for the Leech cohomology are here obtained by composing $\mathcal{A}$ with canonical functor $\mathbb{D}M\to\mathbb{H}M$, $(a,b,c)\mapsto(b,ac)$. Although in dimension one we have that $H^1(M,\mathcal{A})=H^1_{\mathrm{L}}(M,\mathcal{A})$, in higher dimensions the cohomology groups $H^n(M,\mathcal{A})$ and $H^n_{\mathrm{L}}(M,\mathcal{A})$ are, however, different. Indeed, one easily argues that Leech cohomology groups do not take properly account of the commutativity of the monoid, in contrast to what happens with Grillet ones. Thus, for example, while $H^2_{\mathrm{L}}(M,\mathcal{A})$ classifies *all* group coextensions of $M$ by $\mathcal{A}$ [20, §2.4.9], [27, Theorem 2], the symmetric two-dimensional cohomology group $H^2(M,\mathcal{A})$ classifies *commutative* group coextensions of $M$ by $\mathcal{A}$ [15, §V.4].

In [5], we gave a natural interpretation for Leech 3-dimensional cohomology classes in terms of *monoidal abelian groupoids*, that is, of small categories $\mathcal{M}$ all whose arrows are invertible and whose isotropy groups $\mathrm{Aut}_{\mathcal{M}}(x)$ are all abelian, endowed with a monoidal

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structure by a tensor product $\otimes : \mathcal{M}\times\mathcal{M}\to\mathcal{M}$, a unit object I, and coherent and natural associativity and unit constraints $(x\otimes y)\otimes z\cong x\otimes(y\otimes z)$, $x\otimes \mathrm{I}\cong x$, and $\mathrm{I}\otimes x\cong x$ [22]. Thus, in [5, Theorem 4.2], it is stated that monoidal equivalence classes of monoidal abelian groupoids are in one-to-one correspondence with isomorphism classes of triples $(M,\mathcal{A},k)$, consisting of a (non necessarily commutative) monoid $M$, an abelian group valued functor $\mathcal{A}$ on the Leech category $\mathbb{D}M$, and a Leech 3-cohomology class $k\in H^3_{\mathrm{L}}(M,\mathcal{A})$.

In this paper, our goal is to state and prove a similar interpretation for Grillet symmetric 3-cohomology classes, now in terms of *strictly symmetric* (or *strictly commutative*) monoidal abelian groupoids [9, 22, 24], that is, monoidal abelian groupoids, as above, but now endowed with coherent and natural isomorphisms $\boldsymbol{c}_{x,y}: x\otimes y\cong y\otimes x$, satisfying the symmetry and strictness conditions $\boldsymbol{c}_{y,x}\,\boldsymbol{c}_{x,y}=id_{x\otimes y}$ and $\boldsymbol{c}_{x,x}=id_{x\otimes x}$. Our result here can be summarized as follows (see Theorem 8.1):

- *Each symmetric 3-cocycle $h\in Z^3(M,\mathcal{A})$, of a commutative monoid $M$ with coefficients in an abelian group valued functor $\mathcal{A}:\mathbb{H}M\to\mathbf{Ab}$, gives rise to a strictly symmetric monoidal abelian groupoid*
$$\mathcal{S}(M,\mathcal{A},h).$$
- *For any strictly symmetric monoidal abelian groupoid $\mathcal{M}$, there exist a commutative monoid $M$, a functor $\mathcal{A}:\mathbb{H}M\to\mathbf{Ab}$, a symmetric 3-cocycle $h\in Z^3(M,\mathcal{A})$, and a symmetric monoidal equivalence*
$$\mathcal{S}(M,\mathcal{A},h)\simeq\mathcal{M}.$$
- *For any two symmetric 3-cocycles $h\in Z^3(M,\mathcal{A})$ and $h'\in Z^3(M',\mathcal{A}')$, there is a symmetric monoidal equivalence*
$$\mathcal{S}(M,\mathcal{A},h)\simeq\mathcal{S}(M',\mathcal{A}',h')$$
*if and and only if there exist an isomorphism of monoids $i:M\cong M'$ and a natural isomorphism $\psi:\mathcal{A}\cong\mathcal{A}'i$, such that the equality of cohomology classes below holds.*
$$[h]=\psi_*^{-1}i^*[h']\in H^3(M,\mathcal{A})$$

Thus, triples $(M,\mathcal{A},k)$, with $M$ a commutative monoid, $\mathcal{A}:\mathbb{H}M\to\mathbf{Ab}$ a functor, and $k\in H^3(M,\mathcal{A})$ a symmetric 3-cohomology class, provide complete invariants for the classification of strictly symmetric monoidal abelian groupoids, where two of them connected by a symmetric monoidal equivalence are considered the same.

Our result particularizes to strictly commutative Picard categories by giving, as a corollary, Deligne's well-known classification for them [9], also proved independently by Fröhlich and Wall in [10] and by Sinh in [25, 26]. Indeed, in the very special case where $M=G$ is an abelian group, any abelian group valued functor on $\mathbb{H}G$ is naturally equivalent to the constant functor given by an abelian group $A$, and the symmetric 3-dimensional cohomology group $H^3(G,A)$ vanishes, whence Deligne's result follows: *Strictly commutative Picard categories are classified by pairs $(G,A)$ of abelian groups.*

The organization of the paper is simple. After this introduction, it contains two sections. The first is dedicated to stating a minimum of necessary concepts and terminology, by reviewing some facts concerning Grillet cohomology of commutative monoids (Subsection 2.1) and symmetric monoidal groupoids (Subsection 2.2). The second section comprises our classification theorem for strictly symmetric monoidal abelian groupoids by means of symmetric 3-cohomology classes.

## 2. Preliminaries

This section aims to make this paper as self-contained as possible; hence, at the same time as fixing notations and terminology, we also review some necessary aspects and results about cohomology of commutative monoids and symmetric monoidal categories that will be used throughout the paper. However, the material in this preliminary section is perfectly standard by now, so the expert reader may skip most of it. For the cohomology theory of commutative monoids we mainly refer the reader to Grillet [15, Chapters V, XII, XIII, and XIV], and for symmetric monoidal (= tensor) categories to Mac Lane [22, 23] and Saavedra [24].

2.1. **Grillet cohomology of commutative monoids: Symmetric cocycles.** Like most of cohomology theories in Algebra, the cohomology of commutative monoids is a particular instance of the cotriple cohomology by Barr and Beck [1]. Briefly, let us recall that the category of commutative monoids is tripleable over the category of sets and, for any given commutative monoid $M$, the resulting cotriple $(\mathbb{G},\varepsilon,\delta)$ in the comma category $\mathbf{CMon}{\downarrow_M}$, of commutative monoids over $M$, is as follows. For each commutative monoid $X\overset{p}{\to}M$ over $M$,

$$\mathbb{G}(X\overset{p}{\to}M)=\mathbb{N}[X]\overset{\overline{p}}{\to}M,$$

where $\mathbb{N}[X]$ is the free commutative monoid on the underlying set $X$, and $\overline{p}$ is the homomorphism such that $\overline{p}[x]=p(x)$ for any $x\in X$. The counit $\delta:\mathbb{G}\to id$ sends $X\to M$ to the homomorphism in the comma category $\delta:\mathbb{N}[X]\to X$ such that $\delta[x]=x$, and the comultiplication $\varepsilon:\mathbb{G}\to\mathbb{G}^2$ carries each $X\to M$ to the homomorphism $\mathbb{N}[X]\to\mathbb{N}[\mathbb{N}[X]]$ such that $\varepsilon[x]=[[x]]$, for $x\in X$. This cotriple produces a simplicial object $\mathbb{G}_\bullet$ in the category of endofunctors on $\mathbf{CMon}{\downarrow_M}$, which is defined by $\mathbb{G}_n=\mathbb{G}^{n+1}$, with face and degeneracy operators $d_i=\mathbb{G}^{n-i}\delta\mathbb{G}^i:\mathbb{G}_n\to\mathbb{G}_{n-1}$ and $s_i=\mathbb{G}^{n-i}\varepsilon\mathbb{G}^i:\mathbb{G}_n\to\mathbb{G}_{n+1}$, $0\leq i\leq n$. Then, for any abelian group object $\mathbf{A}$ in $\mathbf{CMon}{\downarrow_M}$, one obtains a cosimplicial abelian group $\mathrm{Hom}(\mathbb{G}_\bullet(1_M),\mathbf{A})$, whose associated cochain complex obtained by taking alternating sums of the coface operators

$$0\to\mathrm{Hom}(\mathbb{G}(1_M),\mathbf{A})\overset{\partial^0}{\to}\mathrm{Hom}(\mathbb{G}^2(1_M),\mathbf{A})\overset{\partial^1}{\to}\mathrm{Hom}(\mathbb{G}^3(1_M),\mathbf{A})\overset{\partial^2}{\to}\cdots\qquad\Big(\partial^n=\sum_{i=0}^{n+1}d_i^*\Big)$$

provides the *cotriple cohomology groups of the commutative monoid $M$ with coefficients in* $\mathbf{A}$ by

$$H^n_{\mathbb{G}}(M,\mathbf{A})=H^n(\mathrm{Hom}(\mathbb{G}_\bullet(1_M),\mathbf{A})).$$

In [12], Grillet observes that, for any given commutative monoid $M$, the category of abelian group objects in $\mathbf{CMon}\downarrow_M$, is equivalent to the category of abelian group valued functors

$$\mathcal{A}:\mathbb{H}M\to\mathbf{Ab},$$

where $\mathbb{H}M$ is the category with object set $M$ and arrow set $M\times M$, where $(a,b):a\to ab$. Composition is given by $(ab,c)(a,b)=(a,bc)$, and the identity of an object $a$ is $(a,1)$. An abelian group valued functor, $\mathcal{A}:\mathbb{H}M\to\mathbf{Ab}$, thus consists of abelian groups $\mathcal{A}_a$, $a\in M$, and homomorphisms $b_*:\mathcal{A}_a\to\mathcal{A}_{ab}$, $a,b\in M$, such that, for any $a,b,c\in M$, $b_*c_*=(bc)_*:\mathcal{A}_a\to\mathcal{A}_{abc}$ and, for any $a\in M$, $1_*=id_{\mathcal{A}_a}$. We refers to [15, Chap. XXII, §2] for details but, briefly, let us say that the abelian group object defined by an abelian group valued functor $\mathcal{A}:\mathbb{H}M\to\mathbf{Ab}$ can be written as

$$E(M,\mathcal{A})\to M,$$

where the *crossed product* commutative monoid $E(M,\mathcal{A})$ is the set $\bigcup_{a\in M}\mathcal{A}_a\times\{a\}$ of all ordered pairs $(u_a,a)$ with $a\in M$ and $u_a\in\mathcal{A}_a$, with multiplication given by

$$(u_a,a)(u_b,b)=(a_*u_b+b_*u_a,ab).$$

The monoid homomorphism $E(M,\mathcal{A})\to M$ is the obvious projection $(u_a,a)\mapsto a$, and the internal group operation

$$E(M,\mathcal{A})\times_M E(M,\mathcal{A})\xrightarrow{+}E(M,\mathcal{A})$$

is defined by $(u_a,a)+(v_a,a)=(u_a+v_a,a)$.

Furthermore, in [12, 13, 14], Grillet shows an algebraically more lucid description of the low dimensional cohomology groups

$$H^n(M,\mathcal{A}):=H^{n-1}_{\mathbb{G}}(M,E(M,\mathcal{A})\to M)$$

by means of a specific manageable complex (see also the recent work [18])

$$0\to C^1(M,\mathcal{A})\xrightarrow{\partial}C^2(M,\mathcal{A})\xrightarrow{\partial}C^3(M,\mathcal{A})\xrightarrow{\partial}C^4(M,\mathcal{A}), \qquad (1)$$

called the complex of (normalized on $1\in M$) *symmetric cochains* on $M$ with values in $\mathcal{A}$, which is defined as follows (below, $\bigcup_{a\in M}\mathcal{A}_a$ is the disjoint union set of the groups $\mathcal{A}_a$):

- A *symmetric* 1-*cochain* is a function $f:M\to\bigcup_{a\in M}\mathcal{A}_a$, such that $f(a)\in\mathcal{A}_a$ and $f(1)=0$.
- A *symmetric* 2-*cochain* is a function $g:M^2\to\bigcup_{a\in M}\mathcal{A}_a$, such that $g(a,b)\in\mathcal{A}_{ab}$,

$$g(a,b)=g(b,a),$$

and $g(a,1)=0$.

- A *symmetric* 3-*cochain* is a function $h:M^3\to\bigcup_{a\in M}\mathcal{A}_a$, such that $h(a,b,c)\in\mathcal{A}_{abc}$,

$$h(c,b,a)+h(a,b,c)=0,\quad h(a,b,c)+h(b,c,a)+h(c,a,b)=0 \qquad (2)$$

and $h(a,b,1)=0$.

- A *symmetric* 4-*cochain* is a function $t:M^4\to\bigcup_{a\in M}\mathcal{A}_a$, such that $t(a,b,c,d)\in\mathcal{A}_{abcd}$,

$$\begin{array}{l}t(a,b,b,a)=0,\quad t(d,c,b,a)+t(a,b,c,d)=0,\\ t(a,b,c,d)-t(b,c,d,a)+t(c,d,a,b)-t(d,a,b,c)=0,\\ t(a,b,c,d)-t(b,a,c,d)+t(b,c,a,d)-t(b,c,d,a)=0,\end{array}$$

and $t(a,b,c,1)=0$.

Under pointwise addition, these symmetric $n$-cochains constitute the abelian groups $C^n(M,\mathcal{A})$ in (1), $1\le n\le 4$. The coboundary homomorphisms are defined by

- $(\partial^1 f)(a,b)=a_*f(b)-f(ab)+b_*f(a)$,
- $(\partial^2 g)(a,b,c)=a_*g(b,c)-g(ab,c)+g(a,bc)-c_*g(a,b)$,
- $(\partial^3 h)(a,b,c,d)=a_*h(b,c,d)-h(ab,c,d)+h(a,bc,d)-h(a,b,cd)+d_*h(a,b,c)$. (3)

The groups

$$\begin{array}{l}Z^n(M,\mathcal{A})=\mathrm{Ker}\big(\partial^n:C^n(M,\mathcal{A})\to C^{n+1}(M,\mathcal{A})\big),\\ B^n(M,\mathcal{A})=\mathrm{Im}\big(\partial^{n-1}:C^{n-1}(M,\mathcal{A})\to C^n(M,\mathcal{A})\big),\end{array}$$

are respectively called the groups of *symmetric n-cocycles* and *symmetric n-coboundaries* on $M$ with values in $\mathcal{A}$. By [14, Theorems 1.3 and 2.12], there are natural isomorphisms

$$H^n(M,\mathcal{A}) \cong Z^n(M,\mathcal{A})/B^n(M,\mathcal{A})$$

for $n = 1, 2, 3$.

The elements of $H^1(M,\mathcal{A}) = Z^1(M,\mathcal{A})$ are *derivations* of $M$ in $\mathcal{A}$, that is, functions $f : M \to \bigcup_{a\in M} \mathcal{A}_a$ with $f(a) \in \mathcal{A}_a$, such that $f(ab) = a_*f(b) + b_*f(a)$.

The elements of $H^2(M,\mathcal{A})$ have a natural interpretation in terms of *commutative group coextension* of the commutative monoid $M$ by the functor $\mathcal{A} : \mathbb{H}M \to \mathbf{Ab}$. This is the classification result by Grillet in [15, §V.4], whose proof is a good illustration of the one we give of our result in this paper. We shall not present Grillet's proof here but, briefly, let us recall that in the correspondence between symmetric 2-cohomology classes and isomorphism classes of commutative group coextensions, each symmetric 2-cocycle $g \in Z^2(M,\mathcal{A})$ is carried to the coextension

$$E(M,\mathcal{A}, g) \to M,$$

where the *twisted crossed product* commutative monoid $E(M,\mathcal{A}, g)$ is the set $\bigcup_{a\in M} \mathcal{A}_a \times \{a\}$ of all pairs $(u_a, a)$ with $a \in M$ and $u_a \in \mathcal{A}_a$, with multiplication defined by

$$(u_a, a)(u_b, b) = (a_*u_b + b_*u_a + g(a, b), ab).$$

This multiplication is unitary ($(0, 1)$ is the unit) since $g$ is normalized, that is, $g(a, 1) = 0 = g(1, a)$; and it is associative and commutative due to $g$ being a symmetric 2-cocycle, that is, because of the equalities $a_*g(b, c) + g(a, bc) = g(ab, c) + c_*g(a, b)$ and $g(a, b) = g(b, a)$. The homomorphism $E(M,\mathcal{A}, g) \to M$ is the projection $(u_a, a) \mapsto a$, and, for each $a \in M$, the simply transitive group action of the group $\mathcal{A}_a$ on the fiber set over $a$ is given by

$$u_a \cdot (v_a, a) = (u_a + v_a, a).$$

2.2. **Strictly symmetric monoidal abelian groupoids.** In order to fix some needed notations about those monoidal categories we intend to address, we start by recalling that a *groupoid* is a small category all of whose morphisms are invertible. A groupoid $\mathcal{M}$ whose isotropy (or vertex) groups $\mathrm{Aut}_{\mathcal{M}}(x)$, $x \in \mathrm{Ob}\mathcal{M}$, are all abelian is termed an *abelian groupoid* (cf. [2, Definition 2.11.3 and Example 2.11.4], where the notion of abelian groupoid is discussed under a categorical point of view). We will use additive notation for abelian groupoids, thus, the identity morphism of an object $x$ of an abelian groupoid $\mathcal{M}$ will be denoted by $0_x$; if $u : x \to y$, $v : y \to z$ are morphisms, their composite is written $v + u : x \to z$, while the inverse of $u$ is $-u : y \to x$.

**Example 3.** *Any abelian group $A$ can be regarded as an abelian groupoid $\mathcal{M}$ with only one object, say $a$, and $\mathrm{Aut}_{\mathcal{M}}(a) = A$. For many purposes it is convenient to distinguish $A$ from the one-object groupoid $\mathcal{M}$; the notation $(K(A, 1), a)$ for $\mathcal{M}$ is not bad (its nerve or classifying space* [11, I, Example 1.4] *is precisely the pointed Eilenberg-Mac Lane minimal complex $K(A, 1)$ with base-vertex $a$), and we shall use it below.*

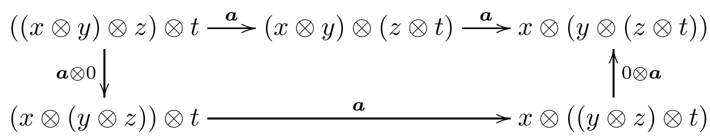
*A groupoid in which there is no morphisms between different objects is called* totally disconnected. *It is easily seen that any totally disconnected abelian groupoid is actually a disjoint union of abelian groups, or, more precisely, of the form $\bigcup_{a\in M}(K(\mathcal{A}_a, 1), a)$, for some family of abelian groups $(\mathcal{A}_a)_{a\in M}$.*

A *strictly symmetric* (or *strictly commutative*) *monoidal abelian groupoid*

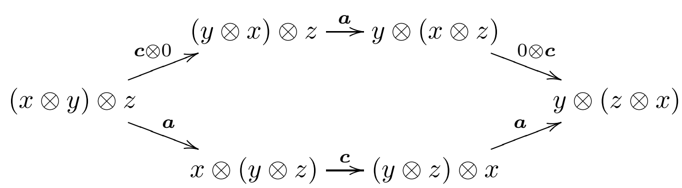
$$\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{c})$$

consists of an abelian groupoid $\mathcal{M}$, a functor $\otimes : \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ (the *tensor product*), an object I (the *unit object*), and natural isomorphisms $\boldsymbol{a}_{x,y,z} : (x\otimes y)\otimes z \to x\otimes(y\otimes z)$, $\boldsymbol{r}_x : x\otimes \mathrm{I}\to x$, and $\boldsymbol{c}_{x,y} : x\otimes y\to y\otimes x$ (called the *associativity, unit, and symmetry constraints*, respectively), such that the following five conditions are satisfied.

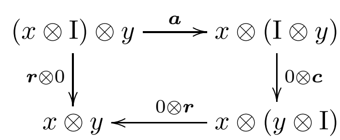
- $\boldsymbol{a}_{x,y,z\otimes t} + \boldsymbol{a}_{x\otimes y,z,t} = (0_x\otimes \boldsymbol{a}_{y,z,t}) + \boldsymbol{a}_{x,y\otimes z,t} + (\boldsymbol{a}_{x,y,z}\otimes 0_t),$ (4)



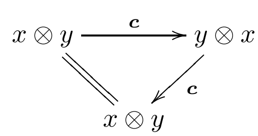

- $(0_y\otimes \boldsymbol{c}_{x,z}) + \boldsymbol{a}_{y,x,z} + (\boldsymbol{c}_{x,y}\otimes 0_z) = \boldsymbol{a}_{y,z,x} + \boldsymbol{c}_{x,y\otimes z} + \boldsymbol{a}_{x,y,z},$ (5)




- $(0_x\otimes \boldsymbol{r}_y) + (0_x\otimes \boldsymbol{c}_{\mathrm{I},y}) + \boldsymbol{a}_{x,\mathrm{I},y} = \boldsymbol{r}_x\otimes 0_y,$ (6)



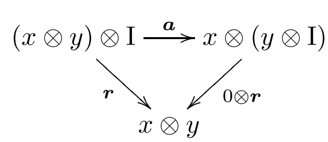

- $\boldsymbol{c}_{y,x} + \boldsymbol{c}_{x,y} = 0_{x\otimes y},$ (7)




- $\boldsymbol{c}_{x,x} = 0_{x\otimes x} : x\otimes x\to x\otimes x.$ (8)

For later reference, let us remark that the conditions above imply the following [19]

- $(0_x\otimes \boldsymbol{r}_y) + \boldsymbol{a}_{x,y,\mathrm{I}} = \boldsymbol{r}_{x\otimes y},$ (9)




These axioms guarantee the coherence of the constraints in the following sense (see Mac Lane [22, Theorem 5.1] and Fröhlich and Wall [10, Theorem (5.2)]).

**Fact 4** (Coherence Theorem)**.** *Let $\mathcal{M}$ be a strictly symmetric monoidal abelian groupoid. Then, commutativity holds in every diagram in $\mathcal{M}$ with vertices iterated instances of the functors $x\mapsto x$, the identity, $*\mapsto \mathrm{I}$, which selects the unit object, $(x,y)\mapsto x\otimes y$, $(x,y)\mapsto y\otimes x$, and $x\mapsto x\otimes x$, the diagonal functor, and whose edges are expanded instances of $\boldsymbol{a}$, $\boldsymbol{a}^{-1}$, $\boldsymbol{c}$, $\boldsymbol{r}$, and $\boldsymbol{r}^{-1}$.*

Below there is a convenient way to express this coherence in practice (see Deligne [9, 1.4.1] and Fröhlich and Wall [10, Theorem (5.3)]). Recall that, for any set $M$, the free commutative monoid $\mathbb{N}[M]$ consists of commutative words in $M$, which are unordered sequences $[a_1, \ldots, a_n]$ of elements of $M$; unordered means that for any permutation $\sigma$, $[a_{\sigma 1}, \ldots, a_{\sigma n}] = [a_1, \ldots, a_n]$. Multiplication in $\mathbb{N}[M]$ is given by by concatenation:

$$[a_1, \ldots, a_n][b_1, \ldots, b_m] = [a_1, \ldots, a_n, b_1, \ldots, b_m],$$

and the unit is $1 = [\ \ ]$, the empty word.

**Lemma 5.** *Let $(x_a)_{a\in M}$ be any family of objects of a strictly symmetric monoidal abelian groupoid $\mathcal{M}$. If $\mathbb{N}[M]$ is the free commutative monoid generated by the index set $M$, then, there exists a map $F : \mathbb{N}[M] \to \mathrm{Ob}\mathcal{M}$ with $F[a] = x_a$, $a \in M$, and isomorphisms $\varphi_{f,g} : Ff \otimes Fg \cong F(fg)$, $f, g \in \mathbb{N}[M]$, and $\varphi_0 : \mathrm{I} \to F1$, satisfying the three equations below.*

- $\varphi_{fg,h} + (\varphi_{f,g} \otimes 0_{Fh}) = \varphi_{f,gh} + (0_{Ff} \otimes \varphi_{g,h}) + \boldsymbol{a}_{Ff,Fg,Fh}$,

$$\begin{array}{ccccc}
(Ff \otimes Fg) \otimes Fh & \xrightarrow{\varphi\otimes 0} & F(fg) \otimes Fh & \xrightarrow{\varphi} & F(fgh) \\
{\scriptstyle \boldsymbol{a}}\downarrow & & & & \| \\
Ff \otimes (Fg \otimes Fh) & \xrightarrow{0\otimes\varphi} & Ff \otimes F(gh) & \xrightarrow{\varphi} & F(fgh)
\end{array}$$

- $\varphi_{g,f} + \boldsymbol{c}_{Ff,Fg} = \varphi_{f,g}$.

$$\begin{array}{ccc}
Ff \otimes Fg & \xrightarrow{\boldsymbol{c}} & Fg \otimes Ff \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\
F(fg) & = & F(gf)
\end{array}$$

- $\varphi_{f,1} + (0_{Ff} \otimes \varphi_0) = \boldsymbol{r}_{Ff}$,

$$\begin{array}{ccc}
Ff \otimes \mathrm{I} & \xrightarrow{0\otimes\varphi_0} & Ff \otimes F1 \\
{\scriptstyle \boldsymbol{r}}\downarrow & & \downarrow{\scriptstyle \varphi} \\
Ff & = & Ff
\end{array}$$

*Proof.* Let us now choose a total order for the index set $M$, so that any $f \in \mathbb{N}[M]$ can be uniquely expressed as a sequence in increasing order

$$f = [a_1, \ldots, a_n], \quad a_1 \leq \cdots \leq a_n.$$

Then, we define $F : \mathbb{N}[M] \to \mathrm{Ob}\mathcal{M}$ by putting $F1 = \mathrm{I}$, $F[a] = x_a$, and, recursively,

$$F[a_1, \ldots, a_n] = F[a_1, \ldots, a_{n-1}] \otimes x_{a_n}$$

for $n > 1$. We have the identity isomorphism $\varphi_0 = 0_{\mathrm{I}} : \mathrm{I} \to F1$ and, for any $f, g \in \mathbb{N}[M]$, it is clear that there is an isomorphism

$$\varphi_{f,g} : Ff \otimes Fg \cong F(fg)$$

coming from instances of $\boldsymbol{a}$, $\boldsymbol{a}^{-1}$, $\boldsymbol{c}$, and $\boldsymbol{r}$. It follows from the Coherence Theorem above that these isomorphisms $\varphi_{f,g}$ so obtained satisfy all the requirements in the lemma. □

If $\mathcal{M}$, $\mathcal{M}'$ are strictly symmetric monoidal abelian groupoids, then a *symmetric monoidal functor* $F = (F, \varphi, \varphi_0) : \mathcal{M} \to \mathcal{M}'$ consists of a functor on the underlying groupoids

$F:\mathcal{M}\to\mathcal{M}'$, natural isomorphisms $\varphi_{x,y}: Fx\otimes Fy\to F(x\otimes y)$, and an isomorphism $\varphi_0:\mathrm{I}'\to F\mathrm{I}$, such that the following coherence conditions hold.

- $F\boldsymbol{a}_{x,y,z}+\varphi_{x\otimes y,z}+(\varphi_{x,y}\otimes 0_{Fz})=\varphi_{x,y\otimes z}+(0_{Fx}\otimes\varphi_{y,z})+\boldsymbol{a}'_{Fx,Fy,Fz}$, (10)

$$\begin{array}{ccccc}(Fx\otimes Fy)\otimes Fz & \xrightarrow{\varphi\otimes 0} & F(x\otimes y)\otimes Fz & \xrightarrow{\varphi} & F((x\otimes y)\otimes z)\\ \Big\downarrow{\scriptstyle \boldsymbol{a}'} & & & & \Big\downarrow{\scriptstyle F\boldsymbol{a}}\\ Fx\otimes(Fy\otimes Fz) & \xrightarrow{0\otimes\varphi} & Fx\otimes F(y\otimes z) & \xrightarrow{\varphi} & F(x\otimes(y\otimes z))\end{array}$$

- $F\boldsymbol{r}_x+\varphi_{x,\mathrm{I}}+(0_{Fx}\otimes\varphi_0)=\boldsymbol{r}'_{Fx}$, (11)

$$\begin{array}{ccc}Fx\otimes\mathrm{I}' & \xrightarrow{0\otimes\varphi_0} & Fx\otimes F\mathrm{I}\\ \Big\downarrow{\scriptstyle \boldsymbol{r}'} & & \Big\downarrow{\scriptstyle\varphi}\\ Fx & \xleftarrow{F\boldsymbol{r}} & F(x\otimes\mathrm{I})\end{array}$$

- $\varphi_{y,x}+\boldsymbol{c}'_{Fx,Fy}=F\boldsymbol{c}_{x,y}+\varphi_{x,y}$. (12)

$$\begin{array}{ccc}Fx\otimes Fy & \xrightarrow{\boldsymbol{c}'} & Fy\otimes Fx\\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi}\\ F(x\otimes y) & \xrightarrow{F\boldsymbol{c}} & F(y\otimes x)\end{array}$$

Suppose $F':\mathcal{M}\to\mathcal{M}'$ is another symmetric monoidal functor. Then, a *symmetric isomorphism* $\theta:F\Rightarrow F'$ is a natural isomorphism on the underlying functors, $\theta:F\Rightarrow F'$, such that the coherence equations below are satisfied.

- $\theta_{x\otimes y}+\varphi_{x,y}=\varphi'_{x,y}+(\theta_x\otimes\theta_y),\quad \theta_{\mathrm{I}}+\varphi_0=\varphi'_0.$

$$\begin{array}{ccc}Fx\otimes Fy & \xrightarrow{\varphi} & F(x\otimes y)\\ \Big\downarrow{\scriptstyle\theta\otimes\theta} & & \Big\downarrow{\scriptstyle\theta}\\ F'x\otimes F'y & \xrightarrow{\varphi'} & F'(x\otimes y)\end{array}\qquad \begin{array}{ccc} & \overset{\varphi_0}{\nearrow} & F\mathrm{I}\\ \mathrm{I}' & & \Big\downarrow{\scriptstyle\theta}\\ & \underset{\varphi'_0}{\searrow} & F'\mathrm{I}\end{array}$$

With compositions given in a natural way, strictly symmetric monoidal abelian groupoids, symmetric monoidal functors, and symmetric isomorphisms form a 2-category. A symmetric monoidal functor $F:\mathcal{M}\to\mathcal{M}'$ is called a *symmetric monoidal equivalence* if it is an equivalence in this 2-category, that is, when there exists a symmetric monoidal functor $F':\mathcal{M}'\to\mathcal{M}$ and symmetric isomorphisms $\theta:F'F\cong id_{\mathcal{M}}$ and $\theta':FF'\cong id_{\mathcal{M}'}$.

Our goal is to show a classification for strictly symmetric monoidal abelian groupoids, where two of them that are connected by a symmetric monoidal equivalence are considered the same. To do that, we will use the fact below by Saavedra [24, I, 4.4.5], where it is shown how to transport the symmetric monoidal structure on an abelian groupoid along an equivalence on its underlying groupoid. Recall that a functor between (non necessarily abelian) groupoids $F:\mathcal{M}\to\mathcal{M}'$ is an equivalence (of categories) if and only if the induced map on the sets of iso-classes of objects

$$\mathrm{Ob}\mathcal{M}/_{\cong}\to\mathrm{Ob}\mathcal{M}'/_{\cong},\quad [x]\mapsto[Fx],$$

is a bijection, and the induced homomorphisms on the automorphism groups

$$\mathrm{Aut}_{\mathcal{M}}(x)\to\mathrm{Aut}_{\mathcal{M}'}(Fx),\quad u\mapsto Fu$$

are all isomorphisms [16, Chapter 6, Corollary 2].

**Fact 6** (Transport of Structure)**.** *Let $F : \mathcal{M} \to \mathcal{M}'$ be an equivalence between abelian groupoids, so that there is a functor $F' : \mathcal{M}' \to \mathcal{M}$ with natural equivalences $\theta : id_{\mathcal{M}} \cong F'F$ and $\theta' : FF' \cong id_{\mathcal{M}'}$ satisfying*

$$\theta' F + F\theta = id_F, \quad F'\theta' + \theta F' = id_{F'}.$$

$(i)$ *Any strictly symmetric monoidal structure on $\mathcal{M}$ can be transported to one on $\mathcal{M}'$ such that the functors $F$ and $F'$ underlie symmetric monoidal functors, and the natural equivalences $\theta$ and $\theta'$ turn to be symmetric isomorphisms.*

$(ii)$ *If both $\mathcal{M}$ and $\mathcal{M}'$ have a strictly symmetric monoidal structure, then any symmetric monoidal structure on $F$ can be transported to one on $F'$ such that $\theta$ and $\theta'$ become symmetric isomorphisms. Hence, a symmetric monoidal functor is a symmetric monoidal equivalence if and only if the underlying functor is an equivalence.*

Concerning Fact 6$(i)$, let us point out that for any strictly symmetric monoidal structure $(\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{c})$ on the abelian groupoid $\mathcal{M}$, the structure $(\mathcal{M}', \otimes, \mathrm{I}', \boldsymbol{a}', \boldsymbol{r}', \boldsymbol{c}')$ transported onto the abelian groupoid $\mathcal{M}'$ by means of $(F, F', \theta, \theta')$ is such that the monoidal product $\otimes$ is the dotted functor in the commutative square

$$\begin{array}{ccc} \mathcal{M}' \times \mathcal{M}' & \overset{\otimes}{\dashrightarrow} & \mathcal{M}' \\ {\scriptstyle F' \times F'} \downarrow & & \uparrow {\scriptstyle F} \\ \mathcal{M} \times \mathcal{M} & \overset{\otimes}{\longrightarrow} & \mathcal{M}, \end{array}$$

and the unit object is $F\mathrm{I}$. The functors $F$ and $F'$ are endowed with the isomorphisms

$$\varphi_{x,y} = -F(\theta_x \otimes \theta_y) : Fx \otimes Fy \to F(x \otimes y), \quad \varphi_0 = 0_{FI} : F\mathrm{I} \to F\mathrm{I}, \tag{13}$$
$$\varphi'_{x',y'} = \theta_{F'x' \otimes F'y'} : F'x' \otimes F'y' \to F'(x' \otimes y'), \quad \varphi'_0 = \theta_{\mathrm{I}} : \mathrm{I} \to F'\mathrm{I}',$$

and then the constraints $\boldsymbol{a}', \boldsymbol{r}'$ and the symmetry $\boldsymbol{c}'$ are given by those isomorphisms uniquely determined by the equations (10), (11), and (12), respectively.

Concerning Fact 6$(ii)$, let us recall that if both $\mathcal{M}$ and $\mathcal{M}'$ are strictly symmetric monoidal abelian groupoids, then for any symmetric monoidal structure $(F, \varphi, \varphi_0)$ on $F$, the structure $(F', \varphi', \varphi'_0)$ transported on the functor $F'$ is such that the isomorphisms

$$\varphi'_{x',y'} : F'x' \otimes F'y' \to F'(x' \otimes y'), \quad \varphi'_0 : \mathrm{I} \to F'\mathrm{I}',$$

are the uniquely determined by the dotted arrows making commutative the diagrams below.

$$\begin{array}{ccc} FF'x' \otimes FF'y' & \xrightarrow{\theta'_{x'} \otimes \theta'_{y'}} & x' \otimes y' \\ {\scriptstyle \varphi} \downarrow & & \downarrow {\scriptstyle -\theta'_{x' \otimes y'}} \\ F(F'x' \otimes F'y') & \overset{F\varphi'}{\dashrightarrow} & FF'(x' \otimes y') \end{array} \qquad \begin{array}{ccc} & \overset{\varphi_0}{\nearrow} & F\mathrm{I} \\ \mathrm{I}' & & \downarrow {\scriptstyle F\varphi'_0} \\ & \underset{-\theta'_{\mathrm{I}'}}{\searrow} & FF'\mathrm{I}' \end{array}$$

## 7. The classification theorem

The framework of our discussion below comes suggested by the known classification theorems for strictly commutative Picard categories given in [9], [10] and [26], for categorical groups and Picard categories in [25], for braided categorical groups in [17], for graded categorical groups, braided graded categorical groups, graded Picard categories and strictly commutative graded Picard categories in [6, 7, 8], for braided fibred categorical groups, fibred Picard categories and strictly commutative fibred Picard categories in [4], and for monoidal groupoids in [5].

Let $M$ be a commutative monoid and let $\mathcal{A}: \mathbb{H}M \to \mathbf{Ab}$ be a functor. Each symmetric 3-cocycle $h \in Z^3(M, \mathcal{A})$ gives rise to a strictly symmetric monoidal abelian groupoid

$$\mathcal{S}(M, \mathcal{A}, h) \tag{14}$$

which should be thought of a sort of 2-*dimensional twisted crossed product of* $M$ *by* $\mathcal{A}$, and it is built as follows: Its underlying groupoid is the totally disconnected groupoid

$$\bigcup_{a\in M}(K(\mathcal{A}_a, 1), a),$$

where, recall from Example 3, each $(K(\mathcal{A}_a, 1), a)$ denotes the groupoid having $a$ as its unique object and $\mathcal{A}_a$ as the automorphism group of $a$. Thus, an object of $\mathcal{S}(M, \mathcal{A}, h)$ is an element $a \in M$; if $a \neq b$ are different elements of the monoid $M$, then there is no morphisms in $\mathcal{S}(M, \mathcal{A}, h)$ between them, whereas its isotropy group at any $a \in M$ is $\mathcal{A}_a$.

The tensor functor

$$\otimes : \mathcal{S}(M, \mathcal{A}, h) \times \mathcal{S}(M, \mathcal{A}, h) \to \mathcal{S}(M, \mathcal{A}, h)$$

is given on objects by multiplication in $M$, so $a \otimes b = ab$, and on morphisms by the group homomorphisms

$$\otimes : \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{A}_{ab}, \quad u_a \otimes u_b = b_* u_a + a_* u_b.$$

The unit object is $\mathrm{I} = 1$, the unit element of the monoid $M$, and the structure constraints and the symmetry isomorphisms are

$$\begin{aligned} \boldsymbol{a}_{a,b,c} &= h(a, b, c) : (ab)c \to a(bc), \\ \boldsymbol{c}_{a,b} &= 0_{ab} : ab \to ba, \\ \boldsymbol{r}_a &= 0_a : a1 \to a, \end{aligned}$$

which are easily seen to be natural since $\mathcal{A}$ is an abelian group valued functor. The coherence condition (4) holds thanks to the cocycle condition $\partial^3 h = 0$ in (3), while (5) easily follows from the cochain equations in (2). The normalization condition $h(a, 1, b) = 0$, easily deduced from being $h(a, b, 1) = 0$, implies the coherence condition (6), and those in (7) and (8) are obviously verified.

In the next Theorem 8.1, we observe how any strictly symmetric monoidal abelian groupoid is symmetric monoidal equivalent to such a 2-dimensional crossed product. Previously, we combine the transport process in Fact 6 with the generalization of Brandt's Theorem [3], which asserts that every groupoid is equivalent as a category to a totally disconnected groupoid [16, Chapter 6, Theorem 2], to obtain the following.

**Lemma 8.** *Any strictly symmetric monoidal abelian groupoid is symmetric monoidal equivalent to one which is totally disconnected and whose symmetry and unit constraints are all identities.*

*Proof.* Let $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{c})$ be any given strictly symmetric monoidal abelian groupoid.

Let $M = \mathrm{Ob}\mathcal{M}/_{\cong}$ be the set of isomorphism classes $[x]$ of the objects of $\mathcal{M}$, and let us choose, for each $a \in M$, any representative object $x_a \in a$, with $x_{[\mathrm{I}]} = \mathrm{I}$.

In a first step, let us assume that all the symmetry constraints are identities, that is, $x \otimes y = y \otimes x$ and $\boldsymbol{c}_{x,y} = 0_{x\otimes y}$, for any objects $x, y$ of $\mathcal{M}$, and also that $\mathrm{I} \otimes \mathrm{I} = \mathrm{I}$ and $\boldsymbol{r}_{\mathrm{I}} = 0_{\mathrm{I}}$, the identity of the unit object. Then, let us form the totally disconnected abelian groupoid

$$\mathcal{M}' = \bigcup_{a\in M}(K(\mathcal{A}_a, 1), a),$$

whose set of objects is $M$, and whose isotropy group at any object $a \in M$ is $\mathcal{A}_a = \mathrm{Aut}_{\mathcal{M}}(x_a)$.

This groupoid $\mathcal{M}'$ is equivalent to the underlying groupoid $\mathcal{M}$. To give a particular equivalence $F:\mathcal{M}\to\mathcal{M}'$, let us choose, for each $a\in M$ and each $x\in a$, an isomorphism $\theta_x : x\cong x_a$ in $\mathcal{M}$. In particular, for every $a\in M$, we take $\theta_{x_a\otimes \mathrm{I}}=\boldsymbol{r}_{x_a}$. Note that this selection implies that $\theta_{\mathrm{I}}=\theta_{\mathrm{I}\otimes\mathrm{I}}=\boldsymbol{r}_{\mathrm{I}}=0_{\mathrm{I}}$. Then, let $F:\mathcal{M}\to\mathcal{M}'$ be the functor which acts on objects by $Fx=[x]$, and on morphisms $u:x\to y$ by $Fu=\theta_y+u-\theta_x$. We have also the more obvious functor $F':\mathcal{M}'\to\mathcal{M}$, which is defined on objects by $F'a=x_a$, and on morphisms $u:a\to a$ by $F'u=u$. We have the natural isomorphisms $\theta : id_{\mathcal{M}}\cong F'F$, and $\theta' : FF'\cong id_{\mathcal{M}'}$, where $\theta'_a=-\theta_{x_a}$, which clearly satisfy the equalities $\theta'F+F\theta=id_F$ and $F'\theta'+\theta F'=id_{F'}$.

Therefore, according to Fact 6, we can transport the given symmetric monoidal structure of $\mathcal{M}$ to a corresponding one on $\mathcal{M}'$ by means of $(F,F',\theta,id)$, so that we get a totally disconnected strictly symmetric monoidal abelian groupoid $\mathcal{M}'=(\mathcal{M}',\otimes,\mathrm{I}',\boldsymbol{a}',\boldsymbol{r}',\boldsymbol{c}')$, and a symmetric monoidal equivalence $F=(F,\varphi,\varphi_0):\mathcal{M}\to\mathcal{M}'$. Now, a quick analysis of the structure on $\mathcal{M}'$ points out that its unit object is $F\mathrm{I}=[\,\mathrm{I}\,]$ and that, for any object $a\in \mathrm{Ob}\mathcal{M}'=M$,

$$\begin{aligned}\boldsymbol{r}'_a &\overset{(11)}{=} F(\boldsymbol{r}_{x_a})+\varphi_{x_a,I}+(0_a\otimes\varphi_0)\overset{(13)}{=} F(\boldsymbol{r}_{x_a})+\varphi_{x_a,I}+(0_a\otimes 0_{[\mathrm{I}]}) = F(\boldsymbol{r}_{x_a})+\varphi_{x_a,I}\\ &\overset{(13)}{=} \theta_{x_a}+\boldsymbol{r}_{x_a}-\theta_{x_a\otimes\mathrm{I}}+\theta_{x_a\otimes\mathrm{I}}-(\theta_{x_a}\otimes 0_{\mathrm{I}})-\theta_{x_a\otimes\mathrm{I}}\\ &= \theta_{x_a}+\boldsymbol{r}_{x_a}-(\theta_{x_a}\otimes 0_{\mathrm{I}})-\theta_{x_a\otimes\mathrm{I}} \overset{(\text{naturality of } \boldsymbol{r})}{=} \boldsymbol{r}_{x_a}-\theta_{x_a\otimes\mathrm{I}}=0_{x_a}=0_a,\end{aligned}$$

and, for any $a,b\in M$,

$$\begin{aligned}\boldsymbol{c}'_{a,b} &\overset{(12)}{=} -\varphi_{x_b,x_a}+F(\boldsymbol{c}_{x_a,x_b})+\varphi_{x_a,x_b}\overset{(\boldsymbol{c}=0)}{=} -\varphi_{x_b,x_a}+\varphi_{x_a,x_b}\\ &\overset{(13)}{=} \theta_{x_b\otimes x_a}-\theta_{x_b}\otimes\theta_{x_a}-\theta_{x_b\otimes x_a}+\theta_{x_a\otimes x_b}+\theta_{x_a}\otimes\theta_{x_b}-\theta_{x_a\otimes x_b}\\ &\text{(since } x_a\otimes x_b=x_b\otimes x_a)\\ &=\theta_{x_b\otimes x_a}-\theta_{x_b}\otimes\theta_{x_a}+\theta_{x_a}\otimes\theta_{x_b}-\theta_{x_a\otimes x_b}\\ &\text{(since } \theta_{x_a}\otimes\theta_{x_b}=\theta_{x_b}\otimes\theta_{x_a}, \text{ by the naturality of } \boldsymbol{c}_{x_a,x_b}=0_{x_a\otimes x_b})\\ &=\theta_{x_b\otimes x_a}-\theta_{x_a\otimes x_b}=0_{x_a\otimes x_b}=0_{ab}.\end{aligned}$$

Thus, $\mathcal{M}$ is symmetric monoidal equivalent to $\mathcal{M}'$, which is a totally disconnected strictly symmetric abelian groupoid whose unit and symmetry constraints are all identities.

Hence, it suffices to prove now that the given strictly symmetric monoidal abelian groupoid $\mathcal{M}$ is symmetric monoidal equivalent to another one whose symmetry constraints are all identities and whose unit constraint at the unit object is also the identity. Even more, following Deligne [9], we can prove that there is a symmetric monoidal abelian groupoid $\mathcal{N}=(\mathcal{N},\bar{\otimes})$ whose constraints are all trivial (i.e., $\boldsymbol{a}=0$, $\boldsymbol{c}=0$, and $\boldsymbol{r}=0$) with a symmetric monoidal equivalence $\mathcal{N}\simeq\mathcal{M}$:

Let $\mathbb{N}[M]$ be the free commutative monoid generated by $M$, which we shall regard as a strictly symmetric monoidal discrete groupoid (i.e., with only identities as morphisms). It follows from Lemma 5 that there is a symmetric monoidal functor

$$F=(F,\varphi,\varphi_0):\mathbb{N}[M]\to\mathcal{M}$$

such that $F[a]=x_a$, for any $a\in M$. Then, we define $\mathcal{N}$ to be the abelian groupoid whose set of objects is $\mathbb{N}[M]$, and whose hom-sets are defined by

$$\mathrm{Hom}_{\mathcal{N}}(f,g)=\mathrm{Hom}_{\mathcal{M}}(Ff,Fg).$$

Composition in $\mathcal{N}$ is given by that in $\mathcal{M}$, so that we have a full, faithful, and essentially surjective functor (i.e., an equivalence)

$$F:\mathcal{N}\to\mathcal{M},\quad (f\overset{u}{\to}g)\mapsto(Ff\overset{u}{\to}Fg).$$

The monoidal functor $\bar{\otimes}:\mathcal{N}\times\mathcal{N}\to\mathcal{N}$ is defined by multiplication in $\mathbb{N}[M]$ on objects, and on morphisms by

$$(f\overset{u}{\to}g)\otimes(f'\overset{u'}{\to}g')=(ff'\overset{u\bar{\otimes}u'}{\longrightarrow}gg'),$$

where $u\bar{\otimes}u'$ is the dotted morphism in the commutative square in $\mathcal{M}$

$$\begin{array}{ccc} Ff\otimes Ff' & \xrightarrow{u\otimes u'} & Fg\otimes Fg' \\ {\scriptstyle\varphi_{f,f'}}\downarrow & & \downarrow{\scriptstyle\varphi_{g,g'}} \\ F(ff') & \overset{u\bar{\otimes}u'}{\dashrightarrow} & F(gg'). \end{array} \tag{15}$$

So defined $\mathcal{N}=(\mathcal{N},\bar{\otimes})$ is a strictly symmetric monoidal abelian groupoid with all the constraints being identities. To prove this claim, the following equalities on morphisms in $\mathcal{N}$ should be verified

$$u\bar{\otimes}u'=u'\bar{\otimes}u,\quad u\bar{\otimes}0_1=u,\quad (u\bar{\otimes}u')\bar{\otimes}u''=u\bar{\otimes}(u'\bar{\otimes}u''). \tag{16}$$

But these follow from the naturality of the structure constraints of $\mathcal{M}$, $\boldsymbol{c}$, $\boldsymbol{r}$, and $\boldsymbol{a}$, respectively. For example, given any $u\in\mathrm{Hom}_{\mathcal{N}}(f,g)$, we have the diagram

$$\begin{array}{ccccc} & & \overset{\boldsymbol{r}_{Ff}}{\frown\!\!\longrightarrow} & & \\ & & (C) & & \\ Ff\otimes\mathrm{I} & \xrightarrow{0_{Ff}\otimes\varphi_0} & Ff\otimes F1 & \xrightarrow{\varphi_{f,1}} & Ff \\ {\scriptstyle u\otimes 0_{\mathrm{I}}}\downarrow & (A) & \downarrow{\scriptstyle u\otimes 0_{F1}} & (B) & \downarrow{\scriptstyle u} \\ Fg\otimes\mathrm{I} & \xrightarrow{0_{Fg}\otimes\varphi_0} & Fg\otimes F1 & \xrightarrow{\varphi_{g,1}} & Fg \\ & & (C) & & \\ & & \underset{\boldsymbol{r}_{Fg}}{\smile\!\!\longrightarrow} & & \end{array}$$

where the outside region commutes by naturality of $\boldsymbol{r}$, those labelled with $(C)$ commute because $(F,\varphi,\varphi_0):\mathbb{N}[M]\to\mathcal{M}$ is a symmetric monoidal functor, and the square $(A)$ commutes due to $\otimes:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ being a functor. It follows that the square $(B)$ is also commutative and then that $u\bar{\otimes}0_1=u$. The other two equations in (16) are proved similarly, and we leave them to the reader.

Owing to the commutativity of squares (15), the isomorphisms $\varphi_{f,f'}$ are natural on morphisms of $\mathcal{N}$ and, therefore, $F=(F,\varphi,\varphi_0):\mathcal{N}\to\mathcal{M}$ is actually a symmetric monoidal functor, whence, by Fact 6 $(ii)$, a symmetric monoidal equivalence. □

We are now ready to prove the main result in this paper, namely

**Theorem 8.1** (Classification of Strictly Symmetric Monoidal Abelian Groupoids)**.** $(i)$ *For any strictly symmetric monoidal abelian groupoid $\mathcal{M}$, there exist a commutative monoid $M$, a functor $\mathcal{A}:\mathbb{H}M\to\mathbf{Ab}$, a symmetric 3-cocycle $h\in Z^3(M,\mathcal{A})$, and a symmetric monoidal equivalence*

$$\mathcal{S}(M,\mathcal{A},h)\simeq\mathcal{M}.$$

*(ii) For any two commutative* 3*-cocycles* $h \in Z^3(M, \mathcal{A})$ *and* $h' \in Z^3(M', \mathcal{A}')$*, there is a symmetric monoidal equivalence*

$$\mathcal{S}(M, \mathcal{A}, h) \simeq \mathcal{S}(M', \mathcal{A}', h')$$

*if and only if there exist an isomorphism of monoids* $i : M \cong M'$ *and a natural isomorphism* $\psi : \mathcal{A} \cong \mathcal{A}'i$*, such that the equality of cohomology classes below holds.*

$$[h] = \psi_*^{-1} i^*[h'] \in H^3(M, \mathcal{A})$$

*Proof.* $(i)$ By Lemma 8, we can suppose that $\mathcal{M}$ is totally disconnected and that all its symmetry and unit constraints are identities. In assuming that hypothesis, let us write the underlying groupoid as $\mathcal{M} = \bigcup_{a \in M}(K(\mathcal{A}_a, 1), a)$, where $M = \mathrm{Ob}\mathcal{M}$ and, for each $a \in M$, $\mathcal{A}_a = \mathrm{Aut}_{\mathcal{M}}(a)$. Then, a system of data $(M, \mathcal{A}, h)$, such that $\mathcal{S}(M, \mathcal{A}, h) = \mathcal{M}$ as symmetric monoidal abelian groupoids, is defined as follows:

- *The monoid* $M$. The function on objects of the tensor functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ determines a multiplication on $M$, simply by putting $ab = a \otimes b$, for any $a, b \in M$. If we write $1 \in M$ for the the unit object of $\mathcal{M}$, then this multiplication on $M$ is unitary, since the unit is strict. Furthermore, it is associative and commutative since, being $\mathcal{M}$ totally disconnected, the existence of the associativity and symmetry constraints $(ab)c \to a(bc)$ and $ab \to ba$ forces the equalities $(ab)c = a(bc)$ and $ab = ba$. Thus, $M$ becomes a commutative monoid.

- *The functor* $\mathcal{A} : \mathbb{H}M \to \mathbf{Ab}$. The group homomorphisms $\otimes : \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{A}_{ab}$ have an associative, commutative, and unitary behaviour, in the sense that the equalities

$$(u_a \otimes u_b) \otimes u_c = u_a \otimes (u_b \otimes u_c), \quad u_a \otimes u_b = u_b \otimes u_a, \quad 0_1 \otimes u_a = u_a, \tag{17}$$

hold. These follow from the abelianess of the groups of automorphisms in $\mathcal{M}$, since the diagrams below commute due to the naturality of the structure constraints.

$$\begin{array}{ccc} (ab)c & \xrightarrow{\boldsymbol{a}_{a,b,c}} & a(bc) \\ {\scriptstyle (u_a\otimes u_b)\otimes u_c}\downarrow & & \downarrow{\scriptstyle u_a\otimes(u_b\otimes u_c)} \\ (ab)c & \xrightarrow{\boldsymbol{a}_{a,b,c}} & a(bc) \end{array} \qquad \begin{array}{ccc} ab & \xrightarrow{0_{ab}} & ba \\ {\scriptstyle u_a\otimes u_b}\downarrow & & \downarrow{\scriptstyle u_b\otimes u_a} \\ ab & \xrightarrow{0_{ab}} & ba \end{array} \qquad \begin{array}{ccc} a1 = a & \xrightarrow{0_a} & a \\ {\scriptstyle u_a\otimes 0_1}\downarrow & & \downarrow{\scriptstyle u_a} \\ a1 = a & \xrightarrow{0_a} & a \end{array}$$

Then, if write $b_* : \mathcal{A}_a \to \mathcal{A}_{ab}$ for the homomorphism such that

$$b_* u_a := 0_b \otimes u_a = u_a \otimes 0_b,$$

the equalities

$$\begin{aligned} &(bc)_*(u_a) = 0_{bc} \otimes u_a = (0_b \otimes 0_c) \otimes u_a \overset{(17)}{=} 0_b \otimes (0_c \otimes u_a) = b_*(c_* u_a), \\ &1_* u_a = 0_1 \otimes u_a \overset{(17)}{=} u_a, \end{aligned}$$

show that the assignments $a \mapsto \mathcal{A}_a$, $(a, b) \mapsto b_* : \mathcal{A}_a \to \mathcal{A}_{ab}$, define an abelian group valued functor on $\mathbb{H}M$. Observe that this functor determines the monoidal product $\otimes$ of $\mathcal{M}$, since

$$\begin{aligned} u_a \otimes u_b &= (u_a + 0_a) \otimes (0_b + u_b) = (u_a \otimes 0_b) + (0_a \otimes u_b) \overset{(17)}{=} (0_b \otimes u_a) + (0_a \otimes u_b) \\ &= b_* u_a + a_* u_b. \end{aligned}$$

- *The symmetric* 3*-cocycle* $h \in Z^3(M, \mathcal{A})$. The associativity constraints of $\mathcal{M}$ are necessarily written in the form $\boldsymbol{a}_{a,b,c} = h(a, b, c)$, for some list $\big(h(a, b, c) \in \mathcal{A}_{abc}\big)_{a,b,c \in M}$.

Since the symmetry constraints are all identities, for any $(a,b,c)\in M^3$, equation (5) gives

$$h(a,b,c)+h(b,c,a)=h(b,a,c), \tag{18}$$

which, making the permutation $(a,b,c)\leftrightarrow(b,a,c)$, is written as $h(b,a,c)+h(a,c,b)=h(a,b,c)$. If we carry this to (18), we obtain $h(b,a,c)+h(a,c,b)+h(b,c,a)=h(b,a,c)$, whence it follows the first symmetric cochain condition in (2), that is,

$$h(a,c,b)+h(b,c,a)=0.$$

To get the second one, we replace the term $h(b,a,c)$ with $-h(c,a,b)$ in (18) and we have

$$h(a,b,c)+h(b,c,a)+h(c,a,b)=0,$$

as it is required. Now, from (9) it follows that $h(a,b,1)=0$. Hence $h\in C^3(M,\mathcal{A})$ is a symmetric 3-cochain. Finally, the coherence condition in (4) gives the equations

$$a_*h(b,c,d)+h(a,bc,d)+d_*h(a,b,c)=h(a,b,cd)+h(ab,c,d),$$

which means that $\partial^3h=0$ in (3), so that $h\in Z^3(M,\mathcal{A})$ is a symmetric 3-cocycle.

Since an easy comparison shows that $\mathcal{M}=\mathcal{S}(M,\mathcal{A},h)$, the proof of this part is complete.

$(ii)$ We first assume that there exist an isomorphism of monoids $i:M\cong M'$ and a natural isomorphism $\psi:\mathcal{A}\cong\mathcal{A}'i$, such that $\psi_*[h]=i^*[h']\in H^3(M,\mathcal{A}'i)$. This means that there is a symmetric 2-cochain $g\in C^2(M,\mathcal{A}'i)$ such that the equalities below hold.

$$\psi_{abc}h(a,b,c)=h'(ia,ib,ic)+(ia)_*g(b,c)-g(ab,c)+g(a,bc)-(ic)_*g(a,b) \tag{19}$$

Then, we have a symmetric monoidal isomorphism

$$\mathcal{S}(i,\psi,g)=(F,\varphi,\varphi_0):\mathcal{S}(M,\mathcal{A},h)\to\mathcal{S}(M',\mathcal{A}',h'), \tag{20}$$

whose underlying functor acts by

$$F(a\overset{u}{\to}a)=(ia\overset{\psi_au_a}{\longrightarrow}ia),$$

and whose structure isomorphisms are given by

$$\varphi_{a,b}=g(a,b):(ia)\,(ib)\to i(ab),$$
$$\varphi_0=0_1:1\to i1=1.$$

In effect, so defined, it is easy to see that $F$ is an isomorphism between the underlying groupoids. Verifying the naturality of the isomorphisms $\varphi_{a,b}$, that is, the commutativity of the squares

$$\begin{array}{ccc} (ia)(ib) & \xrightarrow{\varphi_{a,b}} & i(ab) \\ {\scriptstyle (ia)_*\psi_b(u_b)+(ib)_*\psi_a(u_a)}\downarrow & & \downarrow{\scriptstyle \psi_{ab}(a_*u_b+b_*u_a)} \\ (ia)(ib) & \xrightarrow{\varphi_{a,b}} & i(ab), \end{array} \tag{21}$$

for $u_a\in\mathcal{A}_a$, $u_b\in\mathcal{A}_b$, is equivalent (since the groups $\mathcal{A}'_{i(ab)}$ are abelian) to verify the equalities

$$\psi_{ab}(a_*u_b+b_*u_a)=(ia)_*\psi_b(u_b)+(ib)_*\psi_a(u_a), \tag{22}$$

which hold since the naturality of $\psi:\mathcal{A}\cong\mathcal{A}'i$ just says that

$$\psi_{ab}(a_*u_b)=(ia)_*\psi_b(u_b). \tag{23}$$

The coherence condition (10) is verified as follows

$$\begin{aligned}\varphi_{a,b\otimes c} + (0_{Fa}\otimes\varphi_{b,c}) + \boldsymbol{a}'_{Fa,Fb,Fc} &= \varphi_{a,bc} + (ia)_*\varphi_{b,c} + h'(ia,ib,ic) \qquad (24)\\ &= g(a,bc) + (ia)_*g(b,c) + h'(ia,ib,ic) \overset{(19)}{=} \psi_{abc}h(a,b,c) + g(ab,c) + (ic)_*g(a,b)\\ &= \psi_{abc}h(a,b,c) + \varphi_{ab,c} + i(c)_*\varphi_{a,b} = F(\boldsymbol{a}_{a,b,c}) + \varphi_{a\otimes b,c} + (\varphi_{a,b}\otimes 0_{Fc}),\end{aligned}$$

whilst the conditions in (11) and (12) trivially follow from the symmetric cochain conditions $g(a,1) = 0_{ia}$ and $g(a,b) = g(b,a)$, respectively.

Conversely, suppose that

$$F = (F,\varphi,\varphi_0) : \mathcal{S}(M,\mathcal{A},h) \to \mathcal{S}(M',\mathcal{A}',h')$$

is any symmetric monoidal equivalence. By [8, Lemma 3.1], there is no loss of generality in assuming that $F$ is strictly unitary in the sense that $\varphi_0 = 0_1 : 1 \to 1 = F1$.

As the underlying functor establishes an equivalence between the underlying groupoids,

$$F : \textstyle\bigcup_{a\in M}(K(\mathcal{A}_a,1),a) \simeq \bigcup_{a'\in M'}(K(\mathcal{A}'_{a'},1),a'),$$

and these are totally disconnected, it is necessarily an isomorphism. Let us write $i : M \cong M'$ for the bijection describing the action of $F$ on objects; that is, such that $ia = Fa$, for each $a \in M$. Then, $i$ is actually an isomorphism of monoids, since the existence of the structure isomorphisms $\varphi_{a,b} : (ia)(ib) \to i(ab)$ forces the equality $(ia)(ib) = i(ab)$.

Let us write $\psi_a : \mathcal{A}_a \cong \mathcal{A}'_{ia}$ for the isomorphism giving the action of $F$ on automorphisms $u_a : a \to a$, that is, such that $\psi_a u_a = F u_a$, for each $u_a \in \mathcal{A}_a$, and $a \in M$. The naturality of the automorphisms $\varphi_{a,b}$ tell us that the equalities (22) hold (see diagram (21)). These, for the case when $u_a = 0_a$, give the equalities in (23), which amounts to saying that $\psi : \mathcal{A} \cong \mathcal{A}'i$ is a natural isomorphism of abelian group valued functors on $\mathbb{H}M$.

Writing now $g(a,b) = \varphi_{a,b}$, for each $a,b \in M$, the equations $g(a,1) = 0_{ia}$ and $g(a,b) = g(b,a)$ hold just due to the coherence equations (11) and (12), and thus we have a symmetric 2-cochain $g = \big(g(a,b) \in \mathcal{A}'_{i(ab)}\big)_{a,b\in M}$, which satisfies the equations (19) owing to the coherence equations (10), as we can see just by retracting our steps in (24). This means that $\psi_*(h) = i^*(h') + \partial^2(g)$ and, therefore, we have that $\psi_*[h] = i^*[h'] \in H^3(M,\mathcal{A}'i)$, whence $[h] = \psi_*^{-1}i^*[h'] \in H^3(M,\mathcal{A})$, as required. □

**Remark 9.** *Let*

**Symmetric 3-cocycles**

*denote the* category of 3-cocycles of commutative monoids*. That is, the category whose objects are triples $(M,\mathcal{A},h)$ with $M$ a commutative monoid, $\mathcal{A} : \mathbb{H}M \to \mathbf{Ab}$ a functor, and $h \in Z^3(M,\mathcal{A})$ a symmetric 3-cocycle, and whose arrows*

$$(i,\psi,[g]) : (M,\mathcal{A},h) \to (M',\mathcal{A}',h')$$

*are triples consisting of a monoid homomorphism $i : M \to M'$, a natural transformation $\psi : \mathcal{A} \to \mathcal{A}'i$, and the equivalence class $[g]$ of a symmetric 2-cochain $g \in C^2(M,\mathcal{A}'i)$ such that $\psi_*(h) = i^*(h') + \partial^2(g)$ (i.e., equation (19) holds). Two such cochains $g, g' \in C^2(M,\mathcal{A}'i)$ are equivalent if there is a symmetric 1-cochain $f \in C^1(M,\mathcal{A}'i)$ such that $g = g' + \partial^1(f)$. Composition in this category of 3-cocycles is defined in a natural way: The composite of $(i,\psi,[g])$ with $(i',\psi',[g']) : (M',\mathcal{A}',h') \to (M'',\mathcal{A}'',h'')$ is the arrow*

$$(i'i, \psi'i\ \psi, [(\psi'i)_*(g) + i^*(g')]) : (M,\mathcal{A},h) \to (M'',\mathcal{A}'',h''),$$

*where $i'i : M \to M''$ is the composite homomorphism of $i'$ and $i$, $\psi' i\ \psi : \mathcal{A} \to \mathcal{A}'' i' i$ is the natural transformation such that $(\psi' i\ \psi)_a = \psi'_{ia}\psi_a$, the composite homomorphism of $\psi'_{ia} : \mathcal{A}'_{ia} \to \mathcal{A}''_{i'ia}$ with $\psi_a : \mathcal{A}_a \to \mathcal{A}'_{ia}$, for each $a \in M$, and $(\psi' i)_*(g) + i^*(g') \in C^2(M, \mathcal{A}'' i' i)$ is the symmetric 2-cochain given by*

$$((\psi' i)_*(g) + i^*(g'))(a, b) = \psi'_{i(ab)} g(a, b) + g'(ia, ib).$$

*The identity arrow of any object $(M, \mathcal{A}, h)$ is the triple $(id_M, id_{\mathcal{A}}, [0])$.*

*With a slight adaptation of the arguments in the proof of part (ii), Theorem 3.1 can be formulated as an equivalence of categories*

**Symmetric 3-cocycles $\simeq$ Strictly symmetric monoidal abelian groupoids**

*between the category of symmetric 3-cocycles and the category of strictly symmetric monoidal abelian groupoids, $\mathcal{M}$, with iso-classes, $[F] : \mathcal{M} \to \mathcal{M}'$, of symmetric monoidal functors, $F : \mathcal{M} \to \mathcal{M}'$, as arrows. The equivalence of categories is given by the constructions* (14) *on objects and* (20) *on morphisms, that is,*

$$\big((M, \mathcal{A}, h) \xrightarrow{(i,\psi,[g])} (M', \mathcal{A}', h')\big) \ \mapsto\ \big(\mathcal{S}(M, \mathcal{A}, h) \xrightarrow{[\mathcal{S}(i,\psi,g)]} \mathcal{S}(M', \mathcal{A}', h')\big).$$

A *strictly commutative Picard category* [9, Definition 1.4.2] is a strictly symmetric monoidal abelian groupoid $\mathcal{P} = (\mathcal{P}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{c})$ in which, for any object $x$, there is an object $x^*$ with an arrow $x \otimes x^* \to \mathrm{I}$. Actually, the hypothesis of being abelian is superfluous here since a monoidal groupoid in which every object has a quasi-inverse is always abelian [5, Proposition 4.1 (ii)]. Next, we obtain Deligne's classification result for these Picard categories as a corollary of Theorem 8.1 and the lemma below by Mac Lane [21, Theorem 4].

**Lemma 10.** *Let $G$ be any abelian group. For any abelian group $A$, regarded as a constant functor $A : \mathbb{H}G \to \mathbf{Ab}$, the symmetric 3-dimensional cohomology group of $G$ with coefficients in $A$ is zero, that is, $H^3(G, A) = 0$.*

For any abelian groups $G$ and $A$, let $\mathcal{S}(G, A, 0)$ be the strictly symmetric monoidal abelian groupoid built as in (14), for the constant functor $A : \mathbb{H}G \to \mathbf{Ab}$ and the zero 3-cocycle $0 : G^3 \to A$. Since $G$ is a group, $\mathcal{S}(G, A, 0)$ is actually a strictly commutative Picard category. Then, we have

**Corollary 11** (Deligne [9], Fröhlich-Wall [10], Sinh [26])**.** $(i)$ *For any strictly commutative Picard category $\mathcal{P}$, there exist abelian groups $G$ and $A$ and a symmetric monoidal equivalence*

$$\mathcal{S}(G, A, 0) \simeq \mathcal{P}.$$

$(ii)$ *For any abelian groups $G, G', A$ and $A'$, there is a symmetric monoidal equivalence*

$$\mathcal{S}(G, A, 0) \simeq \mathcal{S}(G', A', 0)$$

*if and and only if there are isomorphisms $G \cong G'$ and $A \cong A'$.*

*Proof.* $(i)$ Let $\mathcal{P}$ be a strictly commutative Picard category. By Theorem 8.1, there are a commutative monoid $M$, a functor $\mathcal{A} : \mathbb{H}M \to \mathbf{Ab}$, a 3-cocycle $h \in Z^3(M, \mathcal{A})$, and a symmetric monoidal equivalence $\mathcal{S}(M, \mathcal{A}, h) \simeq \mathcal{P}$.

Then, $\mathcal{S}(M, \mathcal{A}, h)$ is a strictly commutative Picard category as $\mathcal{P}$ is and, therefore, for any $a \in M$, it must exist another $a^* \in M$ with a morphism $a \otimes a^* = aa^* \to \mathrm{I} = 1$ in $\mathcal{S}(M, \mathcal{A}, h)$. Since the groupoid $\mathcal{S}(M, \mathcal{A}, h)$ is totally disconnected, it must be $aa^* = 1$ in $M$, which means that $a^* = a^{-1}$ is an inverse of $a$ in $M$. Therefore, $M = G$ is actually an abelian group.

Let $A = A_1$ be the abelian group attached by $\mathcal{A}$ at the unit of $G$. Then, a natural isomorphism $\phi : A \cong \mathcal{A}$ is defined such that, for any $a \in G$, $\phi_a = a_* : A = A_1 \to \mathcal{A}_a$. Therefore, Theorem 8.1($ii$) and Lemma 10 give the existence of a symmetric monoidal equivalence

$$\mathcal{S}(G, \mathcal{A}, h) \simeq \mathcal{S}(G, A, 0),$$

whence a symmetric monoidal equivalence $\mathcal{S}(G, A, 0) \simeq \mathcal{P}$ follows.

($ii$) This follows directly form Theorem 8.1($ii$). □

**Remark 12.** *As in Remark 9, the classification result above can be formulated in terms of an equivalence between the category of strictly commutative Picard categories, with iso-classes of symmetric monoidal functors as morphisms, and the category of pairs* $(G, A)$ *of abelian groups, with morphisms*

$$(i, \psi, k) : (G, A) \to (G', A')$$

*triples consisting of two group homomorphisms* $i : G \to G'$, $\psi : A \to A'$, *and a cohomology class* $k \in H^2(G, A') = \mathrm{Ext}_{\mathbb{Z}}(G, A')$, *where composition is given by*

$$(i', \psi', k')(i, \psi, k) = (i'i, \psi'\psi, \psi'_*(k) + i^*(k')).$$

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