HAUSDORFF DIMENSION OF METRIC SPACES AND LIPSCHITZ MAPS ONTO CUBES

TAMÁS KELETI, ANDRÁS MÁTHÉ, AND ONDŘEJ ZINDULKA

Abstract. We prove that a compact metric space (or more generally an analytic subset of a complete separable metric space) of Hausdorff dimension bigger than k can be always mapped onto a k-dimensional cube by a Lipschitz map. We also show that this does not hold for arbitrary separable metric spaces.

As an application we essentially answer a question of Urbański by showing that the transfinite Hausdorff dimension (introduced by him) of an analytic subset A of a complete separable metric space is $\lfloor \dim_{\mathsf{H}} A \rfloor$ if $\dim_{\mathsf{H}} A$ is finite but not an integer, $\dim_{\mathsf{H}} A$ or $\dim_{\mathsf{H}} A - 1$ if $\dim_{\mathsf{H}} A$ is an integer and at least ω_0 if $\dim_{\mathsf{H}} A = \infty$.

1. Introduction

Which compact metric spaces X can be mapped onto a k-dimensional cube by a Lipschitz map? Since Lipschitz maps can increase the k-dimensional Hausdorff measure by at most a constant multiple, a necessary condition is $\mathcal{H}^k(X) > 0$, where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. Is this condition sufficient? In 1932 Kolmogorov [12, last sentence of §6] posed a conjecture that would imply an affirmative answer at least for $k = 1$ and $X \subset \mathbb{R}^n$. However, Vitushkin, Ivanov and Melnikov [22] (see also [11] for a less concise proof) constructed a compact subset of the plane with positive 1-dimensional Hausdorff measure that cannot be mapped onto a segment by a Lipschitz map. Konyagin found in the 90's a simpler construction of an abstract compact metric space with the same property but he has not published it.

By proving the following result we show that the condition $\dim_{\mathsf{H}} X > k$, which is just a bit stronger than the necessary condition $\mathcal{H}^k(X) > 0$, is already sufficient.

Theorem 1.1. *If (X, d) is a compact metric space with Hausdorff dimension larger than a positive integer k, then X can be mapped onto a k-dimensional cube by a Lipschitz map.*

In fact we shall prove a more general result (Theorem 2.6) by allowing not only compact metric spaces but also Borel, or even analytic subsets of complete separable metric spaces. But some assumption about the metric space is needed: we show (Theorem 3.1) that there exist separable metric spaces with arbitrarily large Hausdorff dimension that cannot be mapped onto a segment by a uniformly continuous function. Surprisingly, our construction depends on a set theoretical hypothesis that is independent of the standard ZFC axioms: if less than continuum many sets of first category cannot cover the real line then we can give an example in $\mathbb{R}^n$ (Theorem 3.2), otherwise our example is a separable metric space of cardinality less than continuum (Theorem 3.3).

Recall that an *ultrametric space* is a metric space in which the triangle inequality is replaced with the stronger inequality $d(x, y) \le \max(d(x, z), d(y, z))$. For compact

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ultrametric spaces we get the following simple answer to the first question of the introduction.

Theorem 1.2. *A compact ultrametric space can be mapped onto a k-dimensional cube by a Lipschitz map if and only if it has positive k-dimensional Hausdorff measure.*

In fact, we prove (Corollary 2.2) the above result for the following more general (see Lemma 2.3) class of metric spaces.

Definition 1.3. A metric space (X, d) is called *monotone* if there exists a linear order $<$ and a constant C such that

$$\text{diam}([a, b]) \le C \cdot d(a, b) \qquad (\forall a, b \in X), \tag{1}$$

where $[a, b]$ denotes the closed interval $\{x \in X : a \le x \le b\}$.

If this holds for a given C then we can also say that the metric space is *C-monotone*.

This notion has been introduced recently by the third author in [24] and it was studied by him and Nekvinda [19, 20]. Our results about monotone spaces are closely related to some of the results of the mentioned papers but for the sake of completeness we present here brief self-contained proofs. Some further closely related results will be published in [23] by the third author.

In section 4 we use our results to discuss the *Urbański conjecture*: In his paper [21] Urbański defined the *transfinite Hausdorff dimension* $\text{tHD}(X)$ of a separable metric space X as the largest possible topological dimension of a Lipschitz image of a subset of X (cf. (2)). He proved that if the Hausdorff dimension of X is finite then it is an upper bound for the transfinite Hausdorff dimension of X (cf. Theorem 4.1) and conjectured that, roughly speaking, it is close to a lower bound (cf. Conjecture 4.2). We show that the conjecture is correct for analytic subsets of complete separable metric spaces, but consistently fails in general.

2. Nice large metric spaces can be mapped onto cubes

First we prove the following result about mapping monotone metric spaces onto an interval by a Hölder function.

Theorem 2.1. *If (X, d) is a compact monotone metric space with positive s-dimensional Hausdorff measure (for some $s > 0$) then X can be mapped onto a non-degenerate interval by an s-Hölder function.*

Proof. By Frostman lemma (see e.g. in [15, Theorem 8.17]) we can choose a nonzero finite Borel measure μ on X so that $\mu(E) \le (\text{diam}(E))^s$ for any $E \subset X$. Since X is a monotone metric space there exists a linear order $<$ and a constant C such that (1) of Definition 1.3 holds. It is easy to show (see also in [20]) that any open interval $(a, b) = \{x \in X : a < x < b\}$ is open, so any interval of X is Borel. For $x \in X$ let $g(x) = \mu((-\infty, x))$, where $(-\infty, x) = \{y \in X : y < x\})$. Then g is s-Hölder since for any $a, b \in X$, $a < b$ we have

$$0 \le g(b) - g(a) = \mu([a, b)) \le \text{diam}([a, b))^s \le (C \cdot d(a, b))^s.$$

Thus g is continuous, so $g(X) \subset \mathbb{R}$ is compact.

Now we show that $g(X)$ is not a singleton. Since X is compact and the intervals of the form $(-\infty, b)$ and (a, ∞) are open, there exists a minimal element x_- and a maximal element x_+ in $(X, <)$. Then $g(x_-) = \mu(\emptyset) = 0$ and $g(x_+) = \mu((-\infty, x_+)) = \mu(X \setminus \{x_+\})$. Since $\mu(X) > 0$ and $\mu(\{x_+\}) \le (\text{diam}(\{x_+\}))^s = 0$ we get that indeed $g(x_+) > g(x_-)$.

Therefore all we need to prove is that there is no $u, v \in g(X)$ with $u < v$ and $(u,v) \cap g(X) = \emptyset$. Suppose there exist such u and v. Since X is a compact metric space, it is separable, let S be a countable dense subset of X. Let $S_1 = \{s \in S : g(s) \le u\}$ and $S_2 = \{t \in S : g(t) \ge v\}$. Since $(u,v) \cap g(X) = \emptyset$ we must have $S = S_1 \cup S_2$. Since S_1 and S_2 are countable and $g(x) = \mu((-\infty, x))$ we get that $\mu\big(\bigcup_{s\in S_1}(-\infty, s)\big) \le u$ and $\mu\big(\bigcap_{t\in S_2}(-\infty, t)\big) \ge v$. Hence letting $D = \bigcap_{t\in S_2}(-\infty, t) \setminus \bigcup_{s\in S_1}(-\infty, s))$ we get $\mu(D) \ge v - u > 0$. On the other hand, the set D must have at most two points since if $x, y, z \in D$ and $x < y < z$ then (x, z) would be a nonempty open set not containing any point of the dense set $S = S_1 \cup S_2$. Since $\mu(E) \le (\operatorname{diam}(E))^s$ for any $E \subset X$, the singletons must have zero μ measure, so we get $\mu(D) = 0$ contradicting the previously obtained $\mu(D) > 0$. □

Corollary 2.2. *Let X be a compact monotone metric space and let k be a positive integer. Then X can be mapped onto the k-dimensional cube $[0,1]^k$ by a Lipschitz map if and only if X has positive k-dimensional Hausdorff measure.*

Proof. It is clear that $\mathcal{H}^k(X) > 0$ is a necessary condition.

To prove that it is sufficient note that by the previous theorem there exists a k-Hölder map $g : X \to \mathbb{R}$ such that $g(X) = [0,1]$. It is well-known (see e.g. [18, Theorem 4.55]) that there exists a $\frac{1}{k}$-Hölder Peano curve $h : [0,1] \to [0,1]^k$. Then the composition $h \circ g$ is a Lipschitz map that maps X onto $[0,1]^k$. □

Nekvinda and Zindulka [20] proved that an ultrametric space is always a monotone metric space. For compact ultrametric spaces even the following stronger result can be proved fairly easily.

Lemma 2.3. *Any compact ultrametric space (X, d) is 1-monotone.*

Proof. The following tree structure is well-known but as it can be obtained quickly we give a self-contained proof.

Let $D = \operatorname{diam} X$. If $D = 0$ then let $k = 1$ and $X_1 = X$. Otherwise, since d is an ultrametric, the relation $d(x,y) < D$ is an equivalence relation. The equivalence classes are open, X is compact, so there are only finitely many equivalence classes: $X_1, \dots, X_k$. Thus $X_1, \dots, X_k$ are closed and compact as well. Since X is compact, there exist two points with distance D, so $k \ge 2$, unless $D = 0$. Note also that if a and b are from distinct equivalence classes then $d(a,b) = D$.

Since each equivalence class is a compact ultrametric space we can do the same for each of them. This way we get a tree of clopen sets $X_{i_1\dots i_m}$ with the property that for a fixed m these sets give a partition of X and if $a \in X_{i_1\dots i_m j}$, $b \in X_{i_1\dots i_m j'}$ and $j \ne j'$ then $d(a,b) = \operatorname{diam}(X_{i_1\dots i_m})$. The first property implies that for any $x \in X$ there exists a unique sequence $J(x) = (i_1, i_2 \dots)$, so that $x \in X_{i_1\dots i_m}$ for every m. On the other hand, $\lim_{m\to\infty} \operatorname{diam}(X_{i_1\dots i_m}) = 0$, since otherwise picking one point from each $X_{i_1\dots i_m} \setminus X_{i_1\dots i_{m+1}}$ we would get an infinite discrete subspace. Hence $\{x\} = \bigcap_{m=1}^{\infty} X_{i_1\dots i_m}$, therefore the function J is injective.

This injectivity of J ensures that the following pull-back of the lexicographic order via J is an order on X: let $x < y$ if $J(x)$ is smaller than $J(y)$ in the lexicographical order. We need to show that for any $a, b \in X$ we have $\operatorname{diam}([a,b]) = d(a,b)$. Let $(i_1, \dots, i_m)$ be the longest common initial segment of a and b. Then, as we saw above, $d(a,b) = \operatorname{diam}(X_{i_1\dots i_m})$. On the other hand $[a,b] \subset X_{i_1\dots i_m}$, so we get $d(a,b) \le \operatorname{diam}([a,b]) \le \operatorname{diam}(X_{i_1\dots i_m}) = d(a,b)$, which completes the proof. □

The following result is a weaker version of [16, Theorem 1.4].

Theorem 2.4 (Mendel and Naor [16])**.** *For every compact metric space (X, d) and $\varepsilon > 0$ there exists a closed subset $Y \subset X$ such that $\dim_{\mathsf{H}} Y \ge (1 - \varepsilon) \dim_{\mathsf{H}} X$ and (Y, d) is bi-Lipschitz equivalent to an ultrametric space.*

We will also need the following result.

Theorem 2.5 (Howroyd [9])**.** *Let $s > 0$. If an analytic subset of a separable complete metric space is of infinite s-dimensional Hausdorff meausure then it has a compact subset of finite and positive s-dimensional Hausdorff measure.*

Now we are ready to prove the main result of the paper.

Theorem 2.6. *Let A be an analytic subset of a separable complete metric space (X, d), and let k be a positive integer. If $\dim_{\mathsf{H}} A > k$ then A can be mapped onto the k-dimensional cube $[0,1]^k$ by a Lipschitz map.*

Proof. Let $s \in (k, \dim_{\mathsf{H}} A)$. By the above Howroyd theorem, A has a compact subset C with finite and positive s-dimensional Hausdorff measure. By the above Mendel–Naor theorem, C has a closed subset E with $\dim_{\mathsf{H}} E > k$ that is bi-Lipschitz equivalent to an ultrametric space. Applying Howroyd's theorem again we get a compact subset B of E with positive and finite k-dimensional Hausdorff measure. Clearly, B is also bi-Lipschitz equivalent to a compact ultrametric space Y. By Lemma 2.3 and Corollary 2.2, Y can be mapped onto $[0,1]^k$ by a Lipschitz map. Since $B \subset A$ is bi-Lipschitz equivalent to Y, this means that B can be also mapped onto $[0,1]^k$ by a Lipschitz map. Since real valued Lipschitz functions can be always extended as Lipschitz functions, by extending the coordinate functions we get a Lipschitz function that maps A onto $[0,1]^k$. □

Remark 2.7. As we saw in the introduction, in the above theorem the condition $\dim_{\mathsf{H}} A > k$ cannot be replaced by the condition that A has positive k-dimensional Hausdorff measure, not even if A is a compact subset of $\mathbb{R}^n$.

Remark 2.8. The following argument shows that if (X, d) is the Euclidean space $\mathbb{R}^n$, or more generally if it is a complete doubling metric space then we can also prove Theorem 2.6 without using the recent deep theorem of Mendel and Naor (instead we use more classical theorems of Assouad and Mattila, and we still need Howroyd's theorem).

First, let A be an analytic set in $\mathbb{R}^n$ with $\dim_{\mathsf{H}} A > k$. Let $C \subset A$ be compact with $\dim_{\mathsf{H}} C > k$. Choose S to be a self-similar set in $\mathbb{R}^n$ with the strong separation condition (which means that S is the disjoint union of sets similar to S) with $\dim_{\mathsf{H}} S > \max(n - (\dim_{\mathsf{H}} C - k), (n+1)/2)$. By a theorem of Mattila [15, Theorem 13.11] there exists an isometry φ such that $\dim_{\mathsf{H}} C \cap \varphi(S) > k$. It is easy to check that a self-similar set with the strong separation condition is bi-Lipschitz equivalent to an ultrametric space, so $C \cap \varphi(S)$ is bi-Lipschitz equivalent to a compact ultrametric space X. Since ultrametric spaces are monotone this implies by Corollary 2.2 that $C \cap \varphi(S)$ can be mapped onto $[0,1]^k$ by a Lipschitz function. By extending the Lipschitz function onto A we get a Lipschitz function that maps A onto $[0,1]^k$.

By Theorem 2.1 we also get that A can be mapped onto $[0,1]$ by an s-Hölder function for some $s > k$. By the Assouad embedding theorem [2] for any doubling metric space (X, d) and $\varepsilon > 0$ the metric space $(X, d^{1-\varepsilon})$ admits a bi-Lipschitz embedding into a Euclidean space. Combining these results we get that if B is an analytic subset of a complete doubling metric space (X, d) and $\dim_{\mathsf{H}}(B) > k$ then B can be mapped onto $[0,1]$ by a k-Hölder function. Then, composing this map with a $\frac{1}{k}$-Hölder Peano curve as in the proof of Corollary 2.2, we get a Lipschitz map from B onto $[0,1]^k$.

3. Large metric spaces that cannot be mapped onto a segment

The main result (Theorem 2.6) of the previous section said that reasonably nice metric spaces of large Hausdorff dimension can be always mapped onto large dimensional cube by a Lipschitz map. The following result, which is the main result of this section, shows that some assumption on the metric space is necessary, even if we allow not only Lipschitz functions but also uniformly continuous functions.

Theorem 3.1. *There exist separable metric spaces with arbitrarily large Hausdorff dimension that cannot be mapped onto a segment by a uniformly continuous function.*

We obtain the above theorem by proving the following two results.

Theorem 3.2. *If less than continuum many sets of first category cannot cover* $\mathbb{R}$*, then for any* n *there exists a set* $A \subset \mathbb{R}^n$ *of Hausdorff dimension* n *that cannot be mapped onto a segment by a uniformly continuous function.*

Theorem 3.3. *If less than continuum many sets of first category can cover* $\mathbb{R}$*, then there exist separable metric spaces of arbitrarily high Hausdorff dimension such that their cardinality is less than continuum, consequently they cannot be mapped onto a segment by any funtion.*

The hypotheses of the above two theorems are abbreviated by $\operatorname{cov}\mathcal{M} = \mathfrak{c}$ and $\operatorname{cov}\mathcal{M} < \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality continuum, $\mathcal{M}$ is the collection of subsets of $\mathbb{R}$ of first category, and $\operatorname{cov}\mathcal{M}$ stands for the least cardinality of a collection of sets of first category that can cover $\mathbb{R}$. It is well known (see e.g. in [4, Chapter 7]) that both $\operatorname{cov}\mathcal{M} = \mathfrak{c}$ and $\operatorname{cov}\mathcal{M} < \mathfrak{c}$ are consistent with the standard ZFC axioms of set theory. Note that the Continuum Hypothesis (CH) clearly implies $\operatorname{cov}\mathcal{M} = \mathfrak{c}$ but it is also consistent with ZFC that CH fails but $\operatorname{cov}\mathcal{M} = \mathfrak{c}$ holds (see e.g. in [4, Chapter 7]).

First we prove Theorem 3.2. The following result is probably known but for completeness we present a proof.

Theorem 3.4. *If Continuum Hypothesis holds then for any* n *there exists a set* $A \subset \mathbb{R}^n$ *such that for any continuous function* $f : \mathbb{R}^n \to \mathbb{R}$ *the set* $f(A)$ *does not contain any interval and* A *is not a Lebesgue null set.*

In fact, instead of the CH*, it is enough to assume the following hypothesis, which clearly follows from* CH*.*

($\star$) *Less than continuum many closed measure zero sets and a set of measure zero cannot cover* $\mathbb{R}^n$*.*

Proof. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ and $\{N_\alpha : \alpha < \mathfrak{c}\}$ be enumerations of the collection of $\mathbb{R}^n \to \mathbb{R}$ continuous functions and the collection of Lebesgue null Borel subsets of $\mathbb{R}^n$, respectively.

By transfinite induction for every $\alpha < \mathfrak{c}$ we construct points $x_\alpha \in \mathbb{R}^n$ and $y_\alpha \in (0,1)$ such that

(i) $x_\alpha \notin N_\alpha$,
(ii) $x_\alpha \notin \bigcup_{\beta<\alpha} f_\beta^{-1}(\{y_\beta\})$,
(iii) $y_\alpha \notin f_\alpha(\{x_\beta : \beta \le \alpha\})$ and
(iv) $f_\alpha^{-1}(\{y_\alpha\})$ has Lebesgue measure zero.

In the α-th step we suppose that (i)–(iv) hold for smaller indices. Since for each $\beta < \alpha$ the set $f_\beta^{-1}(\{y_\beta\})$ is a closed set of measure zero, ($\star$) implies that we can choose x_α so that (i) and (ii) hold. We can choose $y_\alpha \in (0,1)$ so that (iii) and (iv) hold since more than countably many of the pairwise disjoint closed sets $f_\alpha^{-1}(\{t\})$ ($t \in (0,1)$) cannot have positive measure.

Let $A = \{x_\alpha : \alpha < \mathfrak{c}\}$. This set cannot have zero measure by (i). If there exists a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(A)$ contains an interval then the image of a linear transformation contains $(0,1)$, so there exists an $\alpha < \mathfrak{c}$ for which $f_\alpha(A) \supset (0,1)$. But this is impossible since $f_\alpha(x_\beta) \neq y_\alpha$ for any $\beta \le \alpha$ by (iii) and for any $\beta > \alpha$ by (ii), so $y_\alpha \notin f_\alpha(A)$. □

Since real valued uniformly continuous functions can be always extended we get the following.

Corollary 3.5. *If Continuum Hypothesis (or $(\star)$ of Theorem 3.4) holds then for any n there exists a set $A \subset \mathbb{R}^n$ such that no subset of A can be mapped onto a segment by a uniformly continuous map and A is not a Lebesgue null set, so it has Hausdorff dimension n.*

Corollary 3.5 together with the following result clearly completes the proof of Theorem 3.2. The result itself is known but we have not found it in the literature in this exact form, so for completeness we show how it follows from published theorems.

Theorem 3.6. *The hypothesis* $\operatorname{cov}\mathcal{M} = \mathfrak{c}$ *implies* $(\star)$.

Proof. Let 2^ω be the product space $\{0,1\} \times \{0,1\} \times \dots$ and let us consider the natural uniformly distributed product probability measure on it.

By a well-known theorem of Bartoszyński and Shelah [5] (see also [4, Theorem 2.6.14]) $\operatorname{cov}\mathcal{M}$ is also the least cardinality of a collection of closed sets of measure zero in 2^ω such that the union is not a set of measure zero. (In this theorem $\mathcal{M}$ is meant on 2^ω but it does not change $\operatorname{cov}\mathcal{M}$ since $(\mathbb{R}, \mathcal{M}_R)$ and $(2^\omega, \mathcal{M}_{2^\omega})$ are isomorphic, see e.g. [8, 522V(b)(i)].) By this theorem $\operatorname{cov}\mathcal{M} = \mathfrak{c}$ implies that

$(\star\star)$ in 2^ω less than continuum many closed measure zero sets and a set of measure zero cannot cover 2^ω.

We need to prove that this implies that the same is true in $\mathbb{R}^n$. For this we need a continuous measure preserving map $f : 2^\omega \to [0,1]^n$. Suppose that we have such a function and $(\star)$ is false, so there exists a decomposition $\mathbb{R} = N \cup \bigcup_{\alpha \in I} F_\alpha$, where N is a set of measure zero, I has cardinality less than continuum, and each F_α is a closed set of measure zero. Then we get decomposition $2^\omega = f^{-1}(\mathbb{R}^n) = f^{-1}(N) \cup \bigcup_{\alpha \in I} f^{-1}(F_\alpha)$, which contradicts $(\star\star)$.

So it remains to show a continuous measure preserving map $f : 2^\omega \to [0,1]^n$. This is well known and easy: let the j-th coordinate of $f((a_k))$ be $\sum_{i=0}^{\infty} a_{in+j} 2^{-i-1}$. □

Remark 3.7. Some set theoretical assumption is needed in Theorem 3.2, it cannot be proved in ZFC: Corazza [6] constructed a model in which every positive Hausdorff dimensional subset of a Euclidean space can be mapped onto $[0,1]$ by a uniformly continuous function.

Now it remains to prove Theorem 3.3.

Let $\mathcal{H}^\varphi$ denote the Hausdorff measure with strictly increasing bijective gauge function $\varphi : [0,\infty) \to [0,\infty)$. A metric space X is of *strong measure zero* if $\mathcal{H}^\varphi(X) = 0$ for any gauge function φ.

We construct our example in ω^ω. For $x \neq y$ in ω^ω denote by $|x \wedge y|$ the length of the common initial segment of x and y. Note that, for any decreasing function $g : \{1,2,\dots\} \to (0,\infty)$ with $\lim_n g(n) = 0$, by letting $d_g(x,y) = g(|x \wedge y| + 1)$ we get a separable metric space (ω^ω, d_g). One of the classical metrics on ω^ω is d_{g_0} for $g_0(m) = 1/m$.

The following theorem clearly implies Theorem 3.3, so this will also complete the proof of Theorem 3.1.

Theorem 3.8. *For any gauge function φ there exists a separable metric space (X,d) of cardinality* $\operatorname{cov}\mathcal{M}$ *with* $\mathcal{H}^\varphi((X,d)) > 0$.

Proof. Fremlin and Miller [17, Theorem 5] proved that $\operatorname{cov}\mathcal{M}$ is also the least cardinality of a subset of (ω^ω, d_{g_0}) that is not a strong measure zero subspace. Therefore there exists $H \subset \omega^\omega$ of cardinality $\operatorname{cov}\mathcal{M}$ such that (H, d_{g_0}) is not a strong measure zero metric space, so there exists a gauge function φ_0 such that $\mathcal{H}^{\varphi_0}(H, d_{g_0}) > 0$. Set $g = \varphi^{-1} \circ \varphi_0 \circ g_0$. Then $\varphi_0 \circ g_0 = \varphi \circ g$ and so $\mathcal{H}^{\varphi}(H, d_g) = \mathcal{H}^{\varphi_0}(H, d_{g_0}) > 0$. Therefore (H, d_g) is a separable metric space of cardinality $\operatorname{cov} \mathcal{M}$ with $\mathcal{H}^{\varphi}((H, d_g)) > 0$. □

4. Transfinite Hausdorff dimension

Urbański [21] introduced the transfinite Hausdorff dimension (tHD) of a metric space X in the following way:

(2) $\quad \mathrm{tHD}(X) = \sup\{\mathrm{ind}\, f(Y) : Y \subset X,\ f : Y \to Z \text{ Lipschitz, } Z \text{ a metric space}\},$

where ind denotes the transfinite small inductive topological dimension (see e.g. in [7]). He showed the following connection between Hausdorff dimension and transfinite Hausdorff dimension.

Theorem 4.1 (Urbański [21, Theorem 2.8])**.** *If X is a metric space with finite Hausdorff dimension then* $\mathrm{tHD}(X) \le \lfloor \dim_{\mathsf{H}} X \rfloor$*, where* $\lfloor . \rfloor$ *denotes the floor function.*

After noticing that equality is not always true (if C is a Cantor set with zero Lebesgue measure but Hausdorff dimension 1 then $\mathrm{tHD}(C) = 0$) Urbański stated the following conjecture.

Conjecture 4.2 (Urbański [21, Conjecture 6.1])**.** *If X is a metric space with finite Hausdorff dimension then* $\mathrm{tHD}(X) \ge \lfloor \dim_{\mathsf{H}} X \rfloor - 1$*. Consequently* $\mathrm{tHD}(X) \in \{\lfloor \dim_{\mathsf{H}} X \rfloor - 1, \lfloor \dim_{\mathsf{H}} X \rfloor\}$.

It is not hard to see that one cannot prove this conjecture without some additional assumptions: It is well-known (see e.g. in [4]) that for any n the existence of a set $X \subset \mathbb{R}^n$ with positive Lebesgue outer measure and cardinality less than continuum is consistent with ZFC. Then $\dim_{\mathsf{H}} X = n$ but $\mathrm{tHD}(X) = 0$ since any image of any subset of X has cardinality less than continuum and it is easy to check that any set of cardinality less than continuum must have zero topological dimension (since we can easily find balls of arbitrarily small radius with empty boundary around any point).

Theorem 2.6 immediately implies that if we have some reasonable assumption about the metric space then even the following stronger form of Conjecture 4.2 holds.

Theorem 4.3. *If A is an analytic subset of a complete separable metric space then* $\mathrm{tHD}(A) \ge \lceil \dim_{\mathsf{H}} A \rceil - 1$*, where* $\lceil . \rceil$ *denotes the ceiling function.*

Combining Theorems 4.1 and 4.3 we get the following.

Corollary 4.4. *Let A be an analytic subset of a complete separable metric space.*

- *If* $\dim_{\mathsf{H}} A$ *is finite but not an integer then* $\mathrm{tHD}(A) = \lfloor \dim_{\mathsf{H}} A \rfloor$*,*
- *if* $\dim_{\mathsf{H}} A$ *is an integer then* $\mathrm{tHD}(A)$ *is* $\dim_{\mathsf{H}} A$ *or* $\dim_{\mathsf{H}} A - 1$*, and*
- *if* $\dim_{\mathsf{H}} A = \infty$ *then* $\mathrm{tHD}(A) \ge \omega_0$*.*

Remark 4.5. Corollary 4.4 is sharp in the sense that if $\dim_{\mathsf{H}} A = n \in \mathbb{N}$ then $\mathrm{tHD}(A)$ can be either n or $n-1$: the trivial $A = \mathbb{R}^n$ is clearly an example for the former one and we claim that any A with zero n-dimensional Hausdorff measure is an example for the latter one. It is well-known (see e.g. [10]) that the topological dimension of a metric space with zero n-dimensional Hausdorff measure is at most $n-1$. Therefore if A has zero n-dimensional Hausdorff measure then the Lipschitz image of any of its subsets also has zero n-dimensional Hausdorff measure, so its topological dimension is at most $n-1$, which means that $\mathrm{tHD}(A) \le n-1$.

Our final goal is to show (in ZFC) that Theorem 4.3 and Corollary 4.4 do not hold for a general separable metric space. The proof of the following useful observation uses a well-known argument that can be found for example in [3] or [25, 4.2].

Lemma 4.6. *For any metric space X,* $\mathrm{tHD}(X) = 0$ *if and only if X cannot be mapped onto a segment by a Lipschitz map.*

Proof. It is clear that if $\mathrm{tHD}(X) = 0$ then X cannot be mapped onto a segment by a Lipschitz map. To prove the converse suppose that $Y \subset X$ and $f : Y \to Z$ is a Lipschitz map onto a metric space (Z, ρ) of positive topological dimension. The latter implies that there is $z_0 \in Z$ and $r > 0$ such that every sphere $\{z \in Z : \rho(z_0, z) = s\}$ of radius $s \leq r$ is nonempty. It follows that the Lipschitz function $g(z) = \min(\rho(z_0, z), r)$ maps Z onto $[0, r]$. Thus a Lipschitz extension of $g \circ f$ maps X onto $[0, r]$. □

Combining Theorem 3.1 and Lemma 4.6 we get the following.

Theorem 4.7. *There exist separable metric spaces with zero transfinite Hausdorff dimension and arbitrarily large Hausdorff dimension.*

5. Further directions for research

There are (at least) two possible natural directions of future research:

1. We still do not know the answer to the first question of the introduction: Which compact metric spaces X can be mapped onto a k-dimensional cube by a Lipschitz map?

Our Theorem 1.1 gives that $\dim_{\mathsf{H}} X > k$ is a sufficient condition but this is clearly not a necessary condition (the k-dimensional cube itself is a trivial counterexample).

As we saw in the introduction, the trivial necessary condition that the k-dimensional Hausdorff measure $\mathcal{H}^k(X)$ is positive is not sufficient, not even if X is a compact subset of $\mathbb{R}^n$. However, for $k = n$ this might be sufficient: it is a well-known long standing conjecture of Laczkovich [13] that a measurable set with positive Lebesgue measure in $\mathbb{R}^n$ can be always mapped onto an n-dimensional cube by a Lipschitz map. This is known to be true for $n \leq 2$ (see [1] and [14]), and there is recent (still unpublished) progress for $n \geq 3$ due to Csörnyei and Jones.

Our Corollary 2.2 gives that for compact monotone metric spaces X (in particular for compact ultrametric spaces) the condition $\mathcal{H}^k(X) > 0$ is necessary and sufficient. It would be nice to know more about those compact metric spaces for which the condition $\mathcal{H}^k(X) > 0$ is necessary and sufficient. A full characterization of these spaces seems to be hopeless since, as we saw above, we do not even know if the compact subspaces of $\mathbb{R}^k$ have this property. The other difficulty is that it is hard to construct compact metric spaces that do not have this property, only the two ingenious constructions mentioned in the introduction are known: the construction of Vitushkin, Ivanov and Melnikov [22] (see also [11]) is really complicated, the unpublished construction of Konyagin is not simple either. It would be nice to have simpler examples.

2. Although the main results Theorem 3.1, Corollary 4.4 and Theorem 4.7 of Sections 3 and 4 do not depend on any set theoretical hypothesis (in other words these results are proved in ZFC), there are several statements that we cannot prove in ZFC, only under some hypothesis. As it is explained in Remark 3.7, Theorem 3.2 cannot be proved in ZFC. But for the following weaker and weaker statements we do not know if they can be proved in ZFC or they are independent of ZFC (which is the case if the negation of the statement is also consistent with ZFC).

(i) For any n there exists a set $A \subset \mathbb{R}^n$ with Hausdorff dimension n that cannot be mapped onto a segment by a Lipschitz function.
(ii) There exist separable metric spaces with arbitrarily large finite Hausdorff dimension that cannot be mapped onto a segment by a Lipschitz function.
(ii') There exist separable metric spaces with zero transfinite Hausdorff dimension and arbitrarily large finite Hausdorff dimension.
(iii) The original form (Conjecture 4.2) of Urbanski's conjecture is false.

Note that (ii) $\Leftrightarrow$ (ii') holds by Lemma 4.6, the implications (i) $\Rightarrow$ (ii) and (ii') $\Rightarrow$ (iii) are clear, and by Theorem 3.2 $\operatorname{cov}\mathcal{M} = \mathfrak{c}$ (so in particular CH as well) implies all four statements.

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Institute of Mathematics, Eötvös Loránd University, Pázmány Péter s. 1/c, 1117 Budapest, Hungary

E-mail address: tamas.keleti@gmail.com

URL: http://www.cs.elte.hu/analysis/keleti

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom

E-mail address: A.Mathe@warwick.ac.uk

URL: http://homepages.warwick.ac.uk/~masibe/

Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 160 00 Prague 6, Czech Republic

E-mail address: zindulka@mat.fsv.cvut.cz

URL: http://mat.fsv.cvut.cz/zindulka