arXiv:0706.2431v1 [math.CO] 16 Jun 2007

ON THE CORDIAL DEFICIENCY OF COMPLETE MULTIPARTITE GRAPHS

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Abstract. We calculate the cordial edge deficiencies of the complete multipartite graphs and find an upper bound for their cordial vertex deficiencies. We also give conditions under which the tensor product of two cordial graphs is cordial.

1. Introduction and definitions

Cahit [2] introduced cordial graph labelings as a generalization of both graceful and harmonious labelings. See Gallian [3] for a comprehensive bibliography on the topic. We allow graphs to have multiple edges but not loops. A *binary labeling* of a graph G is a map $f : V(G) \to \{0, 1\}$. Two real numbers are said to be *roughly equal* if $0 \leq |x-y| \leq 1$. A binary labeling of G is *friendly* provided that $|f^{-1}(0)|$ is roughly equal to $|f^{-1}(1)|$. A binary labeling f of G induces a labeling $f_e : E(G) \to \{0, 1\}$ by $f_e(uv) = f(u) + f(v)$ where $uv \in E(G)$ and the sum is calculated modulo 2. A friendly labeling f of G is called *cordial* when $|f_e^{-1}(0)|$ is roughly equal to $|f_e^{-1}(1)|$, and a graph G is called *cordial* if if has a cordial labeling.

In [6] I introduced a new measure of the degree to which a noncordial graph fails to be cordial, inspired by Kotzig's and Rosa's notion of *edge-magic deficiency* [5], and studied another such measure, first defined in [1]. Note first that every friendly labeling f of a graph can be made into a cordial labeling of an augmented graph G' by adding no more than $|f_e^{-1}(0) - f_e^{-1}(1)| - 1$ edges between appropriate pairs of vertices so that $|f_e^{-1}(0)|$ becomes roughly equal to $|f_e^{-1}(1)|$. The minimum number of edges, taken over all friendly labelings of G, which it is necessary to add in order

1991 *Mathematics Subject Classification.* 05C78.
Key words and phrases. Cordial graph.
I would like to thank Sam Patteson and his colleagues for providing an ideal working environment for the writing of this paper.

that G' become cordial is called the *cordial edge deficiency* of G, denoted by $\mathrm{ced}(G)$. This is essentially the same concept as the *index of cordiality* introduced in [1]. If it is possible to find a binary labeling f of G so that $|f_e^{-1}(0)|$ and $|f_e^{-1}(1)|$ are roughly equal, then it is possible to make f into a cordial labeling of an augmented graph G' by adding no more than $|f^{-1}(0) - f^{-1}(1)| - 1$ isolated vertices labeled in such a way as to make f into a friendly labeling. The minimum number of vertices, taken over all such binary labelings of G, which it is necessary to add in order to make G' cordial is called the *cordial vertex deficiency* of G, denoted by cvd(G). If there are no such binary labelings of G we call G *strictly noncordial* and write $\mathrm{cvd}(G) = \infty$. If G is a graph and f is a binary labeling of G then the *cordial deficit* of the pair (G, f) is $||f_e^{-1}(0)| - |f_e^{-1}(1)||$.

Finally, we will be studying the cordiality of the tensor products of graphs. If G and H are graphs, then the tensor product $G \times H$ is the graph whose vertex set is the cartesian product of the vertex sets of G and H, namely $V(G) \times V(H)$, in which $((u_1, u_2), (v_1, v_2)) \in E(G \times H)$ if and only if $u_1v_1 \in E(G))$ and $u_2v_2 \in E(H)$. The tensor product is also known as the weak product and as the categorical product of G and H. Considerable effort has focused on tensor products due to Hedetniemi's conjecture, for a useful survey of which see [7].

2. Comments on Lee's and Liu's constructions

In 1991 Lee and Liu [4] published the following theorem:

Theorem 1. *Let H be a graph with an even number of edges and a cordial labeling such that the vertices of H can be divided into ℓ parts $H_1, H_2, \ldots, H_\ell$ each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let G be any graph and $G_1, G_2, \ldots, G_\ell$ be any ℓ subsets of the vertices of G. Let (G,H) be the graph which is the disjoint union of G and H augmented by edges joining every vertex in G_i to every vertex in H_1 for $1 \leq \ell$. Then G is cordial if and only if (G, H) is.*

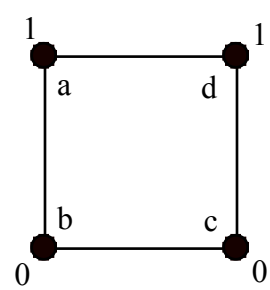
They provide an explicit proof that the cordiality of G implies the cordiality of (G, H) and state that "the converse can be proved in the same way." Unfortunately

this is not the case for the theorem as it is stated. In fact the converse as stated is false. The theorem would in fact be true if it were required that the restriction of the cordial labeling of (G, H) to the vertices of H is a cordial labeling of H This need not be true for every cordial labeling of (G, H) as we will show below with a counterexample. It is extremely important to note that the only problem with this theorem is its statement. In fact, Lee and Liu use the correct version throughout the paper, so that not only are the other theorems in the paper correct, but the proofs are in fact correct as well. In order for the statement of the theorem to be correct, the last sentence must be replaced with:

> Then G is cordial if and only if (G, H) has a cordial labeling which, when restricted to H yields an equal number of vertices labeled 0 and vertices labeled 1 in each H_i

Counterexample 1: Let $H = C_4$. Then (i) H is cordial, (ii) H has an even number of edges, and (iii) H has a cordial labeling such that the vertices of H can be divided into two parts H_1 and H_2 such that the number of vertices labeled 0 is equal to the number of vertices labeled 1 (see Figure 1).

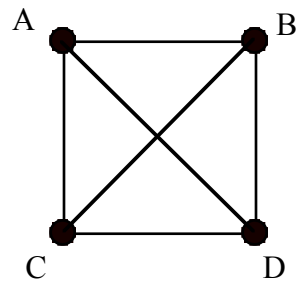
FIGURE 1.



Note that $H_1 = \{a, b\}$ and $H_2 = \{c, d\}$. Now let $G = K_4$. Cahit [2] showed that K_n is cordial if and only if $n \leq 3$. We label the vertices of K_4 as shown in Figure 2.

Let $G_1 = \{A, B\}$ and $G_2 = \{B, C\}$. It is easily checked that the following is a cordial labeling of (G, H):

Figure 2.



a	b	c	d	A	B	C	D
0	1	1	1	0	0	1	0

3. Cordial edge-deficiency of complete multipartite graphs

One of Lee and Liu's most interesting results is:

Theorem 2. *A complete k-partite graph is cordial if and only if the number of parts with an odd number of vertices is at most three.*

We generalize this using the concept of cordial edge-deficiency thus:

Theorem 3. *Let G be a complete multipartite graph with k odd parts. Then ced(G)= max* $\left\{0, \left\lfloor \frac{k}{2} \right\rfloor - 1\right\}$

Proof: Let $f : V(G) \to \{0, 1\}$ be a friendly labeling of G. Let $E_1, \ldots, E_j$ be the even parts of G. Suppose that an even part E_i is assigned fewer zeros than ones by the labeling f. Because f is a friendly labeling there must be another part of G which has more zeros than ones. There are two cases to consider, depending on whether this other part is even or odd. In the first case, suppose that E_ℓ has more zeros than ones. Let E_i have z_i zeros and E_ℓ have z_ℓ zeros. Furthermore, assume that $|E_i| = 2m_i$ and that $|E_\ell| = 2m_\ell$. Note that E_i has $2m_i - z_i$ ones and that E_ℓ has $2m_\ell - z_\ell$ ones. If we switch a zero label from E_i with a zero label from E_ℓ we produce a new friendly labeling $g : V(G) \to \{0, 1\}$ in which there is no change in the number of edges labeled zero nor in the number of edges labeled one except possibly amongst the edges with one end in E_i and the other end in E_ℓ. Note that

$$|f_{e*}^{-1}(0)| = z_i z_\ell + (2m_i - z_i)(2m_\ell - z_\ell)$$

and that

$$|f_{e*}^{-1}(1)| = z_i(2m_\ell - z_\ell) + z_\ell(2m_i - z_i)$$

where f_{e*} represents f_e restricted to the complete bipartite subgraph of G generated by E_i and E_ℓ. The difference between these is therefore

$$||f_{e*}^{-1}(0)| - |f_{e*}^{-1}(1)|| = 4(m_i - z_i)(z_\ell - m_\ell)$$

After the switch is completed, E_i has $z_i + 1$ zeros and $(2m_i - z_i - 1)$ ones and E_ℓ has $z_\ell - 1$ zeros and $2m_\ell - z_\ell + 1$ ones. Hence, as above,

$$|g_{e*}^{-1}(0)| = (z_i + 1)(z_\ell - 1) + (2m_i - z_i - 1)(2m_\ell - z_\ell + 1)$$

and

$$|g_{e*}^{-1}(1)| = (z_i + 1)(2m_\ell - z_\ell + 1) + (z_\ell - 1)(2m_i - z_i + 1)$$

so that

$$||g_{e*}^{-1}(0)| - |g_{e*}^{-1}(1)|| = 4|m_i - (z_i + 1)||m_\ell - (z_\ell - 1)|$$

Furthermore

$$0 \leq m_i - (z_i + 1) < m_i - z_i$$

and

$$0 \leq (z_\ell - 1) - m_\ell < z_\ell - m_\ell$$

and therefore

$$||g_{e*}^{-1}(0)| - |g_{e*}^{-1}(1)|| < ||f_{e*}^{-1}(0)| - |f_{e*}^{-1}(1)||$$

Similar calculations show the analogous results for the other cases where one or both parts are odd. It follows by induction that the cordial deficit for a friendly labeling is minimized when the numbers of zeros and of ones in each part are roughly equal.

Now, we may assume without loss of generality that if k is odd, $|f^{-1}(0)| = |f^{-1}(1)| - 1$. Note also that the cordial deficit of the labeling is equal to the sum of the cordial deficits of pairs of parts over all such pairs. Furthermore, if one or both of the parts in a pair has an even number of vertices then the cordial deficit of that pair is zero. Hence we need only calculate the sum over all pairs of odd parts. If k is odd, then since $|f^{-1}(0)| = |f^{-1}(1)| - 1$ there are $\frac{k+1}{2}$ odd parts which have one more one than zero, and $\frac{k-1}{2}$ parts which have one more zero than one. This makes the net cordial deficit

$$\left| \binom{\frac{k-1}{2}}{2} + \binom{\frac{k+1}{2}}{2} - \frac{k-1}{2}\frac{k+1}{2} \right|$$

and if k is even the net cordial deficit is

$$\left| 2\binom{\frac{k}{2}}{2} - \left(\frac{k}{2}\right)^2 \right|$$

In either case, $\mathrm{ced}(G) = \left\lfloor \frac{k}{2} \right\rfloor - 1$ □

The calculation of the cordial vertex deficiency of the complete multipartite graphs seems to be a more difficult problem. I was able, however, to obtain an upper bound which applies in certain cases. The following theorem from [6] is necessary for the proof:

Theorem 4. *The cordial vertex deficiency of K_n is $j-1$ if $n = j^2 + \delta$, where $\delta \in \{-2, 0, 2\}$. Otherwise K_n is strictly noncordial.*

Theorem 5. *If G is a complete multipartite graph with $n \geq 1$ odd parts and $n = j^2 + \delta$ where $\delta \in \{-2, 0, 2\}$ then $cvd(G) \leq j - 1$.*

Proof: Let v_i be a single vertex from the ith odd part of G for $1 \leq i \leq n$. Label the vertices in each of the even parts with half zeros and half ones. Label the vertices

in each of the odd parts, omitting v_i, with half zeros and half ones. We now have a labeling of all but n of the vertices of G which has the same number of vertices labeled zero as are labeled one and with all the edges in the subgraph H isomorphic to K_n induced by the v_i's labeled half with zeros and half with ones. Now apply the previous theorem to H. □

4. Cordiality of tensor products

Note that if G_1 is connected, simple, and bipartite, and G_2 has q edges, then the tensor product $G_1 \times G_2$ is decomposable into $2q$ edge-disjoint isomorphs of G_1. Also, if G_1 and G_2 are cordially labeled by friendly labelings f and g respectively, then the *induced labeling* of the tensor product is obtained by labeling $(u, v) \in V(G_1 \times G_2)$ by $f(u) + g(v)$, where the sum is calculated modulo 2.

Theorem 6. *Let G_1 and G_2 be cordially labeled simple graphs such that G_1 is connected, bipartite, and has an even number of edges. Then $G_1 \times G_2$ is cordially labeled by the induced vertex labeling.*

Proof: Each of the $2q$ isomorphs of G_1 is generated from a directed edge of G_2 by using it to generate a labeling of the unique bipartition of G_1. The induced labeling restricted to a particular isomorph of G_1 is obtained from the cordial labeling of G_1 by adding the label of the edge which is generating the isomorph to the labeling of each edge of the isomorph as determined by the isomorphism between it and cordially labeled G_1. Since G_1 has an even number of edges, half of the edges of the isomorph end up labeled zero and the other half end up labeled one. Since the isomorphs of G_1 partition the edges of $G_1 \times G_2$, the tensor product itself ends up cordially labeled. □

It is fairly easy to find cordial labelings of such tensor products when G_1 has an odd number of edges. However, it is also easy to show that in none of these cases is the induced labeling cordial. However, I feel it is worth a conjecture to the effect

that the requirement that G_1 have an even number of edges can be dropped from the statement of Theorem 6.

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