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Holographic Renormalization of Asymptotically Flat Spacetimes

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ABSTRACT: A new local, covariant "counter-term" is used to construct a variational principle for asymptotically flat spacetimes in any spacetime dimension $d \geq 4$. The new counter-term makes direct contact with more familiar background subtraction procedures, but is a local algebraic function of the boundary metric and Ricci curvature. The corresponding action satisfies two important properties required for a proper treatment of semi-classical issues and, in particular, to connect with any dual non-gravitational description of asymptotically flat space. These properties are that 1) the action is finite on-shell and 2) asymptotically flat solutions are stationary points under *all* variations preserving asymptotic flatness; i.e., not just under variations of compact support. Our definition of asymptotic flatness is sufficiently general to allow the magentic part of the Weyl tensor to be of the same order as the electric part and thus, for $d = 4$, to have non-vanishing NUT charge. Definitive results are demonstrated when the boundary is either a cylindrical or a hyperbolic (i.e., de Sitter space) representation of spacelike infinity (i^0), and partial results are provided for more general representations of i^0. For the cylindrical or hyperbolic representations of i^0, similar results are also shown to hold for both a counter-term proportional to the square-root of the boundary Ricci scalar and for a more complicated counter-term suggested previously by Kraus, Larsen, and Siebelink. Finally, we show that such actions lead, via a straightforward computation, to conserved quantities at spacelike infinity which agree with, but are more general than, the usual (e.g., ADM) results.

KEYWORDS: Asymptotic flatness, gravitational action.

Contents

1. Introduction

In covariant approaches to quantum mechanics, the action S provides a fundamental link between classical and quantum treatments. Broadly speaking, classical mechanics is recovered through the semi-classical approximation, in which the path integral is dominated by stationary points of the action. Here we note that, in order to dominate the path integral, the action must be stationary under the *full* class of variations corresponding to the space of paths over which the integral is performed. Thus, one must consider *all* variations which preserve any boundary conditions and not just, say, variations of compact support. In particular, requiring the action to be stationary should yield precisely the classical equations of motion, with all boundary terms in the associated computation vanishing on any allowed variation.

Even ignoring the low differentiability of paths in the support of the measure and restricting the discussion to smooth paths, this requirement can be rather subtle. We are interested here in the case of asymptotically flat gravity. Thus, we begin with the familiar covariant action given by the Einstein-Hilbert "bulk" term with Gibbons-Hawking boundary term,

$$S_{EH+GH} = -\frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}R - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{-h}K. \tag{1.1}$$

This action does *not* satisfy the above requirement under all asymptotically flat variations. Indeed, as is well-known, on the space of classical solutions (1.1) satisfies

$$\delta S_{EH+GH} = \frac{1}{16\pi G} \int_{\partial\mathcal{M}} \sqrt{-h}\pi^{ij}\delta h_{ij}, \tag{1.2}$$

where $\pi^{ij} = K^{ij} - Kh^{ij}$ and h_{ij} is the induced metric on a timelike boundary $\partial\mathcal{M}$. Here we have, for simplicity, neglected additional boundary terms associated with past and future boundaries, and we will continue to neglect such terms throughout this work. The metric h_{ij} and its inverse are used to raise and lower indices i, j, k, l, m... associated with the boundary spacetime.

The reader will readily check that, when the boundary $\partial\mathcal{M}$ is taken to spatial infinity, the right-hand side of (1.2) does *not* vanish under standard definitions of asymptotic flatness, e.g., [1, 2, 3, 4, 5]. Instead, for standard choices[1] of ∂M, the variation (1.2) generically either diverges linearly or approaches a non-zero constant as ∂M is taken to spatial infinity.

In contrast, the analogous stationarity requirement has been well-studied within the 3+1 framework. Indeed, Regge and Teietelboim [6] showed that requiring the Hamiltonian to be stationary under all asymptotically flat variations leads directly to the ADM boundary term [7, 8, 9], and thus to the usual ADM definitions of energy, momentum, and angular momentum at spacelike infinity. Similarly, in the Palatini formalism, fixing a gauge at infinity for local Lorentz transformations allows a boundary term [10, 11] which yields a well-defined variational principle for asymptotically flat spacetimes. What we seek is an analogous covariant boundary term for the Einstein-Hilbert action[2]. We will find that it can be specified by a local, covariant term analogous to the counter-terms used to regulate the gravitational action of asymptotically anti-de Sitter spacetimes, see e.g. [12]-[20] .

Within the covariant framework, the "reference background approach" provides a step toward our goal. Here one adds to S_{EH+GH} an additional term

$$S_{Ref} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{-h} K_{Ref}, \tag{1.3}$$

where K_{Ref} is the trace of the extrinsic curvature of the boundary $(\partial\mathcal{M}, h)$ when this boundary spacetime is detached from $\mathcal{M}$ and isometrically embedded in Minkowski space (or, more generally, in another reference background, see e.g. [21, 22, 23].) As with the Gibbons-Hawking term, the term S_{Ref} is to be defined through an appropriate limiting procedure.

[1]Recall that the boundary spacetime $(\partial M, h)$ is not uniquely defined by the bulk spacetime $(\mathcal{M}, g)$. Instead, it also depends on the choice of limiting procedure used to define the Gibbons-Hawking term. This issue will be discussed in detail in section 2 below. The divergence of (1.2) is linear for what will be called "hyperbolic" temporal cut-offs in section 2, and (1.2) approaches a constant under what will be called "cylindrical" temporal cut-offs.

[2]Both the Palatini and canonical variational principles mentioned above are valid only for what are called "cylindrical" temporal cut-offs below. Thus, we also seek extensions to the "hyperbolic" temporal cut-offs.

The term (1.3) and its generalizations were originally suggested (see, e.g. [23]) in order to make the action finite on the space of solutions, but variations of this term are not typically addressed in the literature. Indeed, before one can vary this term one must address the existence and uniqueness of embeddings of $(\partial\mathcal{M}, h)$ into $(\mathcal{M}^{Ref}, g^{Ref})$. Note that, in order to vary S_{Ref}, we will need such an embedding not only for some particular boundary spacetime, but in fact for an open set of boundary spacetimes associated with arbitrary small variations.

In $d = 3$ spacetime dimensions, the desired embeddings may plausibly exist for suitably general choices of $(\partial\mathcal{M}, h)$ near spatial infinity of asymptotically flat spacetimes. For example, in the Euclidean signature, Weyl's embedding theorem states that a 2-manifold $(\partial M, h)$ can be embedded in Euclidean $\mathbb{R}^3$ if the scalar curvature of h is everywhere positive. (It also states that the resulting extrinsic curvature can be chosen to be positive definite.)

However, such an embedding generically fails in higher dimensions: in particular, given any boundary spacetime which can be so embedded, there are spacetimes arbitrarily nearby which cannot. To see this, simply note that any co-dimension one embedding can be specified locally by a single relation among the coordinates of the embedding space; i.e., by a single function on ∂M. In contrast, when the dimension of ∂M is greater than 2, the metric h (after gauge-fixing) contains more than a single degree of freedom[3]. Thus, an open set of such embeddings can exist only for $d \leq 3$, and variations of S_{Ref} (1.3) are ill-defined in $d \geq 4$ dimensions.

Nevertheless, in this work we show that, for asymptotically flat vacuum spacetimes with $d \geq 4$, one may build the desired action by simply replacing K_{Ref} in (1.3) with the trace of the tensor $\hat{K}_{ij}$ (i.e., $\hat{K} = h^{ij}\hat{K}_{ij}$) implicitly defined by solving the equation

$$\mathcal{R}_{ik} = \hat{K}_{ik}\hat{K} - \hat{K}_i^m \hat{K}_{mk}, \tag{1.4}$$

where $\mathcal{R}_{ij}$ is the Ricci tensor of h_{ij} induced on $\partial\mathcal{M}$. Although the definition is implicit, our counter-term is clearly a local algebraic function of the boundary metric h_{ij} and its Ricci tensor $\mathcal{R}_{ij}$. Adding such a counter-term to the Einstein-Hilbert action with Gibbons-Hawking boundary term results in an action which is finite on asymptotically flat solutions, and which has well-defined asymptotically flat variations which vanish on solutions.

The motivation for using equation (1.4) to define $\hat{K}$ arises from considering the Gauss-Codazzi relation (see, e.g. [1]) for a surface with spacelike normal,

$$\mathcal{R}_{ijkl} = R^{Ref}_{ijkl} + K_{ik}K_{jl} - K_{jk}K_{il}. \tag{1.5}$$

In particular, an equation of the form (1.5) would hold if the boundary spacetime $(\partial\mathcal{M}, h)$ had indeed been embedded in a reference background $(\mathcal{M}^{Ref}, g^{Ref})$. Here $\mathcal{R}_{ijkl}$ is the Riemann tensor (formed from h) on $\partial\mathcal{M}$ and R^{Ref}_{ijkl} is the (bulk) Riemann tensor of $(\mathcal{M}^{Ref}, g^{Ref})$ pulled back to $\partial\mathcal{M}$. In the case where the reference background is Minkowski space, we have $R^{Ref}_{ijkl} = 0$. Note that for $d-1 > 3$ the full Gauss-Codazzi relation (1.5) has too many components to generically admit a solution for the extrinsic curvature, K_{ij}. However, taking the trace of (1.4) by contracting with h^{il} provides a symmetric tensor

[3]This argument was suggested to us by Jan de Boer.

equation (1.4), which may then be solved for K_{ij} (also a symmetric tensor) within open sets in the space of possible $\mathcal{R}_{ij}$. In particular, we will see that solutions always exist for suitable definitions of ∂M near spacelike infinity of an asymptotically flat spacetime; i.e., for all spacetimes over which the action should be varied. Note that, although the definition of $\hat{K}$ is somewhat implicit, $\hat{K}$ is nevertheless a local algebraic function of the boundary metric h_{ij} and its Ricci curvature $\mathcal{R}_{ij}$.

The plan of our paper is as follows: Section 3 begins with various preliminaries and definitions (section 2), including our definition of asymptotically flat spacetimes. The reader will note that our notion of asymptotic flatness is less restrictive than that of many standard references [2, 4, 5, 24, 25], as we allow the electric and magnetic parts of the Weyl tensor to be of the same order; in particular, for $d = 4$ we allow non-vanishing NUT charge.

Section 3 contains our main results. In particular, we show that the action

$$S_{renorm} = S_{EH+GH} + S_{new\ CT}, \tag{1.6}$$

with

$$S_{new\ CT} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{-h}\hat{K}, \tag{1.7}$$

where $\hat{K}$ is defined via the solution of (1.4), leads to a finite action on asymptotically flat solutions (section 3.1) and that (section 3.2) the action is stationary under all asymptotically flat variations for common "cylindrical" and "hyperbolic" choices of $(\partial M, h)$. For a large class of more general boundaries, we show that (1.6) is again finite on asymptotically flat solutions, but we reach no conclusion regarding its variations.

We then turn in section 3.3 to two other proposed counter-terms [26, 27] for asymptotically flat spacetimes. We show, for cylindrical and hyperbolic choices of $(\partial M, h)$, that for $d \geq 5$ such counter-terms again define successful covariant variational principles for asymptotically flat spacetimes with cylindrical and hyperbolic boundaries. However, for at least one of these additional two counter-terms, the corresponding result in $d = 4$ holds only for the cylindrical boundaries.

In section 4, we discuss conserved quantities (energy, angular momentum, etc.) constructed from our renormalized actions. We show that, when they provide valid variational principles, all of the above actions lead via the algorithm of [28] to (finite) conserved quantities at spatial infinity for each asymptotic symmetry. Our conserved quantities agree with the usual definitions [2, 4, 5, 6, 7, 8, 9] of energy, momentum, etc., but also generalize such definitions to allow, e.g., non-vanishing NUT charge in four-dimensions[4]. We also demonstrate that these quantities are related to a boundary stress tensor of the sort used in [30, 31] (and related to that of the anti-de Sitter context [12, 13]). Finally, we close with some discussion in section 5.

[4]An independent construction, based on canonical methods, of conserved quantities for $d = 4$ in the presence of NUT charge will appear soon [29].

2. Preliminaries

This section presents various definitions and fixes notation for the rest of the paper. We begin with the definition of asymptotic flatness used in this work.

Consider a $d \geq 4$ dimensional spacetime for which the line element admits an expansion of the form[5]

$$ds^2 = \left(1 + \frac{2\sigma}{\rho^{d-3}} + \mathcal{O}(\rho^{-(d-2)})\right) d\rho^2 + \rho^2 \left(h^0_{ij} + \frac{h^1_{ij}}{\rho^{d-3}} + \mathcal{O}(\rho^{-(d-2)})\right) d\eta^i d\eta^j, \tag{2.1}$$

for large positive ρ. Here, h^0_{ij} and η^i are a metric and the associated coordinates on the unit $(d-2,1)$ hyperboloid $\mathcal{H}^{d-1}$ (i.e., on $d-1$ dimensional de Sitter space) and σ, h^1_{ij} are respectively a smooth function and a smooth tensor field on $\mathcal{H}^{d-1}$. Thus, ρ is the "radial" function associated with some asymptotically Minkowski coordinates x^a through $\rho^2 = \eta_{ab}x^a x^b$. In (2.1), the symbols $\mathcal{O}(\rho^{-(d-2)})$ refer to terms that fall-off at least as fast as $\rho^{-(d-2)}$ as one approaches *spacelike* infinity, i.e., $\rho \to +\infty$ with fixed η. For the purposes of this paper, we shall take (2.1) to define the class of asymptotically flat spacetimes. Here we follow a coordinate-based approach, but a more geometric treatment of this condition can also be given, generalizing to higher dimensions either the treatment of Ashtekar and Hansen [2] or that of Ashtekar and Romano [5].

Note that, for $d = 4$, any metric which is asymptotically flat by the criteria of any of [1, 2, 4, 5] also satisfies (2.1). In $d \geq 5$ dimensions, our definition is more restrictive than that of [32], which for $d \geq 5$ allows additional terms of order ρ^{-k} for $d-4 \geq k \geq 1$ relative to the leading terms. However, our definition is at least as general as the definition which would result by applying the methods of [6]; i.e., by considering the action of the Poincare' group on the Schwarzschild spacetime. We also note that, because Minkowski space itself solves the equations of motion (the Einstein equations), it is clear that (2.1) is consistent with the dynamics of the system.

Consider now the action S_{renorm}, including in particular the "counter-term" $S_{new\ CT}$ of (1.7). We wish to show that S_{renorm} is finite on asymptotically flat solutions (i.e., on Ricci flat spacetimes satisfying (2.1)), and that it is stationary about any such solution under variations preserving (2.1). In general, we will allow any variation compatible with (2.1). The one exception will occur in the case of $d = 4$ spacetime dimensions, and will be mentioned shortly (see equation 2.7).

To derive such results, we must carefully specify the form of the boundary spacetime $(\partial M, h)$. As with any discussion of the more familiar Gibbons-Hawking term in asymptotically flat spacetimes, the term 'boundary spacetime' is an abuse of language which in fact refers to a one-parameter family $(\partial\mathcal{M}_\Omega, h_\Omega)$ of boundaries of regions $\mathcal{M}_\Omega \subset \mathcal{M}$. Here we take $\mathcal{M}_\Omega$ to be an increasing family (i.e., $\mathcal{M}_\Omega \supset \mathcal{M}_{\Omega'}$ whenever $\Omega > \Omega'$) converging to $\mathcal{M}$ (that is, such that $\cup_\Omega \mathcal{M}_\Omega = \mathcal{M}$). Any such family represents a particular way of 'cutting off' the spacetime $\mathcal{M}$ and then removing this cut-off as $\Omega \to \infty$. Thus, expressions such as

[5] The inclusion of NUT-charge requires some changes in the global structure, but these changes have little effect on the arguments below. Instead of presenting the details here, we confine ourselves to brief comments on the NUT case in the relevant places below.

(1.1), (1.3), and (1.7) are to be understood as the $\Omega \to \infty$ limits of families of functionals S_Ω, in which $(\mathcal{M}, \partial\mathcal{M},\ h)$ are replaced by $(\mathcal{M}_\Omega, \partial\mathcal{M}_\Omega, h_\Omega)$. We will take this cut-off to be specified by some given function Ω on $\mathcal{M}$ such that $\Omega \to \infty$ at spatial infinity. We define $\mathcal{M}_{\Omega_0}$ to be the region of $\mathcal{M}$ in which $\Omega < \Omega_0$, so that $(\partial\mathcal{M}_{\Omega_0}, h_{\Omega_0})$ is the hypersurface where $\Omega = \Omega_0$.

Two particular classes of cut-off Ω will be of use below. The first is the class of "hyperbolic cut-offs," in which Ω is taken to be some function of the form:

$$\Omega^{hyp} = \rho + O(\rho^0), \tag{2.2}$$

where the fall-off condition on Ω^{hyp} is chosen so that the metric induced by (2.1) on any surface $\Omega^{hyp} = constant$ takes the form

$$h_{ij} = \rho^2 \left(h^0_{ij} + \frac{h^1_{ij}}{\rho^{d-3}} + \mathcal{O}(\rho^{-(d-2)}) \right), \tag{2.3}$$

where h^0_{ij} and h^1_{ij} were defined previously in equation (2.1). Choosing such a cut-off leads to a hyperbolic representation of spacelike infinity analogous to the construction of Ashtekar and Romano [5], though we will not pursue all details of the geometric structure here. Another useful class of cut-offs is the "cylindrical cut-off"

$$\Omega^{cyl} = r + O(\rho^0). \tag{2.4}$$

In (2.4), the coordinate r is defined by $r^2 = \rho^2 + t^2$ and t is an asymptotically Minkowski time coordinate. More precisely, we may define t through the requirement that the metric (2.1) takes the form

$$ds^2 = -\left(1 + \mathcal{O}(\rho^{-(d-3)})\right) dt^2 + \left(1 + \mathcal{O}(\rho^{-(d-3)})\right) dr^2 + r^2 \left(\mu_{IJ} + \mathcal{O}(\rho^{-(d-3)})\right) d\theta^I d\theta^J, \tag{2.5}$$

where μ_{IJ}, θ^I are the metric and coordinates on the unit $(d-2)$-sphere. As implied by (2.2) and (2.4), the action S_{renorm} will depend only on the asymptotic form of Ω, which we will take to represent a fixed auxiliary structure.

A further subtlety is related to the way in which the spacetime $\mathcal{M}$ is cut off in time. In the Lorentz-signature context (on which we focus), one is typically interested in the region of spacetime between two Cauchy surfaces (say, Σ_- and Σ_+), and we will have in mind that $\mathcal{M}$ represents such a region. However, in the asymptotically flat setting, two rather different such situations may be natural, depending on the physical context. Consider first the special case in which Σ_+ and Σ_- are related by an asymptotic translation. Then the volume of $\partial\mathcal{M}_\Omega$ grows as ρ^{d-2} in the limit $\Omega \to \infty$. We refer to spacetimes $\mathcal{M}$ having past and future boundaries related in this way as corresponding to "a cylindrical temporal cut-off $\mathcal{T}^{cyl}$."

However, it is also natural to consider a more general case in which Σ_+ and Σ_- are allowed to asymptote to any Cauchy surfaces C_+ and C_- of the hyperboloid $\mathcal{H}^{d-1}$; that is, we allow $\Sigma_\pm$ to be defined by any equations of the form

$$0 = f_\pm(\eta) + \mathcal{O}(\rho^{-1}), \tag{2.6}$$

for smooth functions $f_\pm$ on $\mathcal{H}^{d-1}$. One may think of such surfaces Σ_-, Σ_+ as being locally boosted relative to each other at infinity. We refer to spacetimes $\mathcal{M}$ having this latter sort of past and future boundaries as corresponding to "a hyperbolic temporal cut-off $\mathcal{T}^{hyp}$." Note that when $\mathcal{M}$ is defined by such past and future boundaries the volume of $\partial\mathcal{M}_\Omega$ grows as ρ^{d-1}.

Since the volume of $\partial\mathcal{M}$ grows as one power of ρ faster in the case of hyperbolic temporal cut-off ($\mathcal{T}^{hyp}$) than in the case of cylindrical temporal cut-off ($\mathcal{T}^{cyl}$), it is clear that the choice of temporal cut-off can affect conclusions about our action S_{renorm}. One consequence is that, in the particular case of $d = 4$ and for the hyperbolic temporal cut-off, it will be necessary to restrict variations of h^1_{ij} to be of the form

$$\delta h^1_{ij} = \alpha h^0_{ij}, \tag{2.7}$$

for α a smooth function on $\mathcal{H}^3$. The physics of this restriction will be discussed further in section 3.2 and in appendix A. Here we simply note that this restriction is analogous to a condition imposed in [4] in order to arrive at a well-defined covariant phase space formalism.

In physical applications, it is natural to use a cylindrical temporal cut-off $\mathcal{T}^{cyl}$ in conjunction with the cylindrical spatial cut-off Ω^{cyl}; for example when studying evolution between two Cauchy surfaces related by time translations or, in the Euclidean context, with periodic Euclidean time (i.e., at finite temperature). Similarly, it seems natural to use a hyperbolic temporal cut-off $\mathcal{T}^{hyp}$ in conjunction with a hyperbolic spatial cut-off Ω^{hyp} (as in, for example, the covariant phase space treatment of Ashtekar, Bombelli, and Reula [4]). However, in principle one may make independent choices of spatial and temporal cut-off. We find it interesting to do so below in order to probe the possibility of generalizing our results to a more general spatial cut-offs. In particular, one would like to generalize the results below to arbitrary spatial cut-offs of the form $\Omega = \omega\Omega^{hyp}$, with ω a smooth non-vanishing function on the unit hyperboloid. Our results indicate that this may be possible, at least in the case of cylindrical temporal cut-off $\mathcal{T}^{cyl}$.

3. Gravitational counter-terms

In this section we study the new counter-term (1.7) for asymptotically flat spacetimes, as well as those counter-terms suggested previously in [26, 27]. Our main results for the new counter-term are presented in sections 3.1 and 3.2. Similar results are derived for the counter-terms of [26, 27] in section 3.3.

3.1 The on-shell action is finite

We now consider our new counter-term and show that the action is finite on asymptotically flat (see 2.1) solutions of the equations of motion. For a cylindrical temporal cut-off $\mathcal{T}^{cyl}$, this is a straightforward exercise in power counting for (almost) any spatial cut-off $\Omega = \omega\Omega^{hyp}$ with ω a smooth non-vanishing function on the hyperboloid and, in particular, for $\Omega = \Omega^{hyp}$ or $\Omega = \Omega^{cyl}$. To see this, simply note that, since our spacetimes are Ricci flat, the Einstein-Hilbert term vanishes on solutions. As a result, only the boundary terms contribute to the action:

$$S_{renorm} = -\frac{1}{8\pi G}\int \sqrt{-h}(K - \hat{K}). \tag{3.1}$$

To first order, the difference $\Delta K_{ij} := K_{ij} - \hat{K}_{ij}$ can be found by linearizing the trace of the Gauss-Codazzi relations (1.5). Furthermore, since both K_{ij} and $\hat{K}_{ij}$ satisfy such (traced) relations with the same metric h_{ij} and Ricci tensor $\mathcal{R}_{ij}$, the change ΔK_{ij} is sourced entirely by the bulk Riemann tensor:

$$h^{kl}R_{ikjl} = -\Delta K_{kl}\left(h^{kl}\hat{K}_{ij} + \delta_i^k\delta_j^l\hat{K} - \delta_i^k\hat{K}_j^l - \delta_j^k\hat{K}_i^l\right). \tag{3.2}$$

Note that h^{jk} is the inverse of h_{ij}, and that we will always raise and lower indices $i, j, k, l, ...$ with h^{ij} and h_{ij} (as opposed to using $(h^0)^{ij}$ and h^0_{ij}).

Now, the operator

$$L_{ij}^{kl} = h^{kl}\hat{K}_{ij} + \delta_i^k\delta_j^l\hat{K} - \delta_i^k\hat{K}_j^l - \delta_j^k\hat{K}_i^l, \tag{3.3}$$

acting on ΔK_{kl} in (3.2) is generically invertible and of the same order in ρ as K; namely, $\mathcal{O}(\rho^{-1})$. Thus, L_{ij}^{kl} will generically have an inverse of order $\mathcal{O}(\rho)$; we will see this explicitly for Ω^{cyl} and Ω^{hyp} in section 3.2. Since $h^{mn}R_{kmln} = \mathcal{O}(\rho^{-(d-3)})$, we have $\Delta K_{ij} = \mathcal{O}(\rho^{-(d-4)})$, and higher corrections to ΔK_{ij} are sub-leading. As a result, $\Delta K = K - \hat{K} = \mathcal{O}(\rho^{-(d-2)})$ and, with a temporal cut-off $\mathcal{T}^{cyl}$, we have $\int_{\partial\mathcal{M}}\sqrt{-h}\Delta K = \mathcal{O}(1)$.

Let us now address the case of hyperbolic temporal cut-off $\mathcal{T}^{hyp}$. Because the volume element on $\partial\mathcal{M}$ is larger by a power of ρ, a more careful analysis is required to obtain useful results in this case and, in the end, we will obtain such results only for hyperbolic spatial cut-off Ω^{hyp}. Our starting point is the observation that, for Ω^{hyp}, substituting $\hat{K}_{ij} = \frac{1}{\rho}h_{ij} + \mathcal{O}(\rho^{-(d-4)})$ into (3.2), yields

$$\left(\frac{d-3}{\rho} + \mathcal{O}(\rho^{-(d-2)})\right)\Delta K_{ik} = -R_{ijkl}h^{jl} + \frac{1}{2(d-2)}h_{ij}h^{mn}R_{mknl}h^{kl}. \tag{3.4}$$

Now, recall that we are interested in vacuum *solutions*; i.e., Ricci-flat metrics[6]. As a result, we may replace R_{ijkl} with the bulk Weyl tensor, C_{ijkl}. Recall that the Weyl tensor is traceless: $g^{ac}C_{abcd} = 0 = g^{bd}C_{abcd}$, where g_{ab} is the metric on $\mathcal{M}$ and indices $a, b, c...$ will denote coordinates on $\mathcal{M}$. Introducing the unit normal N^a to the boundary, we may define

$$h_{ab} = g_{ab} - N_aN_b, \tag{3.5}$$

which pulls back to the metric h_{ij} on the boundary. Furthermore, h_b^a is a projector onto directions tangent to $\partial\mathcal{M}$. It follows that the electric part of the Weyl tensor (as in [2, 3, 5]) is

$$E_{ac} := C_{abcd}N^bN^d = -C_{abcd}h^{bd} \tag{3.6}$$

[6]More generally, one may consider matter sources with $T_{ab} = \mathcal{O}(\rho^{-d})$ in terms of asymptotically Minkowski Cartesian coordinates. Since the Weyl tensor in such coordinates is typically of order $\rho^{-(d-1)}$, the Riemann and Weyl tensors agree to leading order, so that one may proceed similarly in such cases.

and that, due to the antisymmetry and traceless properties of the Weyl tensor, we have $h^{ac}E_{ac} = 0$. We may then pull this expression back to the boundary to find the familiar result $h^{ij}E_{ij} = 0$. Thus, (3.4) yields

$$K_{ij} - \hat{K}_{ij} = \frac{\rho}{d-3} E_{ij} + \mathcal{O}(\rho^{-(d-3)}). \tag{3.7}$$

But E_{ij} is traceless, so the contribution to $K - \hat{K}$ from E_{ij} vanishes and $K - \hat{K} = \mathcal{O}(\rho^{-(d-1)})$. Thus, for a hyperbolic temporal cut-off $\mathcal{T}^{hyp}$, the integrand in (3.1) is of order ρ^0. We conclude that, neglecting past and future boundary terms, S_{renorm} takes finite values on asymptotically flat solutions with $(\partial\mathcal{M})$ defined by Ω^{hyp}.

We have shown above that S_{renorm} is finite for either i) cylindrical temporal cut-offs ($\mathcal{T}^{cyl}$) and any spatial cut-off $\Omega = \omega\Omega^{hyp}$, where ω is a smooth function on the hyperboloid, or ii) hyperbolic temporal cut-off ($\mathcal{T}^{hyp}$) with hyperbolic spatial cut-off (Ω^{hyp}. However, we have not yet addressed variations of such actions, nor have we discussed whether the numerical value of each action is sensitive to the particular spatial cut-off Ω, say, within the class Ω^{hyp} (2.2). Both of these issues will be addressed in subsection 3.2 below.

3.2 Variations of the action

Having shown that S_{renorm} is finite on solutions, we now consider its first variations. We wish to show that such variations vanish about any asymptotically flat solution using either spatial cut-off, Ω^{cyl} or Ω^{hyp}, and using either temporal cut-off $\mathcal{T}^{cyl}$ or $\mathcal{T}^{hyp}$. For the particular case of $d = 4$ and $\mathcal{T}^{hyp}$, we will require the variations to satsify the extra condition (2.7).

Since the variation of the Einstein-Hilbert action with Gibbons Hawking term is given by (1.2), our task is essentially to consider the variation of our new counter-term $S_{new\ CT}$, which we compute as follows:

$$\delta S_{new\ CT} = \frac{1}{8\pi G}\int_{\partial M} \delta(\sqrt{-h}\hat{K}) = \frac{1}{8\pi G}\int_{\partial M} \sqrt{-h}\left(\frac{1}{2}\hat{K}h^{ij}\delta h_{ij} + \hat{K}_{ij}\delta h^{ij} + h^{ij}\delta\hat{K}_{ij}\right). \tag{3.8}$$

Now, expanding the definition (1.4) of $\hat{K}_{ij}$ to first order, we find

$$\begin{aligned}\delta\mathcal{R}_{ij} &= \delta\hat{K}_{kl}\left(h^{kl}\hat{K}_{ij} + \delta^k_i\delta^l_j\hat{K} - \delta^k_i\hat{K}^l_j - \delta^k_j\hat{K}^l_i\right) \\ &\quad + (\hat{K}_{ij}\hat{K}_{mn} - \hat{K}_{im}\hat{K}_{nj})\delta h^{mn}.\end{aligned} \tag{3.9}$$

For the case where h^1_{ij}, σ and the higher corrections to the metric vanish, (1.4) is easy to solve. This solution is

$$\hat{K}_{ij} = \begin{cases} \frac{1}{\rho}h_{ij} = \rho(h^0)_{ij} & \text{for } \Omega^{hyp} \\ r\mu_{ij} & \text{for } \Omega^{cyl}, \end{cases} \tag{3.10}$$

where μ_{ij} is the pull-back of μ_{IJ} (the unit round metric on S^{d-2}) to $S^{d-2}\times\mathbb{R}$. More generally, we can solve perturbatively around this solution.

By contracting (3.9) with h^{ij} and μ^{ij}, it is straightforward to show that the trace of $\delta\hat{K}_{ij}$ satisfies

$$\delta\hat{K}_{ij}h^{ij} = \begin{cases} -\frac{1}{2}\left[\hat{K}_{ij}\delta h^{ij} - \frac{\rho}{(d-2)}h^{ij}\delta\mathcal{R}_{ij}\right] + \mathcal{O}(\rho^{-(2d-5)}) & \text{for } \Omega^{hyp} \\ -\frac{1}{2}\hat{K}_{ij}\delta h^{ij} + \frac{1}{(2d-6)\hat{K}}\left[(d-4)h^{ij}\delta\mathcal{R}_{ij} + \frac{2}{r^2}\mu^{ij}\delta\mathcal{R}_{ij}\right] + \mathcal{O}(\rho^{-(2d-5)}) & \text{for } \Omega^{cyl}, \end{cases} \tag{3.11}$$

and in fact to iteratively solve for the full δK^{ij}. As an aside we note that, using the inverse function theorem and compactness of $\partial\mathcal{M}$ (due to the temporal cut-off), we may conclude that, given any asymptotically flat $(\mathcal{M}, g)$, solutions for $\hat{K}_{ij}$ exist for sufficiently large Ω.

Now, recall [1] that $\delta\mathcal{R}_{ij}$ can be expressed in the form

$$\delta\mathcal{R}_{ij} = -\frac{1}{2}h^{kl}D_iD_j\delta h_{kl} - \frac{1}{2}h^{kl}D_kD_l\delta h_{ij} + h^{kl}D_kD_{(i}\delta h_{j)l}, \tag{3.12}$$

where D_i is the (torsion-free) covariant derivative on $\mathcal{H}^{d-1}$ compatible with h_{ij}. Thus, when $\delta\mathcal{R}_{ij}$ is contracted with any covariantly constant tensor (e.g., $(h^0)^{ij}$ or, on the cylinder, μ^{ij}), the result is a total derivative. Note that because we are expanding about homogeneous spaces, $\hat{K}$ is constant over the space, so that $\frac{1}{\hat{K}}h^{ij}\delta R_{ij}$ yields a total divergence in the cases of interest.

Inserting the remaining term $-\frac{1}{2}\hat{K}_{ij}\delta h^{ij}$ from (3.11) into (3.8), up to boundary terms at the past and future boundaries and when the equations of motion hold we find

$$\delta S_{renorm} = \frac{1}{16\pi G}\int_{\partial M}\sqrt{-h}(\pi^{ij} - \hat{\pi}^{ij})\delta h_{ij} + \mathcal{O}(\rho^{-(d-4+c)}), \tag{3.13}$$

where $\hat{\pi}^{ij} = \hat{K}^{ij} - \hat{K}h^{ij}$ and we have used that $A^{ij}\delta h_{ij} = -A_{ij}\delta h^{ij}$ for any tensor A_{ij}. The constant c in the exponent of ρ in the error term takes the value $c = 1$ for cylindrical temporal cut-offs ($\mathcal{T}^{cyl}$) and takes the value $c = 0$ for hyperbolic temporal cut-offs ($\mathcal{T}^{hyp}$). Thus, the error term contributes only for $d = 4$ with hyperbolic temporal cut-off ($\mathcal{T}^{hyp}$). Note that $\hat{\pi}^{ij}$ has the same form as the momentum π^{ij} conjugate to h_{ij}, except that it is built from $\hat{K}^{ij}$ instead of from the actual extrinsic curvature K^{ij}.

For some cases, one may show that (3.13) vanishes on asymptotically flat solutions by simply counting powers of ρ. We note from (3.7), (and the analogous result for Ω^{cyl}) that $K^{ij} - \hat{K}^{ij}$ is of order ρ^{-d}. Since δh_{ij} is of order $\rho^{-(d-5)}$, the integral in (3.13) is of order $\rho^{-(d-4)}$ for hyperbolic temporal cut-off $\mathcal{T}^{hyp}$ and is of order $\rho^{-(d-3)}$ for cylindrical temporal cut-off $\mathcal{T}^{cyl}$. Thus, our action is stationary on asymptotically flat solutions with either spatial cut-off (Ω^{hyp} or Ω^{cyl}) for either $d \geq 5$ and hyperbolic temporal cut-off $\mathcal{T}^{hyp}$ or for $d \geq 4$ and cylindrical temporal cut-off $\mathcal{T}^{cyl}$. Note that our argument for Ω^{cyl} generalizes readily to more complicated cylindrical boundaries appropriate to infinitely long strings, branes, etc., whose metric to leading order matches that of the standard embedding of $S^n \times \mathbb{R}^{d-n-1}$ (with $n \geq 2$) into Minkowski space[7], as well as to other products of maximally symmetric manifolds.

[7] Since we work in Lorentz signature, it is convenient to abuse notation and to understand S^n for $n = d-1$ to represent the Hyperboloid $\mathcal{H}^{d-1}$. Boundaries of the form $S^n \times \mathbb{R}^{d-n-1}$ were of interest in [27].

Let us now consider the case of $d = 4$ with hyperbolic temporal cut-off $\mathcal{T}^{hyp}$. The error term not written explicitly in (3.13) must now be calculated to the next order. This calculation is outlined in appendix B, where it is shown that the leading contribution vanishes, so that this term is in fact of order ρ^{-1}. Thus, we may concentrate on the explicit term (involving $\pi^{ij} - \hat{\pi}^{ij}$) on the right-hand-side of (3.13).

Now in $d = 4$ spacetime dimensions one does not in fact expect a covariant quantum path integral to integrate over all metrics of the form (2.1). Instead, one expects the proper domain of integration to reflect the classical covariant phase space. We note that in [4] it was necessary to restrict variations of h^1_{ij} as in (2.7); i.e., to be of the form $\delta h^1_{ij} = \alpha h^0_{ij}$ where α is a smooth function on the hyperboloid $\mathcal{H}^3$. This was done in order to make the symplectic structure finite and to ensure that the symplectic flux through spatial infinity vanishes. Recall that finiteness of the symplectic structure is closely related to the norm of perturbative particle states when one quantizes the theory; this norm is just the symplectic product of a positive frequency solution with its (negative frequency) complex conjugate. Thus, variations not compatible with keeping the symplectic structure finite are properly viewed as a change of boundary conditions, and not as a variation of histories within a given physical system. Similar comments apply to variations not compatible with keeping symplectic flux from flowing outward through spatial infinity. For this reason, we are happy to adopt the restriction (2.7) here.

In fact, [4] imposed $h^1_{ij} = -2\sigma h^0_{ij}$, and imposed as well as a number of other restrictions. We shall have no need for these additional restrictions. However, since [4] justified the condition (2.7) only in the presence of these additional restrictions, it is legitimate to ask whether we may use (2.7) with more generality. In appendix A, we argue that the answer is affirmative by demonstrating that (2.7) is compatible with the equations of motion.

We will now show that, with the restriction (2.7), our action is stationary on asymptotically flat solutions when one chooses *both* the spatial and temporal cut-offs to be hyperbolic ($\mathcal{T}^{hyp}$ and Ω^{hyp}). Note that we have

$$\delta h_{ij} = \frac{\alpha}{\rho^{d-5}} h^0_{ij} + \mathcal{O}(\rho^{-(d-4)}) = \frac{\alpha}{\rho^{d-3}} h_{ij} + \mathcal{O}(\rho^{-(d-4)}). \tag{3.14}$$

For this case, the leading order term in $(\pi^{ij} - \hat{\pi}^{ij})\delta h_{ij}$ is proportional to the trace of $\pi^{ij} - \hat{\pi}^{ij}$. But, as shown in (3.7), this vanishes for hyperbolic spatial cut-off Ω^{hyp}. As a result, up to past and future boundary terms, we have

$$\delta S_{renorm} = \frac{1}{16\pi G} \int_{\partial M} \sqrt{-h} \times \mathcal{O}(\rho^{-(2d-4)}) = \frac{1}{16\pi G} \int_{\partial M} \mathcal{O}(\rho^{-(d-3)}), \tag{3.15}$$

which vanishes for $d \geq 4$. Having shown that S_{renorm} provides a valid variational principle for i) cylindrical temporal cut-off ($\mathcal{T}^{cyl}$) and either cylindrical or hyperbolic spatial cut-off (Ω^{cyl} or Ω^{hyp}) and ii) hyperbolic temporal cut-off $\mathcal{T}^{hyp}$) with hyperbolic spatial cut-off Ω^{hyp}, it is now straightforward to show that for such cases the numerical value of the action on a solution is invariant under a change of spatial cut-off of the form

$$\Omega \to \Omega + \delta\Omega, \tag{3.16}$$

with $\delta\Omega = \mathcal{O}(\rho^0)$; i.e., the value of S_{renorm} depends at most on the choice of Ω^{cyl} or Ω^{hyp}, but not on the precise choice of Ω within either class. To see this, simply note that the change δh_{ij} induced by (3.16) takes the form $\delta h^1_{ij} = 2\delta\Omega\ h^0_{ij}$, together with changes in the higher order terms in h_{ij}. Since we have just shown that δS_{renorm} vanishes under any such variation about a solution, it is clear that that the numerical value of S_{renorm} is invariant under shifts (3.16).

3.3 Other proposed Counter-terms

Our counter-term $S_{new\ CT}$ is not the first counter-term to have been proposed for asymptotically flat spacetimes. In particular, Mann showed [26] that for $d = 4$ a counter-term proportional to $\sqrt{\mathcal{R}}$ leads to a finite on-shell action for Schwarzschild spacetimes with $\Omega = \Omega^{cyl}$ and could, in this context, be related to the known counter-terms [12, 13] for asymptotically AdS spacetimes. This result was generalized to arbitrary spacetime dimension by Kraus, Larsen, and Siebelink [27], who also established similar results for the counter-term

$$S_{KLS} = \frac{1}{8\pi G}\int_{\partial M}\sqrt{-h}\frac{\mathcal{R}^{3/2}}{\sqrt{\mathcal{R}^2 - \mathcal{R}_{ij}\mathcal{R}^{ij}}}, \tag{3.17}$$

for both cylindrical and hyperbolic boundaries, and in fact for any boundary metric which agrees to leading order with the standard metric on $S^n \times \mathbb{R}^{d-n-1}$. Here we proceed further, considering arbitrary asymptotically flat vacuum solutions and addressing both the value of the action and the issue of whether the first variations vanish. We consider both (3.17) and the counter-term

$$S_{\sqrt{\mathcal{R}}} = \frac{1}{8\pi G}\sqrt{\frac{n}{n-1}}\int_{\partial M}\sqrt{-h}\sqrt{\mathcal{R}}, \tag{3.18}$$

which generalizes Mann's counter-term (see also [27]) to $S^n \times \mathbb{R}^{d-n-1}$. Here $S_{\sqrt{\mathcal{R}}}$ depends explicitly on the choice of cut-off Ω through the integer n. Due to this feature, it is not clear how $S_{\sqrt{\mathcal{R}}}$ might be usefully generalized away from the above classes of spatial cut-offs Ω (e.g., to cut-offs $\Omega = \omega\Omega$ for smooth non-vanishing functions ω on the unit hyperboloid).

Let us first consider the counter-terms $S_{\sqrt{\mathcal{R}}}$ and S_{KLS} for cylindrical temporal cut-off $\mathcal{T}^{cyl}$. Here each counter-term is only linearly divergent, so in discussing finiteness of the action it suffices to consider the leading order term. One may readily check that, to leading order, $\hat{K}$ defined by (1.4) agrees with both $\sqrt{\frac{n}{n-1}\mathcal{R}}$ and $\frac{\mathcal{R}^{3/2}}{\sqrt{\mathcal{R}^2-\mathcal{R}_{ij}\mathcal{R}^{ij}}}$ for both cylindrical spatial cut-offs Ω^{cyl} and hyperbolic spatial cut-offs Ω^{hyp}. Thus, under these conditions the counter-terms $S_{\sqrt{\mathcal{R}}}$ and S_{KLS} lead to renormalized actions that are finite on-shell.

The fact that the counter-terms agree to leading order in ρ is essentially true by construction, as both $S_{\sqrt{\mathcal{R}}}$ and S_{KLS} were motivated by the fact that they cancel the leading divergences in the Gibbons-Hawking term. Somewhat surprisingly, we also find that the first order variations of both $S_{\sqrt{\mathcal{R}}}$ and S_{KLS} about such backgrounds exactly match those of our original S_{renorm} (1.7). The key steps in such calculations for $S_{\sqrt{\mathcal{R}}}$ are:

$$\delta\sqrt{\frac{n}{n-1}\mathcal{R}} = \frac{1}{2\sqrt{\mathcal{R}}}\sqrt{\frac{n}{n-1}}\left(\mathcal{R}_{ij}\delta h^{ij} + \delta\mathcal{R}_{ij}h^{ij}\right)$$

$$= -\frac{1}{2}(\hat{K}^{ij} + \mathcal{O}(\tilde{r}^{-d}))\delta h_{ij}, \tag{3.19}$$

where in the last step we have again used (3.12) to show that, to leading order in ρ, the $\delta\mathcal{R}_{ij}$ term is a total divergence. We have also used the Gauss-Codazzi equations (1.4) to show that $\mathcal{R}_{ij} = \frac{n-1}{n}\hat{K}\hat{K}_{ij} + \mathcal{O}(\tilde{r}^{-(d-3)})$, where $\tilde{r}$ is ρ for the hyperboloid, r for the cylinder $\mathbb{R} \times S^{d-2}$, and the analogous radial coordinate in the more general case. The analogous calculation for S_{KLS} yields

$$\delta \frac{\mathcal{R}^{3/2}}{\sqrt{\mathcal{R}^2 - \mathcal{R}_{ij}\mathcal{R}^{ij}}} = -\frac{1}{2}(\hat{K}^{ij} + \mathcal{O}(\tilde{r}^{-d}))\delta h_{ij}. \tag{3.20}$$

Comparing with (3.8) and noting that $\delta\hat{K} = \delta\hat{K}_{ij}h^{ij} + \hat{K}_{ij}\delta h^{ij}$, we see that for cylindrical temporal cut-off $\mathcal{T}^{cyl}$ the counter-terms $S_{\sqrt{\mathcal{R}}}$ and S_{KLS} have the same variations about solutions as does our new counter-term. As a result, with cylindrical temporal cut-off $\mathcal{T}^{cyl}$, both of these counter-terms again yield actions which are stationary on solutions, so long as the spatial cut-off induces a boundary of the form $S^n \times \mathbb{R}^{d-n-1}$.

Let us now consider hyperbolic temporal cut-offs ($\mathcal{T}^{hyp}$). In this case each counter-term is quadratically divergent. Now, we have already established that the counter-terms $S_{\sqrt{\mathcal{R}}}$ and S_{KLS} agree with $S_{new\ CT}$ to leading order in ρ. Furthermore, the behavior at next order in ρ may be considered to be the result of perturbing an original induced metric $h_{ij} = \rho^2(h^0)_{ij}$ by ρh^1_{ij}. Thus, we may compute the next order term using the formulas (3.19) and (3.20), which show that they agree with the corresponding expression for $S_{new\ CT}$ up to total derivative terms. This shows that the counter-terms $S_{\sqrt{\mathcal{R}}}$ and S_{KLS} define a finite action S_{renorm} for hyperbolic temporal cut-off $\mathcal{T}^{hyp}$ when the spatial cut-off is also chosen to be hyperbolic (Ω^{hyp}).

For $d \geq 5$ and hyperbolic spatial cut-off Ω^{hyp}, stationarity of S_{renorm} again follows from (3.19) and (3.20). However, as with $S_{new\ CT}$, the situation is more subtle for $d = 4$. In this case one must calculate the $\mathcal{O}(\tilde{r}^{-d})$ corrections to (3.19) and (3.20). Such calculations for $S_{\sqrt{\mathcal{R}}}$ show that the corresponding S_{renorm} is *not* in general stationary (see appendix C). We have not performed the corresponding calculations for S_{KLS}. Finally, we note that in any case where it provides a valid variational principle, S_{renorm} as defined by $S_{\sqrt{\mathcal{R}}}$ or S_{KLS} is invariant under changes of spatial cut-off of the form (3.16) with $\delta\Omega = \mathcal{O}(\rho^0)$ by the same argument used for $S_{new\ CT}$ in section 3.2. Namely, (3.16) induces a change in h_{ij} equivalent to one for which we have just shown that $\delta S_{renorm} = 0$.

4. Conserved Quantities

Having constructed a variational principle of the desired type, one may expect that conserved quantities (e.g., energy, angular momenta) now follow by a straightforward construction as in Noether's theorem. Such a result is known [28] in a general setting appropriate to gravitational actions constructed from a counter-term prescription and modeled on the case of asymptotically anti-de Sitter spaces. The discussion of [28] is phrased in terms of "boundary fields" and a "boundary metric," but in the current context we may take our boundary metric to be $\gamma_{ij} = \lim_{\Omega\to\infty}\Omega^{-2}h_{ij}$ and all other boundary fields to be zero. For either spatial cut-off (Ω^{hyp} or Ω^{cyl}), we may choose coordinates on $\partial\mathcal{M}$ such that γ_{ij} is non-degenerate.

The treatment in [28] is quite general, and does not specify in detail the way in which either the boundary manifold $\partial\mathcal{M}$ or the boundary fields are to be associated with $\mathcal{M}$ and the dynamical fields. In particular, in our context the $\Omega \to \infty$ limit of $\partial\mathcal{M}$ need not be smoothly attached to any conformal compactification of $\mathcal{M}$, or even to a compactification of the form described by Ashtekar and Romano [5]. What is required is simply that the boundary fields capture the boundary conditions, and that diffeomorphisms on $\mathcal{M}$ induce diffeomorphisms on ∂M. It is clear that both are the case here. The main argument (section III of [28]) then considers any asymptotic Killing field ξ, where an asymptotic Killing field is defined to be a vector field which generates diffeomorphisms that preserve the asymptotic conditions. In particular, the diffeomorphism should preserve the definition (2.1) of asymptotic flatness and, for $d = 4$ with (Ω^{hyp}, $\mathcal{T}^{hyp}$) infinitesimal, such diffeomorphisms should result in a variation satisfying (2.7). Asymptotic symmetries must also preserve any other condition required to define a proper covariant phase space whose symplectic structure is both finite and conserved; see, e.g., [4] for the $d = 4$ covariant phase space containing Minkowski space and a discussion of how the corresponding restrictions remove both supertranslations and logarithmic supertranslations [33, 34] from the list of candidate asymptotic symmetries.

In such cases, [28] considers the operator $\Delta_{f,\xi} = \pounds_{f\xi} - f\pounds_\xi$ on the space of field histories, where $\pounds_\eta$ denotes the Lie derivative along the vector field η and where f is any smooth bounded function which vanishes in a neighborhood of the past boundary and takes the value $f = 1$ in a neighborhood of the future boundary. The argument of [28] then shows that the quantity $-\Delta_{f,\xi} S_{renorm}$ generates asymptotic diffeomorphisms along ξ via the Peierls bracket[8].

While we will not repeat the full proof here, we will pause to demonstrate two important properties of $\Delta_{f,\xi} S_{renorm}$; namely, that it is both finite and differentiable on the space of field histories. To see these properties, note first that $\pounds_{f\xi}$ generates a diffeomorphism. But the space of histories on which S_{renorm} is finite and differentiable is covariant under such diffeomorphisms, so this part of $\Delta_{f,\xi}$ creates no difficulties. Furthermore, since ξ is an asymptotic symmetry, the transformed metric $g_{transformed} = (1 + \pounds_\xi)g$ satisfies the same asymptotic conditions as g and, since these conditions are local, so does $g_{transformed,f} = (1+ f\pounds_\xi)g$. Thus, $(1 + \epsilon\Delta_{f,\xi})S_{renorm}$ is finite and differentiable. Finally, since the Lagrangian is a differentiable function of the fields and their derivatives, $(1+\epsilon\Delta_{f,\xi})S_{renorm}$ will also be linear in ϵ. Thus $\Delta_{f,\xi} S_{renorm}$ is well-defined, finite, and differentiable on the chosen space of histories.

Now, since $-\Delta_{f,\xi} S_{renorm}$ generates asymptotic ξ-translations, it can differ from any Hamiltonian definition of the conserved quantity associated to ξ by at most a "c-number;"

[8]The Peierls bracket [35] is a covariant Poisson structure on the space of solutions modulo gauge transformations. It agrees with the push-forward of the Poisson bracket of Dirac observables under any evolution map which takes gauge orbits on the constraint surface in phase space to such equivalence classes of solutions. See also [36, 37, 38] and see [39] for extensions of the Peierls bracket to algebras of gauge-dependent quantities and [40, 41] for recent related work in quantum field theory.

i.e., by a quantity having trivial Peierls bracket with any observable. Such a quantity may be a function of the auxiliary structure Ω, but must be otherwise constant over the space of solutions. In particular, since the usual ADM charges $H_{ADM}[\xi]$ [7, 8, 9] vanish on Minkowski space, given any metric g on $\mathcal{M}$, if we define

$$Q[\xi] = -\Delta_{f,\xi} S_{renorm}[g] \quad \text{and} \quad Q_0[\xi] = -\Delta_{f,\xi} S_{renorm}[g_{Mink}], \tag{4.1}$$

where g_{Mink} is the Minkowski metric, then $Q[\xi] - Q_0[\xi]$ must agree with the corresponding $H_{ADM}[\xi]$ (or, in fact, with any other standard definition such as [1, 2, 3, 5, 6, 42, 43, 44] of the charge) whenever these more familiar charges are well-defined. We note, however, that (4.1) also extends such definitions to allow for a larger magnetic Weyl tensor at infinity and, in particular, to the case of non-vanishing NUT charge in 4 spacetime dimensions.

Now, the full discussion of [28] does use some structure that is *not* present in our case in order to derive additional results. These additional results would relate $-\Delta_{f,\xi} S_{renorm}$ to a 'boundary stress tensor' given by variations of the action S_{renorm} with respect to γ_{ij}. In the current context, such variations turn out to diverge. However, working for the moment with the regulated action $S_{renorm,\Omega}$ associated with a cut-off region of spacetime $\mathcal{M}_\Omega$ for some finite value of Ω, we may follow [21] and define a 'boundary stress tensor' as a function[9] of Ω through variations of $S_{renorm,\Omega}$ with respect to $(h_\Omega)_{ij}$. This stress tensor admits an expansion of the form

$$\begin{aligned} T_{ij}(\Omega) &:= \frac{-2}{\sqrt{-h}} \frac{\delta S_{renorm,\Omega}}{\delta h^{ij}_\Omega} = \frac{1}{16\pi G}(\pi_{ij} - \hat{\pi}_{ij}) \\ &:= \Omega^{-(d-4)} \left(T^0_{ij} + \Omega^{-1} T^1_{ij} + \textit{ terms vanishing faster than } \Omega^{-1} \right). \end{aligned} \tag{4.2}$$

Note that, since it is local on $\partial\mathcal{M}_\Omega$, the definition of $T_{ij}(\Omega)$ depends only on the spatial cut-off and is independent of the choice off temporal cut-off ($\mathcal{T}^{cyl}$ or $\mathcal{T}^{hyp}$). In general, the definition of T_{ij} will depend on the precise choice of counter-term ($S_{new\ CT}$, $S_{sqrt\mathcal{R}}$, or S_{KLS}) and the choice of spatial cut-off. For definiteness, we will fix our attention on the counter-term $S_{new\ CT}$ below. In this case, for the hyperbolic spatial cut-off (Ω^{hyp}) we see from (3.7) that T^0_{ij} is proportional to the leading term in the electric part of the Weyl tensor:

$$T^0_{ij} = \lim_{\Omega\to\infty} \frac{1}{8\pi G} \frac{\rho^{d-3}}{d-3} E_{ij}; \tag{4.3}$$

Derivation and discussion of the corresponding expression for T^1_{ij} will be left for future study [52].

Despite working at finite Ω, conservation of this boundary stress tensor follows from the usual argument: One notes that the cut-off action $S_{renorm,\Omega}$ is invariant under diffeomorphisms of $\mathcal{M}_\Omega$ preserving $\partial\mathcal{M}_\Omega$. Since any diffeomorphism of $\partial\mathcal{M}_\Omega$ can be extended to such a diffeomorphism of $\mathcal{M}_\Omega$, it follows that $\delta S_{renorm,\Omega} = 0$ whenever $(\delta h_\Omega)_{ij} = D_{(i}\xi_{j)}$

[9] For a given spacetime, this function will be defined for values of Ω large enough that equation (1.4) can be solved for $\hat{K}_{ij}$ so that our counter-term is well-defined. Recall that our counter-term need not be well-defined for small Ω.

for any vector field ξ^j on $\partial\mathcal{M}$. Thus, taking a variational derivative of S with respect to such ξ^j shows that $T_{ij}(\Omega)$ is conserved at each Ω; i.e., that

$$D^i T_{ij}(\Omega) = 0. \tag{4.4}$$

In particular, equation (4.4) holds separately for each finite value of Ω. It turns out that the same general steps as in [28] allow us to relate $Q[\xi]$ to the boundary stress tensor. The argument below holds for either class of spatial cut-off (Ω^{hyp} or Ω^{cyl}), though we remind the reader that our coordinates η^i are defined in terms of the hyperboloid. It also holds for either class of temporal cut-offs ($\mathcal{T}^{cyl}$ or $\mathcal{T}^{hyp}$).

Let us begin by computing $Q[\xi]$ in terms of variations of the action:

$$Q[\xi] := -\Delta_{f,\xi} S_{renorm} = -\lim_{\Omega\to\infty}\left(\int_{\mathcal{M}_\Omega} \frac{\delta S_{renorm}}{\delta g_{ab}} \Delta_{f,\xi} g_{ab} + \frac{1}{2}\int_{\partial\mathcal{M}_\Omega} \sqrt{-h_\Omega}\, T^{ij} \Delta_{f,\xi} h_{ij}.\right) \tag{4.5}$$

Note that the bulk term vanishes, as we evaluate $Q[\xi]$ on a solution. Furthermore, it is straightforward to calculate $\Delta_{f,\xi} h_{ij}$:

$$\Delta_{f,\xi} h_{ij} = (\pounds_{f\xi} g)_{ij} - f(\pounds_\xi g)_{ij} = \xi_i D_j f + \xi_j D_i f, \tag{4.6}$$

where $(\pounds_{f\xi} g)_{ij}$ and $(\pounds_\xi g)_{ij}$ denote quantities evaluated in the bulk of $\mathcal{M}_\Omega$ and then pulled-back to the boundary $\partial\mathcal{M}_\Omega$. We may now use (4.6) to write (4.5) in the form

$$Q[\xi] = -\lim_{\Omega\to\infty}\int_{\partial\mathcal{M}_\Omega} \Omega^{-(d-3)}\sqrt{-h}\left(\Omega T^0_{ij} + T^1_{ij}\right) h^{jk} h^{il} \xi_l D_k f, \tag{4.7}$$

where we have dropped the higher terms in the expansion of the boundary stress tensor; since ξ_i is largest at infinity for a boost, which has $\xi^i = \mathcal{O}(\rho^0)$ and $\xi_i = \mathcal{O}(\rho^2)$, such higher terms in $T^{ij}(\Omega)$ will not contribute to $Q[\xi]$ in the limit $\Omega \to \infty$ for either class of temporal cut-offs $\mathcal{T}^{hyp}$ or $\mathcal{T}^{cyl}$.

In fact, the overall scaling of terms in (4.7) is the same for both $\mathcal{T}^{hyp}$ and $\mathcal{T}^{cyl}$: while for $\mathcal{T}^{cyl}$, the coordinate volume of $\partial\mathcal{M}_\Omega$ scales as $1/\Omega$, but the time derivative of f scales with a compensating factor of Ω. Integrating (4.7) by parts, we may write our charge as[10]

$$Q[\xi] = \lim_{\Omega\to\infty}\int_{C_\Omega} \Omega^{-(d-3)}\sqrt{h_{C_\Omega}}\left(\Omega T^0_{ij} + T^1_{ij}\right) h^{il}\xi_l n^j_\Omega + Q_{vol}[\xi], \tag{4.8}$$

where $C_\Omega = \Sigma_+ \cap \partial\mathcal{M}_\Omega$ is the future boundary of $\partial\mathcal{M}_\Omega$, n^j_Ω is its future (i.e., outward)-pointing unit normal in $\partial\mathcal{M}_\Omega$, and h_{C_Ω} is the induced metric on C_Ω. In addition, we have separated off the 'volume term' which includes an integral over all of $\partial\mathcal{M}_\Omega$:

$$Q_{vol}[\xi] := \lim_{\Omega\to\infty}\int_{\partial\mathcal{M}_\Omega} f\Omega^{-(d-3)}\sqrt{-h}\left(\Omega T^0_{ij} + T^1_{ij}\right) h^{jk} h^{il} D_k \xi_l. \tag{4.9}$$

[10]The rest of this section makes use of the global structure of (2.1), and so does not directly apply to spacetimes with Lorentzian NUT charge. In particular, the boundary of a NUT-charged spacetime does not admit a global cross-section $\mathcal{C}$. However, instead of integrating over a cross-section, one can use the fact that smooth spacetimes with Lorentzian NUT charge are periodic in time to write expressions similar to those below involving integration over all of $\partial\mathcal{M}$.

The first term in (4.8) is of the sort used to define charges in the asymptotically anti-de Sitter context via the so-called boundary counter-term method [12, 13]. This term could itself be taken to give a (slightly different) definition of the charge. Indeed, the second term $Q_{vol}[\xi]$ can be shown to be a "c-number," meaning that it Peierls-commutes with all observables. Thus, the first term in (4.8) alone also generates asymptotic ξ-translations. To see that $Q_{vol}[\xi]$ is a c-number, recall that $Q[\xi]$ generates the asymptotic symmetry ξ for any function f which vanishes on the past boundary and satisfies $f = 1$ on the future boundary. Thus, the difference $Q[\xi; f_1] - Q[\xi; f_2]$ must be a c-number whenever $Q[\xi; f_1]$ and $Q[\xi; f_2]$ are charges of the form (4.1) defined by two such functions f_1, f_2. In computing such a difference from (4.8), the first term cancels and we are left only with the difference $Q_{vol}[\xi; f_1] - Q_{vol}[\xi; f_2]$. But since $D_k\xi_l + D_l\xi_k = \mathcal{O}(\rho^{-(d-5)})$, it is clear that $Q_{vol}[\xi; f_2]$ vanishes in the limit where f_2 is zero everywhere except within a tiny neighborhood of the future boundary. Thus, for any allowed f_1, we see that $Q_{vol}[\xi; f_1]$ must be a c-number.

As a result, $Q_{vol}[\xi]$ can in general depend at most on the auxiliary structure and the particular choice of covariant phase space (i.e., on the choice of boundary conditions) and must be constant over any given covariant phase space. Suppose then that we choose a covariant phase space which contains Minkowski space. Within this context, the second term in (4.8) is given by its value on Minkowski space itself. But the Gauss-Codazzi equations guarantee that $\hat{K}_{ij} = K_{ij}$, and thus that both $T_{ij}(\Omega)$ and (4.8) vanish identically (for either Ω^{hyp} or Ω^{cyl}). We therefore see that our charges $Q[\xi]$ agree precisely with the usual definitions [2, 4, 5, 6, 7, 8, 9] on this covariant phase space.

In a more general context, $Q_{vol}[\xi]$ will still vanish for hyperbolic spatial cut-off Ω^{hyp}. Because this cut-off is invariant under boosts, we see that $D_k\xi_l + D_l\xi_k = \frac{\alpha}{\rho}h^0_{ij}$ for some smooth function α on $\mathcal{H}^{d-1}$. Since for Ω^{hyp} the leading stress-tensor term T^0_{ij} is traceless with respect to h^0_{ij}, this is sufficient to make $Q_{vol}[\xi]$ vanish. Similarly, for cylindrical spatial cut-off Ω^{cyl}, $Q_{vol}[\xi]$ must vanish unless ξ^i contains a boost. The difficulty with boosts is that while, in analogy with Ω^{hyp}, we have $D_k\xi_l + D_l\xi_k = \beta\xi h^0_{ij}$ for some smooth function β on $\mathcal{H}^{d-1}$, for Ω^{cyl} the leading stress-tensor term T^0_{ij} fails to be traceless.

When $Q_{vol}[\xi]$ vanishes, we may formally write (4.8) as

$$Q[\xi] = \int_C \sqrt{h_C}T_{ij}\xi^i n^j, \tag{4.10}$$

where $C = \Sigma_- \cap \partial\mathcal{M}$ and $(h_C)_{ij}, n^j$ are the associated induced metric and future-pointing normal vector field. In fact, the vanishing of $Q_{vol}[\xi]$ for all f is sufficient to guarantee that the charge (4.10) conserved in the sense that its numerical value is independent of the cut C. Thus, for the particular cases described above, we have much of the structure which has become familiar [12, 13] in the anti-de Sitter context and, in particular, the conserved charges may be calculated by an algorithm of the sort described in [31, 30].

For hyperbolic spatial cut-off Ω^{hyp}, using the relation (4.3) between T^0_{ij} and the electric part of the Weyl tensor, expression (4.10) makes manifest the general agreement with standard ADM expressions for energy and momentum[11], as the latter are known from [2, 3, 5] to be expressible via a relation analogous to (4.10) in terms of the leading contribution

[11] In addition, comparing (4.8), (4.10) with the results of [2, 3, 5] for angular momentum strongly suggests,

to E_{ij}. Now, counting powers of Ω in (4.10) may cause the reader to worry that a divergent contribution arises from the term involving $\hat{\xi}^i_B$ and T_0^{ij}. However, we remind the reader that we have already shown $Q[\xi]$ to be finite. In $d = 4$ spacetime dimensions, one may show [2, 3, 5] that the integral over C of the apparently divergent term vanishes due to the fact that the leading contribution to E_{ij} admits a certain scalar potential. In higher dimensions, it is clear that some analogous cancellation must occur.

Note that the above argument also shows that (4.10) holds for the stress tensors computed from $S_{\sqrt{\mathcal{R}}}$ and from S_{KLS}. Furthermore, from the definition (1.4) of $\hat{K}_{ij}$, it is straightforward to show that the stress tensor defined by $S_{new\ CT}$, $S_{\sqrt{\mathcal{R}}}$, and S_{KLS} differ only by terms built from h^1_{ij} and higher order terms in h_{ij}. Thus, these stress tensors all agree (and all vanish) for Minkowski space. Thus, we see that the conserved quantities $Q[\xi]$ as defined through $S_{\sqrt{\mathcal{R}}}$ or S_{KLS} also agree with the standard results [2, 4, 5, 6, 7, 8, 9]. In particular, this observation justifies the calculations performed in [30].

5. Discussion

In the above work, we considered actions S_{renorm} for asymptotically flat vacuum gravity, constructed by adding one of the counter-terms $S_{new\ CT}$ (1.7), $S_{\sqrt{\mathcal{R}}}$ (3.18) [26] , or S_{KLS} (3.17) [27] to the Einstein-Hilbert action with Gibbons-Hawking term. All of these counter-terms are given by local algebraic functions of the boundary metric and Ricci tensor. The new counter-term $S_{new\ CT}$ is constructed from a symmetric tensor $\hat{K}_{ij}$ defined by solving the traced Gauss-Codazzi equations (1.4) which would result if the boundary spacetime $(\partial\mathcal{M}, h)$ were detached from the bulk spacetime $(\mathcal{M}, g)$ and isometrically embedded in Minkowski space. As a result, $S_{new\ CT}$ is directly related to the reference background counter-term S_{Ref}, though with the advantage that, as opposed to S_{Ref}, the new counter-term $S_{new\ CT}$ is well-defined on open sets of boundary metrics in dimesions $d \geq 4$.

The results established depend on the choice of temporal cut-off used to define the system. The simplest case is that of a cylindrical temporal cut-off ($\mathcal{T}^{cyl}$). For this case, we have demonstrated that each of the three counter-terms $S_{new\ CT}, S_{\sqrt{\mathcal{R}}}, S_{KLS}$ leads to an action S_{renorm} such that

- For $d \geq 4$ the action S_{renorm} leads to a fully satisfactory variational principle when $\partial\mathcal{M}$ is defined by either a hyperbolic or a cylindrical spatial cut-off; i.e., by Ω^{hyp} (2.2) or Ω^{cyl} (2.4). In particular, S_{renorm} is both finite and stationary on asymptotically flat vacuum solutions. In addition, on any such solution the action takes a numerical value which is independent of the precise choice of spatial cut-off within the class Ω^{hyp} (2.2) or Ω^{cyl} (2.4).

Also,

- For $d \geq 4$, the action S_{renorm} defined by the counter-term $S_{new\ CT}$ is again finite when $\partial\mathcal{M}$ is defined by any spatial cut-off of the form $\Omega = \omega\Omega^{hyp}$ for ω a smooth

at least for $d = 4$, for spatial cut-off Ω^{hyp}, and when the asymptotic metric takes the form specified in [2, 3, 5], that T^1_{ij} may be expressed in terms of the magnetic part of the Weyl tensor.

non-vanishing function on the unit hyperboloid. Again, the numerical value of the action depends at most on the choice of this ω and not on further details of the cut-off. However, except as stated above, we reached no conclusions with regard to the variations of S_{renorm} in this context. We note that the counter-term $S_{\sqrt{\mathcal{R}}}$ is not defined for such general spatial cut-offs, and we have not investigated the corresponding result for S_{KLS}.

When a hyperbolic temporal cut-off ($\mathcal{T}^{hyp}$) is chosen, both the Gibbons-Hawking term and the counter-terms are more divergent. Thus, the analysis is more subtle. In this case, we have established that

- When defined using our new counter-term $S_{new\ CT}$, the action S_{renorm} leads to a fully satisfactory variational principle for $d \geq 4$ and when $\partial\mathcal{M}$ is defined by a hyperbolic spatial cut-off; i.e., by Ω^{hyp} (2.2). In particular, S_{renorm} is both finite and stationary on asymptotically flat vacuum solutions. In addition, on any such solution the action takes a numerrical value which is independent of the precise choice of spatial cut-off within the class Ω^{hyp} (2.2).

- For the remaining counter-terms $S_{\sqrt{\mathcal{R}}}$ and S_{KLS}, the action S_{renorm} is again finite on vacuum solutions for $d \geq 4$ when when $\partial\mathcal{M}$ is defined by a hyperbolic spatial cut-off; i.e., by Ω^{hyp} (2.2). However, we have demonstrated that the action is stationary only for $d \geq 5$. For $S_{\sqrt{\mathcal{R}}}$, we have demonstrated that the action is *not* in general stationary for $d = 4$, while we have not performed the calculation for S_{KLS} in $d = 4$ to the relevant order in ρ. However, whenever these actions are stationary on such solutions, the numerical value of S_{renorm} is again independent of the particular spatial cut-off Ω chosen within the class Ω^{hyp} (2.2).

All of these results hold up to possible terms localized at the past and future boundaries (Σ_- and Σ_+), as such terms have been neglected in our treatment. It is clearly of interest to address such terms in future work.

While our focus has been on Lorentz signature, analogous results follow immediately in the Euclidean setting. We note that, in the thermal context, periodicity of Euclidean time automatically imposes a cylindrical temporal cut-off and removes the possibility of boundary terms on Σ_+ and Σ_-.

Under the conditions stated above for which each action S_{renorm} is both finite and stationary, we showed that S_{renorm} leads via a straightforward algorithm to the usual conserved quantities $Q[\xi]$ (energy, angular momentum, etc.) at spatial infinity for each asymptotic symmetry ξ. We find this to be a significant conceptual simplification over the textbook calculations of such quantities. Furthermore, our construction generalizes the usual definitions to spacetimes in which the magnetic and electric parts of the Weyl tensor are of the same order and, in particular, for non-zero NUT charge[12] when $d = 4$. We have

[12] A canonical definition of conserved charges in the presence of NUT charge will appear soon [29]. Due to the Peierls argument of [28], the canonical definitions should agree with the covariant ones given here. However, we remind the reader that we have not actually constructed the phase space for such solutions

also shown that our quantities can be expressed in terms of a 'boundary stress tensor,' though both the leading term (T^0_{ij}) and a sub-leading term (T^1_{ij}) are required to construct all conserved quantities. In most cases, $Q[\xi]$ can be expressed in the form

$$Q[\xi] = -\int_C \sqrt{h_C} T_{ij} \xi^i n^j, \tag{5.1}$$

with the one (possible) exception occurring when ξ generates a non-trivial boost in the presence of cylindrical spatial cut-off Ω^{cyl} with boundary conditions chosen such that Minkowski space is *not* part of the covariant phase space. Thus we have much of the structure which has become familiar [12, 13] in the anti-de Sitter context, and which is of much use in the AdS/CFT correspondence (see e.g., [45, 46]).

One notable difference from the anti-de Sitter case is, however, that we have established the above properties only when the leading behavior of the metric at infinity has a specific form (2.1), where in particular h^0_{ij} is an $SO(d-1,1)$ invariant metric on the unit hyperboloid $\mathcal{H}^{d-1}$. Though we have not discussed it here, a major obstacle to considering other h^0_{ij} arises from the form of the Einstein equations themselves near i^0. In particular, suppose for the moment that we attempt to allow h^0_{ij} in to be an arbitrary Lorentz signature boundary metric on $\mathbb{R} \times S^{d-2}$ but to otherwise leave our definition (2.1) of asymptotic flatness unchanged. This might seem natural if one sought a non-gravitating theory associated with i^0 and dual to asymptotically flat gravity[13]. We will, of course, wish to impose the Einstein equations. However, it turns out that the Einstein equations alone require h^0_{ij} to be an Einstein metric [5, 24, 25, 32], which for $d = 4$ implies that it is in fact a constant multiple of the metric on the unit hyperboloid $\mathcal{H}^3$. Using any other metric for h^0_{ij} would thus necessitate the inclusion of more singular terms in our ansatz for the metric, and such terms appear difficult to control.

A less ambitious goal would be to maintain the asymptotic conditions (i.e., (2.1)) used in this work, but to allow more general spatial cut-offs Ω. For example, we have noted that, for spacetimes (2.1), with the counter-term $S_{new\ CT}$ and cylindrical temporal cut-off $\mathcal{T}^{cyl}$, the action S_{renorm} continues to take finite values on shell for a general class of spatial cut-offs Ω. However, we have not been able to establish this result for hyperbolic temporal cut-offs $\mathcal{T}^{hyp}$, nor have we established that the action is fully stationary on asymptotically flat solutions with cylindrical temporal cut-off $\mathcal{T}^{cyl}$. It would be interesting to explore this further, and also to investigate the corresponding properties of S_{KLS} (3.17).

Finally, we note that our discussion has been restricted to asymptotically flat spacetimes in dimensions $d \geq 4$. The case $d = 3$ is also of significant interest with 'asymptotically flat' boundary conditions such as those of [53], as are various other boundary conditions for $d \geq 4$. A particularly interesting case is that of asymptotically Melvin spacetimes [54, 55] associated with black hole pair creation [56, 57, 58]. The action for such spacetimes is

---

here, and have thus not kept track of any conditions required to ensure that the symplectic structure is finite. As a result, though given any asymptotic symmetry our results will yield the correct conserved quantity, we have not determined the precise asymptotic symmetries of any system. Instead, these are taken as an input in the present work.

[13]In contrast, see [47]-[51] for work considering a possible dual theory at null infinity.

known to be finite when defined by either the reference background subtraction prescription [23] or the counter-term $S_{\sqrt{\mathcal{R}}}$ [59]. However, the status of the variational principle has not been addressed. It would be very interesting to discover if $S_{new\ CT}$ can provide a suitable variational principle in such a context.

A. On the restriction of the of the variations for $\mathcal{T}^{hyp}$ and $d = 4$.

In section 2 we imposed the restriction $\delta h^1_{ij} = \alpha h^0_{ij}$ (2.7) on variations about asymptotically flat solutions in $d = 4$ spacetime dimensions. Here α is some smooth function on the hyperboloid. The purpose of this appendix is to argue that this restriction is compatible with the equations of motion. By this we mean that, if an infinitesimal tangent vector to the space of solutions has initial data satisfying (2.7) on some Cauchy surface C of $\mathcal{H}^{d-1}$, then the tangent vector can be taken to have this form on all of $\mathcal{H}^{d-1}$.

Furthermore, we will show that the initial data of this form is sufficiently general. We choose our criterion for "sufficient generality" by comparison with [4], which allowed only one degree of freedom in h^1_{ij} and, furthermore, imposed a single relation between such h^1_{ij} and σ. What we show below is that for any variation satisfying both $\delta h^1_{ij} = \alpha h^0_{ij}$ and $\delta\sigma = -\frac{1}{2}\alpha$, all equations of motion hold to the relevant order in ρ when $\delta\sigma$ satisfies its equation of motion:

$$D^2\delta\sigma + 3\delta\sigma/\rho^2 = \text{ higher order in } \rho, \tag{A.1}$$

see equation (3.29) of [24]. We base our argument on the results of [24, 25], who studied the $d = 4$ Einstein equations expanded near spatial infinity. They found that each correction h^n_{ij} satisfies an equation on $\mathcal{H}^{d-1}$ of the form:

$$L_n h^n_{ij} = s^n_{ij}, \tag{A.2}$$

where L_n is a hyperbolic linear partial differential operator and s^n_{ij} is a source term built (not necessarily linearly) from h^m_{ij} for $m < n$, as well as from σ and the corresponding higher corrections. Equation (A.1) is the corresponding equation for σ, and there is a similar hierarchy of equations for corrections to σ, as well as certain constraints. However, it was shown that this system of equations has solutions whenever the constraints are satisfied on some Cauchy surface of $\mathcal{H}^{d-1}$. Infinitesimal tangent vectors satisfy the linearization of this system of equations.

We will now show that, given initial data of the form $\delta h^1_{ij} = \alpha h^0_{ij}, \delta\sigma = -\frac{1}{2}\alpha$ for a tangent vector to the space of solutions, there is a solution to the (linearization of the) equations of [24, 25], which again takes the form $\delta h^1_{ij} = \alpha h^0_{ij}, \delta\sigma = -\frac{1}{2}\alpha$. As we have already imposed (A.1), the only obstacles are the equation involving $L_1 h^1_{ij}$ and the corresponding constraints. We begin by transcribing the linearization of the equation of motion involving $L_1 \delta h^1_{ij}$ (see (3.28) of [24]), as

$$\begin{aligned}\frac{1}{2}D_k D^k \delta h^1_{ij} - \frac{1}{2}D_i D_j \delta h^1 - 3\rho^{-2} D_i D_j \delta\sigma - 3\rho^{-2}\delta\sigma\ h^0_{ij} - \frac{3}{2}(\rho^{-2}\delta h^1_{ij} - \frac{1}{3}\delta h^1 h^0_{ij})\\ = \text{ higher order in } \rho,\end{aligned} \tag{A.3}$$

where to the desired order $\delta h^1 = (h^0)^{ij}\delta h^1_{ij}$. Substituting $\delta h^1_{ij} = \alpha h^0_{ij}, \delta\sigma = -\frac{1}{2}\alpha$, one finds

$$\frac{1}{2}h^0_{ij}\left(D_i D^i \alpha + 3\alpha/\rho^2\right) = 0, \tag{A.4}$$

where in the last step we have used the linearization of (A.1) with $\delta\sigma = -\frac{1}{2}\alpha$.

It remains only to check the constraint equations. When linearized, these become:

$$-D^2\delta h^1 + D^i D^j \delta h^1_{ij} + 12\frac{\delta\sigma}{\rho^4} = -\frac{2}{\rho^2}(D^2\alpha + 3\alpha/\rho^2) = 0, \tag{A.5}$$

and

$$-\frac{1}{2}D_i(\delta h^1)^i_j + \frac{1}{2}D_j\delta h^1 + 2D_j\delta\sigma = 0. \tag{A.6}$$

Thus, we see that, to this order in ρ, solutions to the linearized equations of motion exist with any initial data of the form $\delta h^1_{ij} = \alpha h^0_{ij}, \delta\sigma = -\frac{1}{2}\alpha$.

B. Sub-leading terms in the variation $\delta S_{new\ CT}$

This appendix outlines the calculation that

$$\delta\hat{K}_{ij}h^{ij} = -\frac{1}{2}\left[\hat{K}_{ij}\delta h^{ij} - \frac{\rho}{(d-2)}h^{ij}\delta\mathcal{R}_{ij}\right] + \mathcal{O}(\rho^{-4}), \tag{B.1}$$

for $d = 4$ with hyperbolic spatial cut-off Ω^{hyp} and $\delta h^1_{ij} = \alpha h^0_{ij}$. This result establishes that our new counter-term yields an action S_{renorm} which is in fact stationary under such conditions. For much of this calculation it will be convenient to work in general dimension d, though we treat only the case of hyperbolic spatial cut-off Ω^{hyp}.

Let us begin by defining

$$\epsilon_{ij} = \hat{K}_{ij} - \frac{1}{\rho}h_{ij}. \tag{B.2}$$

Note that ϵ_{ij} is of order $\mathcal{O}(\rho^0)$, and that it receives contributions at this order from both $\hat{K}_{ij}$ and h_{ij}. For $\epsilon = 0$, equation (3.9) is easy to solve and gives exactly $\delta\hat{K}_{ij}h^{ij} = -\frac{1}{2}(\hat{K}_{ij}\delta h^{ij} - \frac{\rho}{(d-2)}h^{ij}\delta\mathcal{R}_{ij})$. Thus, our task is to solve (3.9) to first order in epsilon. As with all other objects, indices on ϵ_{ij} will be raised and lowered with h_{ij}. In particular, $\epsilon := \epsilon_{ij}h^{ij}$.

To do so, write (3.9) as

$$L_{ij}{}^{kl}\delta\hat{K}_{kl} = \delta\mathcal{R}_{ij} - M_{ijkl}\delta h^{kl}. \tag{B.3}$$

We now expand

$$L = L^0 + L^1 + \cdots, \tag{B.4}$$

$$L^{-1} = (L^{-1})^0 + (L^{-1})^1 + \cdots, \tag{B.5}$$

$$M = M^0 + M^1 + \cdots, \tag{B.6}$$

where terms with superscripts n are homogeneous in ϵ_{ij} of order n and where $(L^{-1})_{ij}{}^{kl}(L)_{kl}{}^{mn} = \delta^m_i\delta^n_j$.

One finds

$$
\begin{aligned}
[(L^{-1})^0]_{ij}{}^{kl} &= \frac{\rho}{d-3}\left[\delta_i^k\delta_j^l - \frac{1}{2(d-2)}h^{kl}h_{ij}\right], && \text{(B.7)}\\
h^{ij}[(L^{-1})^1]_{ij}{}^{kl} &= \frac{\rho^2}{(d-2)(d-3)}\left[\epsilon^{kl} - \frac{1}{2}\epsilon h^{kl}\right], && \text{(B.8)}\\
M^0_{ijkl} &= \rho^{-2}(h_{ij}h_{kl} - h_{ik}h_{jl}), && \text{(B.9)}\\
h^{ij}M^1_{ijkl} &= \rho^{-1}\left(\epsilon h_{kl} + (d-3)\epsilon_{kl}\right), \text{ or,} && \text{(B.10)}\\
h^{ij}M_{ijkl} &= \frac{(d-2)}{\rho}\hat{K}_{kl} + \frac{\epsilon h_{kl}}{\rho} - \frac{\epsilon_{kl}}{\rho} + \mathcal{O}(\epsilon^2). && \text{(B.11)}
\end{aligned}
$$

As a result,

$$
\begin{aligned}
h^{ij}\delta\hat{K}_{ij} &= \frac{\rho}{2(d-2)}\delta\mathcal{R}_{ij}h^{ij} - \frac{1}{2}\hat{K}_{ij}\delta h^{ij}\\
&- \frac{\rho^2}{(d-2)(d-3)}\left[\frac{1}{2}\epsilon\ \delta\mathcal{R}_{ij}h^{ij} - \epsilon^{ij}\delta\mathcal{R}_{ij}\right]\\
&- \frac{\epsilon h_{ij}\delta h^{ij}}{2(d-2)(d-3)} + \frac{(d-1)\epsilon_{ij}\delta h^{ij}}{2(d-2)(d-3)} + \mathcal{O}(\rho^{-(3d-8)}). && \text{(B.12)}
\end{aligned}
$$

Next we observe that, to the desired order in ρ, in any term in (B.15) containing ϵ, we may substitute

$$
\begin{aligned}
\delta h_{ij} &\to \frac{\delta h^1_{ij}}{\rho^{d-5}} = \frac{\alpha h^0_{ij}}{\rho^{d-5}} \to \frac{\alpha h_{ij}}{\rho^{d-3}},\\
\delta h^{ij} &\to -\frac{\alpha h^{ij}}{\rho^{d-3}}. && \text{(B.13)}
\end{aligned}
$$

where we have used (2.7). In addition, we recall that $\delta\mathcal{R}_{ij}$ is given by expression (3.12) in terms of second covariant derivatives (D_k) of δh_{ij}. Thus, in any term containing ϵ we may substitute

$$
\delta\mathcal{R}_{ij} \to -\frac{1}{2\rho^{(d-3)}}\left[(d-3)D_iD_j\alpha + D^2\alpha\ h_{ij}\right]. \tag{B.14}
$$

Recalling that we wish to integrate $\sqrt{-h}h^{ij}\delta\hat{K}_{ij}$ over $\partial\mathcal{M}$, any total divergence will contribute only a boundary term. It is useful to use such integrations by parts to move all of the derivatives to act on ϵ_{ij}. Thus, we write

$$
\begin{aligned}
h^{ij}\delta\hat{K}_{ij} &= \frac{\rho}{2(d-2)}\delta\mathcal{R}_{ij}h^{ij} - \frac{1}{2}\hat{K}_{ij}\delta h^{ij} + \frac{\alpha}{2(d-2)\rho^{(d-5)}}\left[D^2\epsilon - D_iD_j\epsilon^{ij}\right]\\
&\quad + \text{ total divergence terms } + \mathcal{O}(\rho^{-(3d-8)}), && \text{(B.15)}
\end{aligned}
$$

where the explicit terms on the final line of (B.12) have summed to zero after applying (B.13).

At this stage, we need to express ϵ_{ij} in terms of h^1_{ij}. We may do so by treating $\frac{h^1_{ij}}{\rho^{(d-5)}}$ as a perturbation of $\rho^2 h^0_{ij}$ and once again using (3.9) to solve perturbatively for $\hat{K}_{ij}$ to the desired order. The result simplifies greatly when one uses the equations of motion. It is at this stage that we impose $d = 4$, as the equations of motion for this case were expanded in a convenient form in [24]. In particular, we make use of equations (3.27) from [24], which allows us to express the change $\Delta\mathcal{R}_{ij}$ in $\mathcal{R}_{ij}$ associated with changing the induced metric from $\rho^2 h^0_{ij}$ to $\rho^2 h^0_{ij} + \rho h^1_{ij}$ as

$$\Delta\mathcal{R}_{ij} = \frac{1}{2\rho}\left(3h^1_{ij} - h^1 h_{ij} + 2D_i D_j \sigma - 6\frac{\sigma h_{ij}}{\rho^2}\right). \tag{B.16}$$

Using this expression and (3.9), one may readily compute $\beta_{ij} = \hat{K}_{ij} - \rho h^0_{ij}$. Upon using

$$(D^2\sigma + 3\sigma/\rho^2) = 0, \tag{B.17}$$

which is the equation of motion for σ (equation (3.29) of [24]), we find

$$\epsilon_{ij} = \beta_{ij} - h^1_{ij} = D_i D_j \sigma - \frac{1}{2}h^1_{ij}. \tag{B.18}$$

We may now compute:

$$D^2\epsilon - D_i D_j \epsilon = D^4\sigma - \frac{1}{2}D^2 h^1 - D_i D_j D^i D^j \sigma + \frac{1}{2}D^i D^j h^1_{ij}. \tag{B.19}$$

Finally, we use the equation of motion (3.25) from [24],

$$-D^2 h^1 + D^i D^j h^1_{ij} = -\frac{12\sigma}{\rho^4} + \text{ higher order terms}, \tag{B.20}$$

as well as their relation (A.2),

$$[D_i, D_j]\omega_k = h^0_{ki}\omega_j - h^0_{kj}\omega_i + \text{ higher order terms}, \tag{B.21}$$

which follows from the commutator of covariant derivatives on the hyperboloid, to find

$$D^2\epsilon - D_i D_j \epsilon = -\frac{2}{\rho^2}(D^2\sigma + 3\sigma/\rho^2) = 0, \tag{B.22}$$

Substituting this result into (B.15) yields (B.1), as desired.

C. Sub-leading terms in the variation $\delta S_{\sqrt{\mathcal{R}}}$

This appendix outlines the calculation showing that S_{renorm} as defined by the counter term $S_{\sqrt{\mathcal{R}}}$ is *not* stationary for $d = 4$ when one chooses both the temporal and spatial cut-offs to be hyperbolic. This result contrasts with the result derived in appendix B, showing that the action defined by $S_{new\ CT}$ is indeed stationary.

We proceed by calculating (3.19) to the next order in ρ. We begin by noting that

$$\delta\sqrt{\mathcal{R}} = \frac{1}{2}\mathcal{R}^{-1/2}\ \delta\mathcal{R}. \tag{C.1}$$

Now, as in appendix B, it is useful to compare quantities such as $\mathcal{R}$ computed from the metric $h_{ij} = \rho^2 h^0_{ij}$ with those computed from $h_{ij} = \rho^2 h^0_{ij} + \rho^{5-d} h^1_{ij}$. Let us denote the corresponding change in any such quantity by Δ; i.e., $\Delta\mathcal{R}$ is the change in $\mathcal{R}$. Then we have

$$\delta\sqrt{\mathcal{R}} = \frac{1}{2}\left(\frac{\rho}{\sqrt{(d-1)(d-2)}}\left[h^{ij}\delta\mathcal{R}_{ij} + \mathcal{R}_{ij}\delta h^{ij}\right] + \Delta\mathcal{R}^{-1/2}\delta\mathcal{R}\right). \tag{C.2}$$

Let us now examine each term in turn. Using (3.12), the first term (involving $h^{ij}\delta\mathcal{R}_{ij}$) can be written as a total derivative. Thus, it does not contribute.

To evaluate the second term to the desired order, we use

$$\begin{aligned}\mathcal{R}_{ij} &= (d-2)h^0_{ij} + \Delta\mathcal{R}_{ij} \\ &= \frac{(d-2)}{\rho}\hat{K}_{ij} - \frac{(d-2)}{\rho}D_iD_j\sigma - \frac{(d-2)}{2\rho}h^1_{ij} + \Delta\mathcal{R}_{ij},\end{aligned} \tag{C.3}$$

where in the last step we have used (B.18). Specializing to $d=4$ and using (B.16) we have

$$\mathcal{R}_{ij}\delta h^{ij} \approx \frac{2}{\rho}\hat{K}_{ij}\delta h^{ij} + \frac{2\alpha}{\rho^2}D^2\sigma + \frac{\alpha}{\rho^2}h^1 + \frac{12\alpha\sigma}{\rho^4}, \tag{C.4}$$

where the symbol $\approx$ indicates that we have dropped terms which are high enough order in ρ^{-1} that they will not contribute to our final expression. In addition, we have used (B.13) and (B.16) in the subleading terms.

Finally, we compute the third term in (C.2). From (3.12) and (B.13) we find

$$\delta\mathcal{R} \approx -\frac{2}{\rho}(D^2\alpha + 3\alpha/\rho^2). \tag{C.5}$$

Furthermore, (B.16) yields

$$\Delta\mathcal{R} = -\frac{12\sigma}{\rho^3} - \frac{2}{\rho}h^1. \tag{C.6}$$

Thus, we have

$$\frac{1}{2}\Delta\mathcal{R}^{-1/2}\delta\mathcal{R} \approx -\frac{\rho}{12}\alpha(D^2h^1 + 3h^1/\rho^2) + \text{ total derivative terms}, \tag{C.7}$$

where we have also used the equation of motion (A.1).

Putting these results together we have

$$\delta\sqrt{\mathcal{R}} = \frac{1}{\sqrt{6}}\left(\hat{K}_{ij}\delta h^{ij} - \frac{\alpha\rho}{6}D^2h^1 - +3\frac{\alpha\sigma}{\rho^3}\right) + \text{total derivative terms} \tag{C.8}$$

Comparing with (B.1), we see that when the counter-term $S_{\sqrt{\mathcal{R}}}$ is used, the variation δS_{renorm} does not generically vanish on solutions. However, it does vanish on the covariant phase space defined by [4], where $h^1 = -6\frac{\sigma}{\rho^2}$.

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References

[1] R. M. Wald, *General Relativity*, (U. Chicago, Chicago, 1984).

[2] A. Ashtekar and R. O. Hansen, "A Unified Treatment of Null and Spatial Infinity in General Relativity. I. Universal Structure, Asymptotic Symmetries, and Conserved Quantities at Spatial Infinity," *J. Math. Phys.*, **19** 1542 (1978).

[3] A. Ashtekar "Asymptotic structure of the gravitational field at spatial infinity" in *General relativity and gravitation : one hundred years after the birth of Albert Einstein*, edited by A. Held (New York, Plenum Press, 1980).

[4] A. Ashtekar, L. Bombelli and O. Reula, "The Covariant Phase Space Of Asymptotically Flat Gravitational Fields," in *Analysis, Geometry and Mechanics: 200 Years After Lagrange*, edited by M. Francaviglia and D. Holm (North-Holland, Amsterdam, 1991).

[5] A. Ashtekar and J. D. Romano, "Spatial infinity as a boundary of space-time," Class. Quant. Grav. **9**, 1069 (1992).

[6] T. Regge and C. Teitelboim, "Role Of Surface Integrals In The Hamiltonian Formulation Of General Relativity," Annals Phys. **88**, 286 (1974).

[7] R. Arnowitt, S. Deser and C. W. Misner, Nuovo Cimento, **19** (1961) 668.

[8] R. Arnowitt, S. Deser and C. W. Misner, J. Math. Phys. **1** (1960) 434.

[9] R. Arnowitt, S. Deser and C. W. Misner, "The Dynamics Of General Relativity," arXiv:gr-qc/0405109.

[10] A. Ashtekar, S. Fairhurst and B. Krishnan, "Isolated horizons: Hamiltonian evolution and the first law," Phys. Rev. D **62**, 104025 (2000) [arXiv:gr-qc/0005083].

[11] A. Ashtekar, C. Beetle and J. Lewandowski, "Mechanics of Rotating Isolated Horizons," Phys. Rev. D **64**, 044016 (2001) [arXiv:gr-qc/0103026].

[12] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087];

[13] V. Balasubramanian and P. Kraus, "A stress tensor for anti-de Sitter gravity," Commun. Math. Phys. **208**, 413 (1999) [arXiv:hep-th/9902121].

[14] K. Skenderis and S. N. Solodukhin, "Quantum effective action from the AdS/CFT correspondence," Phys. Lett. B **472**, 316 (2000) [arXiv:hep-th/9910023].

[15] M. Bianchi, D. Z. Freedman and K. Skenderis, "How to go with an RG flow," JHEP **0108**, 041 (2001) [arXiv:hep-th/0105276].

[16] K. Skenderis, "Lecture notes on holographic renormalization," Class. Quant. Grav. **19**, 5849 (2002) [arXiv:hep-th/0209067].

[17] S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence," Commun. Math. Phys. **217**, 595 (2001) [arXiv:hep-th/0002230].

[18] K. Skenderis, "Asymptotically anti-de Sitter spacetimes and their stress energy tensor," Int. J. Mod. Phys. A **16**, 740 (2001) [arXiv:hep-th/0010138].

[19] M. Bianchi, D. Z. Freedman and K. Skenderis, "Holographic renormalization," Nucl. Phys. B **631**, 159 (2002) [arXiv:hep-th/0112119].

[20] I. Papadimitriou and K. Skenderis, "AdS/CFT correspondence and geometry," arXiv:hep-th/0404176.

[21] J. D. Brown and J. W. . York, "Quasilocal energy and conserved charges derived from the gravitational action," Phys. Rev. D **47**, 1407 (1993).

[22] J. D. Brown, J. D. .E. Creighton and R. B. Mann, "Temperature, Energy, and Heat Capacity of Asymptotically Anti-De Sitter Black Holes," Phys.Rev. D **50** 6394 (1994).

[23] S. W. Hawking and G. T. Horowitz, "The Gravitational Hamiltonian, action, entropy and surface terms," Class. Quant. Grav. **13**, 1487 (1996) [arXiv:gr-qc/9501014].

[24] R. Beig and B. Schmidt, "Einstein's equations near spatial infinity," 1982 Commun. Math. Phys. 87 65.

[25] R. Beig, "Integration of Einstein's Equations Near Spatial Infinity," 1984 Proc. R. Soc. A 391 295.

[26] R. B. Mann, "Misner string entropy," Phys. Rev. D **60**, 104047 (1999) [arXiv:hep-th/9903229].

[27] P. Kraus, F. Larsen and R. Siebelink, "The gravitational action in asymptotically AdS and flat spacetimes," Nucl. Phys. B **563**, 259 (1999) [arXiv:hep-th/9906127].

[28] S. Hollands, A. Ishibashi and D. Marolf, "Counter-term charges generate bulk symmetries," arXiv:hep-th/0503105.

[29] M. Hennaux and C. Teitelboim, in preparation.

[30] D. Astefanesei and E. Radu, "Quasilocal formalism and black ring thermodynamics," arXiv:hep-th/0509144.

[31] R. B. Mann, "Expanding the Area of Gravitational Entropy," Found. Phys. **33**, 65 (2003) [arXiv:gr-qc/0211047].

[32] S. de Haro, K. Skenderis and S. N. Solodukhin, "Gravity in warped compactifications and the holographic stress tensor," Class. Quant. Grav. **18**, 3171 (2001) [arXiv:hep-th/0011230].

[33] P. G. Bergmann, "Gauge-Invariant Variables in General Relativity," Phys. Rev. **124**, 274 (1961).

[34] A. Ashtekar, "Logarithmic Ambiguities in the Description of Spatial Infinity," Found. Phys. **15**, 419 (1985).

[35] R. E. Peierls, "The commutation laws of relativistic field theory," Proc. Roy. Soc. (London) 214 (1952), 143-157.

[36] B. S. DeWitt, *Dynamical Theory of Groups and Fields*

[37] B. S. DeWitt, "The spacetime approach to Quantum Field Theory," in *Relativity, Groups, and Topology II: Les Houches 1983*, (North-Holland, Amsterdam, 1984) edited by B. S. DeWitt and R. Stora.

[38] B. S. DeWitt, *The Global Approach to Quantum Field Theory*, (Oxford University Press, 2003).

[39] D. M. Marolf, "The Generalized Peierls bracket," Annals Phys. **236**, 392 (1994) [arXiv:hep-th/9308150].

[40] M. Duetsch and K. Fredenhagen, "Algebraic quantum field theory, perturbation theory, and the loop expansion," Commun. Math. Phys. **219**, 5 (2001) [arXiv:hep-th/0001129].

[41] M. Duetsch and K. Fredenhagen, "Causal perturbation theory in terms of retarded products, and a proof of the action Ward identity," arXiv:hep-th/0403213.

[42] L. F. Abbott and S. Deser, "Stability Of Gravity With A Cosmological Constant," Nucl. Phys. B **195**, 76 (1982).

[43] S. Deser and B. Tekin, "Energy in generic higher curvature gravity theories," Phys. Rev. D **67**, 084009 (2003) [arXiv:hep-th/0212292].

[44] S. Deser and B. Tekin, "Gravitational energy in quadratic curvature gravities," Phys. Rev. Lett. **89**, 101101 (2002) [arXiv:hep-th/0205318].

[45] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[46] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[47] J. de Boer and S. N. Solodukhin, "A holographic reduction of Minkowski space-time," Nucl. Phys. B **665**, 545 (2003) [arXiv:hep-th/0303006].

[48] S. N. Solodukhin, "Reconstructing Minkowski space-time," arXiv:hep-th/0405252.

[49] G. Arcioni and C. Dappiaggi, "Exploring the holographic principle in asymptotically flat spacetimes via the BMS group," Nucl. Phys. B **674**, 553 (2003) [arXiv:hep-th/0306142].

[50] C. Dappiaggi, V. Moretti and N. Pinamonti, "Rigorous steps towards holography in asymptotically flat spacetimes," arXiv:gr-qc/0506069.

[51] E. Alvarez, J. Conde and L. Hernandez, "Goursat's problem and the holographic principle," Nucl. Phys. B **689**, 257 (2004) [arXiv:hep-th/0401220].

[52] R.B. Mann and D. Marolf, in preparation.

[53] A. Ashtekar and M. Varadarajan, "A Striking property of the gravitational Hamiltonian," Phys. Rev. D **50**, 4944 (1994) [arXiv:gr-qc/9406040].

[54] M. A. Melvin, "Pure Magnetic And Electric Geons," Phys. Lett. **8**, 65 (1964).

[55] F. J. Ernst, "Removal of the nodal singularity of the C-metric," J. Math. Phys. 17 (1976) 515.

[56] G.W. Gibbons, in *Fields and Geometry*, proceedings of 22nd Karpacz Winter School of Theoretical Physics: Fields and Geometry, Karpacz, Poland, Feb 17 - Mar 1, 1986, ed. A. Jadczyk (World Scientific, 1986).

[57] D. Garfinkle and A. Strominger, "Semiclassical Wheeler Wormhole Production," Phys. Lett. B **256**, 146 (1991).

[58] F. Dowker, J. P. Gauntlett, S. B. Giddings and G. T. Horowitz, "On pair creation of extremal black holes and Kaluza-Klein monopoles," Phys. Rev. D **50**, 2662 (1994) [arXiv:hep-th/9312172].

[59] "A note on Schwarzschild black hole thermodynamics in a magnetic universe," Mod. Phys. Lett. A **17**, 2277 (2002) [arXiv:gr-qc/0211035].