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The hypergeneralized Heun equation in QFT in curved space-times

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Abstract. In this article we show for the first time the role played by the hypergeneralized Heun equation (HHE) in the context of Quantum Field Theory in curved space-times. More precisely, we find suitable transformations relating the separated radial and angular parts of a massive Dirac equation in the Kerr-Newman-deSitter metric to a HHE.

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1. Introduction

The hypergeneralized Heun equation (HHE) is a second order differential equation of the form

$$y^{''}(z)+\sum_{i=0}^{4}\frac{1-\mu_i}{z-z_i}\;y^{'}(z)+\frac{\beta_0+\beta_1 z+\beta_2 z^2+\beta_3 z^3}{\prod\limits_{i=0}^{4}(z-z_i)}\;y(z)=0. \tag{1.1}$$

where $z_0=0,\ z_1=1,\ z_2,\ z_3,\ z_4\in\mathbb{C}\backslash\{0,1\}$ and $\mu_0,\cdots\mu_4,\ \beta_0,\cdots,\beta_3$ are arbitrary complex numbers. Moreover, z_i is a simple singularity with exponents $\{0,\mu_i\}$ for each $i=0,\cdots,4$ and ∞ is also a simple singularity. To underline the importance of equation (1.1) we recall that it represents a generalization of the equation studied by Schmidt in [1] and it contains the generalized Heun equation [2, 3] as a confluent special case. The aim of our work is to provide the first example where the HHE arises in physics. In particular, we are interested in the role played by (1.1) in QFT theory in curved space-times.

2. The Dirac equation in the Kerr-Newman-deSitter metric

In Boyer-Lindquist coordinates (t,r,ϑ,φ) with $r>0,\ 0\leq\vartheta\leq\pi,\ 0\leq\varphi<2\pi$ the Kerr-Newman-deSitter metric is [4]

$$g^{\mu\nu}=\begin{pmatrix}
\frac{\Xi^2}{\Sigma}\left[\frac{(r^2+a^2)^2}{\Delta_r}-\frac{a^2\sin^2\vartheta}{\Delta_\vartheta}\right] & 0 & 0 & \frac{a\Xi^2}{\Sigma}\left[\frac{r^2+a^2}{\Delta_r}-\frac{1}{\Delta_\vartheta}\right] \\
0 & -\frac{\Delta_r}{\Sigma} & 0 & 0 \\
0 & 0 & -\frac{\Delta_\vartheta}{\Sigma} & 0 \\
\frac{a\Xi^2}{\Sigma}\left[\frac{r^2+a^2}{\Delta_r}-\frac{1}{\Delta_\vartheta}\right] & 0 & 0 & -\frac{\Xi^2}{\Sigma\sin^2\vartheta}\left[\frac{1}{\Delta_\vartheta}-\frac{a^2\sin^2\vartheta}{\Delta_r}\right]
\end{pmatrix}$$

with

$$\Sigma:=\Sigma(r,\theta)=r^2+a^2\cos^2\theta,\qquad \Delta_r:=\Delta_r(r)=(r^2+a^2)\left(1-\frac{\Lambda}{3}r^2\right)-2Mr+Q^2,$$

$$\Delta_\vartheta:=\Delta_\vartheta(\vartheta)=1+\frac{\Lambda}{3}a^2\cos^2\vartheta,\qquad \Xi=1+\frac{\Lambda}{3}a^2$$

where Λ is the cosmological constant and M, a and Q are the mass, the angular momentum per unit mass and the charge of the black hole, respectively. In what follows we assume that the function Δ_r has four distinct zeros. According to Penrose and Rindler [5] the Dirac equation for two spinors P^A and Q^A is given by

$$\nabla_{AA'}P^A+\frac{\mathrm{i}m_e}{\sqrt{2}}\;\overline{Q}_{A'}=0,\qquad \nabla_{AA'}Q^A+\frac{\mathrm{i}m_e}{\sqrt{2}}\;\overline{P}_{A'}=0 \tag{2.1}$$

where we used Planck units $\hbar=c=G=1$. Furthermore, $\nabla_{AA'}$ is the symbol for covariant differentiation and m_e is the particle mass. The Dirac equation in the Kerr-

Newman-deSitter geometry was computed and separated by Khanal [4]. If we make the following ansatz for the spinor Ψ

$$\Psi(t,r,\vartheta,\varphi)=\begin{pmatrix} +P^0(t,r,\vartheta,\varphi)\\ +P^1(t,r,\vartheta,\varphi)\\ +\overline{Q}^{1'}(t,r,\vartheta,\varphi)\\ -\overline{Q}^{0'}(t,r,\vartheta,\varphi)\end{pmatrix}=\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\widehat{k}\varphi}\begin{pmatrix} F_1(r,\vartheta)\\ F_2(r,\vartheta)\\ G_1(r,\vartheta)\\ G_2(r,\vartheta)\end{pmatrix},\quad \widehat{k}=k+\frac{1}{2},$$

with $\omega\in\mathbb{R}$ and $k\in\mathbb{Z}$ the energy and the azimuthal quantum number of the particle,respectively the equations in (2.1) lead to the following coupled linear system of first order PDEs

$$\mathcal{D}_0F_1+\sqrt{\Delta_\vartheta/2}\;\mathcal{L}_{1/2}F_2=\frac{m_e}{\sqrt{2}}(\mathrm{i}r+a\cos\vartheta)G_1, \tag{2.2}$$

$$\Delta_r\mathcal{D}^\dagger_{1/2}F_2-\sqrt{2\Delta_\vartheta}\;\mathcal{L}^\dagger_{1/2}F_1=-\sqrt{2}m_e(\mathrm{i}r+a\cos\vartheta)G_2, \tag{2.3}$$

$$\mathcal{D}_0G_2-\sqrt{\Delta_\vartheta/2}\;\mathcal{L}^\dagger_{1/2}G_1=\frac{m_e}{\sqrt{2}}(\mathrm{i}r-a\cos\vartheta)F_2, \tag{2.4}$$

$$\Delta_r\mathcal{D}^\dagger_{1/2}G_1+\sqrt{2\Delta_\vartheta}\;\mathcal{L}_{1/2}G_2=-\sqrt{2}m_e(\mathrm{i}r-a\cos\vartheta)F_1 \tag{2.5}$$

where

$$\mathcal{D}_0=\frac{d}{dr}+\mathrm{i}\frac{\Xi K(r)}{\Delta_r},\qquad K(r)=a\widehat{k}-\omega(r^2+a^2), \tag{2.6}$$

$$\mathcal{D}_{1/2}=\frac{d}{dr}+\mathrm{i}\frac{\Xi K(r)}{\Delta_r}+\frac{1}{2\Delta_r}\frac{d\Delta_r}{dr},\qquad \mathcal{D}^\dagger_{1/2}=\frac{d}{dr}-\mathrm{i}\frac{\Xi K(r)}{\Delta_r}+\frac{1}{2\Delta_r}\frac{d\Delta_r}{dr}, \tag{2.7}$$

$$\mathcal{L}_{1/2}=\frac{d}{d\vartheta}+\frac{\Xi H(\vartheta)}{\Delta_\vartheta}+\frac{1}{2\sqrt{\Delta_\vartheta}\sin\vartheta}\frac{d(\sqrt{\Delta_\vartheta}\sin\vartheta)}{d\vartheta},\qquad H(\vartheta)=a\omega\sin\vartheta-\frac{\widehat{k}}{\sin\vartheta}, \tag{2.8}$$

$$\mathcal{L}^\dagger_{1/2}=\frac{d}{d\vartheta}-\frac{\Xi H(\vartheta)}{\Delta_\vartheta}+\frac{1}{2\sqrt{\Delta_\vartheta}\sin\vartheta}\frac{d(\sqrt{\Delta_\vartheta}\sin\vartheta)}{d\vartheta}. \tag{2.9}$$

By means of the further ansatz

$$F_1(r,\vartheta)=R_-(r)S_-(\vartheta),\qquad F_2(r,\vartheta)=R_+(r)S_+(\vartheta),$$
$$G_1(r,\vartheta)=R_+(r)S_-(\vartheta),\qquad G_2(r,\vartheta)=R_-(r)S_+(\vartheta)$$

the Dirac equation decouples into the following systems of linear first order differential equations for the radial $R_\pm$ and angular components $S_\pm$ of the spinor Ψ

$$\begin{pmatrix}\mathcal{D}_0 & -\mathrm{i}m_er-\lambda\\ \mathrm{i}m_er-\lambda & \Delta_r\mathcal{D}^\dagger_{1/2}\end{pmatrix}\begin{pmatrix}R_-\\ R_+\end{pmatrix}=0, \tag{2.10}$$

$$\begin{pmatrix}\lambda-am_e\cos\theta & \sqrt{\Delta_\vartheta}\;\mathcal{L}_{1/2}\\ \sqrt{\Delta_\vartheta}\;\mathcal{L}^\dagger_{1/2} & -(\lambda+am_e\cos\theta)\end{pmatrix}\begin{pmatrix}S_-\\ S_+\end{pmatrix}=0 \tag{2.11}$$

where λ is a separation constant. In the case $\Lambda=a=0$ the components $S_\pm$ of the angular eigenfunctions are spin-weighted spherical harmonics [6, 7] whereas for $\Lambda=0$ and $a\neq 0$ the radial and angular eigenfunctions satisfy a generalized Heun equation [3, 8]. In the next section we show that in the more general case $\Lambda\neq 0\neq a$ the functions $R_\pm$ and $S_\pm$ satisfy the hypergeneralized Heun equation (1.1).

3. Reduction to the HHE

By means of a suitable transformation we show that the radial and angular eigenfunctions satisfy a hypergeneralized Heun equation. Decoupling the radial system (2.10) in favor of R_- we get

$$\left[\Delta_r \mathcal{D}_{1/2}^{\dagger}\mathcal{D}_0 - \frac{\mathrm{i}m_e\Delta_r}{\lambda+\mathrm{i}m_e r}\mathcal{D}_0 - (\lambda^2+m_e^2r^2)\right] R_- = 0. \tag{3.1}$$

Analogously, if we decide to eliminate S_+ in (2.11) we obtain

$$\left[\sqrt{\Delta_\vartheta}\ \mathcal{L}_{1/2}\sqrt{\Delta_\vartheta}\ \mathcal{L}_{1/2}^{\dagger} + \frac{am_e\sin\vartheta\ \Delta_\vartheta}{\lambda+am_e\cos\vartheta}\ \mathcal{L}_{1/2}^{\dagger} + \lambda^2 + a^2m_e^2\cos^2\vartheta\right] S_- = 0. \tag{3.2}$$

3.1. The radial equation (3.1)

Let $r_1,\cdots,r_4\in\mathbb{C}$ be the roots of the polynomial equation $\Delta_r=0$. We suppose that $r_i\neq r_j$ for each $i\neq j$ with $i,j=1,\cdots,4$. Moreover, let us introduce the notation $r_{ij}:=r_i-r_j$. For the reduction to the HHE it is convenient to make the variable transformation

$$z=\frac{r-r_1}{r_2-r_1}$$

mapping r_1 to zero and r_2 to one. This allows to write the polynomial Δ_r in a very compact form as follows

$$\Delta_r = r_{21}^4\Delta_z,\qquad \Delta_z=-\frac{\Lambda}{3}z(z-1)(z-z_3)(z-z_4),\qquad z_3=\frac{r_{31}}{r_{21}},\qquad z_4=\frac{r_{41}}{r_{21}}.$$

Finally, taking into account that $\mathcal{D}_0$, $\mathcal{D}_{1/2}^{\dagger}$ are given by (2.6) and (2.7) equation (3.1) becomes

$$\frac{d^2R_-}{dz^2}+P(z)\frac{dR_-}{dz}+Q(z)R_-=0 \tag{3.3}$$

where for future convenience we write $P(z)$ and $Q(z)$ in terms of partial fractions as follows

$$P(z)=\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}+\frac{1}{z-z_4}\right)-\frac{1}{z-z_5},\qquad z_5=\frac{r_{51}}{r_{21}},\qquad r_5:=\mathrm{i}\frac{\lambda}{m_e},$$

$$Q(z)=\frac{B_1}{z^2}+\frac{B_2}{(z-1)^2}+\frac{B_3}{(z-z_3)^2}+\frac{B_4}{(z-z_4)^2}+\frac{T_1}{z}+\frac{T_2}{z-1}+\frac{T_3}{z-z_3}+\frac{T_4}{z-z_4}+\frac{T_5}{z-z_5}$$

with

$$B_1=-\frac{3\mathrm{i}\Xi K(r_1)}{2\Lambda r_{21}r_{31}r_{41}},\qquad B_2=-\frac{3\mathrm{i}\Xi\left[K(r_1)-\omega(r_2^2-r_1^2)\right]}{2\Lambda r_{12}r_{32}r_{42}},$$

$$B_3=-\frac{3\mathrm{i}\Xi\left[K(r_1)-\omega(r_3^2-r_1^2)\right]}{2\Lambda r_{13}r_{23}r_{43}},\qquad B_4=-\frac{3\mathrm{i}\Xi\left[K(r_1)-\omega(r_4^2-r_1^2)\right]}{2\Lambda r_{14}r_{24}r_{34}},$$

$$T_1=\frac{3}{\Lambda r_{21}^4r_{31}r_{41}r_{51}}\left\{\Xi^2r_{51}K(r_1)+\mathrm{i}\Xi r_{21}^4\left[K(r_1)-\omega r_1r_{51}\right]-r_{21}^4r_{51}(\lambda^2+m_e^2r_1^4)\right\},$$

$$T_2=\frac{3}{\Lambda r_{21}^4r_{23}r_{24}r_{25}}\left\{\Xi^2r_{52}\left[K(r_1)-\omega r_{21}(r_2+r_1)\right]-\mathrm{i}\Xi r_{21}^4\left[K(r_1)-\omega(r_5r_2-r_1^2)\right]+\right.$$

$$-r_{21}^4 r_{52}\left[\lambda^2+m_e^2(r_1^2+r_{21})^2\right]\Big\},$$

$$T_3=\frac{3}{\Lambda r_{21}^3 r_{31}r_{32}r_{34}r_{35}}\Big\{\Xi^2 r_{53}\left[K(r_1)-\omega r_{31}(r_3+r_1)\right]+\mathrm{i}\Xi r_{21}^4\left[K(r_1)-\omega(r_5r_3-r_1^2)\right]+$$
$$-r_{21}^4 r_{53}\left[\lambda^2+m_e^2(r_1^2+r_{31})^2\right]\Big\},$$

$$T_4=\frac{3}{\Lambda r_{21}^3 r_{41}r_{42}r_{43}r_{54}}\Big\{\Xi^2 r_{45}\left[K(r_1)-\omega r_{41}(r_4+r_1)\right]-\mathrm{i}\Xi r_{21}^4\left[K(r_1)-\omega(r_5r_4-r_1^2)\right]+$$
$$-r_{21}^4 r_{45}\left[\lambda^2+m_e^2(r_1^2+r_{41})^2\right]\Big\},$$

$$T_5=\frac{3\mathrm{i}\Xi r_{21}\left[K(r_1)-\omega r_{51}(r_5+r_1)\right]}{\Lambda r_{51}r_{52}r_{53}r_{54}}.$$

It can be checked that $T_1+T_2+T_3+T_4+T_5=0$. This property of the coefficients T_i will ensure at the end that the point at infinity is a regular singular point. Finally, by transforming R_- according to

$$R_-(z)=z^{\alpha_1}(z-1)^{\alpha_2}(z-z_3)^{\alpha_3}(z-z_4)^{\alpha_4}\widehat{R}_-(z)$$

and requiring that $2\alpha_i^2-\alpha_i+2B_i=0$ for each $i=1,\cdots,4$ equation (3.3) becomes

$$\frac{d^2\widehat{R}_-}{dz^2}+\widetilde{P}(z)\frac{d\widehat{R}_-}{dz}+\widetilde{Q}(z)\widehat{R}_-=0 \tag{3.4}$$

with

$$\widetilde{P}(z)=\frac{1+4\alpha_1}{2z}+\frac{1+4\alpha_2}{2(z-1)}+\frac{1+4\alpha_3}{2(z-z_3)}+\frac{1+4\alpha_4}{2(z-z_4)}-\frac{1}{z-z_5},$$

$$\widetilde{Q}(z)=\frac{C_1}{z}+\frac{C_2}{z-1}+\frac{C_3}{z-z_3}+\frac{C_4}{z-z_4}+\frac{C_5}{z-z_5}.$$

where

$$C_1=\frac{\alpha_1}{z_5}-\frac{\alpha_1+4\alpha_1\alpha_4+\alpha_4}{2z_4}+\frac{(2T_1-\alpha_1-4\alpha_1\alpha_2-\alpha_2)z_3-(\alpha_1+4\alpha_1\alpha_3+\alpha_3)}{2z_3},$$

$$C_2=\frac{\alpha_2}{z_5-1}-\frac{\alpha_2+4\alpha_2\alpha_4+\alpha_4}{2(z_4-1)}+\frac{(2T_2+\alpha_1+4\alpha_1\alpha_2+\alpha_2)(z_3-1)-(\alpha_2+4\alpha_2\alpha_3+\alpha_3)}{2(z_3-1)},$$

$$C_3=T_3-\frac{\alpha_3}{z_3-z_5}+\frac{\alpha_1+4\alpha_1\alpha_3+\alpha_3}{2z_3}+\frac{\alpha_2+4\alpha_2\alpha_3+\alpha_3}{2(z_3-1)}+\frac{\alpha_3+4\alpha_3\alpha_4+\alpha_4}{2(z_3-z_4)},$$

$$C_4=T_4-\frac{\alpha_4}{z_4-z_5}+\frac{\alpha_1+4\alpha_1\alpha_4+\alpha_4}{2z_4}+\frac{\alpha_2+4\alpha_2\alpha_4+\alpha_4}{2(z_4-1)}+\frac{\alpha_3+4\alpha_3\alpha_4+\alpha_4}{2(z_4-z_3)},$$

$$C_5=T_5-\frac{\alpha_1}{z_5}-\frac{\alpha_2}{z_5-1}-\frac{\alpha_3}{z_5-z_3}-\frac{\alpha_4}{z_5-z_4}.$$

Since

$$\sum_{n=1}^{5}C_n=T_1+T_2+T_3+T_4+T_5=0$$

equation (3.4) reduces to the hypergeneralized Heun equation (1.1).

3.2. The angular equation (3.2)

Let $\alpha := \Lambda a^2/3$ and $\beta := \lambda/(am_e)$. If we introduce the variable transformation $z = (1+\cos\vartheta)/2$ equation (3.2) becomes

$$\frac{d^2S_-}{dz^2} + \left[\frac{\Delta_z^{'}}{\Delta_z} + \frac{2z-1}{z(z-1)} - \frac{1}{z-z_5}\right]\frac{dS_-}{dz} - \frac{1}{z(z-1)\Delta_z}\left\{-\Xi\left[a\omega(2z-1) + \frac{\widehat{k}(2z-1)}{4z(z-1)}\right] + \right.$$

$$+\frac{\Xi\Delta_z^{'}}{4\Delta_z}\left[4a\omega z(z-1)+\widehat{k}\right] - \frac{1}{4}z(z-1)\Delta_z^{''} - \frac{3}{8}\Delta_z^{'} + \frac{1}{8}z(z-1)\frac{\Delta_z^{'\,2}}{\Delta_z} + \frac{(2z-1)^2\Delta_z}{16z(z-1)} +$$

$$+\frac{\Xi^2}{\Delta_z}\left[4a^2\omega^2 z(z-1) + \frac{\widehat{k}^2}{4z(z-1)} + 2a\omega\widehat{k}\right] + \frac{\Delta_z^{'\,2}}{64\Delta_z} + \frac{\Xi}{2(z-z_5)}\left[4a\omega z(z-1)+\widehat{k}\right] +$$

$$\left.+\frac{z(z-1)\Delta_z^{'}}{4(z-z_5)} + \frac{(2z-1)\Delta_z}{4(z-z_5)} - \frac{1}{2}\Delta_z + \lambda^2 + a^2m_e^2(2z-1)^2\right\}S_- = 0$$

with

$$\Delta_z = 4\alpha(z-z_3)(z-z_4), \qquad z_3 = \frac{1}{2} - \frac{1}{2\sqrt{-\alpha}}, \qquad z_4 = \frac{1}{2} + \frac{1}{2\sqrt{-\alpha}}, \qquad z_5 = \frac{1-\beta}{2}.$$

The above differential equation can be written in a more amenable form if we make a partial fraction expansion of its coefficient functions. Thus, we obtain

$$\frac{d^2S_-}{dz^2} + \mathcal{P}(z)\frac{dS_-}{dz} + \mathcal{Q}(z)S_- = 0 \tag{3.5}$$

with

$$P(z) = \frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-z_3} + \frac{1}{z-z_4} - \frac{1}{z-z_5},$$

$$Q(z) = \frac{\mathcal{B}_1}{z^2} + \frac{\mathcal{B}_2}{(z-1)^2} + \frac{\mathcal{B}_3}{(z-z_3)^2} + \frac{\mathcal{B}_4}{(z-z_4)^2} + \frac{\mathcal{T}_1}{z} + \frac{\mathcal{T}_2}{z-1} + \frac{\mathcal{T}_3}{z-z_3} + \frac{\mathcal{T}_4}{z-z_4} + \frac{\mathcal{T}_5}{z-z_5}$$

where

$$B_1 = -\frac{(\alpha+1-2\widehat{k}\,\Xi)^2}{16(\alpha+1)^2}, \qquad B_2 = -\frac{(\alpha+1+2\widehat{k}\,\Xi)^2}{16(\alpha+1)^2},$$

$$B_3 = \frac{1}{64\alpha\sqrt{-\alpha}(\alpha+1)^2}\left[16\Xi^2\sqrt{-\alpha}\Psi_\alpha^2 - 16\Xi\alpha(1+\alpha)\Psi_\alpha - \alpha(1+\alpha)^2\right], \quad \Psi_\alpha := \left[a\omega(1+\alpha) - ka\right],$$

$$B_4 = B_3 + \frac{1}{32\sqrt{-\alpha}(1+\alpha)}\left[(1+\alpha)(1+16\Xi a\omega) - 16\Xi ka\right],$$

$$\mathcal{T}_1 = \tau_0 + \tau_1 + \frac{24z_5(z_4-1) - 4z_4^2(6z_5-4)}{64z_5z_4^2} + \frac{8z_5(\lambda^2+a^2m_e^2) + \alpha z_5(1+12z_4) + 4\Xi(2z_5a\omega - \widehat{k})}{32\alpha z_3z_4z_5}$$

$$\mathcal{T}_2 = \tau_2 + \tau_3 + \frac{8z_4^2(3z_5-1) + 8z_4(5-9z_5) + 47z_5 - 31}{64(z_5-1)(z_4-1)^2)} +$$

$$+\frac{(1-z_5)[\alpha(12z_4-11) + 8(\lambda^2+a^2m_e^2 - a\omega\Xi)] + 4\Xi\widehat{k}}{32\alpha(z_3-1)(z_4-1)(z_5-1)},$$

$$\mathcal{T}_3 = \tau_0 - \tau_1 + \tau_2 - \tau_3 + \tau_4 + \tau_5^- - \frac{\Xi^2\widehat{k}(2z_4-1)[4a\omega z_4(z_4-1)+\widehat{k}]}{32\alpha^2 z_4^3(z_4-z_3)^2(z_4-1)^3} +$$

$$-\frac{z_5[8(\lambda^2+a^2m_e^2+\Xi a\omega)+\alpha(1+12z_4)]-4\Xi\widehat{k}}{32\alpha z_3 z_4 z_5} +$$

$$-\frac{(z_5-1)[8(\lambda^2+a^2m_e^2-\Xi a\omega)+\alpha(12z_4-11)]-4\Xi\widehat{k}}{32\alpha(z_3-1)(z_4-1)(z_5-1)} +$$

$$+\frac{(z_4-z_5)[8(\lambda^2+a^2m_e^2)+\alpha+8z_4(1-z4)(\alpha-4a^2m_e^2)]-8\Xi a\omega(z_4+z_5+2z_4z_5)+4\Xi\widehat{k}}{32\alpha z_4(z_4-1)(z_4-z_3)(z_4-z_5)},$$

$$\mathcal{T}_4 = \tau_5^+ - \frac{16z_4^4-56z_4^3+24z_4^2z_5+38z_4^2-22z_4z_5+z_4-z_5}{64z_4^2(z_4-z_5)(z_4-1)^2} +$$

$$+\frac{8(z_4-z_5)[\lambda^2+a^2m_e^2(2z_4-1)^2]+\alpha(z_4-1)(1+8z_4z_5-8z_4^2)-8\Xi a\omega(z_4+z_5-2z_4z_5)+4\Xi\widehat{k}}{32\alpha z_4(z_4-1)(z_4-z_5)(z_3-z_4)}$$

$$+\frac{\Xi^2\widehat{k}[4a\omega z_4(z_4-1)(2z_4-1)+\widehat{k}(2z_4-1)]}{32\alpha^2 z_4^3(z_4-z_3)^2(z_4-1)^3}.$$

Moreover,

$$\tau_0 := -\frac{\Xi^2\widehat{k}^2}{32\alpha^2 z_4^2 z_3^3}, \quad \tau_1 := \frac{z_4(\alpha^2 z_4^2+8\Xi^2 a\omega\widehat{k})-2\Xi^2\widehat{k}^2(1+z_4)}{64\alpha^2 z_3^2 z_4^3}, \quad \tau_2 := \frac{\Xi^2\widehat{k}^2}{32\alpha^2(z_3-1)^3(z_4-1)^2},$$

$$\tau_3 := \frac{(1-z_4)[\alpha^2(1-z_4)^2+2\Xi^2\widehat{k}(4a\omega-\widehat{k})]-2\Xi^2\widehat{k}^2}{64\alpha^2(z_3-1)^2(z_4-1)^3},$$

$$\tau_4 := \frac{2z_5(1-z_5)[\alpha(z_5-z_4)+2\Xi a\omega]-\Xi\widehat{k}}{8\alpha z_5(1-z_5)(z_3-z_5)(z_4-z_5)}, \quad \tau_5^\pm := \frac{\Xi^2\{8a\omega z_4(1-z_4)[2a\omega(1-z_4)\pm\widehat{k}]+\widehat{k}^2\}}{32\alpha^2 z_4^2(z_3-z_4)^3(1-z_4)^2}.$$

Also in this case we have that $\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5=0$ which ensures that the point at infinity is a regular singular point. Finally, if we transform S_- according to

$$S_-(z) = z^{\gamma_1}(z-1)^{\gamma_2}(z-z_3)^{\gamma_3}(z-z_4)^{\gamma_4}\widehat{S}_-(z)$$

together with the requirement $\gamma_i^2+B_i=0$ for each $i=1,\cdots,4$ equation (3.5) becomes

$$\frac{d^2\widehat{S}_-}{dz^2}+\widetilde{\mathcal{P}}(z)\frac{d\widehat{S}_-}{dz}+\widetilde{\mathcal{Q}}(z)\widehat{S}_-=0 \tag{3.6}$$

with

$$\widetilde{P}(z) = \frac{1+2\gamma_1}{z}+\frac{1+2\gamma_2}{z-1}+\frac{1+2\gamma_3}{z-z_3}+\frac{1+2\gamma_4}{z-z_4}-\frac{1}{z-z_5},$$

$$\widetilde{Q}(z) = \frac{\widetilde{C}_1}{z}+\frac{\widetilde{C}_2}{z-1}+\frac{\widetilde{C}_3}{z-z_3}+\frac{\widetilde{C}_4}{z-z_4}+\frac{\widetilde{C}_5}{z-z_5}$$

where

$$\widetilde{C}_1 = \frac{\gamma_1}{z_5} - \frac{\gamma_1 + 2\gamma_1\gamma_4 + \gamma_4}{z_4} + \frac{(\mathcal{T}_1 - \gamma_1 - 2\gamma_1\gamma_2 - \gamma_2)z_3 - (\gamma_1 + 2\gamma_1\gamma_3 + \gamma_3)}{z_3},$$

$$\widetilde{C}_2 = \frac{\gamma_2}{z_5 - 1} - \frac{\gamma_2 + 2\gamma_2\gamma_4 + \gamma_4}{z_4 - 1} + \frac{(\mathcal{T}_2 + \gamma_1 + 2\gamma_1\gamma_2 + \gamma_2)(z_3 - 1) - (\gamma_2 + 2\gamma_2\gamma_3 + \gamma_3)}{z_3 - 1},$$

$$\widetilde{C}_3 = \mathcal{T}_3 - \frac{\gamma_3}{z_3 - z_5} + \frac{\gamma_1 + 2\gamma_1\gamma_3 + \gamma_3}{z_3} + \frac{\gamma_2 + 2\gamma_2\gamma_3 + \gamma_3}{z_3 - 1} + \frac{\gamma_3 + 2\gamma_3\gamma_4 + \gamma_4}{z_3 - z_4},$$

$$\widetilde{C}_4 = \mathcal{T}_4 - \frac{\gamma_4}{z_4 - z_5} + \frac{\gamma_1 + 2\gamma_1\gamma_4 + \gamma_4}{z_4} + \frac{\gamma_2 + 2\gamma_2\gamma_4 + \gamma_4}{z_4 - 1} + \frac{\gamma_3 + 2\gamma_3\gamma_4 + \gamma_4}{z_4 - z_3},$$

$$\widetilde{C}_5 = \mathcal{T}_5 - \frac{\gamma_1}{z_5} - \frac{\gamma_2}{z_5 - 1} - \frac{\gamma_3}{z_5 - z_3} - \frac{\gamma_4}{z_5 - z_4}.$$

Finally, since

$$\sum_{n=1}^{5} \mathcal{C}_n = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 = 0$$

equation (3.6) reduces to the hypergeneralized Heun equation (1.1).

4. Conclusions

In this paper we have related the radial and angular Dirac equations for a massive fermion in the Kerr-Newman-deSitter geometry to a HHE. In view of the results obtained in [3] since the Kerr-Newman-deSitter metric becomes the Kerr-Newman metric in the limit of a vanishing cosmological constant it follows that the generalized Heun equation is a confluent case of the HHE for $\Lambda \to 0$. Finally, since it has been proved in [9] that the neutrino equation can be separated in the class $\mathcal{D}_0$ of algebraically special Petrov type-D vacuum solutions with cosmological constant whereas the massive Dirac equation can be separated only in the Carter's subclass $\widetilde{A}$ our result rises the question if the separated neutrino and massive Dirac equations can be also reduced to a HHE in such more general metrics. We reserve the study of this problem for future investigations.

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