On algebraically integrable outer billiards

Serge Tabachnikov
Department of Mathematics, Penn State University
University Park, PA 16802, USA
e-mail: *tabachni@math.psu.edu*

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Abstract

We prove that if the outer billiard map around a plane oval is algebraically integrable in a certain non-degenerate sense then the oval is an ellipse.

In this note, an outer billiard table is a compact convex domain in the plane bounded by an oval (closed smooth strictly convex curve) C. Pick a point x outside of C. There are two tangent lines from x to C; choose one of them, say, the right one from the view-point of x, and reflect x in the tangency point. One obtains a new point, y, and the transformation $T : x \mapsto y$ is the outer (a.k.a. dual) billiard map. We refer to [3, 4, 5] for surveys of outer billiards.

If C is an ellipse then the map T possesses a 1-parameter family of invariant curves, the homothetic ellipses; these invariant curves foliate the exterior of C. Conjecturally, if an outer neighborhood of an oval C is foliated by the invariant curves of the outer billiard map then C is an ellipse – this is an outer version of the famous Birkhoff conjecture concerning the conventional, inner billiards.

In this note we show that ellipses are rigid in a much more restrictive sense of algebraically integrable outer billiards; see [2] for the case of inner billiards.

We make the following assumptions. Let $f(x, y)$ be a (non-homogeneous) real polynomial such that zero is its non-singular value and C is a component of the zero level curve. Thus f is the defining polynomial of the curve C, and if a polynomial vanishes on C then it is a multiple of f. Assume that a neighborhood of C is foliated by invariant curves of the outer billiard map T, and this foliation is algebraic in the sense that its leaves are components of the level curves of a real polynomial $F(x, y)$. Since C itself is an invariant curve, we assume that $F(x, y) = 0$ on C, and that dF is not identically zero on C. Thus $F(x, y) = g(x, y)f(x, y)$ where $g(x, y)$ is a polynomial, not identically zero on C. Under these assumptions, our result is as follows.

Theorem 1 *C is an ellipse.*

Proof. Consider the tangent vector field $v = F_y\ \partial/\partial x - F_x\ \partial/\partial y$ (the symplectic gradient) along C. This vector field is non-zero (except, possibly, a finite number of points) and tangent to C. The tangent line to C at point (x, y) is given by $(x + \varepsilon F_y, y - \varepsilon F_x)$, and the condition that F is T-invariant means that the function

$$F(x + \varepsilon F_y, y - \varepsilon F_x) \tag{1}$$

is even in ε for all $(x, y) \in C$. Expand in a series in ε; the first non-trivial condition is cubic in ε:

$$W(F) := F_{xxx}F_y^3 - 3F_{xxy}F_y^2F_x + 3F_{xyy}F_yF_x^2 - F_{yyy}F_x^3 = 0 \tag{2}$$

on C. We claim that this already implies that C is an ellipse. The idea is that otherwise the complex curve $f = 0$ would have an inflection point, in contradiction with identity (2).

Consider the polynomial

$$H(F) = \det \begin{pmatrix} F_y, & -F_x \\ F_{yy}F_x - F_{xy}F_y, & F_{xx}F_y - F_{xy}F_x \end{pmatrix}.$$

Lemma 2 *One has:*
1). $v(H(F)) = W(F)$;
2). $H(F) = H(gf) = g^3H(f)$ on C;
3). If C' is a non-singular algebraic curve with a defining polynomial $f(x, y)$ and (x, y) is an inflection point of C' then $H(f)(x, y) = 0$.

Proof of Lemma 2. The first two claims follow from straightforward computations. To prove the third, note that $H(f)$ is the second order term in ε of the Taylor expansion of the function $f(x+\varepsilon f_y, y-\varepsilon f_x)$ (cf. (1)), hence $H(f)=0$ at an inflection point. □

It follows from Lemma 2 and (2) that $H(F) = \text{const}$ on C. Since C is convex, $H(F) \neq 0$, and we may assume that $H(F)=1$ on C. It follows that $g^3 H(f) - 1$ vanishes on C and hence

$$g^3 H(f) - 1 = hf \tag{3}$$

where $h(x,y)$ is some polynomial.

Now consider the situation in $\mathbf{CP}^2$. We continue to use the notation C for the complex algebraic curve given by the homogenized polynomial $\bar{f}(x:y:z) = f(x/z, y/z)$. Unless C is a conic, this curve has inflection points (not necessarily real). Let d be the degree of C.

Lemma 3 *Not all the inflections of C lie on the line at infinity*

Proof of Lemma 3. Consider the Hessian curve given by

$$\det \begin{pmatrix} \bar{f}_{xx}, & \bar{f}_{xy}, & \bar{f}_{xz} \\ \bar{f}_{yx}, & \bar{f}_{yy}, & \bar{f}_{yz} \\ \bar{f}_{zx}, & \bar{f}_{zy}, & \bar{f}_{zz} \end{pmatrix} = 0.$$

The intersection points of the curve C with its Hessian curve are the inflection points of C (recall that C is non-singular). The degree of the Hessian curve is $3(d-2)$, and by the Bezout theorem, the total number of inflections, counted with multiplicities, is $3d(d-2)$. Furthermore, the order of intersection equals the order of the respective inflection and does not exceed $d-2$, see, e.g, [6]. The number of intersection points of C with a line equals d, hence the inflection points of C that lie on a fixed line contribute, at most, $d(d-2)$ to the total of $3d(d-2)$. The remaining inflection point lie off this line. □

To conclude the proof of Theorem 1, consider a finite inflection point of C. According to Lemma 2, at such a point point, we have $f = H(f) = 0$ which contradicts (3). This is proves that C is a conic. □

Remarks. 1). It would be interesting to remove the non-degeneracy assumptions in Theorem 1.

2). A more general version of Birkhoff's integrability conjecture is as follows. Let C be a plane oval whose outer neighborhood is foliated by closed curves. For a tangent line ℓ to C, the intersections with the leaves of the foliation define a local involution σ on ℓ. Assume that, for every tangent line, the involution σ is projective. Conjecturally, then C is an ellipse and the foliation consists of ellipses that form a pencil (that is, share four – real or complex – common points). For a pencil of conics, the respective involutions are projective: this is a Desargues theorem, see [1]. It would be interesting to establish an algebraic version of this conjecture.

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