

Configuration spaces with summable labels

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Abstract

Let M be an n -manifold, and let A be a space with a partial sum behaving as an n -fold loop sum. We define the space $C(M; A)$ of configurations in M with summable labels in A via operad theory. Some examples are symmetric products, labelled configuration spaces, and spaces of rational curves. We show that $C(I^n, \partial I^n; A)$ is an n -fold classifying space of $C(I^n; A)$, and for $n = 1$ it is homeomorphic to the classifying space by Stasheff. If M is compact, parallelizable, and A is path connected, then $C(M; A)$ is homotopic to the mapping space $Map(M, C(I^n, \partial I^n; A))$.

Introduction

The interest in labelled configuration spaces in homotopy theory dates back to the seventies. May [15] and Segal [20] showed that the ‘electric field map’ $C(\mathbb{R}^n; X) \rightarrow \Omega^n \Sigma^n(X)$ is a weak homotopy equivalence if X is path connected, and in general is the group completion. Segal showed later [21] that the inclusion $Rat_*(S^2) \hookrightarrow \Omega^2 S^2$ of the space of based rational selfmaps of the spheres into all based selfmaps is the group completion. He used the identification of $Rat_*(S^2)$ with a space of configurations in \mathbb{C} with partially summable labels in $\mathbb{N} \vee \mathbb{N}$, by counting zeros and roots multiplicities. Guest has recently extended his framework in [9] to the space of based rational curves on projective toric varieties. Labelled configuration spaces on manifolds have been studied by Bödigeimer in [3], where the labels are in a based space, and by Kallel in [10], where the summable labels belong to a discrete partial abelian monoid. In both cases the authors have theorems of equivalence between configuration and mapping spaces. We define configuration spaces on a manifold M with labels in A , where A need not to be abelian. It is sufficient that A has a partial sum that is homotopy commutative up to level $dim(M)$. The definition is not trivial and involves tensor products over the Fulton-MacPherson operad. A substantial part of the paper introduces the necessary tools. We generalize the results listed above to the non-abelian setting, and construct a geometric n -fold delooping in one step. Here is a plan of the paper:

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In the first section we define the preliminary notion of a partial algebra over an operad and its completion. In the second section we introduce the Fulton-MacPherson operad F_n . We call an algebra over F_n an n -monoid. A 1-monoid is exactly an A_∞ -space [12]. In the third section we describe the homotopical algebra of topological operads and their algebras. The main results characterize the homotopy type of F_n .

Corollary 3.8. *The unbased Fulton-MacPherson operad \tilde{F}_n is cofibrant.*

Proposition 3.9. *The operad of little n -cubes is weakly equivalent to F_n .*

This implies that the structure of n -monoid is invariant under based homotopy equivalences, and any connected n -monoid has the weak homotopy type of a n -fold loop space. In the fourth section we recall from [14] that a partial compactification $C(M)$ of the ordered configuration space on an open parallelizable n -manifold M is a right module over F_n . We define the configuration space $C(M; A)$ on M with summable labels in a partial n -monoid A , by tensoring $C(M)$ and A over the operad F_n .

The definition of $C(M; A)$ is extended to a general open n -manifold M when A is framed, in the sense that A has a suitable $GL(n)$ -action. In the fifth section we define $C(M, N; A)$ for a relative manifold (M, N) by ignoring the particles in N . For $n = 1$ we obtain the well known construction by Stasheff:

Proposition 5.11. *If A is a 1-monoid, then $C(I, \partial I; A)$ is homeomorphic to the classifying space $B(A)$ by Stasheff.*

The n -monoid completion of a partial n -monoid A is $C(I^n; A)$, up to homotopy. We obtain its n -fold delooping in one step.

Theorem 6.3. *If A is framed, then the group completion of $C(I^n; A)$ is $\Omega^n C(I^n, \partial I^n; A)$.*

Finally we characterize configuration spaces on manifolds under some conditions.

Theorem 6.6. *If M is a compact closed parallelizable n -manifold and A is a path connected partial framed n -monoid, then there is a weak equivalence $C(M; A) \simeq \text{Map}(M; B_n(A))$.*

As corollary we obtain a model for the free loop space on a suspension built out of cyclohedra. This answers a question by Stasheff [22].

Corollary 6.7. *If X is path connected and well pointed, then there is a weak homotopy equivalence $C(S^1; X) \simeq \text{Map}(S^1, \Sigma X)$.*

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1 Partial modules over operads

Let \mathcal{C} be a symmetric closed monoidal category, with tensor product \otimes and unit element e . We assume that \mathcal{C} has small limits and colimits.

Definition 1.1. *A Σ -object X in \mathcal{C} is a collection of objects $X(n)$, for $n \in \mathbb{N}$, such that $X(n)$ is equipped with an action of the symmetric group Σ_n .*

The category of Σ -objects in \mathcal{C} will be denoted by $\Sigma\mathcal{C}$. We observe as in [8] that $\Sigma\mathcal{C}$ is a monoidal category as follows: given two objects A and B , their tensor product $A \otimes B$ is defined by

$$(A \otimes B)(n) = \prod_{k=0}^{\infty} A(k) \otimes_{\Sigma_k} \left(\prod_{\pi \in \text{Map}(n,k)} \bigotimes_{i=1}^k B_{\pi^{-1}(i)} \right),$$

where $\text{Map}(n,k)$ is the set of maps from $\{1, \dots, n\}$ to $\{1, \dots, k\}$. Here each element B_S , where S is a set of numbers, is identified to $B_{\#S}$ by the order preserving bijection, and the action of Σ_n is given accordingly. There is a natural embedding functor $j : \mathcal{C} \hookrightarrow \Sigma\mathcal{C}$ considering an object X as a Σ -object concentrated in degree 0, so that

$$j(X)(n) = \begin{cases} X & \text{if } n = 0, \\ \emptyset & \text{if } n \neq 0. \end{cases}$$

with the trivial actions of the symmetric groups. Here \emptyset denotes the initial object of \mathcal{C} . The functor j is left adjoint to the forgetful functor $B \mapsto B(0)$ from $\Sigma\mathcal{C}$ to \mathcal{C} . More generally we have an embedding functor $j_n : \Sigma_n\mathcal{C} \rightarrow \Sigma\mathcal{C}$, that is left adjoint to the forgetful functor $B \mapsto B(n)$.

The unit element ι of $\Sigma\mathcal{C}$ is defined by

$$\iota(n) = \begin{cases} e & \text{if } n = 1, \\ \emptyset & \text{if } n \neq 1. \end{cases}$$

Definition 1.2. [8] *An operad in \mathcal{C} is a monoid in the monoidal category $\Sigma\mathcal{C}$. We denote the category of operads in \mathcal{C} by $\mathcal{OP}(\mathcal{C})$.*

This means that an operad (F, μ, η) is a Σ -object F together with composition morphism $\mu : F \otimes F \rightarrow F$ and unit morphism $\eta : \iota \rightarrow F$, such that the associativity property $\mu(\mu \otimes F) = \mu(F \otimes \mu) : F \otimes F \otimes F \rightarrow F$ and the unit property $\mu(F \otimes \mu) = \mu(\mu \otimes F) = id_F : F \rightarrow F$ hold. Note that the functor $F \otimes _$ forms a triple.

Example 1.3. The category \mathcal{CH}_R of non negatively graded chain complexes over a commutative ring R is monoidal by the tensor product. The operads in \mathcal{CH}_R are called differential graded operads over R .

Definition 1.4. [14] Given an operad F in \mathcal{C} , a left F -module A is a Σ -object A , with a morphism $\rho : F \otimes A \rightarrow A$ of Σ -objects such that

$$\rho(F \otimes \rho) = \rho(\mu \otimes A) : F \otimes F \otimes A \rightarrow A \text{ and } \rho(\eta \otimes A) = id_A : A \rightarrow A.$$

In other words A is an algebra over the triple $F \otimes _$. Dually we define the notion of right F -module. We denote the category of left F -modules by Mod_F , and the category of right F -modules by ${}_F Mod$.

Definition 1.5. If F and G are operads in \mathcal{C} , then a F - G -bimodule A is a left F -module and a right G -module such that the left F -module structure map is a right G -homomorphism.

Definition 1.6. An algebra X over an operad F , or F -algebra, is an object X of \mathcal{C} together with a left F -module structure on $j(X)$.

We denote the category of F -algebras by Alg_F . Moreover we will denote by $F(Y)$ the free F -algebra generated by the object Y in \mathcal{C} . This object is defined by $j(F(Y)) = F \otimes j(Y)$.

Definition 1.7. A partial left F -module A is a Σ -object A in \mathcal{C} together with a monomorphism $i : Comp \hookrightarrow F \otimes A$ in $\Sigma\mathcal{C}$ and a composition map $\rho : Comp \rightarrow A$ such that

1. The unit $\eta \otimes A : A \rightarrow F \otimes A$ factors uniquely through $\tilde{\eta} : A \rightarrow Comp$ and the composition $\rho(\tilde{\eta}) = id_A$ is the identity.
2. The pullbacks in $\Sigma\mathcal{C}$ of $i : Comp \hookrightarrow F \otimes A$ along the two maps $(\mu \otimes A)(F \otimes i) : F \otimes Comp \rightarrow F \otimes A$ and $F \otimes \rho : F \otimes Comp \rightarrow F \otimes A$ coincide. Moreover the compositions of the two pullback maps with $\rho : Comp \rightarrow A$ coincide.

Partial right F -modules are defined dually. A partial F -algebra is an object X of \mathcal{C} such that $j(X)$ is a partial left F -module. We denote the categories of partial left F -modules, right F -modules and F -algebras respectively by $PartMod_F$, ${}_F PartMod$, and $PartAlg_F$.

A morphism $g : (A, Comp_A) \rightarrow (B, Comp_B)$ of partial left F -modules is a morphism in $\Sigma\mathcal{C}$ such that $(F \otimes g)i_A : Comp_A \rightarrow F \otimes B$ factors through $\tilde{g} : Comp_A \rightarrow Comp_B$, and $g\rho_A = \rho_B\tilde{g}$.

We exhibit a functor from partial to total left modules that is left adjoint to the forgetful functor. The analogous construction for right modules is exactly dual.

If A is a partial F -left module, then define \hat{A} by the coequalizer in the category $\Sigma\mathcal{C}$

$$F \otimes Comp \begin{array}{c} \xrightarrow{(\mu \otimes A)(F \otimes i)} \\ \xrightarrow{F \otimes \rho} \end{array} F \otimes A \dashrightarrow \hat{A}.$$

Proposition 1.8. *There is a left F -module structure on \hat{A} .*

Proof. The proof is modelled on Lemma 1.15 in [8]. The coequalizer above is reflexive because the input arrows admit the common section $F \otimes \tilde{\eta} : F \otimes A \rightarrow F \otimes \text{Comp}$. Now \hat{A} admits the structure of left F -module, because by Lemma 2.3.8 in [17] $F \otimes _$ preserves reflexive coequalizers. Moreover \hat{A} is the coequalizer of the pair above in the category of left F -modules. \square

Proposition 1.9. *The completion $A \mapsto \hat{A}$ induces a functor that is left adjoint to the forgetful functor $U : \text{Mod}_F \rightarrow \text{PartMod}_F$.*

Definition 1.10. *For any right F -module C with structure map $\sigma : C \otimes F \rightarrow C$ and a partial left F -module (A, Comp, i) we define the tensor product $C \otimes_F A$ as coequalizer in $\Sigma\mathcal{C}$*

$$C \otimes \text{Comp}_A \begin{array}{c} \xrightarrow{(\sigma \otimes A)(C \otimes i)} \\ \xrightarrow{C \otimes \rho} \end{array} C \otimes A \dashrightarrow C \otimes_F A.$$

Dually we define the tensor product of a partial right F -module and a left F -module.

Proposition 1.11. *Given a partial right F -module A , an F - G -bimodule B , and a partial left G -module C , there are natural isomorphisms*

$$(A \otimes_F B) \otimes_G C \cong A \otimes_F (B \otimes_G C).$$

The isomorphism holds because the tensor product is a left adjoint and preserves colimits.

2 The Fulton-MacPherson operad

The category \mathcal{CG} of compactly generated weak Hausdorff topological spaces is a closed monoidal category with all limits and colimits, hence it satisfies the assumptions of the previous section. We note however that in general the forgetful functor to the category of sets does not preserve colimits. Operads and modules in \mathcal{CG} shall be called simply topological operads and topological modules.

The key topological operads in this paper are the Fulton-MacPherson operads, that are suitable cofibrant versions of the little cubes operads. They were introduced in [8]. We recall their definition. Consider the differential-geometric blow-up of $(\mathbb{R}^n)^k$ along the small diagonal $\Delta = \{x_1, \dots, x_k \mid x_1 = \dots = x_k\}$. The blow-up is explicitly obtained if we replace the diagonal by its normal sphere bundle. The fiber of the trivial normal bundle at the origin is $F = \{y_1, \dots, y_k \mid \sum_{i=1}^{i=k} y_i = 0\}$ and the sphere bundle $PF = (F - 0)/(\mathbb{R}^+)$ can be seen as the space of closed half-lines in F . Then the blow-up is

$$\text{Bl}_\Delta((\mathbb{R}^n)^k) = \{(x, y) \in (\mathbb{R}^n)^k \times PF \mid x - \pi_\Delta(x) \in y\},$$

where the orthogonal projection is $\pi_\Delta(x_1, \dots, x_k) = (\sum_{i=1}^{i=k} x_i/k, \dots, \sum_{i=1}^{i=k} x_i/k)$. For any set $S \subseteq \{1, \dots, k\}$ let us denote by $\text{Bl}_\Delta((\mathbb{R}^n)^S)$ the blow-up of $(\mathbb{R}^n)^S$ along its small diagonal.

Let $C_k^0(\mathbb{R}^n) \subset \text{Map}(\{1, \dots, k\}, \mathbb{R}^n)$ be the space of ordered pairwise distinct k -tuples in \mathbb{R}^n . There is a natural right Σ_k -action on this space, and we consider it as left Σ_k -space by the opposite action. As $C_k^0(\mathbb{R}^n)$ does not intersect any diagonal, there is a natural embedding

$$j : C_k^0(\mathbb{R}^n) \rightarrow \prod_{S \subseteq \{1, \dots, n\}, \#S \geq 2} \text{Bl}_\Delta((\mathbb{R}^n)^S).$$

Definition 2.1. *The Fulton-MacPherson configuration space $C_k(\mathbb{R}^n)$ is the closure of the image of j .*

We note that $GL(n)$ acts diagonally on each blowup, j is a $GL(n)$ -equivariant map and therefore $C_k(\mathbb{R}^n)$ is a $GL(n)$ -space.

In a similar way we define the Fulton-MacPherson configuration space $C_k(M)$ of a smooth open manifold M . In this case one builds the differential-geometric blowups of M^k along the diagonal Δ_M by gluing together $M^k - \Delta_M$ and the normal sphere bundle via a tubular neighbourhood of Δ_M in M^k . It turns out that $C_k(M)$ is a manifold with corners Σ_k -equivariantly homotopy equivalent to its interior, the ordinary configuration space $C_k^0(M)$ of ordered pairwise distinct k -tuples in M .

There is a blow-down map $b : C_k(M) \rightarrow M^k$ such that the composite $C_k^0(M) \xrightarrow{j} C_k(M) \xrightarrow{b} M^k$ is the inclusion. We will say that the blow-down map gives the *macroscopic locations* of the particles.

There is a characterization of the Fulton-MacPherson configuration space by means of trees due to Kontsevich. For us a *tree* is an oriented finite connected graph with no cycles such that each vertex has exactly one outgoing edge. An *ordered tree* is a tree together with an ordering of the incoming edges of each vertex. The ordering is equivalent to the assignation of a planar embedding. The only edge with no end vertex is the *root*, the edges with no initial vertex are the *twigs*, and all other edges are *internal*. A tree on a set I is a tree together with a bijection from the set of its twigs to I . The *valence* of a vertex is the number of incoming edges. Let $G(n)$ be the group of affine transformations of \mathbb{R}^n generated by translations and positive dilatations.

Proposition 2.2. [12] *Let M be an open manifold. Then each element in $C_k(M)$ is uniquely determined by:*

1. *Distinct macroscopic locations $P_1, \dots, P_l \in M$, with $1 \leq l \leq k$.*
2. *For each $1 \leq i \leq l$ a tree T_i with f_i twigs, so that $\sum_{i=1}^l f_i = k$, and for each vertex in T_i of valence m an element in $C_m^0(\tau_{P_i}(M))/G(n)$, where $\tau_{P_i}(M)$ is the tangent plane at P_i .*

3. A global ordering of the k twigs of the trees.

Definition 2.3. *If $b : C_k(\mathbb{R}^n) \rightarrow (\mathbb{R}^n)^k$ is the blowdown map, then the Fulton-MacPherson space is $F_n(k) = b^{-1}(\{0\}^k)$.*

This space contains all configurations macroscopically located at the origin.

Proposition 2.4. [8] *The space $F_n(k)$ is a manifold with corners, and it is a compactification of $C_k^0(\mathbb{R}^n)/G(n)$.*

The *faces* of $F_n(k)$ are indexed by trees on $\{1, \dots, k\}$, and the codimension of a face is equal to the number of internal edges of the indexing tree.

Proposition 2.5. [14] *The spaces $F_n(k)$ $k \geq 0$ assemble to form a topological operad. Moreover $F_n(k)$ is a Σ_k -equivariant deformation retract of $C_k(\mathbb{R}^n)$.*

The composition law is easily described in terms of trees: each element in F_n is described by a single tree by Proposition 2.2. If $a \in F_n(k)$ and $b_j \in F_n(i_j)$ for $j = 1, \dots, k$ then $a \circ (b_1, \dots, b_k) \in F_n(i_1 + \dots + i_k)$ corresponds to the tree obtained by merging the j -th twig of the tree of a with the root of the tree of b_j for $j = 1, \dots, k$, and assigning the new twigs the induced order. This operation on trees will be called *grafting*. Note that $F_n(1)$ is a point, the unit ι of the operad, and is represented by the trivial tree. We assume that $F_n(0)$ is a point, the empty configuration. We stress the fact that Getzler and Jones in [8] consider the unpointed version \tilde{F}_n such that $\tilde{F}_n(k) = F_n(k)$ for $k > 0$ and $\tilde{F}_n(0) = \emptyset$. Their paper focuses on the differential graded operad $e_n = H_*(\tilde{F}_n; \mathbb{Q})$, the rational homology of \tilde{F}_n . They denote $H_*(F_n; \mathbb{Q})$ by e_n^+ . The deformation retraction $r : C_k(\mathbb{R}^n) \times I \rightarrow C_k(\mathbb{R}^n)$ such that $r(C_k(\mathbb{R}^n) \times \{1\}) = F_n(k)$ is defined for $t \neq 0$ and $x \in C_k^0(\mathbb{R}^n)$ by $r(x, t) = xt$.

Definition 2.6. *We call an algebra over F_n an n -monoid, and an algebra over \tilde{F}_n an n -semigroup.*

Example 2.7. [12] *The 1-monoids are the A_∞ -spaces.*

In fact $F_1(i) = K_i \times \Sigma_i$, where K_i denotes the associahedron by Stasheff [22], so F_1 is the symmetric operad generated by the non-symmetric Stasheff operad K . But an A_∞ space is by definition an algebra over K .

3 Homotopical algebra and the little discs

We describe the closed model category structure of the categories of topological operads and their algebras.

Definition 3.1. [6] *A cofibrantly generated model category is a closed model category [16], together with a set I of generating cofibrations, and a set J of generating trivial cofibrations, so that the fibrations and the trivial fibrations are respectively the maps satisfying the right lifting property with respect to the maps in J and I .*

Consider the free operad functor $\mathbb{T} : \Sigma(\mathcal{CG}) \rightarrow \mathcal{OP}(\mathcal{CG})$, left adjoint to the forgetful functor $\mathbb{U} : \mathcal{OP}(\mathcal{CG}) \rightarrow \Sigma(\mathcal{CG})$. Let \mathcal{S}_n be the family of subgroups of Σ_n . The simplicial version of the following proposition is 3.2.11 in [17].

Proposition 3.2. [18] *The category of topological operads is a cofibrantly generated model category, with the following structure:*

1. *The set of generating cofibrations is $I = \{\mathbb{T}(\partial I^i \times H \setminus \Sigma_n \hookrightarrow I^i \times H \setminus \Sigma_n) \mid i, n \in \mathbb{N}, H \in \mathcal{S}_n\}$;*
2. *The set of generating trivial cofibrations is $J = \{\mathbb{T}((I^{i-1} \times \{0\}) \times H \setminus \Sigma_n \hookrightarrow I^i \times H \setminus \Sigma_n) \mid i, n \in \mathbb{N}, H \in \mathcal{S}_n\}$;*
3. *A morphism f is respectively a weak equivalence or a fibration if for any $n \in \mathbb{N}$ and $H \in \mathcal{S}_n$ the restriction f_n^H of f_n to the H -invariant subspaces is respectively a weak homotopy equivalence or a Serre fibration.*

There is a functorial cofibrant resolution for operads, introduced in [1]. Let A be a topological operad. Let M_k be the set of isomorphism classes of *ordered trees* on $\{1, \dots, k\}$, and for each tree t let $V(t)$ be the set of its vertices and $E(t)$ the set of its internal edges. For each vertex $x \in V(t)$ let $|x|$ be its valence.

Definition 3.3. *The space of ordered trees on $\{1, \dots, k\}$ with vertices labelled by elements of A , and with internal edges labelled by real numbers in $[0, 1]$ is*

$$T_k(A) = \coprod_{t \in M_k} \left(\prod_{x \in V(t)} A(|x|) \times [0, 1]^{\#E(t)} \right).$$

Let T_t be the summand indexed by a tree $t \in M_k$. For each internal edge $e \in E(t)$ the operad composition induces a map $\partial_e : T_t \rightarrow T_{t-e}$, where $t-e$ is obtained from t by collapsing e to a vertex. If e goes from x to y , $|y| = n$, and e is the i -th incoming edge of y , then $\partial_e(x)$ is the multiplication of the composition $\theta_i : A(|x|) \times A(|y|) \rightarrow A(|x| + |y| - 1)$ by the identity maps of the vertices in $V(t) - \{x \cup y\}$.

Definition 3.4. *The space $WA(k)$ is the quotient of $T_k(A)$ under the following relations:*

1. *Suppose that $t \in T_k(A)$, v is a vertex of t of valence n labelled by $\alpha \in A(n)$, the subtrees stemming from v are $t_1 < \dots < t_n$, and $\sigma \in \Sigma_n$. Then t is equivalent to the element obtained from t by replacing α by $\sigma^{-1}\alpha$ and by permuting the order of the subtrees to $t_{\sigma_1} < \dots < t_{\sigma_n}$.*
2. *If $t \in T_k(A)$ has an edge e of length 0, then t is equivalent to the labelled tree obtained by collapsing e to a vertex, and composing the labels of its vertices.*
3. *If $t \in T_k(A)$ has a vertex w of valence 1 labelled by the unit $\iota \in A(1)$ of the operad A , then t is equivalent to the labelled tree obtained by removing w . If w is between two internal edges of lengths s and t , then we assign length $s + t - st$ to the merged edge.*

There is an action of Σ_k on $WA(k)$ induced by permuting the labelling of the twigs of elements in $T_k(A)$.

Proposition 3.5. [1] *There is an operad structure on WA , defined by grafting trees, and by assigning length 1 to the new internal edges. A natural ordering of the twigs of the composite is induced. The trivial tree consisting of an edge with no vertices is the identity of WA .*

For us a cofibration of topological spaces is the retract of a generalized CW-inclusion [6]. We say that a pointed space (X, x_0) is well-pointed if the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration. The following proposition is essentially proved in [1].

Proposition 3.6. *Let A be a topological operad such that $(A(1), \iota)$ is well-pointed and each $A(n)$ is a cofibrant space. Then WA is a cofibrant resolution of A .*

Proposition 3.7. *There is an isomorphism of topological operads $W(\tilde{F}_n) \cong \tilde{F}_n$.*

Proof. We observe that $W(\tilde{F}_n)(i)$ is obtained by gluing together for each face S of $F_n(i)$ of codimension d a copy of $S \times [0, 1]^d$. This is true because the codimension of a face of the manifold with corners $F_n(i)$ is equal to the number of internal edges of the associated tree. Then $W(\tilde{F}_n)(i)$ admits the structure of manifold with corners diffeomorphic to $F_n(i)$, and the composition maps of both operads are described by grafting of trees. \square

Corollary 3.8. *The operad \tilde{F}_n is cofibrant.*

Let D_n be the operad of little n -discs. The space $D_n(k)$ is the space of k -tuples of direction preserving affine selfembeddings of the unit n -disc with pairwise disjoint images. The operad structure is defined by multicomposition of the embeddings. There is a sequence of Σ_k -equivariant homotopy equivalences $D_n(k) \rightarrow C_k^0(I^n) \hookrightarrow C_k^0(\mathbb{R}^n) \hookrightarrow C_k(\mathbb{R}^n) \rightarrow F_n(k)$. The first map sends the little discs to their centers, the last is a deformation retraction. The inclusion $C_k^0(I^n) \hookrightarrow C_k^0(\mathbb{R}^n)$ is a Σ_k -equivariant homotopy equivalence because the inclusion $I^n \hookrightarrow \mathbb{R}^n$ is isotopic to a homeomorphism. The image of the composite r_k is the interior of the manifold with corners $F_n(k)$. It follows that the Σ -map $r = \{r_k\}$ is not an operad map because all elements in the boundary of $F_n(k)$ are composite.

Proposition 3.9. *The operad D_n is weakly equivalent to F_n .*

Proof. We build an extension $R : WD_n \rightarrow F_n$ of $r : D_n \rightarrow F_n$ that is a map of operads and a weak equivalence.

An element a in $WD_n(k)$ is represented by a labelled tree $\tau \in T_k(D_n)$ on $\{1, \dots, k\}$. If the i -tuple (f_1^v, \dots, f_i^v) labels a vertex v of valence i , then for each j we associate the embedding f_j^v to the j -th incoming edge $e_j(v)$ of v . The

equivalence relation defining WD_n preserves this association. We observe incidentally that the multicomposition of the labels of τ is the k -tuple of embeddings (g_1, \dots, g_k) such that for each j g_j is the composition of the embeddings associated to the edges along the unique path from the j -th twig to the root. Suppose that the internal edges of τ are labelled by numbers in $(0, 1)$. Let $l(e)$ denote the length of an edge e and if $r \in (0, 1]$ let δ_r be the dilatation of the n -disc by r . Consider the labelled tree τ' obtained from τ by replacing for each vertex v and for each $j = 1, \dots, |v|$ the embedding f_j^v by the rescaling $f_j^v \circ \delta_{l(e_j(v))}$. Let b be the multicomposition of the labels of τ' , and set $R_k(a) = r_k(b)$. We have defined R_k on a dense subspace of $WD_n(k)$. The map R_k extends to $WD_n(k)$ and R is an operad map, because the boundary and the composition of F_n are described by a limit procedure. Let $i_k : D_n(k) \rightarrow WD_n(k)$ be the inclusion such that $i_k(a)$ is represented by the tree on $\{1, \dots, k\}$ with a single vertex labelled by a . The map $R_k : WD_n(k) \rightarrow F_n(k)$ is a Σ_k -equivariant homotopy equivalence for each k because i_k is such [1] and $R_k i_k = r_k$. In particular R is a weak equivalence of topological operads, and $D_n \simeq F_n$. \square

The simplicial analogue of the following proposition is 3.2.5 in [17].

Proposition 3.10. [18] [19] *Let F be a topological operad. Then the category Alg_F is a cofibrantly generated model category with the following structure:*

1. *The set of generating cofibrations is $I = \{F(\partial I^i) \hookrightarrow F(I^i) \mid i \in \mathbb{N}\}$.*
2. *The set of trivial generating cofibrations is $J = \{F(I^{i-1} \times \{0\}) \hookrightarrow F(I^i) \mid i \in \mathbb{N}\}$.*
3. *A F -homomorphism is a weak equivalence or a fibration if it is respectively a weak homotopy equivalence or a Serre fibration.*

Under mild conditions there is a functorial cofibrant resolution of topological algebras over operads, introduced in [1]. Let A be a topological operad. Consider the Σ -space W^+A defined similarly as WA , except that relation 3 is not applied if w is the root vertex. It turns out that W^+A is an A - WA -bimodule, by action of A on the label of the root, and by grafting trees representing elements of WA . Let X be an A -algebra. It has a WA -algebra structure induced by the projection $\varepsilon : WA \rightarrow A$. We define the A -algebra $U_A(X) = W^+A \otimes_{WA} X$. The projection $W^+A \rightarrow WA$, obtained by extending relation 3 to the root vertex, induces an A -homomorphism $\pi : U_A(X) \rightarrow X$, that is a deformation retraction, see p. 51 of [1].

Proposition 3.11. [18] *If X is a cofibrant space then $U_A(X)$ is a cofibrant A -algebra.*

Definition 3.12. *If A is a topological operad, X and Y are A -algebras, then a homotopy A -morphism from X to Y is an A -homomorphism from $U_A(X)$ to Y .*

Proposition 3.13. *If F is a topological operad, X is the retract of a generalized CW-space, and $Ho(Alg_F)$ is the homotopy category, then $Ho(Alg_F)(X, Y) = Alg_F(U_F(X), Y) / \simeq$.*

Proof. The set of right homotopy classes $[X, Y]$ in the sense of [16] is the set of F -homomorphisms from a cofibrant model of X to a fibrant model of Y modulo right homotopy. Now $U_F(X)$ is a cofibrant resolution of X , and Y is fibrant because every object is such. It is easy to see that the the right homotopy classes of F -homomorphisms from $U_F(X)$ to Y are ordinary homotopy classes, because Y^I is a path object. \square

This result is consistent with the formulation of the homotopy category of F -algebras in [1].

Proposition 3.14. *If Z is an n -semigroup and $p : Y \rightarrow Z$ is a homotopy equivalence, then Y has a structure of n -semigroup such that p extends to a homotopy \tilde{F}_n -morphism.*

Proof. It is sufficient to observe that $W\tilde{F}_n$ is homeomorphic to \tilde{F}_n , and apply the homotopy invariance theorem 8.1 in [1]. \square

4 Modules and configuration spaces with summable labels

Proposition 4.1. [14] *For any parallelizable open manifold M of dimension n , the space of configurations $C(M) = \coprod_{k \in \mathbb{N}} C_k(M)$ is a right F_n -module.*

Proof. We choose a trivialization of the tangent bundle $\tau(M) \cong M \times \mathbb{R}^n$. Then the composition $C(M) \otimes F_n \rightarrow C(M)$ is described by grafting of trees representing elements as in 2.2. \square

Markl in [14] gives a similar picture for generic open manifolds by introducing the framed Fulton-MacPherson operads.

Definition 4.2. *Let G be a topological group, and let F be a topological operad such that $F(i)$ is a $G \times \Sigma_i$ -space for each i , and the structure map μ of F is G -equivariant. The semidirect product $F \rtimes G$ is the operad defined by $(F \rtimes G)(i) = F(i) \times G^i$, with structure map*

$$\begin{aligned} \tilde{\mu}((x, g_1, \dots, g_k); (x_1, g_1^1, \dots, g_1^{m_1}), \dots, (x_k, g_k^1, \dots, g_k^{m_k})) = \\ = (\mu(x; g_1 x_1, \dots, g_k x_k), g_1 g_1^1, \dots, g_k g_k^{m_k}). \end{aligned}$$

Definition 4.3. *The framed Fulton-MacPherson operad is the semidirect product $fF_n = F_n \rtimes GL(n)$.*

Definition 4.4. *Let M be an open n -manifold. The $GL(n)$ -bundle of frames on M induces a $GL(n)^k$ -bundle $fC_k(M)$ on $C_k(M)$, acted on by Σ_k , that we call the framed configuration space of k frames in M .*

Proposition 4.5. [14] *The Σ -space $fC(M)$ of framed configurations is a right module over fF_n .*

An element of the framed configuration space $fC_k(M)$ is uniquely determined by labelled trees as in Proposition 2.2, and by additional k frames of the tangent planes associated to the k twigs. A smooth embedding $i : M \hookrightarrow N$ of open n -manifolds induces a right fF_n -homomorphism $fC(i) : fC(M) \hookrightarrow fC(N)$.

Remark 4.6. *If M is a Riemannian n -manifold, then we can define for each k a $O(n)^k$ -bundle $f^O C_k(M)$ over $C_k(M)$, so that $f^O C(M)$ is a right $F_n \rtimes O(n)$ -module. If M is oriented then we can define a $SO(n)^k$ -bundle $f^{SO} C_k(M)$ on $C_k(M)$ so that $f^{SO} C(M)$ is a right $F_n \rtimes SO(n)$ -module.*

Definition 4.7. *We call an algebra over fF_n a framed n -monoid.*

Hence a framed n -monoid is an n -monoid equipped with an action of $GL(n)$, that is compatible with the n -monoid structure map.

Definition 4.8. *A partial framed n -monoid is a partial n -monoid with an action of $GL(n)$, such that $GL(n)$ preserves $Comp$ and respects the partial composition.*

Definition 4.9. *Let $fD_n(k)$ be the space of k -tuples of affine selfembeddings of the unit n -disc that preserve angles and have pairwise disjoint images. The multicomposition gives fD_n the structure of an operad, that we call the operad of framed little n -discs.*

Remark 4.10. *Consider the iterated loop space $\Omega^n(X, x_0)$ as the space of maps from the closed unit n -disc to X , sending the boundary to the base point x_0 . This space is an algebra over fD_n .*

Proposition 4.11. *The operad of framed little n -discs fD_n is weakly equivalent to the framed Fulton-MacPherson operad fF_n .*

Proof. We apply the same proof of Proposition 3.9 to show that $fD_n \simeq F_n \rtimes O(n)$, and conclude by the homotopy equivalence $O(n) \hookrightarrow GL(n)$. \square

If we restrict to the suboperad $\bar{f}D_n \subset fD_n$ containing orientation preserving embeddings, then we obtain a weak equivalence $\bar{f}D_n \simeq F_n \rtimes SO(n)$.

Definition 4.12. *Let A be a partial n -monoid, and let M be an open parallelizable manifold of dimension n . Then the space of configurations in M with partially summable labels in A is $C(M; A) := C(M) \otimes_{F_n} A$.*

An element of $C(M) \otimes A = \coprod_k C_k(M) \times_{\Sigma_k} A^k$ consists by 2.2 of a finite set of trees based at distinct points in M , with vertices labelled by F_n and twigs labelled by A . The equivalence relation defining $C(M; A)$ says that if some twigs labelled by a_1, \dots, a_k are departing from a vertex labelled by $c \in F_n(k)$ in $t \in C(M) \otimes A$ and $\rho(c; a_1, \dots, a_k)$ is defined, then we identify t with the forest obtained from t by cutting such twigs, and by replacing their vertex by a twig labelled by $\rho(c; a_1, \dots, a_k)$. Furthermore if the i -th twig departing from a vertex labelled by c in t is labelled by the base point a_0 then we identify t to forest obtained by cutting the twig and by replacing the label c by $s_i(c)$, where $s_i : F_n(k) \rightarrow F_n(k-1)$ is the projection induced by forgetting the i -th coordinate.

If A is an n -monoid, then by iterated identifications any element in $C(M; A)$ has a unique representative consisting of a finite set of trivial trees in M , or points, with labels in $A - \{a_0\}$.

We denote by $|\cdot| : \mathcal{CG} \rightarrow \mathbf{Set}$ be the forgetful functor.

Proposition 4.13. *Suppose that the inclusion $Comp \hookrightarrow F(A)$ is a cofibration, and A is well-pointed. Then*

1. $|C(M; A)| = |C(M)| \otimes_{|F_n|} |A|$;
2. the space $C(M; A)$ has the weak topology with respect to the filtration $C_k(M; A) = Im(\coprod_{i \leq k} C(M)_i \times_{\Sigma_i} A^i)$, $k \in \mathbb{N}$.

Proof. If A is a proper n -monoid, then we have relative homeomorphisms

$$(C_k(M), \partial C_k(M)) \times_{\Sigma_k} (A, a_0)^k \longrightarrow (C_k(M; A), C_{k-1}(M; A))$$

for $k \geq 1$, and we conclude by 8.4, 9.2 and 9.4 in [23]. If A is a partial n -monoid, then we denote by $R_i \subset C_i(M) \times_{\Sigma_i} A^i$ the space of reducible elements that are equivalent to an element of some $C_j(M) \times_{\Sigma_j} A^j$ with $j < i$. For example,

$$R_1 = M \times \{a_0\};$$

$$R_2 = (C_2(M) \times_{\Sigma_2} (A \vee A)) \cup (M \times Comp_2);$$

$$R_3 = (C_3(M) \times_{\Sigma_3} (A \vee A \vee A)) \cup (C_2(M) \times_{\Sigma_2} (Comp_2 \times A)) \cup (M \times Comp_3).$$

We have relative homeomorphisms $(C_i(M) \times_{\Sigma_i} A^i, R_i) \rightarrow (C_i(M; A), C_{i-1}(M; A))$, and we argue similarly. \square

Definition 4.14. *Suppose that M is an open n -dimensional smooth manifold, and A is a partial framed n -monoid. Then the space of configurations in M with labels in A is $C(M; A) := fC(M) \otimes_{fF_n} A$.*

Note that if M is parallelizable then the definition is consistent with 4.12. In fact the framed configurations in M are given by $fC(M) = C(M) \otimes_{F_n} fF_n$ and by 1.11

$$fC(M) \otimes_{fF_n} A = C(M) \otimes_{F_n} fF_n \otimes_{fF_n} A = C(M) \otimes_{F_n} A.$$

Proposition 4.15. *Let A be a partial framed n -monoid with base point a_0 such that the inclusions $\text{Comp} \hookrightarrow fF_n(A)$ and $\{a_0\} \hookrightarrow A$ are cofibrations of $GL(n)$ -spaces. Let M be an open n -manifold. Then*

1. $|C(M; A)| = |fC(M)| \otimes_{|fF_n|} |A|$;
2. the space $C(M; A)$ has the weak topology with respect to the filtration $C_k(M; A) = \text{Im}(\coprod_{i \leq k} fC(M)_i \times_{\Sigma_i} A^i)$, $k \in \mathbb{N}$.

We give some examples of configuration spaces with summable labels. Let us denote by \hat{A}^n the completion of a partial n -monoid A .

Proposition 4.16. *If A is a partial n -monoid, then there is a strong deformation retraction $w_A : C(\mathbb{R}^n; A) \rightarrow \hat{A}^n$.*

Proof. It is sufficient to observe that there is a deformation retraction of right F_n -modules $w : C(\mathbb{R}^n) \rightarrow F_n$. If an element $x \in C(\mathbb{R}^n; A)$ is represented by a finite number of labelled trees τ_1, \dots, τ_k based at distinct points $P_1, \dots, P_k \in \mathbb{R}^n$, then $w_A(x)$ is represented by the single tree obtained by connecting τ_1, \dots, τ_k to a root vertex labelled by the class $[P_1, \dots, P_k] \in C_k^0(\mathbb{R}^n)/G(n) \subset F_n(k)$. \square

Example 4.17. *If M is a discrete partial monoid, then \hat{M}^1 has the homotopy type of its monoid completion. If M is abelian then \hat{M}^∞ has the homotopy type of its abelian monoid completion.*

Definition 4.18. *Let A be a partial abelian monoid and M an n -manifold. We denote by $C^0(M; A)$ the quotient of $\coprod_k C_k^0(M) \times_{\Sigma_k} A^k$ under the following relation \sim : if $(m_1, \dots, m_k) \in C_k^0(M)$, $a_1, \dots, a_k \in A$, $m_1 = m_2$ and $a_1 + a_2$ is defined, then*

$$(m_1, \dots, m_k; a_1, \dots, a_k) \sim (m_2, \dots, m_k; a_1 + a_2, \dots, a_k).$$

Lemma 4.19. *If A is a partial abelian monoid, and M is an n -dimensional open manifold, then the inclusion $C^0(M; A) \hookrightarrow C(M; A)$ is a weak equivalence.*

Proof. The proof makes use of the fact that a copy of the manifold with corners $C_k(M)$ lies inside its interior $C_k^0(M)$, so the retraction $r : C_k(M) \rightarrow C_k^0(M)$ is a Σ_k -equivariant homeomorphism onto its image. We compare via this retraction the pushout diagram for $C_k^0(M; A)$

$$\begin{array}{ccc} C_k^0(M) \times_{\Sigma_k} T_k(X, x_0) \cup (C^0(M) \times_{\tau} \text{Comp}(A))_k & \longrightarrow & C_{k-1}^0(M; A) \\ \downarrow & & \downarrow \\ C_k^0(M) \times_{\Sigma_k} A^k \cup (C^0(M) \times_{\tau} \text{Comp}(A))_k & \longrightarrow & C_k^0(M; A), \end{array}$$

and the pushout diagram for $C_k(M; A)$

$$\begin{array}{ccc}
C_k(M) \times_{\Sigma_k} T_k(X, x_0) \cup (C^0(M) \times_{\tau} \text{Comp}(A))_k & \longrightarrow & C_{k-1}(M; A) \\
\downarrow & & \downarrow \\
C_k(M) \times_{\Sigma_k} A^k & \longrightarrow & C_k(M; A).
\end{array}$$

Here we denote by $(C^0(M) \times_{\tau} \text{Comp}(A))_k$ the subspace of $C(M)_k \times_{\Sigma_k} A^k$ of those labelled configurations such that several points are concentrated in the same macroscopic location if and only if their labels are summable. The inclusion of the space on the left hand top corner of the first diagram into that of the second diagram is a homotopy equivalence, because r induces a common retraction onto a copy of the second space. The same holds for the spaces on the left hand bottom corner. We conclude by induction and the gluing lemma [4]. □

If we regard a pointed space (A, a_0) as a partial abelian monoid with $x + a_0 = x$ as the only defined sums, for $x \in A$, then $C^0(M; A)$ is the configuration space with labels studied in [3].

Corollary 4.20. *Let (A, a_0) be a well-pointed space. Then for any open n -manifold M there is a weak equivalence $C^0(M; A) \simeq C(M; A)$.*

Proof. The space A is a partial n -monoid by $\text{Comp} = \coprod_k F_n(k) \times_{\Sigma_k} \bigvee_{i=1}^k A$. □

For some background about toric varieties we refer to [7].

Corollary 4.21. *If V is a projective toric variety such that $H_2(V)$ is torsion free, then there exists a partial discrete abelian monoid Δ_V , such that the union of some components of $(\hat{\Delta}_V)^2$ is homotopy equivalent to the space $\text{Rat}(V)$ of based rational curves on V .*

Proof. Guest has shown in [9] that if Δ_V is the fan associated to the variety V [7] then the union of some components of $C^0(\mathbb{R}^2; \Delta_V)$ is homeomorphic to $\text{Rat}(V)$. The corollary follows from the theorem and from proposition 4.16. □

Remark 4.22. *It is possible to define labelled configurations with support in a manifold with corners M . It is sufficient to choose an embedding $M \hookrightarrow M'$, with M' open, consider the right F_n -submodule $C(M) \hookrightarrow C(M')$ of configurations macroscopically located at points of M , and carry through the discussion as for open manifolds.*

5 The relative case

We define relative labelled configuration spaces on relative manifolds.

Let (X, x_0) be a pointed topological space. Let M be a manifold with corners and $N \hookrightarrow M$ a cofibration such that $M - N$ is an open manifold. We obtain easily from 2.2 that each element $c \in C(M; X)$ is uniquely determined by a finite set $S(c) \subset M$, and for each $P \in S(c)$ a labelled tree T_P as in 2.2, with the only difference that the twigs of the tree are labelled by $X - x_0$.

Definition 5.1. *The based space $C(M, N)(X)$ is the quotient $C(M; X)/\sim$ by the equivalence relation such that $a \sim a'$ if and only if $S(a) \cap (M - N) = S(a') \cap (M - N)$ and the trees indexed by these intersections coincide. The base point is the class $[a]$ such that $S(a) \subset N$.*

If we regard pointed spaces as partial n -monoids, then the n -monoid completion induces a monad (F_n^*, η_*, μ_*) on the category of pointed compactly generated spaces \mathcal{CG}_* . Each element in the completion $F_n^*(X) = \hat{X}^n$ is represented by a tree with vertex labels in F_n and twigs labels in $X - x_0$. The product μ_* is given by grafting of trees, and the unit η_* sends an element x to the trivial tree labelled by x .

Proposition 5.2. *If M is a parallelizable n -manifold, and $N \hookrightarrow M$ is a cofibration such that $M - N$ is open, then the functor $C(M, N)$ has a structure of right algebra over F_n^* .*

Proof. We need to exhibit a natural transformation $\lambda : C(M, N)F_n^* \rightarrow C(M, N)$ such that $\lambda \circ C(M, N)\eta_*$ is the identity and the diagram

$$\begin{array}{ccc} C(M, N)F_n^*F_n^* & \xrightarrow{C(M, N)\mu_*} & C(M, N)F_n^* \\ \downarrow \lambda F_n^* & & \downarrow \lambda \\ C(M, N)F_n^* & \xrightarrow{\lambda} & C(M, N). \end{array}$$

commutes. The morphism λ is obtained by grafting of trees. □

Definition 5.3. *If (A, ρ) is an n -monoid, and M, N are as before, then the space $C(M, N; A)$ of configurations in (M, N) with summable labels in A is the coequalizer*

$$C(M, N)F_n^*(A) \begin{array}{c} \xrightarrow{C(M, N)\rho} \\ \xrightarrow{\lambda_A} \end{array} C(M, N)A \cdots \rightarrow C(M, N; A).$$

Definition 5.4. *A partial n -monoid A is good if the inclusion $\text{Comp}(A) \rightarrow F_n(A)$ is a cofibration, and the partial composition $\rho : \text{Comp}(A) \rightarrow A$ induces a map on the quotient $\text{Comp}^*(A) \subset F_n^*(A)$ of $\text{Comp}(A)$.*

The definition of a good framed partial n -monoid is similar. From now on we will assume implicitly that all partial (framed) n -monoids are good.

By means of the framed Fulton-MacPherson operad we can define similarly $C(M, N; A)$, if M is an n -dimensional manifold with corners, $N \hookrightarrow M$ is a cofibration, and A is a good partial framed n -monoid, and as in 4.13 we obtain:

Proposition 5.5. *Define a filtration so that $[a] \in C_k(M, N; A)$ if and only if k is the number of twigs of trees in $S(a) \cap (M - N)$. Then $C(M, N; A)$ has the weak topology with respect to the filtration and it is compactly generated.*

Definition 5.6. *If A is a partial framed n -monoid, then $B_k(A) = C((I^k, \partial I^k) \times I^{n-k}; A)$ for $i = 1, \dots, n$.*

If A is a partial abelian monoid and (M, N) is any pair then we define the relative labelled configuration space $C^0(M, N; A)$ as quotient of $C^0(M; A)$, by identifying configurations that coincide on $M - N$. We state the relative version of 4.19.

Proposition 5.7. *If M is a manifold, $N \hookrightarrow M$ is a cofibration, and $M - N$ is open, then there is a weak equivalence $C^0(M, N; A) \simeq C(M, N; A)$.*

Proof. The proof is similar to that of 4.19. In this case we use for each k a Σ_k -equivariant retraction $r_k : C_k(M) \rightarrow C_k^0(M)$ such that r_k preserves $b^{-1}(\overline{M^k - N^k})$, where $b : C_k(M) \rightarrow M^k$ is the blowdown. \square

Corollary 5.8. *If V is a projective toric variety such that $H_2(V)$ is torsion free, with torus T and fan Δ_V , then there is a weak equivalence $B_2(\Delta_V) \simeq V \times_T ET$.*

Proof. Guest has shown in [9] that $V \times_T ET$ is homotopy equivalent to $C^0(I^2, \partial I^2; \Delta_V)$. \square

The relative version of 4.20 is:

Corollary 5.9. *For any well-pointed space X there is a weak equivalence $C^0(M, N; X) \simeq C(M, N; X)$.*

Corollary 5.10. *Let X be a well-pointed space considered as partial n -monoid. Then there is a weak equivalence $\Sigma^n(X) \xrightarrow{\simeq} B_n(X)$.*

Proof. The space of open configurations $C^0(I^n, \partial I^n; X)$ retracts onto $\Sigma^n(X)$, considered as space of configurations of a single labelled point in $(I^n, \partial I^n)$. The retraction is achieved [5] by pushing radially the particles away onto the boundary. But the inclusion $C^0(I^n, \partial I^n; X) \hookrightarrow C(I^n, \partial I^n; X) = B_n(X)$ is a weak equivalence by 5.9. \square

By means of configuration spaces we obtain the classifying space constructed by Stasheff.

Proposition 5.11. *Let (A, a_0) be a well-pointed A_∞ space. The quotient map $C(I, \{0\}; A) \rightarrow C(I, \partial I; A) = B_1(A)$ is canonically homeomorphic to the universal arrow $E(A) \rightarrow B(A)$.*

Proof. It is sufficient to carry out the discussion in the non-symmetric case: in fact $C_k([0, 1]) = S_k([0, 1]) \times \Sigma_k$, where $S_k([0, 1])$ compactifies the space of strictly ordered maps from $\{1, \dots, k\}$ to $I = [0, 1]$.

Let $S_k(I)\{0, 1\} \subseteq S_k(I)$ be the closure of the subspace of maps $\alpha : \{1, \dots, k\} \rightarrow I$ such that $\alpha(1) = 0, \alpha(k) = 1$. Its elements are described by appropriate trees as in 2.2. For $k \geq 2$, we have homeomorphisms $r : S_k(I)\{0, 1\} \xrightarrow{\cong} S_k(0) : j$, where $S_k(0)$ is the space of configurations in \mathbb{R} macroscopically concentrated at the point 0.

If $\alpha_i \rightarrow \alpha \in S_k(I)\{0, 1\}$, $\alpha_i \in C_k^0(I)$, then $r(\alpha) = \lim_i \frac{\alpha_i - \alpha_i(0)}{i(\alpha_i(1) - \alpha_i(0))}$.

If $\beta_i \rightarrow \beta \in S_k(0)$, $\beta_i \in C_k^0(\mathbb{R})$, then $j(\beta) = \lim_i \frac{\beta_i - \beta_i(0)}{\beta_i(1) - \beta_i(0)}$.

We have seen in 2.2 that $S_k(0) = K_k$ is the associahedron. Under the identification $K_k \cong S_k(I)\{0, 1\}$ the Stasheff space $B(A)$ is defined to be the quotient of $\coprod S_k(I)\{0, 1\} \times A^{k-2}$, seen as space of forests labelled by A , under the following steps:

1. We replace a tree on i twigs by a point having as label the action of the tree on its twigs via $K_i \times A^i \rightarrow A$.
2. We can cut twigs labelled by a_0 .
3. We identify any two labelled forests coinciding outside 0 and 1.

But this quotient is exactly $B_1(A) = C(I, \partial I; A)$. In a similar way one shows that $E(A) = \coprod S_k(I)\{0, 1\} \times A^{k-1} / \sim$ is homeomorphic to $C(I, \{0\}; A)$. In this case in 3 we identify forests coinciding outside 0. \square

6 Approximation theorems

We say that a partial framed n -monoid A has homotopy inverse if the H-space \hat{A}^n has homotopy inverse.

Lemma 6.1. *Let M be a connected compact n -manifold, $M' \subset M$ a compact n -submanifold, $N \subset M$ a closed submanifold, and A a partial framed n -monoid. Suppose that either A has a homotopy inverse or the pair $(M', N \cap M')$ is connected. Then there is a quasifibration*

$$C(M', N \cap M'; A) \longrightarrow C(M, N; A) \xrightarrow{\pi} C(M, M' \cup N; A).$$

This holds in particular if A is path connected.

Proof. We follow the proof of proposition 3.1 in [5]. The space $C(M, M' \cup N; A)$ has a filtration by $C_k := C_k(M, M' \cup N; A)$. There is a homeomorphism $\alpha_k : \pi^{-1}(C_k - C_{k-1}) \cong C(M', N \cap M'; A) \times (C_k - C_{k-1})$ such that $\pi\alpha_k^{-1}$ is the projection onto the factor $C_k - C_{k-1}$. Choose a collared neighbourhood U of

M' in M and a smooth isotopy retraction $r : U \rightarrow M'$ such that $r(U \cap N) \subset N$. For each k there is an open neighbourhood U_k of C_k in C_{k+1} such that r induces a smooth isotopy retraction $r_k : U_k \times I \rightarrow C_k$, and a smooth isotopy retraction $\tilde{r}_k : \pi^{-1}(U_k) \times I \rightarrow \pi^{-1}(C_k)$ covering r_k . For any point $P \in U_k$ we need to show that the restriction $t : \pi^{-1}(P) \rightarrow \pi^{-1}(r_1(P))$ of \tilde{r}_1 is a weak homotopy equivalence. If we identify domain and range of t to $C(M', N \cap M'; A)$ by α_k , then t pushes the labelled particles away from N , and adds a finite set of trees T in proximity to N . But if the pair $(M', N \cap M')$ is connected, then the trees in T can be moved continuously to N , where they vanish, and t is homotopic to a homeomorphism. On the other hand, if A has a homotopy inverse, then t has a homotopy inverse that pushes the particles away from N and adds some homotopy inverses of the trees in T in proximity to N . \square

Proposition 6.2. *Let A be a partial framed n -monoid. Then for $i = 1, \dots, n$ there are maps $s_i : B_{i-1}(A) \longrightarrow \Omega B_i(A)$, such that s_i is a weak homotopy equivalence for $i > 1$, and s_1 is a weak homotopy equivalence if A has a homotopy inverse.*

Proof. Note that $B_0(A)$ is homotopic to the framed n -monoid completion of A . For each i the base point of $B_i(A)$ is the empty configuration. The translation $\tau_1(t) : I^n \rightarrow \mathbb{R} \times I^{n-1}$ of the first coordinate by t induces a map $\pi_{\tau_1(t)} : B_0(A) \rightarrow B_1(A)$, composite of the induced map $C(I^n; A) \xrightarrow{C(\tau_1(t); A)} C(\mathbb{R} \times I^{n-1}; A)$ and the projection $C(\mathbb{R} \times I^{n-1}; A) \rightarrow C((I, \partial I) \times I^{n-1}; A)$. Then the ‘scanning’ map s_1 is defined for $x \in B_0(A) = C(I^n; A)$ by $s_1(x)(t) = \pi_{\tau_1(2t-1)}(x) \in B_1(A)$. For $i > 0$ the translation of the $(i+1)$ -th coordinate by t induces similarly a map $\pi_{\tau_{i+1}(t)} : B_i(A) \rightarrow B_{i+1}(A)$, and $s_{i+1} : B_i(A) \rightarrow \Omega B_{i+1}(A)$ is given by $s_{i+1}(x)(t) = \pi_{\tau_{i+1}(2t-1)}(x)$. We define $M = I^k \times [0, 2] \times I^{n-k-1}$, $N = (\partial I^k \times [0, 2] \times I^{n-k-1}) \cup (I^k \times 0 \times I^{n-k-1})$, and we identify $B_k(A)$ to $C(I^k \times [1, 2] \times I^{n-k-1}, \partial I^k \times [1, 2] \times I^{n-k-1}; A)$ via $\tau_{k+1}(1)$. We consider for $1 \leq k \leq n-1$ a commutative diagram

$$\begin{array}{ccccc}
B_k(A) & \longrightarrow & C(M, N; A) & \longrightarrow & B_{k+1}(A) \\
\downarrow s_{k+1} & & \downarrow s & & \Downarrow \\
\Omega B_{k+1}(A) & \longrightarrow & PB_{k+1}(A) & \longrightarrow & B_{k+1}(A) .
\end{array}$$

The top row is a quasifibration and the bottom row a fibration. The scanning map s is defined on the total space $C(M, N; A)$ by $s(x)(t) = \pi_{\tau_{k+1}(2t)}(x)$ and is consistent with s_{k+1} . Now the space $C(M, N; A)$ is contractible. In fact by excision $C(M, N; A) \cong C(M', N'; A)$, with $M' = \mathbb{R}^k \times (-\infty, 2] \times \mathbb{R}^{n-k-1}$ and $N' = M' - (M - N)$. Moreover there is a smooth isotopy $H_t : (M, N) \rightarrow (M', N')$, such that H_0 is the inclusion and $H_1(M) \subset N'$. For example define H_t as the dilatation by $3t$ centered in $(\frac{1}{2}, \dots, \frac{1}{2}, 3, \frac{1}{2}, \dots, \frac{1}{2})$, with 3 at the $(k+1)$ -st position. We conclude by comparing the long exact sequences in homotopy and by induction on k . \square

The spaces $B_0(A) = C(I^n; A)$ and $\Omega^n B_n(A)$ are both fD_n -algebras. The map $s : B_0(A) \rightarrow \Omega^n B_n(A)$ constructed by looping and composing the scanning maps in proposition 6.2 can be extended to a homotopy fD_n -morphism by rescaling suitably the scanning maps on the labels of trees in $U_{fD_n}(B_0(A))$. By 6.2 we obtain:

Theorem 6.3. *If A is a partial framed n -monoid, then $s : B_0(A) \rightarrow \Omega^n B_n(A)$ is the group completion. If A has homotopy inverse, then s is a weak homotopy equivalence.*

Actually s is the group completion in the homotopy category of fD_n -algebras.

Corollary 6.4. [15] *If X is a well-pointed space, then $s : C^0(\mathbb{R}^n; X) \rightarrow \Omega^n \Sigma^n X$ is the group completion. If X is path connected, then s is a weak homotopy equivalence.*

Proof. Consider X as a partial n -monoid as in corollary 4.20. Now $B_0(A) = C(I^n; A) \simeq \hat{A}^n$ by the same argument of proposition 4.16. Moreover $\hat{A}^n \simeq C(\mathbb{R}^n; A)$ by 4.16, $C(\mathbb{R}^n; A) \simeq C^0(\mathbb{R}^n; A)$ by 4.20 and $\Sigma^n X \simeq B_n(X)$ by 5.10. Now we can apply the theorem. \square

Corollary 6.5. [9] *If V is a projective toric variety such that $H_2(V)$ is torsion free, then $s : \text{Rat}(V) \rightarrow \Omega^2(V)$ is the group completion.*

Proof. Apply corollaries 4.21 and 5.8, and restrict to the relevant components. \square

Given a manifold M , and its tangent bundle τ , there is a bundle $\gamma = C(\tau, \partial\tau; A)$ on M with fiber $B_n(A) = C(I^n, \partial I^n; A)$, consisting of relative fiberwise configurations in the fiberwise one-point compactification modulo the section at infinity $(\hat{\tau}, \infty)$. Whether ∂M is empty or not we can define a map $s : C(M, \partial M; A) \rightarrow \Gamma(M; B_n A)$ to the space of sections of γ . Note that if M is parallelizable then $\Gamma(M; B_n A) = \text{Map}(M; B_n A)$. The scanning map s is constructed by the exponential map: if $x \in C(M, \partial M; A)$, then $s(x)$ sends a point $P \in M$ to the restriction of x to a small disc neighbourhood of P modulo its boundary.

Theorem 6.6. *Let A be a partial framed n -monoid. Let M be a compact connected n -manifold with boundary. Then the scanning map $s : C(M, \partial M; A) \rightarrow \Gamma(M; B_n A)$ is a weak homotopy equivalence. If A has homotopy inverse and N is a compact connected n -manifold without boundary then $s : C(N; A) \rightarrow \Gamma(N; B_n A)$ is a weak homotopy equivalence.*

Proof. We follow the proof of 10.4 in [10]. There is a finite handle decomposition of M with no handles of index n . If M' is obtained from M'' by attaching a handle H of index i , then we apply lemma 6.1 and we obtain a quasifibration $C(H, \overline{\partial H - \partial H \cap M''}; A) \rightarrow C(M', \partial M'; A) \rightarrow C(M'', \partial M''; A)$. On the other hand we have a fibration $\Gamma(H/(H \cap M''); B_n A) \rightarrow \Gamma(M'; B_n A) \rightarrow \Gamma(M''; B_n A)$.

But $C(H, \overline{\partial H - \partial H \cap M''}; A) \cong B_{n-i}(A)$, and $\Gamma(H/(H \cap M''); B_n A) \simeq \Omega^i B_n(A)$. We compare the two sequences by the scanning maps and we conclude by proposition 6.2 and induction on the number of handles. In the case of N we have even a handle of index n and we apply the second part of proposition 6.2. \square

Corollary 6.7. *If X is a well-pointed path connected space then $s : C(S^1; X) \rightarrow \text{Map}(S^1, \Sigma X)$ is a weak homotopy equivalence.*

Proof. We consider X as partial framed 1-monoid as in corollary 4.20. By corollary 5.10 $B_1(X) \simeq \Sigma X$. We apply the second part of the theorem, and note that $\Gamma(S^1, \Sigma X) \simeq \text{Map}(S^1, \Sigma X)$ because S^1 is parallelizable. \square

This answers a question raised by Stasheff in [22] p. 10. The analogous result for $C^0(S^1; X)$ is in [3].

Any partial framed n -monoid gives an approximation theorem for mapping spaces, and the homotopy theorist is tempted to discover new examples. It might be worth considering colimits of abelian monoids in the category of n -monoids.

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