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Selfgravitating Yang-Mills solitons and their Chern-Simons numbers

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We present a classification of the possible regular, spherically symmetric solutions of the Einstein-Yang-Mills system which is based on a bundle theoretical analysis for arbitrary gauge groups. It is shown that such solitons must be of magnetic type, at least if the magnetic Yang-Mills charge vanishes. Explicit expressions for the Chern-Simons numbers of these selfgravitating Yang-Mills solitons are derived, which involve only properties of irreducible root systems and some information about the asymptotics of the solutions. It turns out, as an example, that the Chern-Simons numbers are always half-integers or integers for the gauge groups $SU(n)$. Possible physical implications of these results, which are based on analogies with the unstable sphaleron solution of the electroweak theory, are briefly indicated.

1 Introduction

Already at the time when some of us showed that the remarkable Bartnik-McKinnon (BK) solutions [1] of the Einstein-Yang-Mills (EYM) system are unstable [2, 3, 4], we wondered whether these solitons might retain some physical interest, in spite of their instability. This is, after all, the case for the sphaleron [5, 6], which is an unstable static solution of the classical equations of the bosonic sector of the electroweak theory, and which is believed to play an important role in non-perturbative baryon violating processes at high temperatures [7, 8, 9, 10].

As a physical motivation to the present mathematical investigations we recall that the baryon and lepton currents of the standard model have anomalies which are proportional to a second order characteristic density of the gauge fields. This, together with the non-trivial vacuum structure in non-Abelian gauge theories leads to baryon and lepton number violations at the quantum level. Classically, the infinite set of zero energy states are classified by integer Chern-Simons numbers N_{CS}, and states of different N_{CS} differ also in their baryon number. Topologically distinct vacua are separated by a potential barrier whose minimal height is given by the energy of the spaleron. This static solution of the classical bosonic equations has a single unstable mode and its Chern-Simons number $1/2$ is half way between different vacua whose topological number changes by 1 or -1. It is now widely accepted, that at high temperatures there are rapid transitions between different vacua whose rate is proportional to the Boltzmann factor determined by the spaleron mass. (For a short recent review see, e.g., [10].)

There is a similarity between the sphaleron and the BK solution of the EYM equations which was used by Galt'sov and Volkov [11] to give another demonstration of the instability of the BK soution. It is conceivable that these unstable classical solutions may also be responsible for fast baryon and lepton violating processes at extremely high temperatures. Since the energies of the EYM solitons are set by the Planck scale, we enter the domain of quantum gravity and therefore do not want to become more specific. Such considerations were, however, at the origin of our attempt to determine the Chern-Simons numbers of regular solutions of the EYM equations for an arbitrary gauge group. In addition, we were hoping that we might then be able to give a general instability proof of all these solutions.

For the BK solution of $SU(2)$ the Chern-Simons number is equal to $1/2$, as was shown recently by Moss and Wray [12] in an explicit calculation. We shall present in this paper a thorough analysis that is heavily based on our previous work [13], which implies in particular that the Chern-Simons numbers of EYM solitons are always half-interger or integer for the groups $SU(n)$. The final expressions for N_{CS} involve only properties of irreducible root systems and some information about the asymptotics of the soliton solutions, and can easily been worked out for any particular case.

In sect. 2 we present a bundle theoretical classification for all spherically symmetric EYM solitons, which implies that the possible principal bundles are in one-to-one correspondence with a few points of the integral lattice of the gauge group which have to be also in the (closed) fundamental Weyl chamber. This will lead us in sect. 3 to a useful description of the gauge fields which we have derived in detail already in [13]. In sect. 4 we show that regular static and asymptotically flat solutions of the EYM equations must be of magnetic type, at least if the magnetic YM charge of the soliton vanishes. (This assumption is probably not necessary, but so far we were not able to get rid of it for a general gauge group.) The proof of this is based on an adapted form of the field equations and makes also use of some specific group theoretical facts which are derived in an Appendix. Before we can compute the Chern-Simons numbers, we must construct a useful global gauge. (A global gauge always exists since all pricipal bundles for any compact connected gauge group over $\mathcal{R}^3$ or the compactified space S^3 are trivial.) This construction is presented in sect. 5 and is then used in the next section, in which we arrive at explicit formulae for N_{CS}. In a first step we derive a remarkably simple general expression (eq. (60)), which is then further reduced with the result collected in eqs. (74)-(76). These explicit expressions are illustrated for $SU(n)$, but can be worked out similarly for any other gauge group. Several steps and detailed proofs are deferred to Appendix A. In Appendix B we give further information related to the bundle classification for spherically symmetric solitons.

2 Bundle classification for EYM solitons

We begin this section by recalling results of our previous study [13] of the classification of principal bundles $P(M, G)$ over a space-time manifold M for a compact connected gauge group G which admit the symmetry group

$K = SO(3)$ or $SU(2)$, acting by bundle automorphisms.

We fix a maximal torus T of G with the corresponding integral lattice I (= kernel of the exponential map restricted to the Lie algebra LT of the torus). Furthermore, we choose a basis S of the root system R of real roots, which defines the fundamental Weyl chamber

$$K(S) = \{\, H \in LT \mid \alpha(H) > 0 \ \text{ for all } \alpha \in S \,\}. \tag{1}$$

Let us first consider the subbundle over a single orbit of the induced action of K on M. We fix a point x_0 on this orbit. Its isotropy group K_{x_0} maps, of course, the fiber $\pi^{-1}(x_0)$ into itself. Therefore, if we chose a point $u_0 \in \pi^{-1}(x_0)$ and act on this with $k \in K_{x_0}$ there exists an element $\lambda(k) \in G$ such that $k \cdot u_0 = u_0 \cdot \lambda(k)$. It is easy to see that $\lambda\colon K_{x_0} \to G$ is a group homomorphism. Changing u_0 to another point on the same fiber leads to a conjugate homomorphism and the same is true for a K-isomorphic bundle. We have thus a map which associates to equivalence classes of principal G-bundles admitting a fiber transitive K-action a conjugacy class of homomorphisms $\lambda\colon K_{x_0} \to G$. It has been shown in [14] that this is a bijection.

The orbits of the induced action of K by isometries on M with Lorentz metric g will generically have $U(1)$ as stabilizer. From Proposition 1 of our previous work [13] we know that there is a one-to-one relation between the set $H \in I \cap \overline{K(S)}$ ($\overline{K(S)}$ denotes the closure of (1)) and the conjugacy classes of homomorphisms from $U(1)$ to G which is given as follows: To each $H \in I \cap \overline{K(S)}$ there corresponds the class belonging to the homomorphism λ which is determined by $L\lambda(2\pi i) = H$. ($L\lambda$ denotes the induced homomorphism of the Lie algebras.)

At least in a neighbourhood of the origin of a spherically symmetric EYM soliton M is foliated by orbits which are 2-spheres (we leave out the origin):

$$M = \tilde{M} \times S^2 \qquad \text{(locally)}. \tag{2}$$

For each orbit (y, S^2), $y \in \tilde{M}$, the stabilizer of $(y, \text{northpole})$ is the group $U(1)$ and the corresponding conjugacy class of homomorphisms $\lambda_y\colon U(1) \to G$ is determined by an element from the set $I \cap \overline{K(S)}$. Clearly, these lattice points cannot jump from one orbit to the next. This implies that we can choose λ independent of y, a result which follows also from more general considerations [14]; it was pointed out in [15] that this follows also from the so-called "slice theorem" (see [16]).

For the origin of the regular solution the stabilizer is K and we have a corresponding conjugacy class of homomorphisms from K into the gauge group. By continuity we can choose a representative $\tilde{\lambda}\colon K \to G$ such that $\tilde{\lambda}|_{U(1)} = \lambda$.

Thus we arrive at an important restriction for the possible bundles of regular solutions. The classifying homomorphism λ, defined by $L\lambda(2\pi i) = H$, has an extension to a homomorphism $\tilde{\lambda}$ from K to G. This limits the possible lattice points H very much. As an example, consider $G = SU(n)$. Then $\tilde{\lambda}$ can be regarded as an n-dimensional unitary representation of K and is thus a direct sum from the list D^j, $j = 0, \ldots, (n-1)/2$ (with only integer j for $K = SO(3)$). In particular, there is only one non-trivial possibility for $G = SU(2)$ and $K = SU(2)$ (and none if $K = SO(3)$). This is the choice which is behind the BK solution [1]. For $G = SU(3)$ there are only two possibilities, corresponding to the three-dimensional representations D^1, $D^{1/2} \oplus D^0$.

In Appendix B we give a systematic discussion for arbitrary gauge groups which is based on extensive work by Dynkin [17].

3 Spherically symmetric gauge fields

A spherically symmetric gauge field is given by a K-invariant connection form ω on $P(M, G)$. As before, we consider first the subbundle over a single orbit. Using previous notations, we associate to ω the linear map $\Lambda\colon LK \to LG$ defined by

$$\Lambda(X) = \omega_{u_0}(\tilde{X}), \qquad X \in LK, \tag{3}$$

where $\tilde{X}$ is the induced vector field on P determined by the Lie algebra element X. One can show [18] that the linear map $\Lambda\colon LK \to LG$ satisfies

$$\Lambda|_{LK_{x_0}} = L\lambda, \tag{4}$$

$$\Lambda \circ \mathrm{Ad}(k) = \mathrm{Ad}\lambda(k) \circ \Lambda, \qquad k \in K_{x_0}. \tag{5}$$

According to a well-known theorem of Wang [19] there is a one-to-one relation between the K-invariant connections on $P(M, G)$ and linear maps $\Lambda\colon LK \to LG$ which satisfy (4) and (5). The correspondence is given by eq. (3).

For the symmetry group $K = SU(2)$ we can describe these linear maps in more detail. Let

$$H = 2\pi i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{6}$$

be the canonical basis for the complex extension $LSU(2)_{\mathcal{C}}$. H generates the Lie algebra of the isotropy group $U(1)$. Condition (4) and our previous results imply

$$\Lambda(H) = L\lambda(H) \in I \cap \overline{K}. \tag{7}$$

It is easy to see that the second condition (5) is satisfied if and only if Λ_+, $\Lambda_\pm := \Lambda(E_\pm)$, is contained in the following direct sum of root spaces L_α:

$$\Lambda_+ \in \bigoplus_{\substack{\alpha\in R_+ \\ \alpha(L\lambda(H))=2}} L_\alpha, \tag{8}$$

$$L_\alpha = \{\, X \in LG_{\mathcal{C}} \mid [H, X] = 2\pi i \alpha(H) X \ \text{ for all } H \in LT \,\}. \tag{9}$$

Here, R_+ denotes the positive roots of the set R of real roots of G. (For the normalisation of real roots we use the same convention as in the Ref. [20].)

From this we conclude that $\Lambda_\pm \neq 0$ only if the classifying homomorphism $\lambda\colon U(1) \to G$ of $SU(2)$ symmetric principal bundles $P(M, G)$ satisfies

$$H_\lambda := L\lambda(2\pi i\,) \in (I \cap \overline{K}) \cap (\bigcup_{\alpha\in R_+} L_{\alpha 2}\,), \tag{10}$$

where $L_{\alpha n} = \alpha^{-1}\{n\}$, $\alpha \in R_+$, $n \in \mathcal{Z}$. We recall that the union of these planes

$$LT_s = \bigcup_{\alpha,n} L_{\alpha n} \tag{11}$$

is the Stiefel diagram of G. It consists of the inverse image of the set of singular elements of T under the exponential map and plays an important role in the theory of compact Lie groups (see, e.g., Ref. [20]).

We summarize these conclusions in the following

Proposition 1 *With the notations introduced above the equivalence classes of principal G-bundles admitting a fiber transitive $SU(2)$ action characterized by the elements $L\lambda(2\pi i) \in I \cap \overline{K}$ have only "Abelian" connections with $\Lambda(E_\pm) = 0$ (see eq. (3)) if $L\lambda(2\pi i)$ is not in the part $\bigcup_{\alpha\in R_+} L_{\alpha 2}$ of the Stiefel diagram.*

Let the decomposition of the gauge potential relative to (2) be $A = \tilde{A} + \hat{A}$. The following Proposition was proven in [13]:

Proposition 2 *One can always find a gauge such that*

$$\hat{A} = \Lambda \circ \theta, \tag{12}$$

where θ is a pull-back of the Maurer-Cartan form of K to the homogeneous space K/K_{x_0} with any cross section, and $\tilde{A}$ is a one-form on $\tilde{M}$ which is invariant under $\mathrm{Ad}\,\lambda(K_{x_0})$.

It should be noted that Λ varies now from orbit to orbit, $\Lambda\colon \tilde{M} \times LK \to LG$, except for $\Lambda|_{LK_{x_0}}$ which can be chosen as $L\lambda$ for a constant λ as noted above.

For $K = SU(2)$, $K/K_{x_0} \cong S^2$, we choose the cross section σ from S^2 to $SU(2)$ as follows:

$$\sigma(\vartheta, \varphi) = \exp(\varphi\tau_3)\exp(\vartheta\tau_2), \tag{13}$$

where $\tau_k := \sigma_k/2i$ $(k = 1, 2, 3)$ and ϑ, φ are the standard coordinates of S^2. We have

$$\begin{aligned} \theta &= \sigma^{-1} d\sigma \\ &= \tau_2\, d\vartheta + (\cos\vartheta\,\tau_3 - \sin\vartheta\,\tau_1)\, d\varphi, \end{aligned} \tag{14}$$

and thus (12) leads with $\Lambda_k := \Lambda(\tau_k)$ to

$$\hat{A} = \Lambda_2\, d\vartheta + (\Lambda_3 \cos\vartheta - \Lambda_1 \sin\vartheta)\, d\varphi. \tag{15}$$

4 Magnetic structure of EYM solitons

In this section we prove that the YM-fields of a regular static and asymptotically flat solution of the EYM equations must be of magnetic type for any gauge group. So far we were, unfortunately, only able to prove this under the assumption that the magnetic charge of the soliton vanishes. (We are currently trying to generalize the arguments in Ref. [21] for the gauge group $SU(2)$). The proof given below relies on the detailed form of the coupled field equations which we have derived in [13].

For later use we specialize the relevant results of [13] to static fields. Using intrinsically defined Schwarzschild like coordinates we can adapt the metric to (2) in the form

$$g = \tilde{g} + r^2 \hat{g}, \qquad \tilde{g} = -NS^2 dt^2 + N^{-1} dr^2, \tag{16}$$

where $\hat{g}$ is the standard metric on S^2. The metric functions N and S depend only on the radial Schwarzschild coordinate r.

From now on we consider only the *generic case* for which the classifying element H_λ in (10) is in the *open* Weyl chamber $K(S)$. The centralizer of H_λ (and thus the centralizer of Λ_3) is then the infinitesimal torus LT (see sect. V.2 of Ref. [20]). The component $\tilde{A}$ of the gauge potential is thus Abelian and we can use a temporal gauge:

$$A = \tilde{A} + \hat{A}, \qquad \tilde{A} = (NS\mathcal{A})\,dt, \tag{17}$$

where $\mathcal{A}$ satisfies $[\Lambda_3, \mathcal{A}] = 0$ and $\hat{A}$ is given by (15). Introducing the quantities

$$\check{\mathcal{F}} = -r^2\frac{(NS\mathcal{A})'}{S}, \qquad \hat{\mathcal{F}} = [\Lambda_1, \Lambda_2] - \Lambda_3 \tag{18}$$

one can write a particular component of the YM equations as

$$2\check{\mathcal{F}}' + [\Lambda_+, \dot{\Lambda}_-] + [\Lambda_-, \dot{\Lambda}_+] = 0, \tag{19}$$

where

$$\dot{\Lambda}_\pm := [\mathcal{A}, \Lambda_\pm]. \tag{20}$$

(See eq. (34) of Ref. [13].)

We shall need also the Einstein equation

$$m' = 4\pi G\, r^2 \varrho, \tag{21}$$

where $m(r)$ is the usual mass fraction defined by $N = 1 - 2m/r$, and ϱ is the energy-mass density given by

$$\varrho = \frac{N}{2r^2}\Big\{|\dot{\Lambda}_+|^2 + |\Lambda_+'|^2\Big\} + \frac{1}{2r^4}\Big\{|\check{\mathcal{F}}|^2 + |\hat{\mathcal{F}}|^2\Big\}. \tag{22}$$

(See eqs. (39)-(41) in [13].) The norm of any element $Z = X + iY$ $(X, Y \in LG)$ in the complex extension $LG_{\mathcal{C}}$ is defined by $|Z|^2 = \langle X + iY, X - iY\rangle$, where $\langle .\,, .\rangle$ denotes an AdG -invariant scalar product in LG.

According to (8) we know that

$$\Lambda_+ \in \bigoplus_{\alpha\in S(\lambda)} L_\alpha, \tag{23}$$

where

$$S(\lambda) = \{\, \alpha \in R_+ \mid \alpha(H_\lambda) = 2 \,\}. \tag{24}$$

Now, since $\mathcal{A}$ is LT-valued we can split it with respect to the decomposition

$$LT = \langle S(\lambda) \rangle \oplus \langle S(\lambda) \rangle^\perp, \tag{25}$$

where $\langle S(\lambda) \rangle$ denotes the linear span of $S(\lambda)$. We emphasize that the decomposition (25) is independent of the choice of the AdG -invariant scalar product (see Appendix A). In obvious notation we set

$$\mathcal{A} = \mathcal{A}_\parallel + \mathcal{A}_\perp \tag{26}$$

and show first, that $\mathcal{A}_\perp$ can be gauged away.

To prove this we consider the perpendicular component of (19) and make use of the fact

$$[\Lambda_+, \dot{\Lambda}_-] \in \langle S(\lambda) \rangle, \tag{27}$$

which follows from (20), (23) and standard commutation relations for root spaces (using the property that for $\alpha, \beta \in S(\lambda)$, $\alpha - \beta$ is never a root; see Appendix A). This leads to $\check{\mathcal{F}}'_\perp = 0$. From (22) one sees that $\check{\mathcal{F}}$ must vanish at the origin and hence $\check{\mathcal{F}}_\perp$ must vanish identically. Together with (18) we obtain thus

$$NS\mathcal{A}_\perp = \text{const}. \tag{28}$$

A gauge transformation

$$A \to \text{Ad}(g)^{-1} A + g^{-1}\, dg \tag{29}$$

with $g = \exp(-NS\mathcal{A}_\perp t)$ does not affect $\hat{A}$ in (17) (because of (23)), but eliminates the piece $\tilde{A}_\perp$.

Next we show that $\mathcal{A}_\parallel$ must vanish for a regular solution. This time we take the scalar product of (19) with $NS\mathcal{A}_\parallel$. With (20) and the AdG -invariance of the scalar product we obtain

$$\langle \check{\mathcal{F}}', NS\mathcal{A}_\parallel \rangle + NS|\dot{\Lambda}_+|^2 = 0 \tag{30}$$

or with (18)

$$\langle \check{\mathcal{F}}, NS\mathcal{A}_\parallel \rangle' + \frac{S}{r^2}|\check{\mathcal{F}}_\parallel|^2 + NS|\dot{\Lambda}_+|^2 = 0. \tag{31}$$

Integrating this gives

$$-\left.\langle \check{\mathcal{F}}, NS\mathcal{A}_\| \rangle\right|_0^\infty = \int_0^\infty \left(\frac{S}{r^2} |\check{\mathcal{F}}_\||^2 + NS|\dot{\Lambda}_+|^2 \right) dr. \tag{32}$$

The boundary terms on the left hand side vanish: As noted above $\check{\mathcal{F}}(0) = 0$ and from (21) we see that the total energy $\lim_{r\to\infty} m(r)$ is only finite if $\lim_{r\to\infty} r^2 \varrho = 0$. According to (22) this is only possible if

$$\lim_{r\to\infty} [\mathcal{A}, \Lambda_+] = 0. \tag{33}$$

Now, we shall show in the next section that $\lim_{r\to\infty} \Lambda$ is a homomorphism of $LSU(2)$ to LG if the magnetic YM charge of the soliton vanishes. Using this and Lemma A1 in Appendix A it follows from (33) that $\lim_{r\to\infty} \mathcal{A}_\| = 0$.

Since the integrand in (32) is non-negative, we conclude that $\check{\mathcal{F}}_\| = 0$, implying $(NS\mathcal{A}_\|)' = 0$. Since $\lim_{r\to\infty} \mathcal{A}_\| = 0$ we are led to the desired conclusion that $\mathcal{A}_\|$ is identically equal to zero.

5 Construction of a useful global gauge

For the calculation of the Chern-Simons number of a static EYM soliton it is necessary to construct an explicit global gauge. The existence of a global gauge for any compact connected gauge group G is clear, since all principal G-bundles over $\mathcal{R}^3$ or the compactified space S^3 are trivial. This is obvious for $\mathcal{R}^3$ because this is a contractible space. On the other hand, the principal G-bundles over S^3 are classified by the second homotopy group $\Pi_2(G)$ which is trivial for any compact connected group. (That $\Pi_2(G)$ classifies the bundles is easy to see, since a principal bundle over S^3 is given by a transition function on the intersection on two hemispherical neighbourhoods and two such transition functions lead to isomorphic bundles, if their restrictions to the equator are in the same homotopy class.)

From sect. 4 we know that the gauge field is purely magnetic

$$\begin{aligned} A &= \Lambda_2\, d\vartheta + (\Lambda_3 \cos\vartheta - \Lambda_1 \sin\vartheta)\, d\varphi \\ &= \Lambda(\sigma^{-1} d\sigma), \end{aligned} \tag{34}$$

where σ is the cross section from S^2 to $SU(2)$ introduced in eq. (13).

Next we show that $\Lambda_{(0)} := \lim_{r\to 0}\Lambda(r)$ is a homomorphism from LK to LG. Indeed, from (22) we see that the energy density is only well-behaved if $\hat{\mathcal{F}}$ goes to zero for $r \to 0$. A look at (18) shows then that the $\Lambda_k := \Lambda(\tau_k)$ satisfy for $r = 0$ the commutation relations of $LSU(2)$ ($[\Lambda_1, \Lambda_2] = \Lambda_3$, cyclic).

Similarily, the total YM charge of the soliton vanishes only if $\lim_{r\to\infty}\hat{\mathcal{F}} = 0$. Again (18) implies that $\Lambda_{(\infty)} := \lim_{r\to\infty}\Lambda(r)$ is a homomorphism $LK \to LG$.

We have thus two homomorphisms $\Lambda_{(0)}$, $\Lambda_{(\infty)}$ which extend the classifying homomorphism $L\lambda\colon LU(1) \to LG$ introduced earlier (see sect. 2). In Appendix A we show that there exists an element $Z \in LT$ such that

$$\Lambda_{(\infty)} = \mathrm{Ad}(\exp Z)\,\Lambda_{(0)}, \tag{35}$$

at least if H_λ in (10) is in the open Weyl chamber. (This is what we called the generic case in sect. 4.)

Now we are ready to transform (34) to a new gauge which is constructed such that the new potential is well defined on all of $\mathcal{R}^3 \cup \infty$. In

$$A \to \mathrm{Ad}(G^{-1})A + G^{-1}dG =: \tilde{A} \tag{36}$$

we choose G of the form

$$G = g\,\Pi(\sigma^{-1}), \tag{37}$$

where Π is the homomorphism from $SU(2)$ to the gauge group such that $L\Pi = \Lambda_{(0)}$, and

$$g = \exp(\chi Z) \tag{38}$$

with Z appearing in (35) which we multiply with a smooth real function χ that satisfies

$$\chi = \begin{cases} 0 & \text{for } r < 1-\epsilon, \\ 1 & \text{for } r > 1+\epsilon, \end{cases} \tag{39}$$

for an $\epsilon > 0$. (For clarity we note that σ^{-1} in (37) does not denote the inverse map but $(\sigma^{-1})(\vartheta,\varphi) := (\sigma(\vartheta,\varphi))^{-1}$.)

For the determination of $\tilde{A}$ we compute first $G^{-1}dG$. Eq. (37) gives

$$G^{-1}dG = \mathrm{Ad}(\Pi(\sigma))\,g^{-1}dg \,+\, \Pi(\sigma)\,d(\Pi(\sigma^{-1})). \tag{40}$$

The last term is easily seen to be

$$\Pi(\sigma)\,d(\Pi(\sigma^{-1})) = -\mathrm{Ad}(\Pi(\sigma))\,L\Pi(\sigma^{-1}d\sigma). \tag{41}$$

Using also (38), eq. (40) leads to

$$G^{-1}dG = \mathrm{Ad}(\Pi(\sigma))\Big\{Z\chi' dr - L\Pi(\sigma^{-1}d\sigma)\Big\}. \tag{42}$$

The first term of $\tilde{A}$ is according to (34)

$$\mathrm{Ad}(G^{-1})\,A = \mathrm{Ad}(\Pi(\sigma)g^{-1})\,\Lambda(\sigma^{-1}d\sigma) \tag{43}$$

Together we find, using also $\mathrm{Ad}(g^{-1})Z = Z$ and $L\Pi = \Lambda_{(0)}$,

$$\tilde{A} = \mathrm{Ad}(\Pi(\sigma)g^{-1})\Big\{\Big(\Lambda - \mathrm{Ad}(g)\Lambda_{(0)}\Big)(\sigma^{-1}d\sigma) + Z\chi' dr\Big\}. \tag{44}$$

The crucial point is now that this expression is well defined both for $r \to 0$ and $r \to \infty$. Indeed, for $r \to 0$ we have with (38), (39): $\Lambda - \mathrm{Ad}(g)\Lambda_{(0)} \to 0$, and for $r \to \infty$ this quantity vanishes because of (35). Moreover, the last term in (44) vanishes for $r \notin [1-\epsilon, 1+\epsilon]$. (Note, that the right hand side of (44) is independent of the section σ, the potential $\tilde{A}$ can thus be extended across the polar axis.)

The gauge pontential $\tilde{A}$ is thus well-defined on the compactified three-space $\mathcal{R}^3 \cup \infty$, and we can now turn to the computation of the Chern-Simons numbers.

6 Computation of the Chern-Simons numbers for regular solitons

Before we come to the derivation of a concise general formula of the Chern-Simons numbers, we recall some well-known facts.

Consider a principal bundle $P(M, G)$ with a connection form ω and a bilinear AdG -invariant form f on the Lie algebra LG. If Ω is the curvature of ω one easily derives with the help of the structure equation $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ the following idendity

$$f(\Omega, \Omega) = d\{f(\omega, d\omega + \tfrac{1}{3}[\omega, \omega])\}. \tag{45}$$

As a special case of the Chern-Weyl theorem $f(\Omega, \Omega)$ projects to a (unique) closed 4-form on M whose de Rham cohomology class is independent of the connection.

The 3-form

$$2Q(\omega) = f(\omega, d\omega + \tfrac{1}{3}[\omega,\omega]) = f(\omega, \Omega - \tfrac{1}{6}[\omega,\omega]) \tag{46}$$

is, however, not horizontal, and therefore does in general not project to the base manifold M. To each local section $\sigma: U \subset M \to P$ we can define a *local* Chern-Simons form in terms of $A = \sigma^*\omega$

$$2Q(A) := f(A, dA + \tfrac{1}{3}[A,A]). \tag{47}$$

Under a gauge transformation one finds the following transformation law for the local Chern-Simons form

$$2Q(\mathrm{Ad}(g^{-1})A + g^{-1}dg) = 2Q(A) - \tfrac{1}{6}f(\theta,[\theta,\theta]) + d\{f(\mathrm{Ad}(g^{-1})A,\theta)\}, \tag{48}$$

where θ stands for the Lie algebra valued 1-form $g^{-1}dg$, which is the pull-back of the Maurer-Cartan form on G by the transition function g.

For the regular solutions considered in this paper the principal G -bundles are trivial and there exist thus global sections. We are only interested in static solitons and can then consider these transition functions as maps from compactified 3-space $\mathcal{R}^3 \cup \{\infty\} \cong S^3$ to G. Within the class of these global sections the integral of $Q(A)$ over S^3 changes according to (48) by

$$\int_{S^3} Q(\mathrm{Ad}(g^{-1})A + g^{-1}dg) = \int_{S^3} Q(A) - \tfrac{1}{12}\int_{S^3} f(\theta,[\theta,\theta]) \tag{49}$$

Here, the last integral has a topological meaning and is an integer if f is suitably normalized. For example, if $G = SU(2)$ this integral is proportional to the winding number of $g: S^3 \to SU(2) \cong S^3$. The Chern-Simons number

$$N_{CS} := \int_{S^3} Q(A) \tag{50}$$

is thus defined up to an integer if the AdG -invariant bilinear form f is properly normalized.

6.1 A general formula for N_{CS} for an arbitrary gauge group

Our next task is to compute N_{CS} for the globally defined gauge potential $\tilde{A}$ given in (44). It would, however, be complicated to start directly from

Figure 1: Integration domain in the limiting procedure.

(44). An easier way is to use the local transformation law (48) for the gauge transformation (36) and to compute the integral of $Q(\tilde{A})$ over all of S^3 by a limiting process, whereby one first excludes conical regions around the polar axis (see Fig. 1). The last term in (48) gives then a boundary contribution which has to be evaluated carefully in the limit when the conical regions shrink to the polar axis.

We show first that $Q(A) = 0$ for A given by (34):

$$A = \Lambda_2 \, d\vartheta + (\Lambda_3 \cos\vartheta - \Lambda_1 \sin\vartheta) \, d\varphi. \tag{51}$$

It is easy to see that $Q(A)$ is proportional to $\langle \Lambda_+, \Lambda_-' \rangle - \langle \Lambda_-, \Lambda_+' \rangle$ if we choose for f an AdG -invariant scalar product $\langle . , . \rangle$ on LG. Now, we can decompose Λ_+ with respect to a basis $\{e_\alpha\}$ of the root spaces L_α, $\alpha \in S(\lambda)$ (see eq. (23)):

$$\Lambda_+ = \sum_{\alpha \in S(\lambda)} w^\alpha e_\alpha, \tag{52}$$

where the amplitudes w^α are functions of r alone. Using the orthogonality properties of the root spaces we see that $Q(A)$ is proportional to

$$\sum_{\alpha\in S(\lambda)} (w^\alpha \overline{w}^{\alpha\prime} - \overline{w}^\alpha w^{\alpha\prime})$$

(for a suitable normalisation of the e_α). This expression vanishes, however, as a consequence of the YM equations. Indeed, eq. (36) of Ref. [13] reduces for a static solution to

$$[\Lambda_+, \Lambda'_-] + [\Lambda_-, \Lambda'_+] = 0, \tag{53}$$

which gives, by inserting (52) and using the fact that $S(\lambda)$ is a simple system of roots (see Appendix A)

$$w^\alpha \overline{w}^{\alpha\prime} - \overline{w}^\alpha w^{\alpha\prime} = 0 \qquad \text{for each } \alpha \in S(\lambda). \tag{54}$$

This equation just says that w^α is real, up to a constant phase, a fact that we shall use againg later on. (We could absorb this phase by a redefinition of the base vectors e_α.)

Next, we compute the second term in (48) for $\theta = G^{-1}dG$ given in (42). Using $L\Pi = \Lambda_{(0)}$ and the AdG -invariance of the scalar product, we find

$$\langle\theta, [\theta,\theta]\rangle = 3\,\langle\, \chi' Z dr\,,\, [\Lambda_{(0)}(\sigma^{-1}d\sigma)\,, \Lambda_{(0)}(\sigma^{-1}d\sigma)]\,\rangle. \tag{55}$$

Since $\Lambda_{(0)}$ is a homomorphism we have

$$[\Lambda_{(0)}(\sigma^{-1}d\sigma)\,, \Lambda_{(0)}(\sigma^{-1}d\sigma)] = 2{\Lambda_{(0)}}_3 \,\mathrm{vol}_{S^2} + 2{\Lambda_{(0)}}_1 \cot\vartheta\, \mathrm{vol}_{S^2}. \tag{56}$$

Only the first term contributes to (55) because $Z \in LT$, giving

$$\langle\theta, [\theta,\theta]\rangle = 6\,\langle Z, \Lambda_3\rangle \chi'\, dr \wedge \mathrm{vol}_{S^2}. \tag{57}$$

(Recall that Λ_3 is independent of r.) Integration gives with (39)

$$-\tfrac{1}{12}\int \langle\theta, [\theta,\theta]\rangle = -2\pi\langle Z, \Lambda_3\rangle \chi|_0^\infty = -2\pi\langle Z, \Lambda_3\rangle. \tag{58}$$

Finally, we have to calculate the surface integral arising from the last term in (48). Using (37) and (42), we have

$$\begin{aligned} &\langle\, \mathrm{Ad}(G^{-1})\Lambda(\sigma^{-1}d\sigma)\,, G^{-1}dG\,\rangle \\ &\quad = \langle\, \Lambda(\sigma^{-1}d\sigma)\,, \mathrm{Ad}(g)\{Z\chi' dr - \Lambda_{(0)}(\sigma^{-1}d\sigma)\}\,\rangle \\ &\quad = \langle\, \Lambda(\sigma^{-1}d\sigma)\,, Z\,\rangle\, \chi'\, dr \,-\, \langle\, \Lambda(\sigma^{-1}d\sigma)\,, \mathrm{Ad}(g)\Lambda_{(0)}(\sigma^{-1}d\sigma)\,\rangle. \end{aligned} \tag{59}$$

The last term is proportional to $d\vartheta \wedge d\varphi$ and gives thus only contributions from the small and from the large spheres in Fig. 1 in the limit $r \to 0$ and $r \to \infty$, respectively. For $r \to 0$ this becomes (see (38), (39))

$$\langle\, \Lambda_{(0)}(\sigma^{-1}d\sigma)\,, \Lambda_{(0)}(\sigma^{-1}d\sigma)\,\rangle,$$

and for $r \to \infty$ we find with (35)

$$\langle\, \Lambda_{(\infty)}(\sigma^{-1}d\sigma)\,, \mathrm{Ad}(\exp Z)\Lambda_{(0)}(\sigma^{-1}d\sigma)\,\rangle = \langle\, \Lambda_{(0)}(\sigma^{-1}d\sigma)\,, \Lambda_{(0)}(\sigma^{-1}d\sigma)\,\rangle.$$

Hence, the two terms cancel.

There remains the first term in (59) which is equal to $-\langle Z, \Lambda_3\rangle \chi' \cos\vartheta\, dr \wedge d\varphi$ and gives the following contribution from the two conical surfaces in Fig. 1 in the limit $\vartheta \to 0, \pi$:

$$\begin{aligned} -\frac{1}{2}\langle Z, \Lambda_3\rangle \int \chi' \cos\vartheta\, dr \wedge d\varphi \quad &\to \quad -\frac{1}{2}\langle Z, \Lambda_3\rangle \cdot 2\pi \cdot 2 \cdot \chi|_0^\infty \\ &= \quad -2\pi\langle Z, \Lambda_3\rangle. \end{aligned}$$

Note, that this is exactly equal to the other piece (58). Together we obtain the remarkably simple result

$$\int Q(\tilde{A}) = -4\pi\langle Z, \Lambda_3\rangle. \tag{60}$$

Before we reduce this expression further, we illustrate the formula for the BK solution of $G = SU(2)$. Then we have

$$\Lambda_1 = w\tau_1, \qquad \Lambda_2 = w\tau_2, \qquad \Lambda_3 = \tau_3 \tag{61}$$

with the boundary conditions

$$w(0) = 1, \qquad w(\infty) = -1. \tag{62}$$

This implies $\Lambda_{(0)} = 1$, $\Lambda_{(\infty)} = \mathrm{Ad}(\exp(\pi\tau_3))\Lambda_{(0)}$, i.e. $Z = -\pi\tau_3$ and thus $\langle Z, \Lambda_3\rangle = -\pi\langle \tau_3, \tau_3\rangle$. If we choose the scalar product as

$$\langle X, Y\rangle = -1/(2\pi)^2 tr(XY)$$

we obtain thus

$$\int Q(\tilde{A}) = \frac{1}{2}. \tag{63}$$

The scalar product is chosen such that the second term in (49), which is proportional to the winding number of g, is an integer. We have

$$\int_{S^3} tr(\theta \wedge \theta \wedge \theta) = -24\pi^2 \deg(g). \tag{64}$$

The result (63) was obtained before by Moss and Wray [12].

6.2 Further reductions

Let us first express the right hand side of (60) in terms of the classifying element H_λ in (10). Since $H_\lambda := L\lambda(2\pi i\sigma_3) = -4\pi L\lambda(\tau_3) = -4\pi\Lambda_3$ we obtain from (60)

$$\int Q(\tilde{A}) = \langle Z, H_\lambda \rangle. \tag{65}$$

In the course of the evaluation of $\langle Z, H_\lambda \rangle$ we make now use of various tools which are developed in Appendix A. There we show that $S(\lambda)$ is a basis of a root system $R(\lambda)$ (Theorem A1) and that

$$H_\lambda = 2\varrho(\lambda)^* := \sum_{\alpha \in R(\lambda)_+} \alpha^*, \tag{66}$$

where α^* denotes the inverse root corresponding to α (see eq. (A18)). If the root system $R(\lambda)$ is reducible with the decomposition

$$R(\lambda) = \bigcup_j R_j(\lambda) \tag{67}$$

into mutually orthogonal irreducible root systems $R_j(\lambda)$ then $\langle Z, H_\lambda \rangle$ splits into the sum

$$\langle Z, H_\lambda \rangle = \sum_j \langle Z, 2\varrho_j(\lambda)^* \rangle, \qquad 2\varrho_j(\lambda)^* = \sum_{\alpha \in R_j(\lambda)_+} \alpha^*. \tag{68}$$

It is clear that $S(\lambda)$ decomposes correspondingly:

$$S(\lambda) = \bigcup_j S_j(\lambda), \qquad S_j(\lambda) = S(\lambda) \cap R_j(\lambda). \tag{69}$$

It suffices, therefore, to compute $\langle Z, H_\lambda \rangle$ for irreducible root systems which are classified by the well-known Dynkin diagrams.

Thus from now on $R(\lambda)$ is assumed to be irreducible. For the computation of $\langle Z, H_\lambda \rangle$ we expand $2\varrho(\lambda)^*$ with respect to the basis $S(\lambda)$ as in (A10)

$$2\varrho(\lambda)^* = \sum_{\alpha \in S(\lambda)} n_\alpha \frac{2\alpha}{\langle \alpha_L, \alpha_L \rangle}, \tag{70}$$

and Z is expanded as in (A19) with respect to the dual basis $\{h_\alpha\}_{\alpha \in S(\lambda)}$ of $S(\lambda)$

$$Z = \sum_{\alpha \in S(\lambda)} Z^\alpha h_\alpha + Z_\perp, \qquad Z_\perp \in \langle S(\lambda) \rangle^\perp. \tag{71}$$

In (70) $\langle\alpha_L,\alpha_L\rangle$ is the square of the length of a long root, and one knows that n_α are all positive integer numbers which are determined by the Cartan matrix $N(\lambda)$ of $R(\lambda)$ (see eq. (A11)). The coefficients Z^α in (71) have to be chosen such that

$$\exp(2\pi i Z^\alpha) = \frac{w^\alpha(\infty)}{w^\alpha(0)} \tag{72}$$

(see (A20)), where the amplitudes w^α are defined by the expansion of Λ_+,

$$\Lambda_+ = \sum_{\alpha\in S(\lambda)} w^\alpha e_\alpha \tag{73}$$

in terms of base elements e_α of the root spaces L_α.

Collecting all this we find

$$\langle Z, H_\lambda\rangle = \frac{2}{\langle\alpha_L,\alpha_L\rangle}\sum_{\alpha\in S(\lambda)} Z^\alpha n_\alpha, \tag{74}$$

$$Z^\alpha = \frac{1}{2\pi i}\ln\frac{w^\alpha(\infty)}{w^\alpha(0)} \quad \in \mathcal{R}, \tag{75}$$

$$n_\alpha = 2\sum_{\beta\in S(\lambda)} (N(\lambda)^{-1})_{\alpha\beta}\frac{\langle\alpha_L,\alpha_L\rangle}{\langle\beta,\beta\rangle} \quad \in \mathcal{N}. \tag{76}$$

The coefficients n_α are listed in Table A1, A2 of Appendix A for all simple root systems.

Clearly, the coefficients $Z^\alpha = \alpha(Z)$ in (75) are only determined up to integer numbers. A corresponding change $Z \mapsto Z + \Delta Z$ induces an additive contribution in (74) of the form

$$\langle \Delta Z, H_\lambda\rangle = \frac{2}{\langle\alpha_L,\alpha_L\rangle} k(\lambda)\, m, \qquad m \in \mathcal{Z}, \tag{77}$$

where $k(\lambda)$ is a positive integer which is determined by the root system $R(\lambda)$ and which we have listed in Table 1 for the simple root systems.

We show now that

$$Z^\alpha = \alpha(Z) \in \frac{1}{2}\mathcal{Z}. \tag{78}$$

$$A_l: \quad k = \begin{cases} 2 & \text{for } l = 2m, \\ 1 & \text{otherwise,} \end{cases} \qquad B_l: \quad k = 2,$$

$$C_l: \quad k = \begin{cases} 2 & \text{for } l = 2m, \\ 1 & \text{otherwise,} \end{cases} \qquad D_l: \quad k = \begin{cases} 2 & \text{for } l = 4m, 4m+1, \\ 1 & \text{otherwise,} \end{cases}$$

$$E_l: \quad k = \begin{cases} 2 & \text{for } l = 6, 8, \\ 1 & \text{for } l = 7, \end{cases} \qquad F_4: \quad k = 2,$$

$$G_2: \quad k = 2.$$

Table 1: The factors $k(\lambda)$ for the simple root systems

This relies on the consequence (54) of the YM equations, according to which the phases of all w^α, $\alpha \in S(\lambda)$, are constant. Hence (72) reduces to

$$\exp(2\pi i Z^\alpha) = \pm 1,$$

i.e.

$$Z^\alpha \in \frac{1}{2}\mathcal{Z}.$$

We can now use this fact in (74) and find, using also (77),

$$\langle Z, H_\lambda \rangle \in \frac{2}{\langle \alpha_L, \alpha_L \rangle} \frac{k(\lambda)}{2} \mathcal{Z}. \tag{79}$$

Let us work this out for $G = SU(n+1)$ and $R(\lambda) = A_n$, with the scalar product

$$\langle X, Y \rangle = -\frac{1}{(2\pi)^2} tr(XY). \tag{80}$$

Then all roots have length $\sqrt{2}$. According to Table 1 $k(\lambda) = 1$ for n odd, and $k(\lambda) = 2$ for n even. Thus

$$\langle Z, H_\lambda \rangle \in \begin{cases} \frac{1}{2}\mathcal{Z} & \text{for } n \text{ odd,} \\ \mathcal{Z} & \text{for } n \text{ even.} \end{cases} \tag{81}$$

The definition of $Q(\tilde{A})$ depends on the normaliszation of the inner product, and hence also the combined result (65) and our various formulae for $\langle Z, H_\lambda \rangle$:

$$\begin{aligned}\int Q(\tilde{A}) &= \sum_{\text{irr.}} \langle Z, 2\varrho(\lambda)^* \rangle &(82)\\ &= \sum_{\text{irr.}} \frac{2}{\langle \alpha_L, \alpha_L \rangle} \sum_{\alpha \in S(\lambda)} \alpha(Z)\, n_\alpha &(83)\\ &\in \sum_{\text{irr.}} \frac{2}{\langle \alpha_L, \alpha_L \rangle} \frac{k(\lambda)}{2} \mathcal{Z}.\end{aligned}$$

($\sum_{\text{irr.}}$ indicates the sum over the irreducible components in the decomposition (67) of the root system $R(\lambda)$.)

For a simple gauge group and a given normalisation of the scalar product one will define the Chern-Simons number as

$$N_{CS} := \# \int Q(\tilde{A}), \tag{84}$$

where the prefactor is chosen such that the changes ΔN_{CS} of N_{CS} under global gauge transformations can be any integer. This was illustrated for $SU(2)$ at the end of section 6.1 (see eq. (63)). The normalization factor 1 in (63) remains the same for all $SU(n)$, if the scalar product is chosen according to (80). We do not discuss here the normalization question for the other compact Lie groups.

7 Concluding remarks

In concluding, we emphasize, that most of the results of the present paper do not rely on the detailed form of the coupled field equations, but are mainly based on bundle and group theoretical considerations. This is in particular the case for our classification of the possible regular, spherically symmetric solutions of the EYM equations (sect. 2 and Appendix B). We have started a detailed study of the field equations which we have brought into an explicit and well-adapted form. We expect that the set of A_1-vectors in the open Weyl chamber (see Appendix B), for which there exist regular solutions, is further constraint because one is dealing with a singular boundary value problem. This does, however, not seem to be the case for the gauge groups $SU(n)$, as we learned recently from H.P. Künzle [23], who has been able to construct numerically new regular solutions for $SU(3)$.

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APPENDIX

A Mathematical tools and detailed proofs

In this Appendix we prove some mathematical facts which are used in the main body of the paper.

For the formulation of the next theorem we use the following notation which was also adopted in the main text.

We consider a compact connected Lie group G with a fixed maximal torus T and the correspondig integral lattice I. R denotes the set of real roots and S a basis of R with corresponding fundamental (open) Weyl chamber $K(S)$.

For the evaluation of the Chern-Simons number in sect. 6 we need the following

Theorem A1 *Let $H \in K(S)$ and*

$$S(H) = \{\, \alpha \in R \mid \alpha(H) = 2 \,\} \subset R_+. \tag{A1}$$

If $S(H)$ is not empty then $S(H)$ is the basis of a root system $R(H) \subset R$.

We shall show that $S(H)$ is an abstract fundamental system in the sense of the following definition, and then use known results from the mathematical literature.

Definition A1 *An* abstract fundamental system *is a non-empty, finite, linearly independent subset $F = \{\alpha_1, \dots, \alpha_l\}$ of a Euclidean space such that for any $\alpha_i, \alpha_j \in F$ with $\alpha_i \neq \alpha_j$ the value $2\langle\alpha_i, \alpha_j\rangle/\langle\alpha_j, \alpha_j\rangle$ is a non-positive integer.*

Proof of Theorem A1 First, we establish the following three properties of $S(H)$:

(i) If $\alpha, \beta \in S(H)$ then $\alpha - \beta$ is not a root.

(ii) For $\alpha, \beta \in S(H)$ with $\alpha \neq \beta$, we have $\langle\alpha, \beta\rangle \leq 0$.

(iii) The set $S(H)$ is linearly independent.

The first statement is obvious. If $\alpha - \beta$ would be a root it would have to vanish on H, which is , however, impossile because H is in the open Weyl chamber $K(S)$ on which all positive roots have strictly positive values.

Turning to (ii), suppose that $\langle\alpha, \beta\rangle > 0$ for a pair $\alpha, \beta \in S(H)$. Then the Cartan number $n_{\alpha\beta}$ for the roots $\alpha, \beta \in S(H)$ would also be strictly positive and thus also $n_{\beta\alpha}$. Therefore, the product $n_{\alpha\beta} n_{\beta\alpha}$ must have one of the values $1, 2, 3$ (4 is excluded for $\alpha \neq \beta$). This leaves two possibilities: either $n_{\alpha\beta} = 1$ or $n_{\beta\alpha} = 1$. In the second case we have $\alpha - \beta = \alpha - n_{\beta\alpha}\beta = \alpha - \langle\beta^*, \alpha\rangle\beta = s_\beta\alpha \in R$, where s_β is the Weyl reflection belonging to β, and this is impossible according to (i). Similarly, the first possibility is also excluded.

Having established $\langle\alpha, \beta\rangle \leq 0$ for $\alpha, \beta \in S(H)$, we can prove the linear independence of $S(H)$ by a standard argument. A linear dependence of $S(H)$ would imply an equation of the form

$$\sum_\beta m_\beta \beta = \sum_\gamma n_\gamma \gamma, \qquad m_\beta, n_\gamma \geq 0, \tag{A2}$$

where β and γ run through disjoint subsets of $S(H)$. Denoting one side of this equation by ϕ, we have

$$|\phi|^2 = \sum_{\beta,\gamma} m_\beta n_\gamma \langle\beta, \gamma\rangle \leq 0 \tag{A3}$$

and thus $\phi = 0$. But then we have

$$0 = \langle\phi, H\rangle = \sum_\beta m_\beta \langle\beta, H\rangle = \sum_\gamma n_\gamma \langle\gamma, H\rangle. \tag{A4}$$

Since $H \in K(S)$ and $S(H) \subset R_+$ this is only possible for $m_\beta = n_\gamma = 0$ for all β, γ.

From what has been shown so far it is clear that $S(H)$ is an abstract fundamental system in the sense of the definition given above. To such a system F we can consider the Weyl group W generated by all reflections. One can prove quite easily that $W \cdot F$ is a root system. In addidion, it turns out that F is also a fundamental system of roots of $W \cdot F$. Proofs of these facts can be found in sect. 2.12 of [22]. Another possibility to establish these statements is to construct all possible abstract fundamental systems and to verify them in each case: The simple (indecomposable) such systems are described Dynkin diagrams. (To each of these belongs an irreducible root system of which F is basis.)

Before we draw a useful corollarry to Theorem A1 we note that the decomposition of LT into the linear span $\langle S(H) \rangle$ of $S(H)$ and its orthogonal complement

$$LT = \langle S(H) \rangle \oplus \langle S(H) \rangle^\perp \tag{A5}$$

is independent of the AdG -invariant metric. Indeed, $\langle S(H) \rangle$ is also equal to the linear span of the set of inverse roots $S(H)^* := \{\, \alpha^* \mid \alpha \in S(H) \,\}$ and $\langle S(H) \rangle^\perp = \bigcap_{\alpha \in S(H)} \ker \alpha$. (We recall that the inverse roots $\alpha^* \in LT$ are independent of the choice of the inner product; see, e.g., chapt. V of Ref. [20].)

In sect. 6.2 we have used the following

Corollary A1 *With the same notation as in theorem A1 we have for $H_\parallel$ in the decomposition $H = H_\parallel + H_\perp$ with respect to (A5)*

$$H_\parallel = 2\varrho(H)^* := \sum_{\alpha \in R(H)_+} \alpha^* \in I. \tag{A6}$$

Proof From Theorem A1 it follows that $S(H)^*$ is a basis of the root system $R(H)^*$ and therefore the sum of positive inverse roots, $2\varrho(H)^*$, satisfies (see sect. V.4 in [20])

$$\langle 2\varrho(H)^*, \alpha \rangle = 2 \qquad \text{for all } \alpha \in S(H). \tag{A7}$$

On the other hand, by (A1) we have $\alpha(H_\parallel) = \alpha(H) = 2$ for all $\alpha \in S(H)$. This proofs (A6).

The quantity $2\varrho(H)^*$ plays an inportant role in our evaluation of the Chern-Simons numbers and we need a more explicit expression for it.

First of all we note that if the root system $R(H)$ is reducible with the decomposition into mutually orthogonal irreducible root systems

$$R(H) = \bigcup_j R_j(H), \tag{A8}$$

then $S(H)$ and $\varrho(H)^*$ split correspondingly:

$$\begin{aligned} S(H) &= \textstyle\bigcup_j S_j(H), & S_j(H) &= S(H) \cap R_j(H), \\ 2\varrho(H)^* &= \textstyle\sum_j 2\varrho_j(H)^*, & 2\varrho_j(H)^* &= \textstyle\sum_{\alpha\in R_j(H)_+} \alpha^*. \end{aligned} \tag{A9}$$

We expand now $2\varrho_j(H)^*$ in terms of the basis $S(H)_j$

$$2\varrho_j(H)^* = \sum_{\alpha\in S_j(H)} n_\alpha \frac{2\alpha}{\langle \alpha_L, \alpha_L\rangle}. \tag{A10}$$

Here, $\langle \alpha_L, \alpha_L\rangle$ is the square of the length of a long root. (We recall that at most two root lengths occur in an irreducible root system.) Since $\alpha^* = 2\alpha/\langle\alpha,\alpha\rangle$ it is clear that the coefficients n_α in (A10) are positive integers. In addition, they are independent of the choice of the scalar product. An explicit formula in therms of the Cartan matrix $N_j(H)$ of $S_j(H)$ follows by writing out the equation $\langle 2\varrho_j(H)^*, \beta\rangle = 2$, for every $\beta \in S_j(H)$. One finds

$$n_\alpha = 2 \sum_{\beta\in S_j(H)} (N_j(H)^{-1})_{\alpha\beta} \frac{\langle \alpha_L, \alpha_L\rangle}{\langle \beta, \beta\rangle}. \tag{A11}$$

In Table A1, A2 we have listed the expansion coefficients n_α for all irreducible (simple) root systems.

Another tool which is used in sect. 5 is

Proposition A1 *Let $\Lambda, \tilde\Lambda$ be two homomorphisms from $LSU(2)$ to LG with $H := \Lambda(2\pi i\sigma_3) = \tilde\Lambda(2\pi i\sigma_3) \in K(S)$, then these are conjugated by an element in T, i.e. there exists an element $Z \in LT$ such that*

$$\tilde\Lambda = \mathrm{Ad}(\exp Z) \circ \Lambda. \tag{A12}$$

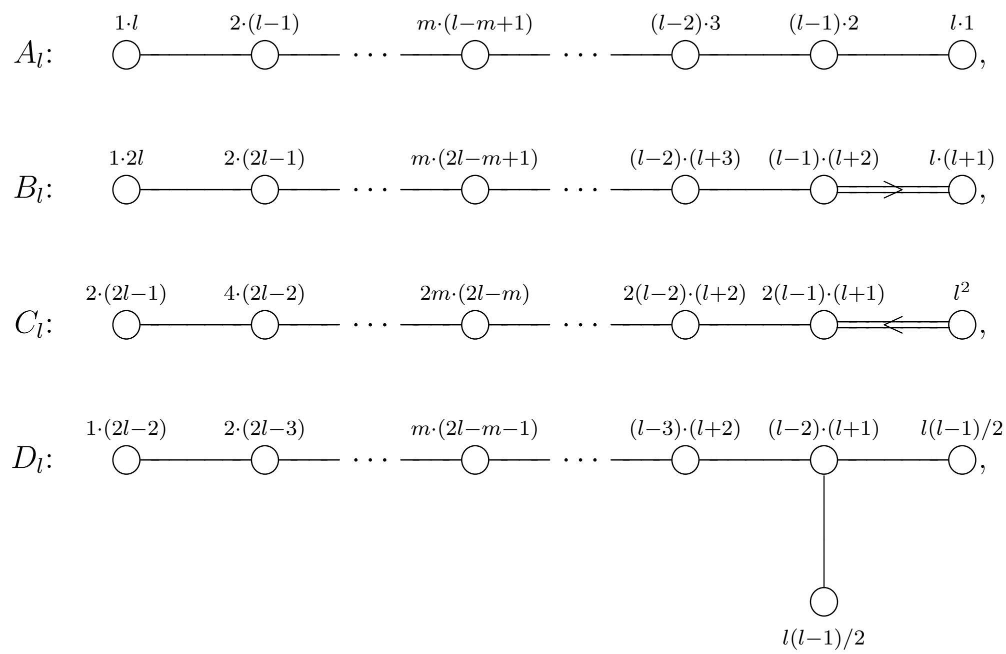

Table A1: Expansion coefficients n_α in (A10) of the sum of positive inverse roots for the classical root systems.

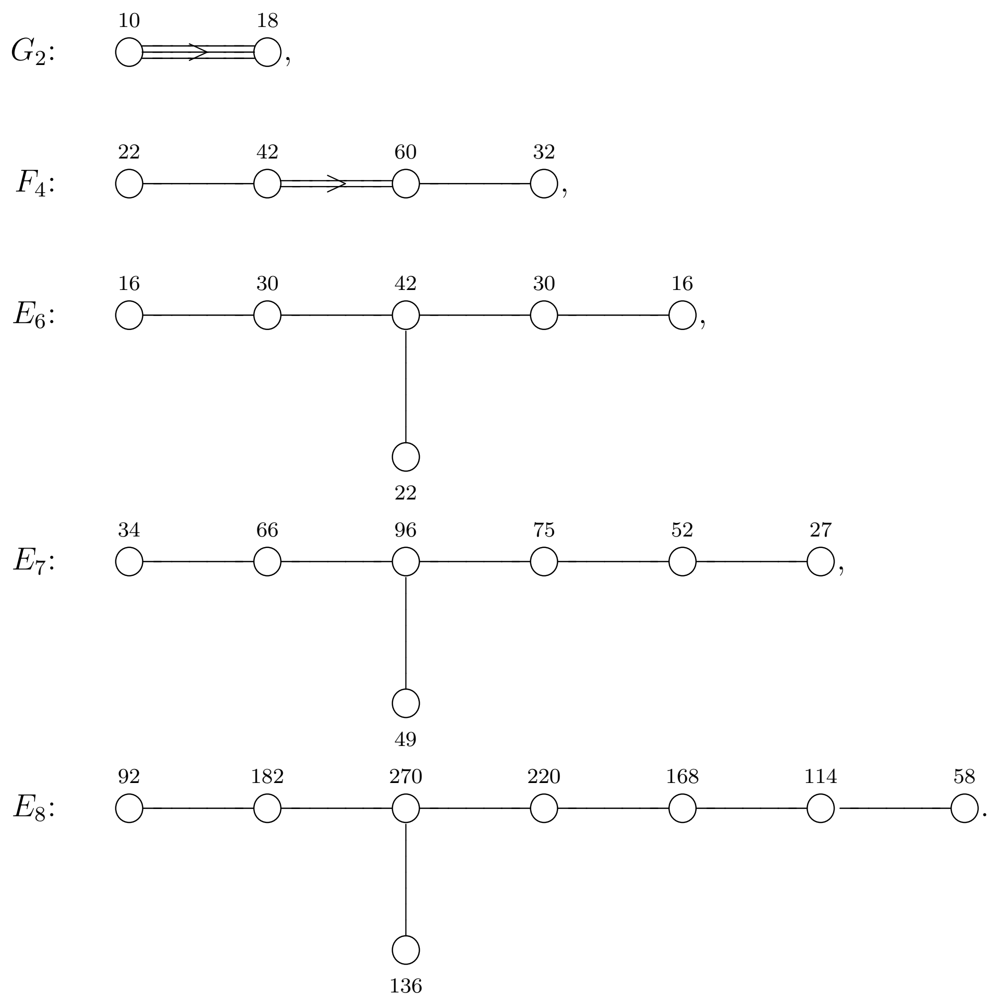

Table A2: Expansion coefficients n_α in (A10) of the sum of positive inverse roots for the exceptional root systems.

PROOF We use the following standard basis of $LSU(2)_{\mathcal{C}}$:

$$2\pi i\sigma_3 = 2\pi i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{A13}$$

Since Λ is a homomorphism we have with the notation $\Lambda_\pm := \Lambda(E_\pm)$

$$\frac{1}{2\pi i}[H, \Lambda_\pm] = \pm 2\,\Lambda_\pm, \qquad [\Lambda_+, \Lambda_-] = -\frac{1}{2\pi i} H. \tag{A14}$$

From the first two equations and the assumption about H we conclude as in sect. 3 that Λ_+ is in the direct sum of root spaces L_α for $\alpha \in S(H)$. (Note, that $S(H)$ is not empty since $\Lambda_\pm = 0$ would imply $H = 0 \notin K(S)$.) Hence we can expand Λ_+ relative to a basis $\{e_\alpha\}_{\alpha\in S(H)}$

$$\Lambda_+ = \sum_{\alpha\in S(H)} w^\alpha e_\alpha. \tag{A15}$$

If we insert this and $\Lambda_- = c(\Lambda_+)$ (c = conjugation in $LG_{\mathcal{C}}$) in the third relation of (A14) and use that $S(H)$ is a simple system of roots (Theorem A1), then we obtain with the well-known orthogonality and commutation relations

$$\sum_{\alpha\in S(H)} |w^\alpha|^2 \langle e_\alpha, c(e_\alpha)\rangle\, 2\pi i\,\alpha = -\frac{1}{2\pi i} H, \tag{A16}$$

The parallel compontent relative to (A5) gives with (A6) and (A10)

$$|w^\alpha|^2 = \frac{1}{(2\pi)^2} \frac{1}{\langle e_\alpha, c(e_\alpha)\rangle} \frac{2n_\alpha}{\langle \alpha_L, \alpha_L\rangle} \; > 0 \tag{A17}$$

and $H_\perp$ must vanish. (A17) shows that the amplitudes w^α are unique up to a phase and that none of them vanishes. In addition we have with (A6) also

$$H = 2\varrho(H)^* \in I, \qquad H = \Lambda(2\pi i\sigma_3). \tag{A18}$$

Now we can construct the element Z in (A12). It is convenient to introduce the basis $\{h_\alpha\}_{\alpha\in S(H)}$ which is dual to the base $S(H)$. We claim that

$$Z := \sum_{\alpha\in S(H)} Z^\alpha h_\alpha + Z_\perp, \tag{A19}$$

with Z chosen such that

$$\exp(2\pi i Z^\alpha) = \frac{\tilde{w}^\alpha}{w^\alpha} \tag{A20}$$

and $Z_\perp \in \langle S(H)\rangle^\perp$, satisfies (A12). Indeed, (A12) is obviously correct for the base element $2\pi i\sigma_3$. Furthermore,

$$\begin{aligned}
\mathrm{Ad}(\exp Z)\Lambda_+ &= \sum_{\alpha\in S(H)} w^\alpha \mathrm{Ad}(\exp Z)e_\alpha \\
&= \sum_{\alpha\in S(H)} w^\alpha \exp(2\pi i\alpha(Z))e_\alpha \\
&= \sum_{\alpha\in S(H)} w^\alpha \exp(2\pi i Z^\alpha)e_\alpha \\
&= \tilde{\Lambda}_+.
\end{aligned}$$

At this point it is simple to prove the following lemma, which is used at the end of sect. 4.

Lemma A1 *Let $\Lambda\colon LSU(2) \to LG$ be a homomorphism and suppose that $H := \Lambda(2\pi i\sigma_3) \in K(S)$. If an element $A \in LT$ satisfies $[A, \Lambda_+] = 0$ then $A \in \langle S(H)\rangle^\perp$.*

Proof If we insert (A15) in $[A, \Lambda_+] = 0$ we obtain

$$0 = \sum_{\alpha\in S(H)} w^\alpha [A, e_\alpha] = \sum_{\alpha\in S(H)} w^\alpha \alpha(A) e_\alpha. \tag{A21}$$

Since all the $w^\alpha, \alpha \in S(H)$, are different from zero, this implies $\alpha(A) = 0$ for all $\alpha \in S(H)$, i.e. $A \in \bigcap_{\alpha\in S(H)} \ker\alpha = \langle S(H)\rangle^\perp$.

B The classifying homomorphisms for EYM solitons

At the end of sect. 2 we noticed that the classifying homomorphism $\lambda\colon U(1) \to G$ of spherically symmetric principal G-bundles, satisfying $L\lambda(2\pi i) \in I \cap$

$\overline{K(S)}$, is further restricted for regular EYM solutions: λ must have an extension $\tilde{\lambda}: K \to G$, with $K = SO(3)$ or $SU(2)$.

There is a systematic procedure to determine all lattice ponts in $I \cap \overline{K(S)}$ for wich this is possible. A sketch of this is given below, which works for any gauge group G if we restrict ourself to the generic case of points in the open fundamental Weyl chamber $K(S)$.

It is useful to introduce the following terminology: A vector $H \in LT$ is an A_1-*vector* if there exist two elements $\Lambda_{\pm} \in LG_{\mathcal{C}}$ such that H, $\Lambda_{\pm}$ satisfy the A_1-commutation relations (A14). (For instance, any inverse root of a semisimple Lie algebra is an A_1-vector.) From what has been said we have to determine all A_1-vectors in $K(S)$. Below we show that the enumeration of this set can be reduced to quite a different problem which has been solved completely long ago by Dynkin [17]. In his classification of so-called regular subalgebras of semisimple Lie algebras, Dynkin had to describe certain subsets of the root system which he called Π-systems. These are defined as follows:

Definition B1 *A subset Σ of a root system R is a* Π-system *if:*

(i) $\alpha, \beta \in \Sigma$ implies that $\alpha - \beta$ is not a root;

(ii) the set Σ is linearly independent.

It is important to note that the proof of Theorem A1 implies that any Π-system Σ is a basis of a root system $R(\Sigma) \subset R$.

Dynkin has given an explicit description of all Π-systems of the root system of a semisimple Lie algebra which we do not want to repeat here. We shall show, however, that the knowledge of the Π-systems also enables one to determine all A_1-vectors in $K(S)$. In order to avoid unimportant complications we assume that the gauge group G is semisimple.

As a useful tool we consider the map κ, which associates to any Π-system Σ the following element in the $\mathcal{Z}$-subspace $\mathrm{Span}_{\mathcal{Z}}(R^*)$, spanned by the set R^* of inverse roots:

$$\kappa(\Sigma) := \sum_{\alpha \in R(\Sigma)_+} \alpha^*. \tag{B1}$$

This map respects the natural actions of the Weyl group W,

$$\kappa(w \cdot \Sigma) = w \cdot \kappa(\Sigma) \qquad \text{for all } w \in W, \tag{B2}$$

and induces thus a map between the corresponding orbit spaces. Now we use the fact that the orbit of an arbitrary point in the $\mathcal{R}$-span $\langle R \rangle$ meets the closed fundamental Weyl chamber $\overline{K(S)}$ in exactly one point. Taken together, the following map ψ from the set of W-orbits of Π-systems into $\overline{K(S)} \cap \mathrm{Span}_{\mathcal{Z}}(R^*)$ is thus well defined:

$$\psi(W \cdot \Sigma) := \text{point of the orbit of } \kappa(\Sigma) \text{ in } \overline{K(S)}. \tag{B3}$$

Now we can formulate the announced correspondence between Π-systems and A_1-vectors in $K(S)$:

Theorem B1 *With the notation introduced obove we have*

$$\mathrm{Im}(\psi) \cap K(S) = \{A_1\text{-}vectors\ in\ K(S)\}. \tag{B4}$$

Note that this theorem implies also that the set on the right hand side of (B4) is determined entirely by the root system R.

Proof We show first that $\kappa(\Sigma)$ is an A_1-vector of LT for any Π-system Σ. From what has been done in Appendix A it is easy to guess the elements $\Lambda_\pm$ which lead to the correct commutation relations (A14). If we express $H := \kappa(\Sigma)$ in the form

$$\kappa(\Sigma) = \sum_{\alpha \in \Sigma} n(\Sigma)_\alpha \frac{2\alpha}{\langle \alpha, \alpha \rangle},$$

then $\Lambda_\pm$ can be chosen as (compare with (A15) and (A17))

$$\Lambda_+ = \sum_{\alpha \in S(H)} w^\alpha e_\alpha, \qquad \Lambda_- = c(\Lambda_+)$$

with the normalization

$$|w^\alpha|^2 = \frac{1}{(2\pi)^2} \frac{1}{\langle e_\alpha, c(e_\alpha) \rangle} \frac{2n(\Sigma)_\alpha}{\langle \alpha, \alpha \rangle}.$$

The orthogonality and commutation relations imply indeed that (A14) is fulfilled.

It remains to show that for each A_1-vector H in $K(S)$ there exists a Π-system Σ with $\kappa(\Sigma) = H$. Now, we know from the proof of Theorem

A1 that $S(H)$, defined in (A1), is a Π-system, and eq. (A18) tells us that $\kappa(S(H)) = 2\varrho(H)^* = H$. This completes the proof of Theorem B1.

One can actually show quite easily that $\Sigma = S(H)$ in the last step is the only Π-system whose image is H.

At this point we could show that Theorem B1, together with Dynkin's description of the Π-systems of semisimple Lie algebras, provides an efficient method to determine all A_1-vectors in $K(S)$. The details of this will be described elsewhere. As an example, the method leads quickly to the following result for the Lie algebra A_n corresponding to $SU(n+1)$: For n *odd* there is only one A_1-vector in the open fundamental Weyl chamber, namely the sum of positive inverse roots $2\varrho^* = \sum_{\alpha \in R_+} \alpha^*$. For n *even* there are more possibilities, corresponding to the following types of Π-systems: A_n, A_{n-1}, $A_{n-2} + A_1$, $A_{n-3} + A_2, \ldots, A_{n/2} + A_{n/2-1}$.

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