

A GENUS ZERO LEFSCHETZ FIBRATION ON THE AKBULUT CORK

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ABSTRACT. We first construct a genus zero positive allowable Lefschetz fibration over the disk (a genus zero PALF for short) on the Akbulut cork and describe the monodromy as a positive factorization in the mapping class group of a surface of genus zero with five boundary components. We then construct genus zero PALFs on infinitely many exotic pairs of compact Stein surfaces such that one is a cork twist of the other along an Akbulut cork. The difference of smooth structures on each of exotic pairs of compact Stein surface is interpreted as the difference of the corresponding positive factorizations in the mapping class group of a common surface of genus zero.

1. INTRODUCTION

Gompf [Go] proved that compact Stein surfaces can be characterized in terms of handle decompositions, or more precisely, Kirby diagrams. Akbulut and Yasui [AY1] introduced corks and plugs, which are compact Stein surfaces themselves, and constructed various exotic smooth structures on Stein surfaces by using cork twists and plug twists together with Gompf's characterization and Seiberg-Witten invariants. On the other hand, Loi and Piergallini [LP] proved that every compact Stein surface admits a positive allowable Lefschetz fibration over D^2 (a PALF for short), which enables us to investigate compact Stein surfaces in terms of positive factorizations in mapping class groups (see also Akbulut and Ozbagci [AO], Akbulut and Arıkan [AA]).

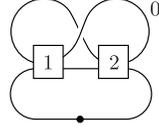
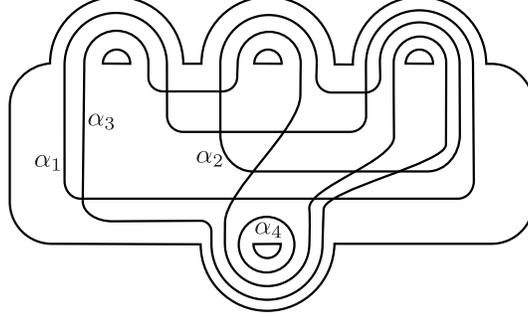
In this paper we first construct a genus zero PALF on the Akbulut cork and describe the monodromy as a positive factorization in the mapping class group of a fiber. The Akbulut cork is the pair (W_1, f_1) of the manifold W_1 shown in Figure 1 and an involution f_1 on W_1 (see Definition 2.11). The manifold W_1 is often called the Mazur manifold.

Theorem 1.1. *The manifold W_1 admits a genus zero PALF. The monodromy of the PALF is described by the factorization $t_{\alpha_4}t_{\alpha_3}t_{\alpha_2}t_{\alpha_1}$, where t_α is a right-handed Dehn twist along a simple closed curve α on a fiber and $\alpha_4, \dots, \alpha_1$ are simple closed curves shown in Figure 2.*

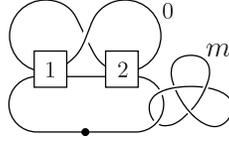
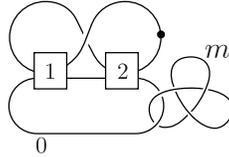
Note that the genus of a PALF on the manifold W_1 which is obtained by applying any known method (cf. [AO] and [AA]) is much larger than zero.

Akbulut and Yasui [AY2] proved that the compact Stein surfaces $C_1(m, 1, 3, 0)$ and $C_2(m, 1, 3, 0)$ ($m \leq -5$) shown in Figure 3 and Figure 4 are homeomorphic but not diffeomorphic to each other. It is easily seen that $C_2(m, 1, 3, 0)$ is a cork twist of $C_1(m, 1, 3, 0)$ along an obvious Akbulut cork.

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FIGURE 1. Kirby diagram for W_1 .FIGURE 2. Vanishing cycles of a genus zero PALF on W_1 .

We next construct PALFs with the same fiber on $C_1(m, 1, 3, 0)$ and $C_2(m, 1, 3, 0)$ for each integer m less than -4 . The common fiber is a surface of genus zero with $-m + 5$ boundary components.

FIGURE 3. Kirby diagram for $C_1(m, 1, 3, 0)$.FIGURE 4. Kirby diagram for $C_2(m, 1, 3, 0)$.

Theorem 1.2. *The manifolds $C_1(m, 1, 3, 0)$ and $C_2(m, 1, 3, 0)$ ($m \leq -5$) shown in Figure 3 and Figure 4 admit genus zero PALFs. The monodromy of the PALF on $C_1(m, 1, 3, 0)$ is described by the positive factorization*

$$t_{\delta_{-m+5}} \cdots t_{\delta_{11}} t_{\delta_{10}} t_{\delta_9} t_{\delta_8} t_{\delta_7} t_{\beta_6} t_{\beta_5} t_{\beta_4} t_{\beta_3} t_{\beta_2} t_{\beta_1}$$

while that for $C_2(m, 3, 1, 0)$ is described by the positive factorization

$$t_{\delta_{-m+5}} \cdots t_{\delta_{11}} t_{\delta_{10}} t_{\delta_9} t_{\delta_8} t_{\delta_7} t_{\gamma_6} t_{\gamma_5} t_{\gamma_4} t_{\gamma_3} t_{\gamma_2} t_{\gamma_1}$$

, where β_i, γ_j are simple closed curves shown in Figure 5 and Figure 6.

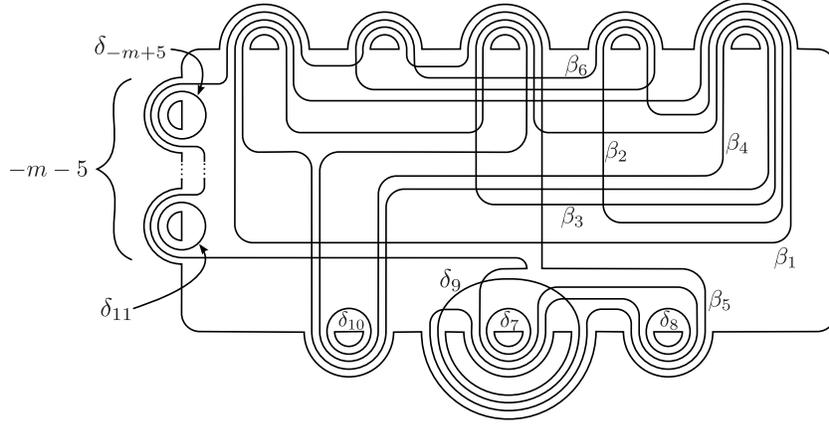


FIGURE 5. Vanishing cycles of a genus zero PALF on $C_1(m, 1, 3, 0)$.

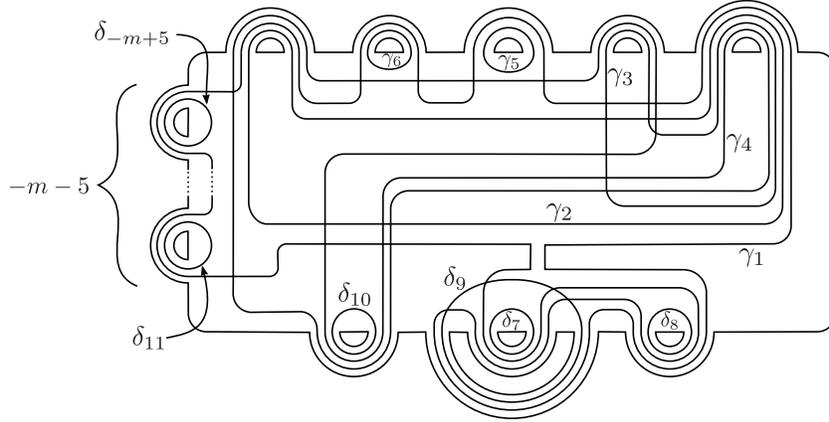


FIGURE 6. Vanishing cycles of a genus two PALF on $C_2(m, 1, 3, 0)$.

The difference of smooth structures on $C_1(m, 1, 3, 0)$ and $C_2(m, 1, 3, 0)$ (or the effect of cork twisting the former to obtain the latter) is reflected in the corresponding positive factorizations as the difference between partial factorizations $t_{\beta_6} t_{\beta_5} t_{\beta_4} t_{\beta_3} t_{\beta_2} t_{\beta_1}$ and $t_{\gamma_6} t_{\gamma_5} t_{\gamma_4} t_{\gamma_3} t_{\gamma_2} t_{\gamma_1}$.

In Section 2 we briefly review definitions of Mapping class groups, PALF, Stein surfaces, and corks, and recall several known results. We prove our main theorems in Section 3.

2. PRELIMINARIES

2.1. Mapping class groups. In this subsection we review a precise definition of mapping class groups of surfaces with boundary and that of Dehn twists along simple closed curves on surfaces.

Definition 2.1. Let F be a compact oriented connected surface with boundary. Let $\text{Diff}^+(F, \partial F)$ be the group of all orientation-preserving self-diffeomorphisms of F fixing the boundary ∂F point-wise. Let $\text{Diff}_0^+(F, \partial F)$ be the subgroup of $\text{Diff}^+(F, \partial F)$ consisting of self-diffeomorphisms isotopic to the identity. The quotient group $\text{Diff}^+(F, \partial F)/\text{Diff}_0^+(F, \partial F)$ is called the mapping class group of F and it is denoted by $\text{Map}(F, \partial F)$.

Definition 2.2. A *positive (or right-handed) Dehn twist* along a simple closed curve α , $t_\alpha : F \rightarrow F$ is a diffeomorphism obtained by cutting F along α , twisting 360° to the right and regluing.

2.2. PALF.

Definition 2.3. Let M^4 and B^2 be compact oriented smooth manifolds of dimensions 4 and 2. Let $f : M \rightarrow B$ be a smooth map. f is called a *positive Lefschetz fibration* over B if it satisfies the following conditions (1) and (2):

- (1) There are finitely many critical values b_1, \dots, b_m of f in the interior of B and there is a unique critical point p_i on each fiber $f^{-1}(b_i)$, and
- (2) The map f is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some local complex coordinates around p_i and b_i compatible with the orientations of M and B .

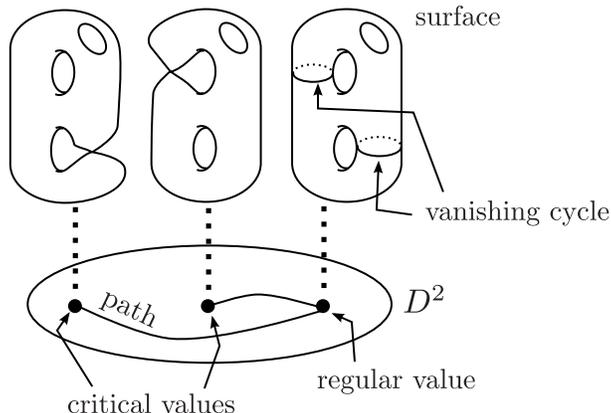


FIGURE 7. PALF.

Definition 2.4. A positive Lefschetz fibration is called *allowable* if its all vanishing cycles are homologically non-trivial on the fiber. A positive allowable Lefschetz fibration over D^2 with bounded fibers is called a *PALF* for short.

The following Lemma is useful to prove Theorem 1.1.

Lemma 2.5 (cf. Akbulut-Ozbagci [AO, Remark 1]). *Suppose that a 4-manifold X admits a PALF. If a 4-manifold Y is obtained from X by attaching a Lefschetz 2-handle, then Y also admits a PALF.*

The Lefschetz 2-handle is defined as follows.

Definition 2.6. Suppose that X admits a PALF. A *Lefschetz 2-handle* is a 2-handle attached along a homologically non-trivial simple closed curve in the boundary of X with framing -1 relative to the product framing induced by the fiber structure.

2.3. Stein surfaces. In this section, we recall a definition of Stein surfaces. The question of which smooth 4-manifolds admit Stein structures can be completely reduced to a problem in handlebody theory.

Definition 2.7. A complex manifold is called a *Stein manifold* if it admits a proper biholomorphic embedding to \mathbb{C}^n .

Definition 2.8. Let W be a compact manifold with boundary. The manifold W is called a *Stein domain* if it satisfies following condition: There is a Stein manifold X and a plurisubharmonic function $\varphi : X \rightarrow [0, \infty)$ such that $W = \varphi^{-1}([0, a])$ for a regular value a of φ .

Definition 2.9. A Stein manifold or a Stein domain is called a *Stein surface* if its complex dimension is 2.

2.4. Corks. Corks are Stein surfaces and they are useful for constructing exotic manifolds.

Definition 2.10. Let C be a Stein domain. Let $\tau : \partial C \rightarrow \partial C$ be an involution on the boundary ∂C of C .

- (1) (C, τ) is called a *cork* if τ extends to a self-homeomorphism of C , but does not extend to any self-diffeomorphism of C .
- (2) Suppose that C is embedded in a smooth 4-manifold X . The manifold obtained from X by removing C and regluing it via τ is called a *cork twist* of X along (C, τ) .
- (3) The pair (C, τ) is called a *cork of X* if the cork twist of X along (C, τ) is homeomorphic but not diffeomorphic to X .

In this paper, we investigate Akbulut cork (W_1, f_1) ([Ak]).

Definition 2.11. Let W_1 be a smooth 4-manifold given by Figure 1. Let $f_1 : \partial W_1 \rightarrow \partial W_1$ be the obvious involution obtained from first surgering $S^1 \times D^3$ to $D^2 \times S^2$ in the interiors of W_1 , then surgering the other imbedded $D^2 \times S^2$ back to $S^1 \times D^2$.

Theorem 2.12 (Akbulut [Ak]). *The pair (W_1, f_1) is a cork.*

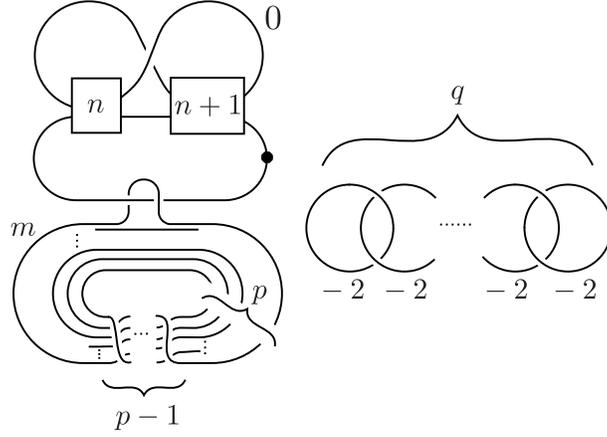
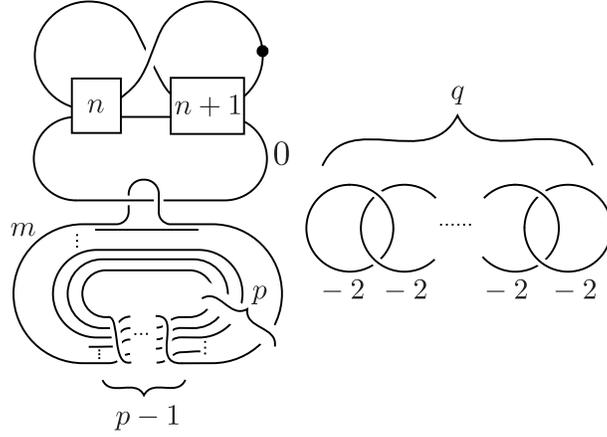
Akbulut and Yasui constructed infinitely many exotic pairs.

Theorem 2.13 (Akbulut-Yasui [AY2, Theorem 3.3(2)]). *$C_1(m, n, p, 0)$ and $C_2(m, n, p, 0)$ are homeomorphic but not diffeomorphic to each other, for $1 \leq n \leq 3, p \geq 3$ and $m \leq p^2 - 3p + 1$.*

3. PROOFS OF THEOREMS 1.1 AND 1.2.

Proof of Theorem 1.1. Let F be the compact oriented surface of genus zero with 5 boundary components and $\alpha_1, \dots, \alpha_4$ be the curves on F shown in Figure 2. We denote the right-handed Dehn twists along $\alpha_1, \dots, \alpha_4$ by $t_{\alpha_1}, \dots, t_{\alpha_4}$, respectively. Let $f : X \rightarrow D^2$ be a Lefschetz fibration over D^2 with monodromy representation $(t_{\alpha_4}, \dots, t_{\alpha_1})$. Since each curve α_i is homologically non-trivial on F , we see that f is a PALF with fiber F .

We now show that X is diffeomorphic to W_1 . The obvious Kirby diagram for W_1 is given by Figure 1. We draw it as in Figure 10(a), and create the cancelling

FIGURE 8. Kirby diagram for $C_1(m, n, p, q)$.FIGURE 9. Kirby diagram for $C_2(m, n, p, q)$.

pair to get Figure 10(b). We slide the 0-framed 2-handle over a 1-framed 2-handle to get Figure 10(c). We get Figure 10(d) by sliding the -3 -framed 2-handle over a 1-framed 2-handle. By 1-handle slide, we get Figure 10(e). We slide the -4 -framed 2-handle over a 1-framed 2-handle, and erase a cancelling 1-handle/2-handle pairs to get Figure 10(f). We create the cancelling pairs to get Figure 10(g). We get Figure 10(h) by by creating the cancelling pairs.

The Kirby diagram for X corresponding to the monodromy representation $t_{\alpha_4}, \dots, t_{\alpha_1}$ is given by Figure 10(h).

Therefore we conclude that X is diffeomorphic to W_1 , which implies the theorem. \square

Proof of Theorem 1.2. Let $F_{C_i(m,1,3,0)}$ ($i = 1, 2$) be the compact oriented surface of genus zero with $-m + 5$ boundary components and $\beta_1, \dots, \beta_{-m+5}$ and

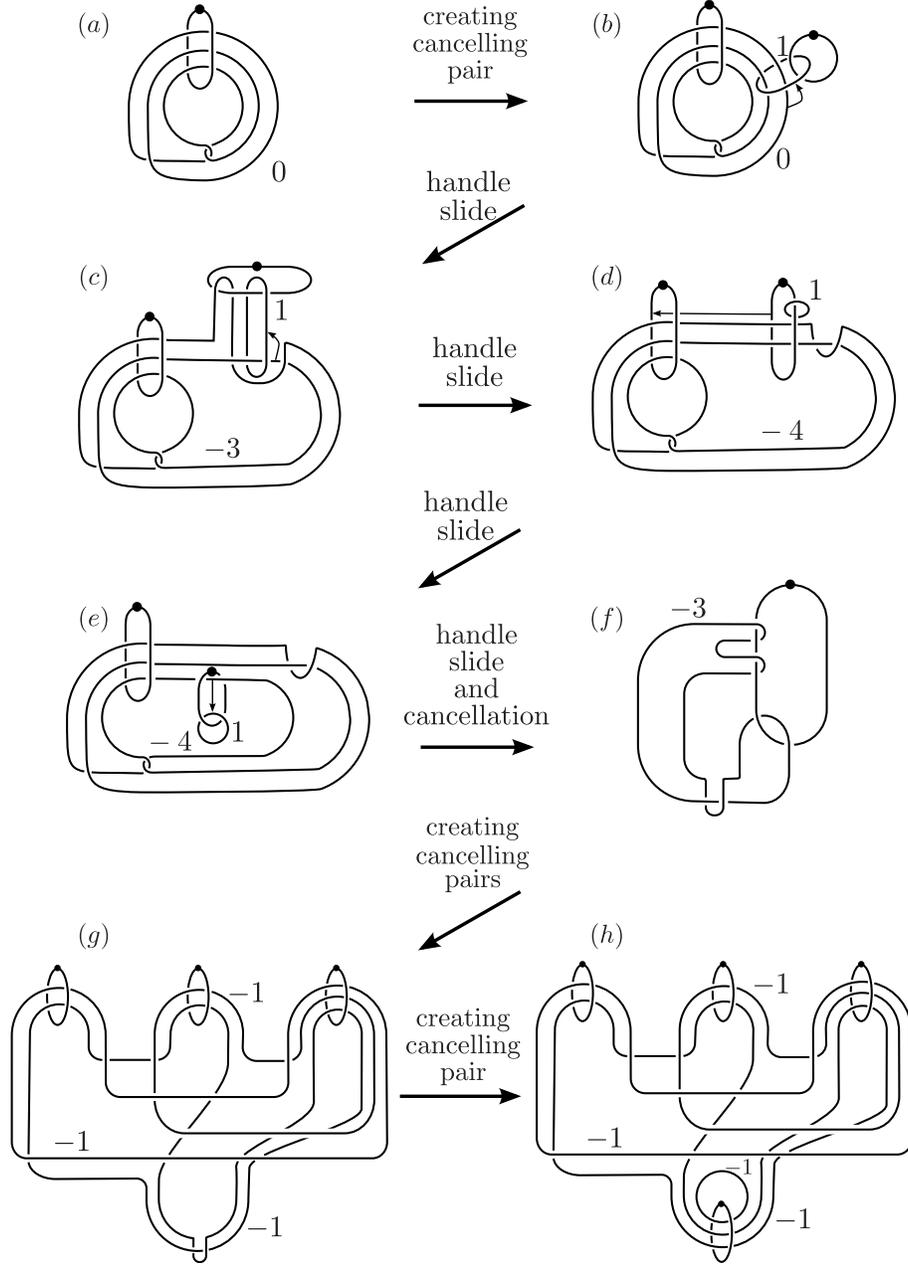


FIGURE 10.

$\gamma_1, \dots, \gamma_{-m+5}$ be the curves on $F_{C_i(m,1,3,0)}$ shown in Figure 5 and Figure 6, respectively. Let $g_1 : X_{C_1(m,1,3,0)} \rightarrow D^2$ (resp. $g_2 : X_{C_2(m,1,3,0)} \rightarrow D^2$) be a Lefschetz fibration over D^2 with monodromy representation $(t_{\beta_{-m+5}}, \dots, t_{\beta_1})$ (resp. $(t_{\gamma_{-m+5}}, \dots, t_{\gamma_1})$). Since each curve β_i (resp. γ_i) is homologically non-trivial on

$F_{C_1(m,1,3,0)}$ (resp. $F_{C_2(m,1,3,0)}$), we see that g_1 (resp. g_2) is a PALF with fiber $F_{C_1(m,1,3,0)}$ (resp. $F_{C_2(m,1,3,0)}$).

The Kirby diagram for $C_1(m, 1, 3, 0)$ is given by Figure 3. We get Figure 11(a) by isotopy. We create a cancelling pair to get Figure 11(b). We slide the 0-framed 2-handle over a 1-framed 2-handle to get Figure 11(c). By sliding the -3 -framed 2-handle over a 1-framed 2-handle, we get Figure 11(d). We get Figure 11(e) by 1-handle slide. We obtain Figure 11(f) by handle slides and cancellation. By handle slide and creating cancelling pair, we get Figure 12(g). We get Figure 12(h) by handle slide and creating cancelling pair. We slide the m -framed 2-handle over -1 -framed 2-handle to get Figure 12(i). In Figure 12(i), creating cancelling pair gives Figure 12(j). We create cancelling pairs to get Figure 13(k).

The Kirby diagram for $X_{C_1(m,1,3,0)}$ corresponding to the monodromy representation $t_{\beta_{-m+5}}, \dots, t_{\beta_1}$ is given by 13(k).

The Kirby diagram for $C_2(m, 1, 3, 0)$ is given by Figure 4. We get Figure 14(a) by isotopy. We obtain Figure 14(b) by handle slides and cancellation (see the proof of Theorem 1.1). By creating cancelling pairs, we get Figure 14(c). We slide the m -framed 2-handle under a 1-handle to get Figure 14(d). In Figure 14(d), handle slides gives Figure 14(e). We create cancelling pairs to get Figure 15(f).

The Kirby diagram for $X_{C_2(m,1,3,0)}$ corresponding to the monodromy representation $t_{\gamma_{-m+5}}, \dots, t_{\gamma_1}$ is given by Figure 15(f).

Therefore we conclude that the manifolds $C_1(m, 1, 3, 0)$ and $C_2(m, 1, 3, 0)$ admit genus zero PALFs. \square

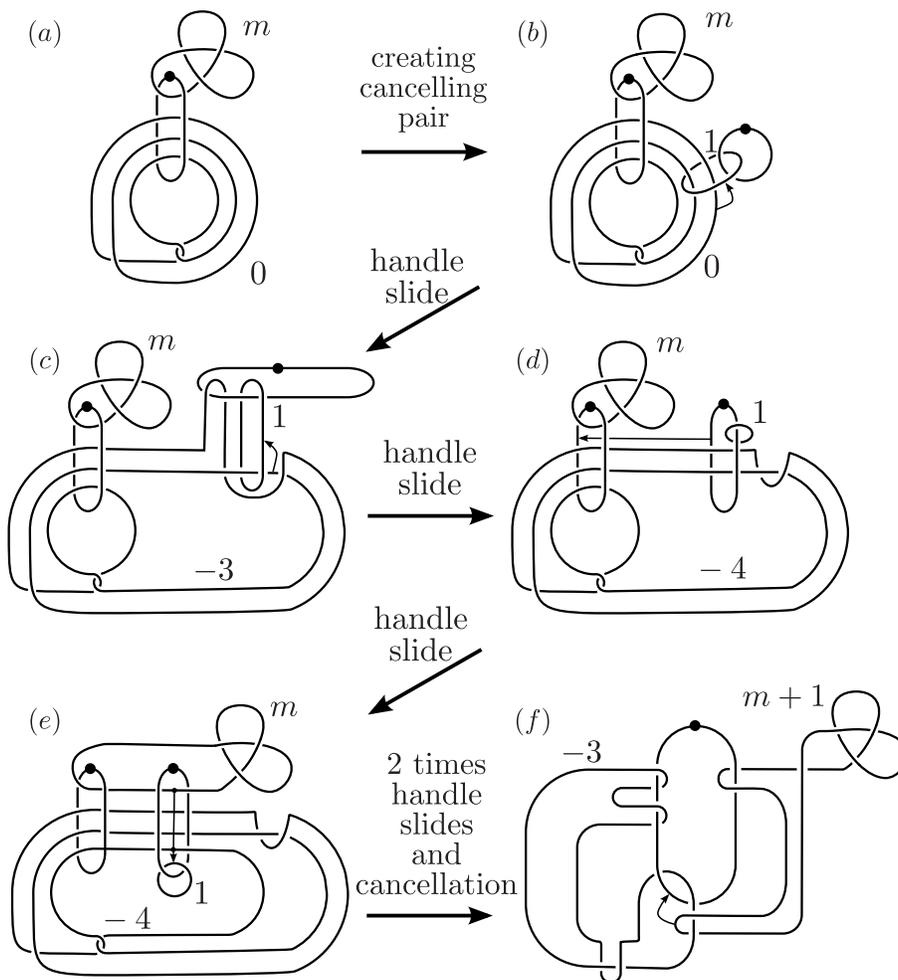


FIGURE 11.

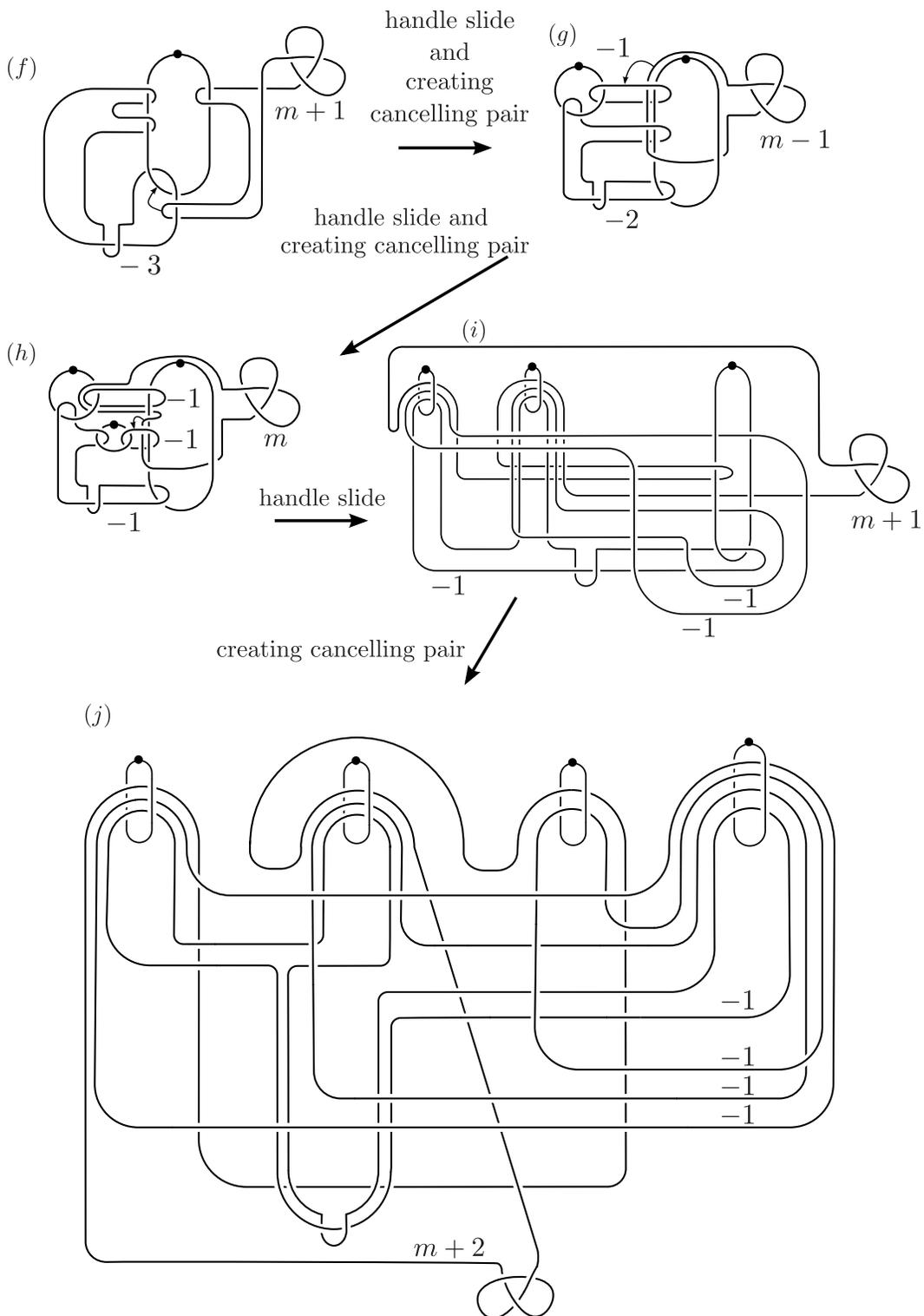


FIGURE 12.

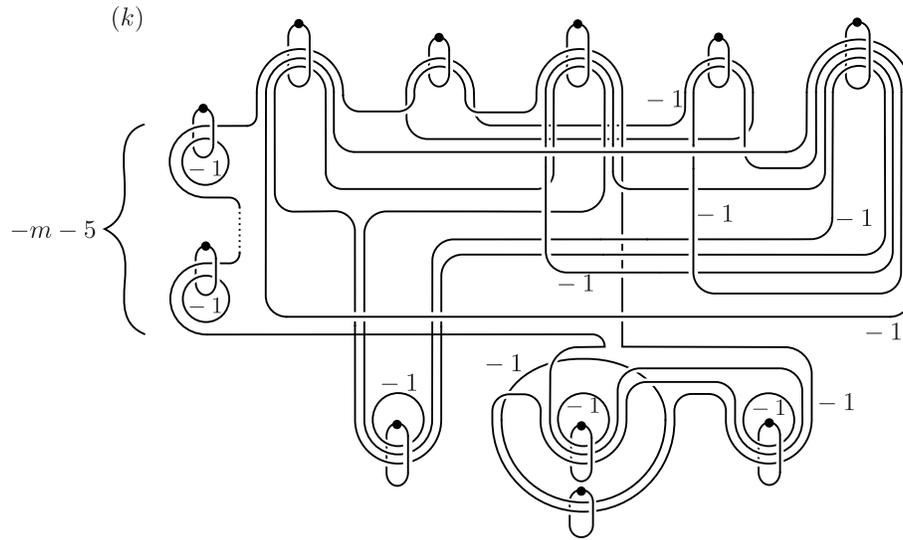


FIGURE 13.

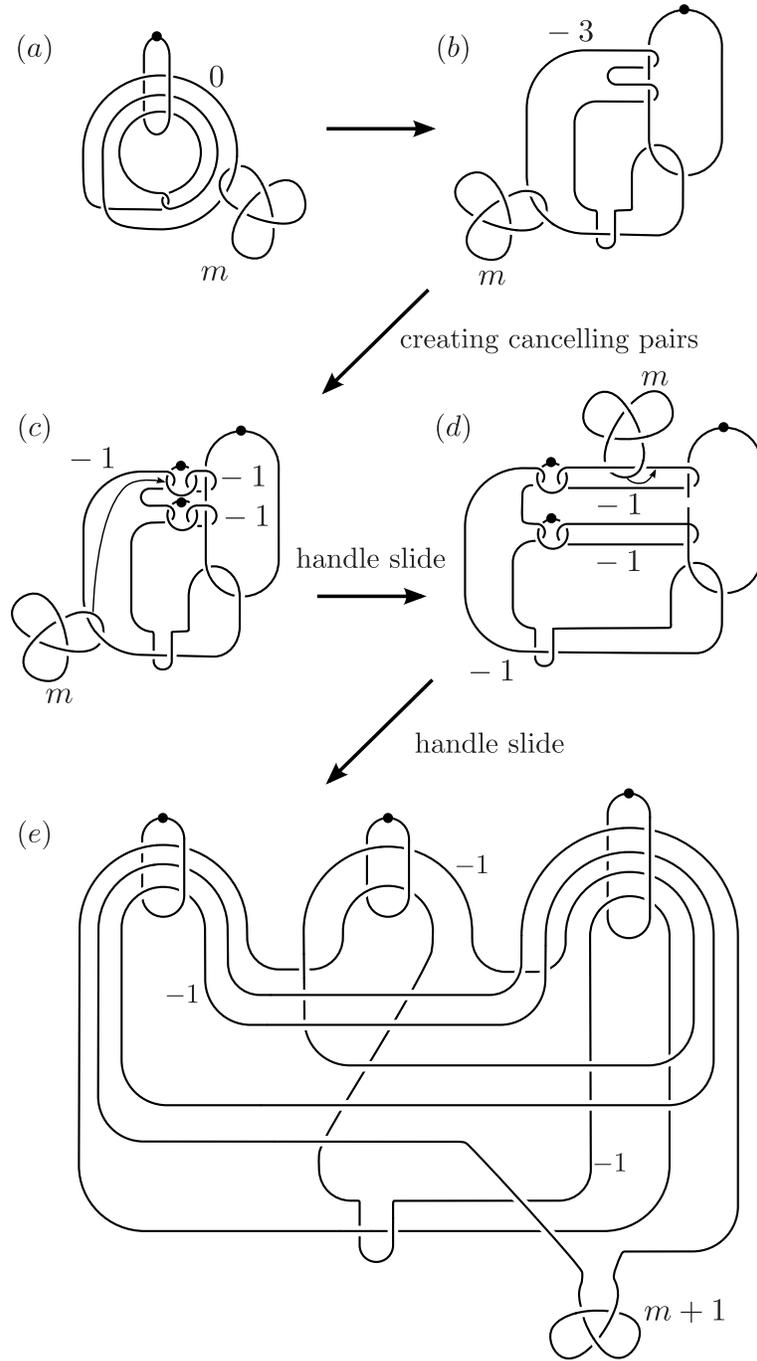


FIGURE 14.

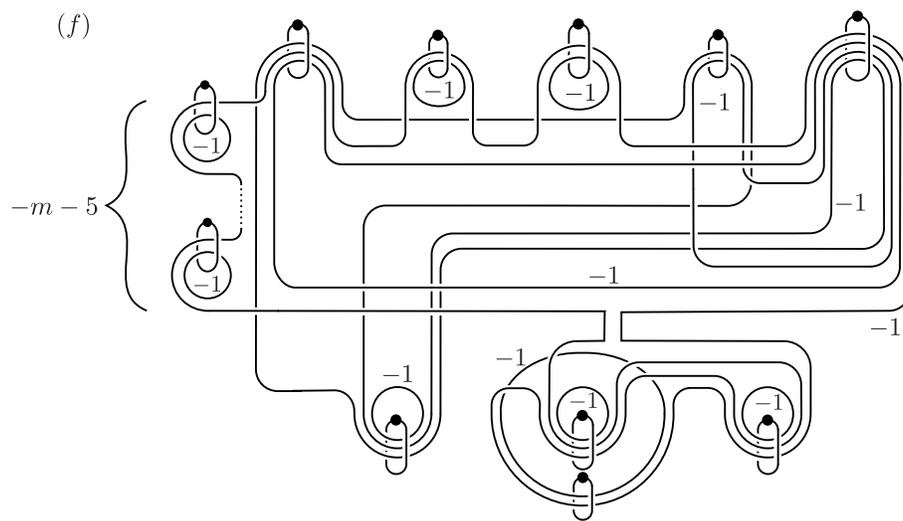


FIGURE 15.

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