IRREDUCIBLE REPRESENTATIONS OF SOLVABLE LIE SUPERALGEBRAS

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Abstract. The description of irreducible finite dimensional representations of finite dimensional solvable Lie superalgebras over complex numbers given by V. Kac is refined. In reality these representations are not just induced from a polarization but twisted, as infinite dimensional representations of solvable Lie algebras. Various cases of irreducibility (general and of type Q) are classified.

Introduction

Hereafter the ground field is $\mathbb{C}$ and all the modules and superalgebras are finite dimensional; $\mathfrak{g}$ is a solvable Lie superalgebra.

A description of irreducible representations of solvable Lie superalgebras given in [K] (Theorem 7) contains an error. In reality to give such a description one has to imitate the description of infinite dimensional solvable Lie algebras [D], i.e., we must consider *twisted* induced representations. In what follows I give a correct description of irreducible representations of solvable Lie superalgebras. I also show where a mistake crept into [K] and give a counterexample to Theorem 7 from [K].

The proof given in what follows was delivered at Leites' *Seminar on Supermanifolds* in 1983 and is preprinted in [L] in a form considerably edited by I. Shchepochkina and D. Leites. My acknowledgements are due to them and also to the Department of Mathematics of Stockholm University that financed publication of [L].

§1. Main result

1.1. Polarizations. Set

$$L = \{\lambda \in \mathfrak{g}^* : \lambda(\mathfrak{g}_{\bar 1}) = 0 \text{ and } \lambda([\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}]) = 0\}.$$

It is convenient to represent the functionals from L as elements of an isomorphic space

$$L_0 = (\mathfrak{g}_{\bar 0}/[\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}])^*.$$

The natural isomorphism of these spaces is given by the formula

$$' : L_0 \longrightarrow L, \ \ \lambda' \mid \mathfrak{g}_{\bar 0} = \lambda, \ \ \lambda' \mid \mathfrak{g}_{\bar 1} = 0, \qquad (')$$

where λ denotes a character and also, by the usual abuse of language, the $(1, 0)$-dimensional representation of the Lie algebra $\mathfrak{g}_{\bar 0}$ determined by the character λ. Every functional $\lambda \in L$ determines a symmetric form f_λ on $\mathfrak{g}_{\bar 1}$ by the formula

$$f_\lambda(\xi_1, \xi_2) = \lambda([\xi_1, \xi_2]).$$

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a *polarization* for $\lambda \in L$ (and is often denoted by $\mathfrak{p}(\lambda)$) if

$$\lambda([\mathfrak{h}, \mathfrak{h}]) = 0, \ \ \mathfrak{h} \supset \mathfrak{g}_{\bar 0} \ \text{ and } \mathfrak{h}_{\bar 1} \ \text{ is a maximal isotropic subspace for } f_\lambda.$$

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Lemma . *For every $\lambda \in L$ there exists a polarization $\mathfrak{h}$.*

If $\mathfrak{h}$ is a polarization for $\lambda \in L$, then, clearly, λ determines a (1, 0)-dimensional representation of $\mathfrak{h}$.

1.2. Twisted representations. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subsuperalgebra that contains $\mathfrak{g}_{\bar 0}$. Define a functional $\theta_{\mathfrak{h}} \in L$ by setting

$$\theta_{\mathfrak{h}(g)} = \begin{cases} \frac{1}{2}\mathrm{str}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{ad}\ g) & \text{for } g \in \mathfrak{g}_{\bar 0} \\ 0 & \text{for } g \in \mathfrak{g}_{\bar 1} \end{cases}$$

Note that $\theta_{\mathfrak{h}}([\mathfrak{h},\mathfrak{h}]) = 0$ by definition of the supertrace. Therefore, $\theta_{\mathfrak{h}}$ is a character of a $(1,0)$-dimensional representation of $\mathfrak{h}$.

Let $\mathfrak{h}$ be a polarization for $\lambda \in L$. Define the twisted (by the character $\theta_{\mathfrak{h}}$) induced and coinduced representations by setting

$$I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda + \theta_{\mathfrak{h}}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} (\lambda + \theta_{\mathfrak{h}});$$
$$CI_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda - \theta_{\mathfrak{h}}) = \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), \lambda - \theta_{\mathfrak{h}}).$$

Lemma . 1) *$I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ is finite dimensional and irreducible.*

2) *$I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ does not depend on the choice of a polarization $\mathfrak{h}$; therefore, notation $I(\lambda)$ ($= I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ for some $\mathfrak{h}$) is well-defined.*

3) *$CI(\lambda) \cong I(\lambda)$.*

1.3. Main Theorem. Let $Z = \{(\lambda, \mathfrak{h}) : \lambda \in L$ and let $\mathfrak{h}$ be a polarization for $\lambda\}$. Define an equivalence relation on Z by setting

$$(\lambda, \mathfrak{h}) \sim (\mu, \mathfrak{t}) \Longleftrightarrow \lambda - \theta_{\mathfrak{h}} = \mu - \theta_{\mathfrak{t}}$$

Clearly, this relation is well-defined.

Recall ([BL]) that the representation of a Lie superalgebra $\mathfrak{g}$ is called irreducible of *G-type* if it has no invariant subspaces; it is called irreducible of *Q-type* if it has no invariant sub*super*spaces.

Theorem . 1) *Every irreducible finite dimensional representation of $\mathfrak{g}$ is isomorphic up to application of the change of parity functor Π to a representation of the form $I(\lambda)$ for some λ.*

2) *The map $\lambda \longrightarrow I(\lambda)$ is (up to Π) a 1-1 correspondence between elements of L and the irreducible finite dimensional representations of $\mathfrak{g}$.*

3) *Let $(\lambda, \mathfrak{h}), (\mu, \mathfrak{t}) \in Z$. Then $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) \cong \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu)$ if and only if $(\lambda, \mathfrak{h}) \sim (\mu, \mathfrak{t})$.*

4) *If $\mathrm{rk} f_\lambda$ is even, then $I(\lambda)$ is a G-type representation; if $\mathrm{rk} f_\lambda$ is odd, then $I(\lambda)$ is a Q-type representation.*

§2. Prerequisites for the proof of Main theorem

Let $\mathfrak{k} \subset \mathfrak{g}$ be a subsuperalgebra, codim $\mathfrak{k} = \varepsilon$, and μ the character of the representation of $\mathfrak{g}_{\bar 0}$ in $\mathfrak{g}/\mathfrak{k}$. For the definition of the isomorphism $'$ see sec. 1.1.

2.1. Lemma . *μ' is a character of $\mathfrak{g}$.*

Proof. Let $\xi \in \mathfrak{g}$ and $\xi \notin \mathfrak{k}$. Since in $\mathfrak{g}/\mathfrak{k}$ there is a $\mathfrak{k}$-action that coincides with $\mu'|\mathfrak{k}$, it suffices to prove that $\mu'([\mathfrak{k}, \xi]) = \mu'([\xi, \xi]) = 0$. By the Jacobi identity $[[\xi, \xi], \xi] = 0$ which proves that $\mu'([\xi, \xi]) = 0$. Since $p(\mu) = \bar 0$, we have $\mu([\xi, \xi]) = 0$. Let $\eta \in \mathfrak{k}_{\bar 1}$. Then $[[\eta, \xi], \xi] = \frac{1}{2}[\eta, [\xi, \xi]] \in \mathfrak{k}$ and, therefore, $\mu([\mathfrak{k}_{\bar 1}, \xi]) = 0$. ∎

2.2. Corollary . *Let* $\mathfrak{g}_{\bar 0} \subset \mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ *so that* $\dim \mathfrak{k}/\mathfrak{h} = \varepsilon$. *Let* λ *be the weight of a vector in the* $\mathfrak{g}_{\bar 0}$*-module* $\mathfrak{g}/\mathfrak{h}$. *Then* λ' *is a character of* $\mathfrak{k}$.

2.3. Corollary . ([K]). *A Lie superalgebra* $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ *is solvable if and only if so is* $\mathfrak{g}_{\bar 0}$.

2.4. Corollary . *If* $\mathfrak{h}$ *is a polarization for* λ, *then* $\mathfrak{h}$ *is also polarization for* $\lambda + \alpha\theta_{\mathfrak{h}}$ *for any* $\alpha \in \mathbb{C}^*$.

Let us recall three well-known lemmas.

2.5. Lemma . (see [K]). *Let* $\mathfrak{g} = \mathfrak{k} \oplus Span(g)$, *where* $\mathfrak{k}$ *is a Lie subalgebra and* $p(g) = \bar 1$. *If* (V, ρ) *is an irreducible representation of* $\mathfrak{k}$ *in a superspace* V, *then* $W = \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(V)$ *is reducible if and only if* V *admits a* $\mathfrak{g}$*-module structure that extends* ρ.

2.6. Lemma . (see [K]). *Let* $\mathfrak{k} \subset \mathfrak{g}$ *be a Lie subsuperalgebra,* $\dim \mathfrak{g}/\mathfrak{k} = (0, 1)$. *If* W *is an irreducible* $\mathfrak{g}$*-module and* $V \subset W$ *is an irreducible proper* $\mathfrak{k}$*-submodule, then* $W = \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(V)$.

2.7. Lemma . *Let* W *be a finite dimensional* $\mathfrak{g}$*-module,* f *a symmetric* $\mathfrak{g}$*-invariant form on* W *and* V *a* $\mathfrak{g}$*-invariant isotropic subspace. Then there exists a maximal* $\mathfrak{g}$*-invariant* f*-isotropic subspace in* W *containing* V.

Proof follows from linear algebra. □

2.8. Corollary . *Lemma* 1.1 *holds.* □

§3. Description of irreducible modules

3.1. Proposition . *Let* $\lambda \in L$, *let* $\mathfrak{h} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{p}$ *be a polarization for* λ, $\mathfrak{n}$ *the kernel of* f_λ *and* $F \subset \mathfrak{p}$ *a subspace such that* $\mathfrak{p} = F \oplus \mathfrak{n}, \xi_0 \in \mathfrak{p}^\perp$ *and* $\xi_0 \notin \mathfrak{p}$ *(if* rk f_λ *is even, then we set* $\xi_0 = 0$*),* v *a generator of* $\mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(\lambda)$. *If* $u \in \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(\lambda)$ *and* $Fu = 0$, *then* $u \in Span(v, \xi_0 v)$.

Proof is carried out by induction on rk f_λ. If rk $f_\lambda = 0$, then $F = 0$ and the statement is obvious.

Let rk $f_\lambda > 0$. Select a subalgebra $\mathfrak{h} \supset \mathfrak{b}$ such that and $\dim \mathfrak{g}_{\bar 1}/\mathfrak{h}_{\bar 1} = 1$. The two cases are possible: $\mathfrak{h}_{\bar 1}^\perp \not\subset \mathfrak{h}_{\bar 1}$ and $\mathfrak{h}_{\bar 1}^\perp \subset \mathfrak{h}_{\bar 1}$.

i) $\mathfrak{h}_{\bar 1}^\perp \not\subset \mathfrak{h}_{\bar 1}$. Then $\mathfrak{g}_{\bar 1} = \mathfrak{h}_{\bar 1} \oplus Span(\xi)$, where $\xi \perp \mathfrak{h}_{\bar 1}$. Hence, $\xi \perp \mathfrak{b}_{\bar 1}$ and $\xi \notin \mathfrak{b}_{\bar 1}$. Therefore, we may assume that $\xi = \xi_0$ and rk f_λ is an odd number. Clearly, $\mathfrak{b}$ is a polarization for the restriction f'_λ onto $\mathfrak{h}_{\bar 1}$ and rk f'_λ is an even number. Further on,

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{h}}(\lambda) \oplus \xi_0 \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda).$$

Let $u = u_0 + \xi_0 u_1 \in \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ and $pu = 0$ for any $p \in F$. Then

$$0 = pu = pu_0 + [p, \xi_0]u_1 - \xi_0 p u_1,$$

therefore, $pu_1 = 0$. By induction, $u_1 \in Span(v)$, where v is the generator of $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$. Since $\xi_0 \perp \mathfrak{b}_{\bar 1}$, it follows that $[p, \xi_0]u_1 = f_\lambda(p, \xi_0)u_1 = 0$. Therefore, $pu_0 = 0$ and $u_0 \in Span(v)$. Hence, $u \in Span(v, \xi_0 v)$.

Let us show now that the weight of $\xi_0 v$ with respect to $\mathfrak{g}_{\bar 0}$ is also equal to λ. Indeed, since $[x, \xi_0] \perp \mathfrak{b}_{\bar 1}$ for any $x \in \mathfrak{g}_{\bar 0}$, it follows that $[x, \xi_0] = \mu(x)\xi_0 + p$ for some $p \in \mathfrak{b}_{\bar 1}$. Furthermore, $[x, [\xi_0, \xi_0] = 2[[x, \xi_0], \xi_0]$; hence,

$$0 = \lambda([x, [\xi_0, \xi_0]) = 2\lambda([\mu(x)\xi_0 + p, \xi_0]) =$$
$$2\mu(x)\lambda([\xi_0, \xi_0]) + 2\lambda([p, \xi_0]) = 2\mu(x).$$

So, $\mu(x) = 0$ and the weight of $\xi_0 v$ is equal to λ.

ii) $\mathfrak{h}_{\bar{1}}^{\perp} \subset \mathfrak{h}_{\bar{1}}$. Then the *restriction* of the form f_λ onto $\mathfrak{h}_{\bar{1}}$ is of rank by 2 less than that of f_λ itself.

Select $\xi \perp \mathfrak{h}_{\bar{1}}$, $\eta \in \mathfrak{h}_{\bar{1}}^{\perp}$ and $F_1 \subset \mathfrak{b}_{\bar{1}}$ so that

$$F = F_1 \oplus Span(\eta);\ \xi \perp F_1;\ f_\lambda(\xi, \eta) \neq 0.$$

Let

$$u = u_0 + \xi u_1, \text{ where } u \in \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda),\ u_0, u_1 \in \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{h}}(\lambda) \text{ and } pu = 0 \text{ for any } p \in F.$$

Then

$$0 = pu = pu_0 + [p, \xi]u_1 + \xi p u_1,$$

hence, $pu_1 = 0$ and by induction $u_1 \in Span(v, \xi_0 v)$. Thanks to i) $[p, \xi]u_1 = f_\lambda(p, \xi)u_1$ and if $p \in F_1$, then $f_\lambda(p, \xi)u_1 = 0$; hence, $pu_0 = 0$ for any $p \in F_1$. By induction we deduce that $u_0 \in Span(v, \xi_0 v)$. Further,

$$0 = \eta u = \eta u_0 + \eta \xi u_1 = [\eta, \xi]u_1 = f_\lambda(\eta, \xi)u_1$$

and since $f_\lambda(\eta, \xi) \neq 0$, then $u_1 = 0$ and $u = u_0 \in Span(v, \xi_0 v)$. □

3.2. Corollary . *If* $\mathfrak{h} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}$ *is a polarization for* λ, *then* $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ *is an irreducible module.*

Proof Irreducibility is equivalent to the absence of singular vectors that do not lie in $Span(v, \xi_0 v)$). □

3.3. Corollary . *heading 1) of Lemma 1.2 holds.*

Proof follows from the definition of $I_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ and sect. 2.4. □

3.4. Corollary . *Let* U *be an irreducible finite dimensional* $\mathfrak{g}$*-module. Then* $U = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ *for some* $\lambda \in L$ *and a polarization* $\mathfrak{h}$.

Proof will be carried out by induction on $\dim \mathfrak{g}_{\bar{1}}$. If $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$, then this is Lie's theorem. Let $\mathfrak{k} \subset \mathfrak{g}$ and $\dim \mathfrak{g}_{\bar{1}}/\mathfrak{k}_{\bar{1}} = 1$.

Let U be irreducible as a $\mathfrak{k}$-module. Then there exist $\lambda \in L$ and a polarization $\mathfrak{h} \subset \mathfrak{k}$ for $\lambda \in L$ such that $U = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{k}}(\lambda)$. If $\mathfrak{h}$ were a polarization for λ in $\mathfrak{g}$, too, then by Corollary 3.2 the representation

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{k}}(\lambda))$$

would have been irreducible contradicting Lemma 2.5.

Let $\hat{\mathfrak{h}} \supset \mathfrak{h}$ be a polarization for λ in $\mathfrak{g}$ and $\xi \in \hat{\mathfrak{h}}$ so that $\xi \notin \mathfrak{h}$. If v is a generator of $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{k}}(\lambda)$ and $p \in \mathfrak{h}_{\bar{1}}$, then

$$p\xi v = [p, \xi]v = f_\lambda(p, \xi)v = 0,$$

$$\xi\xi v = \frac{1}{2}[\xi, \xi]v = \frac{1}{2}f_\lambda(\xi, \xi)v = 0.$$

Therefore, there exists a non-zero $\mathfrak{g}$-module homomorphism $\mathrm{Ind}_{\hat{\mathfrak{h}}}^{\mathfrak{g}}\lambda \longrightarrow \mathrm{Ind}_{\mathfrak{h}}^{\hat{\mathfrak{k}}}(\lambda) = U$ and since both modules are irreducible, this is an "odd isomorphism", i.e., the composition of an isomorpism with the change of parity.

Now let U be reducible as a $\mathfrak{k}$-module. Then by Lemma 2.6 $U = \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}} V$, and, by induction, $V = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{k}}(\lambda)$ for a polarization $\mathfrak{h} \subset \mathfrak{k}$ and $\lambda \in L$. If $\mathfrak{h}$ is not a polarization for λ in $\mathfrak{g}$, then let $\hat{\mathfrak{h}} \supset \mathfrak{h}$ be a polarization. We have a non-zero $\mathfrak{g}$-module homomorphism $U = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) \longrightarrow \mathrm{Ind}_{\hat{\mathfrak{h}}}^{\mathfrak{g}}(\lambda)$ and since both modules are irreducible, this is an isomorphism which is impossible because $\dim \mathrm{Ind}_{\hat{\mathfrak{h}}}^{\mathfrak{g}}(\lambda) < \dim \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$. Therefore, $\mathfrak{h}$ is a polarization for λ in $\mathfrak{g}$ and $U = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$. □

3.5. Corollary . *Heading* 1) *of Theorem holds.*

3.6. A subsuperalgebra subordinate for $l \in L$. Recall, see [K], that if

$$\mathfrak{g}_l = \{g \in \mathfrak{g} \mid l([g, g_1]) = 0 \text{ for all } \quad g_1 \in \mathfrak{g}\},$$

then a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called *subordinate to* λ if $l([\mathfrak{p}, \mathfrak{p}]) = 0$ and $\mathfrak{p} \supset \mathfrak{g}_l$.

Corollary . *Let $l \in L$, $\mathfrak{b}$ a subalgebra subordinate to l. Then $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(l)$ is irreducible if and only if $\mathfrak{b}$ is a polarization for l.*

§4. Classification of modules $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$

4.1. Lemma . *If $(\lambda, \mathfrak{h}) \sim (\mu, \mathfrak{t})$, then $\mathfrak{h}$ is a polarization for μ.*

Proof. By 2.4 $\mathfrak{h}$ is a polarization for $\lambda - \theta_{\mathfrak{h}}$. Since $\lambda - \theta_{\mathfrak{h}} = \mu - \theta_{\mathfrak{t}}$, then $\mathfrak{h}$ is a polarization for $\mu - \theta_{\mathfrak{t}}$ and since $\mathfrak{t}$ is a polarization for $\mu - \theta_{\mathfrak{t}}$, it follows that $\dim \mathfrak{h} = \dim \mathfrak{t}$. Let $\mathfrak{n}$ be the kernel of $f_{\mu-\theta_{\mathfrak{t}}}$. Then $\mathfrak{h} \cap \mathfrak{t} \supset \mathfrak{g}_{\bar{0}} \oplus \mathfrak{n}$ and, therefore, $\mathfrak{g}/\mathfrak{b}$ is a quotient of $\mathfrak{g}_{\bar{1}}/\mathfrak{n}$; hence by sect. 2.2 we have $\theta_{\mathfrak{t}}([\mathfrak{h}, \mathfrak{h}]) = 0$.

Therefore, $\mathfrak{h}$ is completely isotropic with respect to f_μ and since $\dim \mathfrak{h} = \dim \mathfrak{t}$, we see that $\mathfrak{h}$ is a polarization for μ. □

4.2. Proof of heading 3) of Theorem. Let $(\lambda, \mathfrak{h}) \sim (\mu, \mathfrak{t})$. We will carry the proof out by induction on $k = \dim \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{t})$. If $k = 0$ the statement is obvious. Let $k = 1$, then, obviously, $\dim \mathfrak{t}/(\mathfrak{h} \cap \mathfrak{t}) = 1$. Consider the space $\mathfrak{h} + \mathfrak{t}$. By Lemma 4.1 $\mathfrak{t}$ is a polarization for λ and, therefore, the kernel of f_λ on the subspace $\mathfrak{h} + \mathfrak{t}$ is equal to $\mathfrak{h} \cap \mathfrak{t}$. Let $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{t}$ be such that $\bar{\xi} \in \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{t}), \bar{\xi} \neq 0$ and $\bar{\eta} \in \mathfrak{t}/(\mathfrak{h} \cap \mathfrak{t}), \bar{\eta} \neq 0$. We may assume that $f_\lambda(\xi, \eta) = 1$.

Let v be a generator of $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$. Then for $r \in \mathfrak{h} \cap \mathfrak{t}$ we have

$$r\eta v = [r, \eta]v = \lambda([r, \eta])v = 0, \quad \eta\eta v = \frac{1}{2}[\eta, \eta]v = \frac{1}{2}\lambda([\eta, \eta])v = 0,$$

i.e., $\mathfrak{t}_{\bar{1}}(\eta v) = 0$ and, therefore, there exists a non-zero homomorphism $\mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu') \longrightarrow \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$, where μ' is the weight of ηv.

Since $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ is irreducible and $\dim \mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu') = \dim \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$, this homomorphism is an isomorphism. Let $g \in \mathfrak{g}_{\bar{0}}$. Then

$$g(\eta v) = \eta(gv) + [g, \eta]v = [\lambda + \mathrm{tr}_{\mathfrak{t}/(\mathfrak{t}\cap\mathfrak{h})} \mathrm{ad}\ g]\eta v,$$

i.e., $\mu' = \lambda - \mathrm{str}_{\mathfrak{t}/(\mathfrak{t}\cap\mathfrak{h})} \mathrm{ad} g$.

Since $\lambda \in L$, it follows that

$$0 = \lambda([g, [\xi, \eta]]) = \lambda([[g, \xi], \eta]) + \lambda([\xi, [g, \eta]]) =$$
$$= -(\mathrm{str}_{\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h}} \mathrm{ad}\ g + \mathrm{str}_{\mathfrak{h}/\mathfrak{t}\cap\mathfrak{h}} \mathrm{ad}\ g)\lambda([\xi, \eta]).$$

Since $\lambda([\xi, \eta] = 1$, it follows that

$$\mathrm{str}_{\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h}} \mathrm{ad}\ g = -\mathrm{str}_{\mathfrak{h}/\mathfrak{t}\cap\mathfrak{h}} \mathrm{ad}\ g,$$

and $\mu' = \lambda - \theta_{\mathfrak{b}} - \theta_{\mathfrak{t}}$, i.e., $\mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\mu) \cong \pi(\mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(\lambda))$.

Let $k > 1$. Let $\mathfrak{h} = \mathfrak{g}_{\bar{0}} + P$ and $\mathfrak{t} = \mathfrak{g}_{\bar{0}} \oplus Q$ such that $P \cap Q \subset F \subset P$, $F \neq P$ and $F \neq P \cap Q$, where F is a $\mathfrak{g}_{\bar{0}}$-submodule in $\mathfrak{g}_{\bar{1}}$. Set $R = F + (F^\perp \cap Q)$. It is not difficult to verify that $\mathfrak{r} = \mathfrak{g}_{\bar{0}} \oplus R$ is a polarization for λ. Set

$$\nu(x) = \lambda(x) + \mathrm{str}_{P/(P\cap R)}(\mathrm{ad}\ x).$$

Since $P/(P \cap R)$ is a subquotient in $\mathfrak{g}_{\bar{1}}/\mathfrak{n}$, it follows that $\nu([R, R]) = 0$. The same arguments as in Lemma 4.1 show that $\mathfrak{r}$ is a polarization for ν.

Since $P \cap R \supset F \supset P \cap Q$, then $\dim R/(P \cap R) < \dim P/(P \cap Q)$.

Further, the diagram of inclusions

$$\begin{array}{ccc} P \cap R & \longrightarrow & P \\ \downarrow & & \downarrow \\ R & \longrightarrow & \mathfrak{g}_{\bar{1}} \end{array}$$

shows that

$$2\theta_{\mathfrak{h}}(x) + \mathrm{str}_{P/(P\,\cap\,R)}\mathrm{ad}\ (x) = 2\theta_{\mathfrak{r}}(x) + \mathrm{str}_{R/(R\,\cap\,P)}\mathrm{ad}\ (x).$$

By duality, there exists a pairing

$$(P/(P \cap R)) \times (R/(P \cap R)) \longrightarrow \mathbb{C}$$

and since $\mathrm{str}_{P/(P\,\cap\,R)}\mathrm{ad}(x) = -\mathrm{str}_{R/(P\,\cap\,R)}\mathrm{ad}\ (x)$, then $\mathrm{str}_{R/(P\,\cap\,R)}\mathrm{ad}(x) = \theta_{\mathfrak{h}}(x) - \theta_{\mathfrak{r}}(x)$.

Thus,

$$\nu(x) - \theta_{\mathfrak{r}}(x) = \lambda(x) - \mathrm{str}_{R/(P\,\cap\,R)}\mathrm{ad}(x) - \theta_R(x) = \lambda(x) - \theta_{\mathfrak{h}}(x),$$

i.e., $(\lambda, \mathfrak{h}) \sim (\nu, \mathfrak{t})$ and, by induction, $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{r}}(\nu)$. Besides, $\nu - \theta_{\mathfrak{r}} = \lambda - \theta_{\mathfrak{h}} = \mu - \theta_{\mathfrak{t}}$ and $Q \cap R \supset F^{\perp} \cap Q <\supset Q \cap P$; therefore,

$$\dim Q/Q \cap R \leq \dim \bar{Q} = \dim Q/(Q \cap P).$$

By induction, $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{r}}(\nu) \cong \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu)$, therefore, $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu)$.

Conversely, let $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) \cong \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu)$. Then $\lambda = \mu + \alpha_1 + \ldots + \alpha_k$, where the α_i are the weights of $\mathfrak{g}_{\bar{1}}/Q$. Therefore, by sect. 2.2 $\lambda([Q, Q]) = 0$ and since $\dim \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) = \dim \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu)$, it folows that $\dim \mathfrak{h} = \dim \mathfrak{t}$ and $\mathfrak{h}$ is a polarization for λ and for $\mu' = \lambda - \theta_{\mathfrak{h}} + \theta_{\mathfrak{t}}$, too. Since $\mu' - \theta_{\mathfrak{t}} = \lambda - \theta_{\mathfrak{h}}$, then by the above $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu') = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu)$. Let $v' \in \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu')$ be a generator of $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu')$ and $Qv = 0$. By 4.1 $v' \in Span(v, \xi_0 v)$, therefore, $\mu' = \mu$ and $(\mu, \mathfrak{t}) \sim (\lambda, \mathfrak{h})$. □

4.3. Corollary . *Heading* 2) *of Theorem and heading* 2) *of Lemma 1.2 hold.*

Proof. Due to sect. 2.4 it is clear that $\mathfrak{h}$ is a polarization for $\lambda + \theta_{\mathfrak{h}}$ and, therefore, $I^{\mathfrak{g}}_{\mathfrak{h}}(\lambda)$ is irreducible. If $\mathfrak{t}$ is another polarization for λ, then by sect. 4.2

$$I^{\mathfrak{g}}_{\mathfrak{t}}(\lambda) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\lambda + \theta_{\mathfrak{t}}) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda + \theta_{\mathfrak{h}}) = I^{\mathfrak{g}}_{\mathfrak{h}}(\lambda).$$

If U is irreducible, then by sect. 3.4 $U \cong V^{\mathfrak{g}}_{\mathfrak{h}}(\lambda)$ for some λ and $\mathfrak{h}$.

If $I(\lambda) = I(\mu)$, then $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda + \theta_{\mathfrak{h}}) \cong \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{t}}(\mu + \theta_{\mathfrak{t}})$ and by sect. 4.2

$$\lambda = \lambda + \theta_{\mathfrak{h}} - \theta_{\mathfrak{b}} = \mu + \theta_{\mathfrak{t}} - \theta_{\mathfrak{t}} = \mu. \quad \square$$

Proof of heading 3) of Lemma 1.2 Let us prove now that $I^{\mathfrak{g}}_{\mathfrak{b}}(\lambda) \cong CI^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$. To this end, make use of the isomorphisms $(I^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))^* \cong CI^{\mathfrak{g}}_{\mathfrak{b}}(-\lambda)$ and $(I^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))^* \cong I^{\mathfrak{g}}_{\mathfrak{b}}(-\lambda + 2\theta_{\mathfrak{b}})$. The first of these isomorphisms follows from the definitions of the induced and coinduced modules.

Let us prove the other one. Select a basis $\xi_1, \ldots, \xi_n$ in the complement to $\mathfrak{b}_{\bar{1}}$ in $\mathfrak{g}_{\bar{1}}$ and consider the following filtration of $I^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$:

$$I_0 = Span(v) \subset I_1 = Span(v; \xi_1 v, \ldots, \xi_n v) \subset \cdots \subset I_n = I^{\mathfrak{g}}_{\mathfrak{b}}(\lambda).$$

It is clear that the elements ξ can be chosen so that each I_k is a $\mathfrak{g}_{\bar{0}}$-module. Let $l \in (I^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))^*$ be such that $l(I_n) \neq 0$ while $l(I_{n-1}) = 0$. Then it is easy to verify that $\mathfrak{b}_{\bar{1}} l = 0$ and the weight l with respect to $\mathfrak{g}_{\bar{0}}$ is equal to $-\lambda + 2\theta_{\mathfrak{b}}$. Therefore, there exists a nonzero homomorphism $\varphi : I^{\mathfrak{g}}_{\mathfrak{b}}(-\lambda + 2\theta_{\mathfrak{b}}) \longrightarrow (I^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))^*$.

Since the dimensions of these modules are equal and the first of them is irreducible, φ is an isomorphism. Hence,

$$CI(\lambda) = CI^{\mathfrak{g}}_{\mathfrak{b}}(\lambda - \theta_{\mathfrak{b}}) \cong (I^{\mathfrak{g}}_{\mathfrak{b}}(-\lambda + \theta_{\mathfrak{b}}))^* = I^{\mathfrak{g}}_{\mathfrak{b}}(\lambda - \theta_{\mathfrak{b}} + 2\theta_{\mathfrak{b}}) = I(\lambda).$$

□

§5. An example

Set $\mathfrak{g}_{\bar 0} = Span(x, y, z, u)$, $\mathfrak{g}_1 = Span(\xi_{-2}, \xi_{-1}, \xi_1, \xi_2)$ and let the nonzero brackets be:

$$[\xi_{-1}, \xi_1] = u, \quad [\xi_{-2}, \xi_2] = u, \quad [\xi_1, \xi_2] = y$$

$$[\xi_{-1}, \xi_2] = z, \quad [\xi_2, \xi_2] = x,$$

$$[y, \xi_{-2}] = -\frac{1}{2}\xi_{-2}, \quad [y, \xi_2] = \frac{1}{2}\xi_1$$

$$[z, \xi_1] = \frac{1}{2}\xi_{-2}, \quad [z, \xi_2] = -\frac{1}{2}\xi_1$$

$$[x, y] = -y, \quad [x, z] = z, \quad [x, \xi_{-1}] = \xi_{-1}, \quad [x, \xi_1] = -\xi_1.$$

It is not difficult to verify that $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ is a Lie superalgebra. It is solvable since so is $\mathfrak{g}_{\bar 0}$. Consider $\lambda = \lambda_1 u^* + \lambda_2 x^*$ with $\lambda_1 \neq 0$. Then $\mathfrak{h} = \mathfrak{g}_{\bar 0} \oplus Span(\xi_{-1}, \xi_{-2})$ and $\mathfrak{t} = \mathfrak{g}_{\bar 0} \oplus Span(\xi_1, \xi_2)$ are polarizations for λ. As is easy to verify, $I^{\mathfrak{g}}_{\mathfrak{h}}(\lambda) \ncong I^{\mathfrak{g}}_{\mathfrak{t}}(\lambda)$; besides, the weights of the module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(\lambda)$ are λ, $\lambda - x^*$, hence, $\lambda - (\lambda - x^*) = x^*$ but $x^*([\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}]) \neq 0$ contradicting the statement of Theorem 7 of [K].

The error in the proof of Theorem 7 of [K] is not easy to find: it is an incorrect induction in the proof of heading a) on p. 80. Namely, if, in notations of [K], the subalgebra H is of codimension (0, 1) then the irreducible quotients of W considered as G_0-modules belong by inductive hypothesis to one class from L/L_0^H, where

$$L_0^H = \{\lambda \in \mathfrak{g}^* : \lambda([H, H]) = 0\},$$

NOT to one class from L/L_0^G as stated on p. 80, line 13 from below.

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