Moment of Inertia and Quadrupole Response Function of a Trapped Superfluid

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We derive an explicit relationship between the moment of inertia and the quadrupole response function of an interacting gas confined in a harmonic trap. The relationship holds for both Bose and Fermi systems and is well suited to reveal the effects of irrotationality of the superfluid motion. Recent experimental results on the scissors mode are used to extract the value of the moment of inertia of a trapped Bose gas and to point out the deviations from the rigid value due to superfluidity.

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Important evidences of superfluidity in Bose-Einstein condensed gases have become available in a recent series of experiments. These include the occurrence of quantized vortices [1,2], the existence of a critical velocity for the motion of external impurities [3], and the rotational properties associated with the scissors mode [4]. In particular this latter experiment has revealed the consequences [5] of the quenching of the moment of inertia with respect to the classical rigid value. The reduction of the moment of inertia follows from irrotationality which makes it impossible, for a superfluid, to respond to a transverse probe and to rotate in a rigid way. The moment of inertia characterizes the response of the system to a rotational field $-\Omega J_z$ and is defined by the ratio

$$\Theta = \frac{\langle J_z \rangle}{\Omega} \tag{1}$$

between the angular momentum $\langle J_z \rangle$ induced by the rotation (hereafter assumed to take place around the z-axis) and the angular velocity Ω. In superfluid helium the moment of inertia was determined long time ago by rotating the bucket containing the liquid and by measuring the angular momentum of the sample [6]. This experiment pointed out the occurrence of significant deviations from the rigid value for temperatures below the λ-point. Relation (1) naturally defines the moment of inertia in the linear regime. At higher angular velocities new phenomena, associated with the creation of quantized vortices, take place in superfluids [7]. Like in superfluid helium, also in a cold trapped gas the moment of inertia is expected to deviate significantly from the rigid value. The temperature dependence of Θ for a trapped Bose gas below the critical temperature T_c for Bose-Einstein condensation and the corresponding deviations from the rigid value were investigated in [8]. First estimates of the moment of inertia of a trapped Fermi gas below the BCS transition have been also reported [9]. In a trapped gas one can rotate the confining potential using magnetic or optical methods [10,2]. Actually this procedure has proven quite successful in producing quantized vortices in Bose-Einstein condensed gases [2]. However, the direct measurement of the angular momentum induced by the rotation of the trap is difficult in atomic gases, since most of diagnostic techniques, based on optical imaging, provide information on the density profiles, either *in situ* or after the expansion of the gas.

In this work we propose a method to measure the moment of inertia of a trapped gas based on the study of the collective oscillations resulting from the coupling between the rotational and the quadrupole motion. Our analysis, which develops the ideas introduced in [5], is based on the derivation of an explicit relationship between the moment of inertia and the quadrupole response function, and will be applied to the recent experiments on the scissors mode of a trapped Bose gas [4]. The possibility of revealing the effects of superfluidity using this method would be particularly appealing in the case of Fermi systems, where most of diagnostic techniques are insensitive to the occurrence of the BCS transition.

The starting point of our analysis is the commutation rule ($\hbar = 1$)

$$[H, J_z] = -i(x\nabla_y - y\nabla_x)V_{\rm ext}(\mathbf{r}) \ , \tag{2}$$

where $J_z = \sum_{i=1}^{N} x_i p_i^y - y_i p_i^x$ is the third component of the angular momentum operator and

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2M} + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) + V_{\rm ext}(\mathbf{r}) \ , \tag{3}$$

is the Hamiltonian of the system, containing the external confining potential $V_{\rm ext}(\mathbf{r})$. Result (2) is independent of the choice of the two-body potential, and follows from the rotational symmetry of the internal Hamiltonian $H - V_{\rm ext}$. In

the following we will consider harmonic potentials

$$V_{\rm ext}(\mathbf{r}) = \sum_{i=1}^{N} \frac{M}{2}(\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2) \tag{4}$$

with an asymmetry ($\omega_x \neq \omega_y$) in the rotating plane. In this case the commutation rule (2) takes the simple form

$$[H, J_z] = -iM(\omega_x^2 - \omega_y^2)Q \ , \tag{5}$$

where

$$Q = \sum_{i=1}^{N} x_i y_i \tag{6}$$

is the relevant quadrupole operator. Eqs. (5)-(6) explicitly reveal the coupling between the degrees of freedom associated with the angular momentum and quadrupole variables, induced by the deformation of the confining potential.

For a sample in thermal equilibrium at temperature T the moment of inertia is given, according to linear response theory, by the expression

$$\Theta = \frac{1}{\mathcal{Z}} \sum_{n,m} \left[e^{-\beta\omega_m} - e^{-\beta\omega_n}\right] \frac{|\langle m|J_z|n\rangle|^2}{\omega_n - \omega_m} \ , \tag{7}$$

where $|n\rangle$ and ω_n are eigenstates and eigenfrequencies of the Hamiltonian (3), $\mathcal{Z}$ is the canonical partition function and $e^{-\beta\omega_n}$ is the usual Boltzmann factor, with $\beta = 1/k_B T$.

Because of the algebraic relationship (5) the moment of inertia can be also written in the form

$$\Theta = -\frac{M^2}{\pi}(\omega_x^2 - \omega_y^2)^2 \int d\omega \, \frac{\chi_Q''(\omega)}{\omega^3} \tag{8}$$

where

$$\chi_Q''(\omega) = \frac{\pi}{\mathcal{Z}} \sum_{n,m} [e^{-\beta\omega_n} - e^{-\beta\omega_m}] |\langle m|Q|n\rangle|^2 \delta(\omega - \omega_{nm}) \ , \tag{9}$$

is the imaginary part of quadrupole dynamic response function

$$\chi_Q(\omega) = \frac{2}{\mathcal{Z}} \sum_{n,m} e^{-\beta\omega_m} \frac{|\langle m|Q|n\rangle|^2 \omega_{nm}}{(\omega + i\eta)^2 - \omega_{nm}^2} \ , \tag{10}$$

and $\omega_{nm} = \omega_n - \omega_m$. Here and in the following the frequency integrals are taken between $-\infty$ and $+\infty$. It is useful to compare the moment of inertia (8) with the rigid value

$$\Theta_{\rm rig} = MN\langle x^2 + y^2\rangle \tag{11}$$

given by classical theory. One can easily show that the average (11) is proportional to the double commutator of the Hamiltonian (3) with the quadrupole operator (6). In fact the following sum rule holds [11]: $\langle [Q,[H,Q]]\rangle = N\langle x^2+y^2\rangle/M$. This result is the natural extension of the dipole Thomas-Reiche-Kuhn sum rule [12] to the quadrupole operator. On the other hand the double commutator is directly related to the energy weighted moment of the imaginary part (9) of the quadrupole response function, so that one finally obtains the exact relationship $N\langle x^2+y^2\rangle = -M\int d\omega \chi_Q''(\omega)\omega/\pi$, which permits to write the ratio between the moment of inertia (8) and the rigid value (11) in the useful form

$$\frac{\Theta}{\Theta_{\rm rigid}} = (\omega_x^2 - \omega_y^2)^2 \frac{\int d\omega \chi_Q''(\omega)/\omega^3}{\int d\omega \chi_Q''(\omega)\omega} \ . \tag{12}$$

Eq. (12) represents a major result of the present paper. It explicitly shows that the occurrence of superfluidity, characterized by a reduction of the value of the moment of inertia with respect to the rigid value, must show up also in the structure of the quadrupole excitations of the system, characterized by the function $\chi_Q''(\omega)$. Actually the physical consequence of superfluidity is rather clear. The constraint of irrotationality suppresses the modes with low

frequency ($\omega \sim |\omega_x - \omega_y|$). As a consequence in a perfect superfluid the integral (8) approaches a constant value when $\omega_x \to \omega_y$ with the consequent quenching of the value of Θ, which behaves like $(\omega_x^2 - \omega_y^2)^2$. The suppression of the low frequency oscillations in the presence of Bose-Einstein condensation has been experimentally pointed out in [4]. Conversely, in a normal system the occurrence of low frequency modes is crucial to bring the moment of inertia to the rigid value.

It is worth discussing the effects of superfluidity more explicitly. The equations characterizing the macroscopic behaviour of a superfluid at zero temperature have the form of irrotational hydrodynamics

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho \mathbf{v}) = 0 \tag{13}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla}\left(\frac{v^2}{2} + \frac{V_{\rm ext}(\mathbf{r})}{M} + \frac{\mu_{\rm loc}(\rho)}{M}\right) = 0 \tag{14}$$

where $\mu_{\rm loc}(\rho)$ is the chemical potential of a uniform system evaluated at density ρ, and $\mathbf{v}$ is the irrotational superfluid velocity. These equations hold in particular for a cold dilute Bose gas in the Thomas-Fermi limit [13]. In this case $\mu_{\rm loc}(\rho) = g\rho$ is linear in the density, the coupling constant $g = 4\pi a/m$ being fixed by the s-wave scattering length a. In this limit the hydrodynamic equations (13-14) are equivalent to the Gross-Pitaevskii equation for the order parameter and can be easily solved to obtain the collective frequencies of a trapped Bose-Einstein condensed gas [14]. In Ref. [15] the same Eqs. (13-14) have been used to calculate the corrections to the collective frequencies of a trapped Bose gas, due to the inclusion of the Lee-Huang-Yang term [16] in the equation of state. Applicability of Eqs. (13-14) is not however restricted to Bose superfluids. They are in fact expected to hold also in the case of Fermi superfluids. Of course in this case the equation of state can be very different due Pauli effects. For example, in a very dilute Fermi gas the chemical potential is a purely kinetic effect: $\mu_{\rm loc}(\rho) = (3\pi^2\rho)^{2/3}/2m$ (we consider here a Fermi gas of two spin species with equal density). In this limit the hydrodynamic equations of superfluids have been investigated in [17]. In general the applicability of Eqs. (13-14) is subject to the following constraints: (*i*) the non-superfluid component of the system should be negligible and hence the temperature should be much smaller than the transition temperature to the normal phase; (*ii*) the sample should be large enough in order to ensure the validity of the local density approximation for the chemical potential (Thomas-Fermi limit); (*iii*) the length scale of the oscillations should be larger than the healing length.

Starting from the equations of irrotational hydrodynamics it is straightforward to calculate the moment of inertia. To this purpose it is convenient to rewrite these equations in the frame rotating with angular velocity Ω. One finds

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\big(\rho(\mathbf{v} - \boldsymbol{\Omega} \times \mathbf{r})\big) = 0 \tag{15}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot \left(\frac{v^2}{2} + \frac{V_{\rm ext}(\mathbf{r})}{M} + \frac{\mu_{\rm loc}(\rho)}{M} - \mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{r})\right) = 0 \,. \tag{16}$$

The stationary solutions in the rotating frame have the simple irrotational form

$$\mathbf{v} = \Omega \frac{\langle x^2 - y^2 \rangle}{\langle x^2 + y^2 \rangle} \boldsymbol{\nabla}(xy) \,. \tag{17}$$

From Eq. (17) one can easily calculate the angular momentum and, using definition (1), one finds the irrotational result [18]:

$$\Theta = \left[\frac{\langle x^2 - y^2 \rangle}{\langle x^2 + y^2 \rangle}\right]^2 \Theta_{\rm rigid} \tag{18}$$

for the moment of inertia, showing that in a superfluid the value of the moment of inertia is smaller than the rigid value [20]. Results (17-18) are not restricted to the perturbative regime of small Ω. They are simply the consequence of the fact that the equilibrium density, fixed by the stationary solution of the Euler equation (16), depends on the variables x and y through the quadratic combination $\alpha_x x^2 + \alpha_y y^2$ [19]. For small angular velocities the moment of inertia (18) reduces to

$$\Theta = \left[\frac{\omega_x^2 - \omega_y^2}{\omega_x^2 + \omega_y^2}\right]^2 \Theta_{\rm rig} \,. \tag{19}$$

Another important feature emerging from the hydrodynamic Eqs. (13-14) is that also the behaviour of the surface oscillations, characterized by the condition $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, is independent of the equation of state [21]. In particular, by

adding a term of the form $\lambda xy e^{i\omega t}$ + c.c. in the external potential one finds that the quadrupole operator (6) excites, in the linear limit, only one mode with frequency

$$\omega_{\rm HD} = \sqrt{\omega_x^2 + \omega_y^2}\,, \tag{20}$$

so that the imaginary part of the quadrupole response function takes the simple form

$$\chi_Q'' = -\frac{N\pi}{2}\frac{\langle x^2+y^2\rangle}{M\omega_{\rm HD}}\left[\delta(\omega-\omega_{\rm HD}) - \delta(\omega+\omega_{\rm HD})\right]\,. \tag{21}$$

By inserting result (21) into Eq. (12) one immediately recovers the irrotational value (19) for the moment of inertia.

In the second part of the work we show that the deviations of the moment of inertia from the rigid value can be measured through a rather straightforward experiment which consists of a sudden rotation of the confining trap by a small angle, followed by the imaging of the oscillations of the gas. This experiment has been recently carried out [4] in a trapped Bose gas where the effects of superfluidity predicted in [5] for the scissors mode have been directly observed.

The abrupt rotation of the harmonic trap is characterized by the transformation of coordinates $x \to x\cos\phi_0 + y\sin\phi_0$, $y \to -x\sin\phi_0 + y\cos\phi_0$. For small values of ϕ_0, satisfying the condition $\phi_0 \ll (\omega_x^2-\omega_y^2)/(\omega_x^2+\omega_y^2)$, the corresponding change in the harmonic potential (4) produces a time dependent perturbation of the form

$$H_{\rm pert} = \lambda Q\theta(t)\,, \tag{22}$$

where $\lambda = M(\omega_x^2-\omega_y^2)\phi_0$, Q is the quadrupole operator (6) and $\theta(t)$ is the usual step function. Eq. (22) can be rewritten in its Fourier components as

$$H_{\rm pert} = \lambda Q\frac{i}{4\pi}\int d\omega \frac{e^{-i\omega t}}{\omega+i\eta} + \text{H.c.}\,. \tag{23}$$

According to linear response theory the average value $Q(t) \equiv \langle Q\rangle$ of the quadrupole operator evolves, for $t>0$, as

$$Q(t) = \lambda\frac{i}{4\pi}\int d\omega \frac{e^{-i\omega t}}{\omega+i\eta}\chi_Q(\omega) + \text{c.c.}\,, \tag{24}$$

where $\chi_Q(\omega)$ is the quadrupole response (10). Integrating (24) in the complex plane and using the analytic properties of $\chi_Q(\omega)$, one can rewrite Eq. (24) in the form

$$Q(t) = \int d\omega Q(\omega) e^{i\omega t} - \int d\omega Q(\omega)\,, \tag{25}$$

with

$$Q(\omega) = -\frac{\lambda}{\pi}\frac{\chi_Q''(\omega)}{\omega}\,. \tag{26}$$

Combining the above results with Eq. (12), one finally finds the useful relationship

$$\frac{\Theta}{\Theta_{\rm rigid}} = (\omega_x^2-\omega_y^2)^2\frac{\int d\omega Q(\omega)/\omega^2}{\int d\omega Q(\omega)\omega^2}\,, \tag{27}$$

which provides the sought link between the moment of inertia of the system and the Fourier transform of the observable quadrupole signal

$$Q(t) = \int d\mathbf{r}\rho(\mathbf{r},t)xy\,. \tag{28}$$

In actual experiments the quantity which is more easily measured is the angle of rotation $\phi(t)$ rather than the quadrupole moment (28). The angle $\phi(t)$ fixes the position of the symmetry axis of the gas with respect to the new symmetry axis of the confining trap. During the rotation the density of the gas varies according to the law, valid for small angles,

$$\rho(\mathbf{r},t) = \rho_0(\mathbf{r}) + (x\nabla_y - y\nabla_x)\rho_0(\mathbf{r})(\phi(t)-\phi_0)\,, \tag{29}$$

where $\rho_0(\mathbf{r})$ is the density before the sudden rotation of the trap. Using Eqs. (28) and (29) one derives the useful relationship

$$Q(t) = N\langle x^2 - y^2\rangle(\phi(t) - \phi_0) \tag{30}$$

between the angle of rotation and the quadrupole moment.

We are now ready to use our formalism to extract the moment of inertia from the experiment of Ref. [4]. At low temperature the angle $\phi(t)$ was measured at different times after the sudden rotation of the confining potential. The experimental results have confirmed the occurrence of a single frequency in $\phi(t)$, as predicted by hydrodynamic theory:

$$\phi(t) = \phi_0 \cos(\omega_{\rm HD} t) \; . \tag{31}$$

The observed frequency ω coincides, with excellent accuracy, with the hydrodynamic value (20). Using the relationship (30) and inserting the experimental values $\omega_x = 2\pi(90 \pm 0.2)$ sec^{-1}, $\omega_y = 2\pi(249 \pm 0.6)$ sec^{-1} and $\omega = 2\pi(265.6 \pm 0.8)$ sec^{-1} into Eq. (27), one obtains the value $\Theta/\Theta_{\rm rigid} \simeq 0.6$, providing an unambiguous proof of the superfluidity of the sample.

At higher temperature, well above T_c, the situation changes drastically. In first approximation the effects of interactions can be neglected and the time evolution of $\phi(t)$ can be described by the non-interacting model [5]

$$\phi(t) = \frac{1}{2}\phi_0[\cos(\omega_+ t) + \cos(\omega_- t)] \; , \tag{32}$$

where

$$\omega_\pm = |\omega_x \pm \omega_y| \tag{33}$$

are the two frequencies excited by the external perturbation. The experimental results of [4], carried out at $T \sim 5T_c$, well agree with these predictions. In fact the observed frequencies $\omega_+ = 2\pi(338.5 \pm 0.8)$ sec^{-1} and $\omega_- = 2\pi(159.1 \pm 0.8)$ sec^{-1} are very close to the theoretical values (33) and enter the measured signal $\phi(t)$ with practically equal weight. These results, inserted into Eq. (27), give finally $\Theta/\Theta_{\rm rigid} \simeq 1$, confirming the crucial role played by the mode with the low frequency $\omega = |\omega_x - \omega_y|$ in providing the rigid response to the rotational field.

A similar behaviour is expected to occur also in a Fermi gas. In fact at temperatures much smaller that the BCS transition the response is still given by the hydrodynamic predictions (20) and (31). On the other hand, in the normal phase of a very dilute Fermi gas one can use the predictions (32)-(33) of the non-interacting model.

The above procedure can be also used to investigate the effects of superfluidity as a function of temperature. In particular, by investigating the measured signal (28) and inserting its Fourier transform into Eq. (27), one can determine the value of the moment of inertia at any value of T. One should however take into account that, below the critical temperature, the density profile of the oscillating system cannot be simply parametrized through a single angle of rotation, like in Eq. (29), because the normal and the superfluid components will respond in a different way to the rotation of the trap. As a consequence one should introduce two distinct angles and the density of the gas will oscillate, in the linear limit, according to

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + (x\nabla_y - y\nabla_x)\rho_S(\mathbf{r})\phi_S(t) + (x\nabla_y - y\nabla_x)\rho_N(\mathbf{r})\phi_N(t) \; , \tag{34}$$

where $\rho_0(\mathbf{r}) = \rho_S(\mathbf{r}) + \rho_N(\mathbf{r})$ is the sum of the superfluid and normal component at equilibrium. The careful investigation of the time evolution of the density profile would provide unique information on the shape of the superfluid and normal components. In a dilute Bose gas these components are expected to coincide, respectively, with the condensate and the thermal cloud, except for temperature smaller than the chemical potential. The situation is very different in Fermi gases, where the total density profile is scarcely affected by the BCS transition [22], and the separation between the two components is much less trivial.

In conclusion we have shown that the moment of inertia Θ of a gas trapped by harmonic potential can be explicitly measured by analyzing the quadrupole oscillations generated by the a sudden rotation of the trap. An explicit evaluation of Θ has been provided using the recent data on the scissors mode available for a trapped Bose gas either at very low temperature and above the critical temperature. In the first case the method explicitly reveals the quenching of the moment of inertia due to superfluidity. The same method can be naturally applied to a Fermi system in order to reveal the effects of superfluidity associated with the BCS transition. Moreover, with suitable modifications, it can be also extended to the case of non-harmonic trapping.

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