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Gauging Higher Derivatives

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Abstract

The usual prescription for constructing gauge-invariant Lagrangian is generalized to the case where a Lagrangian contains second derivatives of fields as well as first derivatives. Symmetric tensor fields in addition to the usual vector fields are introduced as gauge fields. Covariant derivatives and gauge-field strengths are determined.

1 Introduction

Higher-derivative theories have been investigated in various contexts of physics: higher-derivative terms naturally occur as quantum corrections to lower-derivative theories; nonlocal theories, for example string theories, are considered to be equivalent to higher-derivative theories; gravity theories with R^2 terms are good candidates to renormalizable theories of gravity.[1)]

It is pointed out, however, that higher-derivative theories have some crucial disadvantages:[2),3),4)] they are unbounded from below; they violate unitarity; they break down the initial value problem. Several attempts have been made to remove these faults, but without success.

In the present paper we generalize the usual prescription for constructing gauge-invariant Lagrangian to the case where higher-derivatives of fields are included. We expect that this new type of gauge degeneracy could be useful to settle the above-mentioned problems of higher-derivative theories.

In §2 we consider a generic action that contains arbitrary order derivatives of fields. Requiring the action be invariant under some not only global but also local transformations, we get a series of identities. These are generalizations of the usual Noether's theorems. In §3 and the followings we restrict ourselves to treating a simple case where the action has at most second derivatives. Tensor gauge fields associated with second derivatives are introduced. A series of identities that the tensor gauge fields as well as the usual vector gauge fields should satisfy are writtten down in §3. In §4 first and second derivatives of the two kinds of gauge fields are decomposed to their irreducible components. In §5 we solve the identities to get covariant derivatives and gauge-field strengths. Summary and discussion are briefly given in §6.

2 Identities

In this section we consider a generic higher-derivative theories whose action depends on fields $\varphi_A(x)$ and their derivatives up to Nth order

$$S \stackrel{\mathrm{d}}{\equiv} \int \mathrm{d}^4x \mathcal{L}(\varphi_A, \partial_\mu\varphi_A, \partial_{\mu\nu}\varphi_A, \cdots, \partial_{\mu_1\mu_2\cdots\mu_N}\varphi_A), \tag{1}$$

where we have used the abbreviation

$$\partial_{\mu_1\mu_2\cdots\mu_n}\varphi_A \stackrel{\mathrm{d}}{\equiv} \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_n}\varphi_A. \tag{2}$$

The Euler-Lagrange equations are

$$\frac{\delta S}{\delta\varphi_A} \stackrel{\mathrm{d}}{\equiv} \sum_{n=0}^{N}(-)^n\partial_{\mu_1\cdots\mu_n}(\partial^{A,\mu_1\cdots\mu_n}\mathcal{L}) = 0. \tag{3}$$

Here and hereafter the differentiation

$$\partial^{A,\mu_1\cdots\mu_n}\mathcal{L} \stackrel{\mathrm{d}}{\equiv} \frac{\partial\mathcal{L}}{\partial\partial_{\mu_1\cdots\mu_n}\varphi_A} \tag{4}$$

always stands for that of weight 1: in the case of $\mathcal{L} = C^{A\mu\nu}\partial_{\mu\nu}\varphi_A$, for example, we have $\partial^{A,\mu\nu}\mathcal{L} = \frac{1}{2!}(C^{A\mu\nu} + C^{A\nu\mu})$.

Suppose the action is invariant under a 'global' infinitesimal transformation with p independent parameters ϵ^a

$$\begin{cases} \delta x^\mu &= \epsilon^a X_a^\mu(x), \\ \delta\varphi_A(x) &= \epsilon^a N_{aA}(x,\varphi). \end{cases} \tag{5}$$

Then p identities follow:[5)]

$$\begin{aligned} &\frac{\delta S}{\delta\varphi_A}\left(N_{aA} - \partial_\nu\varphi_A\cdot X_a^\nu\right) \\ &+ \ \partial_\mu\left[\sum_{n=0}^{N-1}\left(\partial^{A,\mu\alpha_1\cdots\alpha_n}\mathcal{L}\right)\overleftrightarrow{\partial}_{\alpha_1\cdots\alpha_n}\left(N_{aA} - \partial_\nu\varphi_A\cdot X_a^\nu\right) + \mathcal{L}X_a^\mu\right] \equiv 0, \end{aligned} \tag{6}$$

where the generalized 'both-side' derivatives for two arbitrary functions F and G

$$F\overleftrightarrow{\partial}_{\mu_1\cdots\mu_n}G \stackrel{\mathrm{d}}{\equiv} \sum_{k=0}^{n}(-)^k\partial_{\mu_1\cdots\mu_k}F\cdot\partial_{\mu_{k+1}\cdots\mu_n}G \tag{7}$$

has been introduced. These identities are also written in the form of

$$\sum_{n=0}^{N}\partial^{A,\alpha_1\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_1\cdots\alpha_n}\left(N_{aA} - \partial_\mu\varphi_A\cdot X_a^\mu\right) + \partial_\mu\left(\mathcal{L}X_a^\mu\right) \equiv 0. \tag{8}$$

Generalized Noether currents are given by the bracketed quantities in Eq.(6):

$$J_a^\mu \stackrel{\mathrm{d}}{\equiv} \sum_{n=0}^{N-1}\left(\partial^{A,\mu\alpha_1\cdots\alpha_n}\mathcal{L}\right)\overleftrightarrow{\partial}_{\alpha_1\cdots\alpha_n}\left(N_{aA} - \partial_\nu\varphi_A\cdot X_a^\nu\right) + \mathcal{L}X_a^\mu, \tag{9}$$

which are conserved when the Euler-Lagrange equations are satisfied

$$\partial_\mu J_a^\mu = 0. \tag{10}$$

Let us now proceed to 'local' infinitesimal transformation. In the present case, since the action contains up-to-Nth order derivatives, the local version of the transformation (5) can be given as follows:

$$\begin{cases} \delta x^\mu &= \lambda^a(x)X_a^\mu(x), \\ \delta\varphi_A(x) &= \displaystyle\sum_{n=0}^{N}\partial_{\mu_1\cdots\mu_n}\lambda^a(x)\cdot N_{aA}^{\mu_1\cdots\mu_n}(x,\varphi), \end{cases} \tag{11}$$

where $\lambda^a(x)$ are p independent arbitrary functions. If we require the action be invariant under the local (infinitesimal) transformation (11), we get the following series of identities:

$$\sum_{n=0}^{N}\partial^{A,\alpha_1\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_1\cdots\alpha_n}\left(N_{aA} - \partial_\mu\varphi_A\cdot X_a^\mu\right) + \partial_\mu\left(\mathcal{L}X_a^\mu\right) \equiv 0, \tag{12}$$

$$
\begin{aligned}
&\sum_{n=1}^{N} {}_{n}C_1 \partial^{A,\nu_1\alpha_2\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_2\cdots\alpha_n}\left(N_{aA}-\partial_\mu\varphi_A\cdot X_a^\mu\right) \\
&\quad + \quad \sum_{n=0}^{N}\partial^{A,\alpha_1\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_1\cdots\alpha_n}N_{aA}^{\nu_1}+\mathcal{L}X_a^{\nu_1}\equiv 0, \qquad (13)\\
&\sum_{n=2}^{N} {}_{n}C_2 \partial^{A,\nu_1\nu_2\alpha_3\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_3\cdots\alpha_n}\left(N_{aA}-\partial_\mu\varphi_A\cdot X_a^\mu\right) \\
&\quad + \quad \frac{1}{2}\sum_{n=1}^{N} {}_{n}C_1\left[\partial^{A,\nu_1\alpha_2\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_2\cdots\alpha_n}N_{aA}^{\nu_2}+(\nu_1\leftrightarrow\nu_2)\right] \\
&\quad + \quad \sum_{n=0}^{N}\partial^{A,\alpha_1\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_1\cdots\alpha_n}N_{aA}^{\nu_1\nu_2}\equiv 0, \qquad (14)\\
&\sum_{n=3}^{N} {}_{n}C_3 \partial^{A,\nu_1\nu_2\nu_3\alpha_4\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_4\cdots\alpha_n}\left(N_{aA}-\partial_\mu\varphi_A\cdot X_a^\mu\right) \\
&\quad + \quad \frac{1}{3}\sum_{n=2}^{N} {}_{n}C_2\left[\partial^{A,\nu_1\nu_2\alpha_3\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_3\cdots\alpha_n}N_{aA}^{\nu_3}+(\nu_1\nu_2\nu_3\text{–cyclic})\right] \\
&\quad + \quad \frac{1}{3}\sum_{n=1}^{N} {}_{n}C_1\left[\partial^{A,\nu_1\alpha_2\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_2\cdots\alpha_n}N_{aA}^{\nu_2\nu_3}+(\nu_1\nu_2\nu_3\text{–cyclic})\right] \\
&\quad + \quad \sum_{n=0}^{N}\partial^{A,\alpha_1\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_1\cdots\alpha_n}N_{aA}^{\nu_1\nu_2\nu_3}\equiv 0, \qquad (15)\\
&\vdots \\
&\partial^{A,\nu_1\cdots\nu_N}\mathcal{L}\cdot\left(N_{aA}-\partial_\mu\varphi_A\cdot X_a^\mu\right) \\
&\quad + \quad \frac{1}{{}_{N}C_{N-1}}\sum_{n=N-1}^{N} {}_{n}C_{N-1} \\
&\qquad\qquad \times\left[\partial^{A,\nu_1\cdots\nu_{N-1}\alpha_N\alpha_n}\mathcal{L}\cdot\partial_{\alpha_N\alpha_n}N_{aA}^{\nu_N}+(\nu_1\cdots\nu_N\text{–combination})\right] \\
&\quad + \quad \cdots \\
&\quad + \quad \frac{1}{{}_{N}C_1}\sum_{n=1}^{N} {}_{n}C_1\left[\partial^{A,\nu_1\alpha_2\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_2\cdots\alpha_n}N_{aA}^{\nu_2\cdots\nu_N}+(\nu_1\cdots\nu_N\text{–combination})\right] \\
&\quad + \quad \sum_{n=0}^{N}\partial^{A,\alpha_1\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_1\cdots\alpha_n}N_{aA}^{\nu_1\cdots\nu_N}\equiv 0, \qquad (16)\\
&\frac{1}{{}_{N+1}C_N}\left[\partial^{A,\nu_1\cdots\nu_N}\mathcal{L}\cdot N_{aA}^{\nu_{N+1}}+(\nu_1\cdots\nu_{N+1}\text{–combination})\right] \\
&\quad + \quad \frac{1}{{}_{N+1}C_{N-1}}\sum_{n=N-1}^{N} {}_{n}C_{N-1} \\
&\qquad\qquad \times\left[\partial^{A,\nu_1\cdots\nu_{N-1}\alpha_N\alpha_n}\mathcal{L}\cdot\partial_{\alpha_N\alpha_n}N_{aA}^{\nu_N\nu_{N+1}}+(\nu_1\cdots\nu_{N+1}\text{–combination})\right] \\
&\quad + \quad \cdots \\
&\quad + \quad \frac{1}{{}_{N+1}C_1}\sum_{n=1}^{N} {}_{n}C_1\left[\partial^{A,\nu_1\alpha_2\cdots\alpha_n}\mathcal{L}\cdot\partial_{\alpha_2\cdots\alpha_n}N_{aA}^{\nu_2\cdots\nu_{N+1}}+(\nu_1\cdots\nu_{N+1}\text{–combination})\right]
\end{aligned}
$$

$$\equiv \quad 0, \tag{17}$$

$$\vdots$$

$$\begin{aligned}
& \frac{1}{{}_{2N-1}C_N}\left[\partial^{A,\nu_1\cdots\nu_N}\mathcal{L}\cdot N_{aA}^{\nu_{N+1}\cdots\nu_{2N-1}} + (\nu_1\cdots\nu_{2N-1}\text{–combination})\right] \\
& + \quad \frac{1}{{}_{2N-1}C_{N-1}}\sum_{n=N-1}^{N} {}_nC_{N-1} \\
& \qquad\qquad \times\left[\partial^{A,\nu_1\cdots\nu_{N-1}\alpha_N\alpha_n}\mathcal{L}\cdot\partial_{\alpha_N\alpha_n}N_{aA}^{\nu_N\cdots\nu_{2N-1}} + (\nu_1\cdots\nu_{2N-1}\text{–combination})\right] \\
& \equiv \quad 0,
\end{aligned} \tag{18}$$

$$\frac{1}{{}_{2N}C_N}\left[\partial^{A,\nu_1\cdots\nu_N}\mathcal{L}\cdot N_{aA}^{\nu_{N+1}\cdots\nu_{2N}} + (\nu_1\cdots\nu_{2N}\text{–combination})\right] \equiv 0. \tag{19}$$

The identities (12) are the same as (8), obtained as the coefficients of $\lambda^a(x)$ in the requirement $\delta S \equiv 0$. The coefficients of $\partial_{\nu_1}\lambda^a, \partial_{\nu_1\nu_2}\lambda^a, \cdots$, and $\partial_{\nu_1\cdots\nu_{2N}}\lambda^a$ give (13), (14), $\cdots$, and (19) respectively.

3 Gauge fields

In this and the following sections, we set $N = 2$ for the sake of simplicity. The starting action is therefore

$$S \stackrel{\mathrm{d}}{\equiv} \int \mathrm{d}^4x\mathcal{L}\left(\varphi_A, \partial_\mu\varphi_A, \partial_{\mu\nu}\varphi_A\right). \tag{20}$$

This action is assumed to be invariant under the global *internal* transformation

$$\left\{\begin{array}{rcl} \delta x^\mu & = & 0, \\ \delta\varphi_A(x) & = & \epsilon^a M_{aA}{}^B\varphi_B(x), \end{array}\right. \tag{21}$$

where $M_{aA}{}^B$ are certain representation matrices of a group G. That means the Lagrangian $\mathcal{L}$ should satisfy the following identities:

$$\partial^A\mathcal{L}\cdot M_{aA}{}^B\varphi_B + \partial^{A,\mu}\mathcal{L}\cdot M_{aA}{}^B\partial_\mu\varphi_B + \partial^{A,\mu\nu}\mathcal{L}\cdot M_{aA}{}^B\partial_{\mu\nu}\varphi_B \equiv 0. \tag{22}$$

Next consider the local version of the transformation (21). The transformation obtained by simply replacing the arbitrary parameters ϵ^a with arbitrary function $\lambda^a(x)$

$$\left\{\begin{array}{rcl} \delta x^\mu & = & 0, \\ \delta\varphi_A(x) & = & \lambda^a(x) M_{aA}{}^B\varphi_B(x) \end{array}\right. \tag{23}$$

does not leave the action invariant. Generalizing the usual prescription for gaugeization, we introduce two kinds of gauge fields $B^a_\mu(x)$ and $B^a_{\mu\nu}(x)$: the vector fields $B^a_\mu(x)$ are to be combined with the first derivative ∂_μ as usual; the symmetric tensor fields $B^a_{\mu\nu}(x)$ are newly introduced to form second-order covariant derivative. It seems natural to assume the following transformation properties for these gauge fields:

$$\left\{\begin{array}{rcl} \delta B^a_\mu(x) & = & \lambda^b(x) f^a_{bc} B^c_\mu(x) + \partial_\mu \lambda^a(x), \\ \delta B^a_{\mu\nu}(x) & = & \lambda^b(x) f^a_{bc} B^c_{\mu\nu}(x) + \frac{1}{2}\left[\partial_\mu \lambda^b(x) f^a_{bc} B^c_\nu(x) + (\mu \leftrightarrow \nu)\right] + \partial_{\mu\nu}\lambda^a(x), \end{array}\right. \tag{24}$$

where f^a_{bc} are structure constants of G

$$[M_b, M_c] = f^a_{bc} M_a. \tag{25}$$

The action S should be extended to incorporate the gauge fields *and* their derivatives up to second order:

$$S_1 \stackrel{\mathrm{d}}{\equiv} \int \mathrm{d}^4 x \mathcal{L}_1 \left(\varphi_A, \partial_\mu \varphi_A, \partial_{\mu\nu}\varphi_A; B^a_\mu, \partial_\mu B^a_\nu, \partial_{\mu\nu} B^a_\rho; B^a_{\mu\nu}, \partial_\mu B^a_{\nu\rho}, \partial_{\mu\nu} B^a_{\rho\sigma}\right). \tag{26}$$

In what form is the Lagrangian $\mathcal{L}_1$ to contain the gauge fields and their derivatives? To answer this question we require that the action S_1 be invariant under the local transformation (23) and (24).

The invariance requirement gives the following series of identities:

$$\begin{aligned} & \frac{\partial \mathcal{L}_1}{\partial \varphi_A} M_{aA}{}^B \varphi_B + \frac{\partial \mathcal{L}_1}{\partial \partial_\mu \varphi_A} M_{aA}{}^B \partial_\mu \varphi_B + \frac{\partial \mathcal{L}_1}{\partial \partial_{\mu\nu}\varphi_A} M_{aA}{}^B \partial_{\mu\nu}\varphi_B \\ & \quad + \frac{\partial \mathcal{L}_1}{\partial B^b_\rho} f^b_{ac} B^c_\rho + \frac{\partial \mathcal{L}_1}{\partial \partial_\mu B^b_\rho} f^b_{ac} \partial_\mu B^c_\rho + \frac{\partial \mathcal{L}_1}{\partial \partial_{\mu\nu} B^b_\rho} f^b_{ac} \partial_{\mu\nu} B^c_\rho \\ & \quad + \frac{\partial \mathcal{L}_1}{\partial B^b_{\rho\sigma}} f^b_{ac} B^c_{\rho\sigma} + \frac{\partial \mathcal{L}_1}{\partial \partial_\mu B^b_{\rho\sigma}} f^b_{ac} \partial_\mu B^c_{\rho\sigma} + \frac{\partial \mathcal{L}_1}{\partial \partial_{\mu\nu} B^b_{\rho\sigma}} f^b_{ac} \partial_{\mu\nu} B^c_{\rho\sigma} \equiv 0, \end{aligned} \tag{27}$$

$$\begin{aligned} & \frac{\partial \mathcal{L}_1}{\partial \partial_\alpha \varphi_A} M_{aA}{}^B \varphi_B + 2\frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\mu}\varphi_A} M_{aA}{}^B \partial_\mu \varphi_B \\ & \quad + \frac{\partial \mathcal{L}_1}{\partial \partial_\alpha B^b_\rho} f^b_{ac} B^c_\rho + 2\frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\mu} B^b_\rho} f^b_{ac} \partial_\mu B^c_\rho \\ & \quad + \frac{\partial \mathcal{L}_1}{\partial \partial_\alpha B^b_{\rho\sigma}} f^b_{ac} B^c_{\rho\sigma} + 2\frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\mu} B^b_{\rho\sigma}} f^b_{ac} \partial_\mu B^c_{\rho\sigma} + \frac{\partial \mathcal{L}_1}{\partial B^a_\alpha} \\ & \quad + \frac{\partial \mathcal{L}_1}{\partial B^b_{\alpha\rho}} f^b_{ac} B^c_\rho + \frac{\partial \mathcal{L}_1}{\partial \partial_\mu B^b_{\alpha\rho}} f^b_{ac} \partial_\mu B^c_\rho + \frac{\partial \mathcal{L}_1}{\partial \partial_{\mu\nu} B^b_{\alpha\rho}} f^b_{ac} \partial_{\mu\nu} B^c_\rho \equiv 0, \end{aligned} \tag{28}$$

$$\begin{aligned} & \frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\beta}\varphi_A} M_{aA}{}^B \varphi_B \\ & \quad + \frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\beta} B^b_\rho} f^b_{ac} B^c_\rho + \frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\beta} B^b_{\rho\sigma}} f^b_{ac} B^c_{\rho\sigma} + \frac{1}{2}\left(\frac{\partial \mathcal{L}_1}{\partial \partial_\alpha B^a_\beta} + \frac{\partial \mathcal{L}_1}{\partial \partial_\beta B^a_\alpha}\right) + \frac{\partial \mathcal{L}_1}{\partial B^a_{\alpha\beta}} \\ & \quad + \frac{1}{2}\left(\frac{\partial \mathcal{L}_1}{\partial \partial_\alpha B^b_{\beta\rho}} + \frac{\partial \mathcal{L}_1}{\partial \partial_\beta B^b_{\alpha\rho}}\right) f^b_{ac} B^c_\rho + \left(\frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\mu} B^b_{\beta\rho}} + \frac{\partial \mathcal{L}_1}{\partial \partial_{\beta\mu} B^b_{\alpha\rho}}\right) f^b_{ac} \partial_\mu B^c_\rho \equiv 0, \end{aligned} \tag{29}$$

$$\begin{aligned} & \frac{1}{3}\left(\frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\beta} B^a_\gamma} + \frac{\partial \mathcal{L}_1}{\partial \partial_{\beta\gamma} B^a_\alpha} + \frac{\partial \mathcal{L}_1}{\partial \partial_{\gamma\alpha} B^a_\beta}\right) + \frac{1}{3}\left(\frac{\partial \mathcal{L}_1}{\partial \partial_\alpha B^a_{\beta\gamma}} + \frac{\partial \mathcal{L}_1}{\partial \partial_\beta B^a_{\gamma\alpha}} + \frac{\partial \mathcal{L}_1}{\partial \partial_\gamma B^a_{\alpha\beta}}\right) \\ & \quad + \frac{1}{3}\left(\frac{\partial \mathcal{L}_1}{\partial \partial_{\alpha\beta} B^b_{\gamma\rho}} + \frac{\partial \mathcal{L}_1}{\partial \partial_{\beta\gamma} B^b_{\alpha\rho}} + \frac{\partial \mathcal{L}_1}{\partial \partial_{\gamma\alpha} B^b_{\beta\rho}}\right) f^b_{ac} B^c_\rho \equiv 0, \end{aligned} \tag{30}$$

$$\frac{1}{6}\left(\frac{\partial\mathcal{L}_1}{\partial\partial_{\alpha\beta}B^a_{\gamma\delta}}+\frac{\partial\mathcal{L}_1}{\partial\partial_{\alpha\gamma}B^a_{\beta\delta}}+\frac{\partial\mathcal{L}_1}{\partial\partial_{\alpha\delta}B^a_{\beta\gamma}}+\frac{\partial\mathcal{L}_1}{\partial\partial_{\gamma\delta}B^a_{\alpha\beta}}+\frac{\partial\mathcal{L}_1}{\partial\partial_{\beta\delta}B^a_{\alpha\gamma}}+\frac{\partial\mathcal{L}_1}{\partial\partial_{\beta\gamma}B^a_{\alpha\delta}}\right)$$
$$\equiv \; 0. \tag{31}$$

Solving these identities will determine the forms of covariant derivatives and gauge-field strengths, through which the gauge fields and their derivatives are contained in the Lagrangian $\mathcal{L}_1$. This is the task of the next two sections.

4 Irreducible decomposition

To solve the identities it is useful to decompose the derivatives of the gauge fields into their irreducible components. This is done by using Young's prescription and by taking into account symmetric properties such as $\partial_{\mu\nu}=\partial_{\nu\mu}$ and $B^a_{\mu\nu}=B^a_{\nu\mu}$.

The first and second derivatives of B^a_μ are decomposed as

$$F^{(i)}_{\alpha\beta} \stackrel{\mathrm{d}}{\equiv} \Phi^{(i)\mu\nu}_{\alpha\beta}\partial_\mu B_\nu, \qquad (i=1,2) \tag{32}$$

$$C^{(i)}_{\alpha\beta\gamma} \stackrel{\mathrm{d}}{\equiv} \Gamma^{(i)\lambda\mu\nu}_{\alpha\beta\gamma}\partial_{\lambda\mu}B_\nu, \qquad (i=1,2) \tag{33}$$

where

$$\begin{cases} \Phi^{(1)\mu\nu}_{\alpha\beta} \stackrel{\mathrm{d}}{\equiv} \frac{1}{2}\left(\delta^\mu_\alpha\delta^\nu_\beta+\delta^\mu_\beta\delta^\nu_\alpha\right) \stackrel{\mathrm{d}}{\equiv} \delta^{\mu\nu}_{\alpha\beta}, \\ \Phi^{(2)\mu\nu}_{\alpha\beta} \stackrel{\mathrm{d}}{\equiv} \frac{1}{2}\left(\delta^\mu_\alpha\delta^\nu_\beta-\delta^\mu_\beta\delta^\nu_\alpha\right); \end{cases} \tag{34}$$

and

$$\begin{cases} \Gamma^{(1)\lambda\mu\nu}_{\alpha\beta\gamma} \stackrel{\mathrm{d}}{\equiv} \frac{1}{3}\left(\delta^{\lambda\mu}_{\alpha\beta}\delta^\nu_\gamma+\delta^{\lambda\mu}_{\beta\gamma}\delta^\nu_\alpha+\delta^{\lambda\mu}_{\gamma\alpha}\delta^\nu_\beta\right), \\ \Gamma^{(2)\lambda\mu\nu}_{\alpha\beta\gamma} \stackrel{\mathrm{d}}{\equiv} \frac{1}{4}\left(2\delta^{\lambda\mu}_{\alpha\beta}\delta^\nu_\gamma-\delta^{\lambda\mu}_{\beta\gamma}\delta^\nu_\alpha-\delta^{\lambda\mu}_{\gamma\alpha}\delta^\nu_\beta\right). \end{cases} \tag{35}$$

The first and second derivatives of $B^a_{\mu\nu}$ are decomposed as

$$D^{(i)}_{\alpha\beta\gamma} \stackrel{\mathrm{d}}{\equiv} \Delta^{(i)\lambda\mu\nu}_{\alpha\beta\gamma}\partial_\lambda B_{\mu\nu}, \qquad (i=1,2) \tag{36}$$

$$G^{(i)}_{\alpha\beta\gamma\delta} \stackrel{\mathrm{d}}{\equiv} \Xi^{(i)\kappa\lambda\mu\nu}_{\alpha\beta\gamma\delta}\partial_{\kappa\lambda}B_{\mu\nu}, \qquad (i=1,2,3) \tag{37}$$

where

$$\begin{cases} \Delta^{(1)\lambda\mu\nu}_{\alpha\beta\gamma} \stackrel{\mathrm{d}}{\equiv} \Gamma^{(1)\mu\nu\lambda}_{\alpha\beta\gamma}, \\ \Delta^{(2)\lambda\mu\nu}_{\alpha\beta\gamma} \stackrel{\mathrm{d}}{\equiv} \Gamma^{(2)\mu\nu\lambda}_{\alpha\beta\gamma}; \end{cases} \tag{38}$$

and

$$\begin{cases} \Xi^{(1)\kappa\lambda\mu\nu}_{\alpha\beta\gamma\delta} \stackrel{\mathrm{d}}{\equiv} \frac{1}{6}\left[\left(\delta^{\kappa\lambda}_{\alpha\beta}\delta^{\mu\nu}_{\gamma\delta}+\delta^{\kappa\lambda}_{\gamma\delta}\delta^{\mu\nu}_{\alpha\beta}\right)+\left(\delta^{\kappa\lambda}_{\alpha\gamma}\delta^{\mu\nu}_{\beta\delta}+\delta^{\kappa\lambda}_{\beta\delta}\delta^{\mu\nu}_{\alpha\gamma}\right)+\left(\delta^{\kappa\lambda}_{\beta\gamma}\delta^{\mu\nu}_{\alpha\delta}+\delta^{\kappa\lambda}_{\alpha\delta}\delta^{\mu\nu}_{\beta\gamma}\right)\right], \\ \Xi^{(2)\kappa\lambda\mu\nu}_{\alpha\beta\gamma\delta} \stackrel{\mathrm{d}}{\equiv} \frac{1}{6}\left[\left(\delta^{\kappa\lambda}_{\alpha\beta}\delta^{\mu\nu}_{\gamma\delta}-\delta^{\kappa\lambda}_{\gamma\delta}\delta^{\mu\nu}_{\alpha\beta}\right)+\left(\delta^{\kappa\lambda}_{\alpha\gamma}\delta^{\mu\nu}_{\beta\delta}-\delta^{\kappa\lambda}_{\beta\delta}\delta^{\mu\nu}_{\alpha\gamma}\right)+\left(\delta^{\kappa\lambda}_{\beta\gamma}\delta^{\mu\nu}_{\alpha\delta}-\delta^{\kappa\lambda}_{\alpha\delta}\delta^{\mu\nu}_{\beta\gamma}\right)\right], \\ \Xi^{(3)\kappa\lambda\mu\nu}_{\alpha\beta\gamma\delta} \stackrel{\mathrm{d}}{\equiv} \frac{1}{8}\left[2\left(\delta^{\kappa\lambda}_{\alpha\beta}\delta^{\mu\nu}_{\gamma\delta}+\delta^{\kappa\lambda}_{\gamma\delta}\delta^{\mu\nu}_{\alpha\beta}\right)-\left(\delta^{\kappa\lambda}_{\alpha\gamma}\delta^{\mu\nu}_{\beta\delta}+\delta^{\kappa\lambda}_{\beta\delta}\delta^{\mu\nu}_{\alpha\gamma}\right)-\left(\delta^{\kappa\lambda}_{\beta\gamma}\delta^{\mu\nu}_{\alpha\delta}+\delta^{\kappa\lambda}_{\alpha\delta}\delta^{\mu\nu}_{\beta\gamma}\right)\right]. \end{cases} \tag{39}$$

The Lagrangian $\mathcal{L}_1$ is a function of these irreducible components:

$$\mathcal{L}_1 \equiv \mathcal{L}_2\left(\varphi,\partial_\mu\varphi,\partial_{\mu\nu}\varphi;B_\rho,F^{(1)},F^{(2)},C^{(1)},C^{(2)};B_{\rho\sigma},D^{(1)},D^{(2)},G^{(1)},G^{(2)},G^{(3)}\right). \tag{40}$$

5 Covariant derivatives and field strengths

For the Lagrangian $\mathcal{L}_2$ the identities (31) reduce to

$$\frac{\partial \mathcal{L}_2}{\partial G^{(1)a}_{\alpha\beta\gamma\delta}} \equiv 0. \tag{41}$$

These identities tell the fact that the Lagrangian $\mathcal{L}_2$ is independent of $G^{(1)}$:

$$\mathcal{L}_2 \equiv \mathcal{L}_3\left(\varphi, \partial_\mu\varphi, \partial_{\mu\nu}\varphi; B_\rho, F^{(1)}, F^{(2)}, C^{(1)}, C^{(2)}; B_{\rho\sigma}, D^{(1)}, D^{(2)}, G^{(2)}, G^{(3)}\right). \tag{42}$$

The identities (30) are rewritten for the Lagrangian $\mathcal{L}_3$ as

$$\begin{aligned}&\frac{\partial \mathcal{L}_3}{\partial C^{(1)a}_{\alpha\beta\gamma}} + \frac{\partial \mathcal{L}_3}{\partial D^{(1)a}_{\alpha\beta\gamma}} \\ &\qquad + \frac{1}{3}\frac{\partial \mathcal{L}_3}{\partial G^{(2)b}_{\mu\nu\rho\sigma}} f^b_{ac}\left(\Xi^{(2)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} + \Xi^{(2)\beta\gamma\alpha\delta}_{\mu\nu\rho\sigma} + \Xi^{(2)\gamma\alpha\beta\delta}_{\mu\nu\rho\sigma}\right) B^c_\delta \equiv 0.\end{aligned} \tag{43}$$

These identities show that the Lagrangian $\mathcal{L}_3$ should have $D^{(1)}$ in the form of

$$\mathcal{L}_3 \equiv \mathcal{L}_4\left(\varphi, \partial_\mu\varphi, \partial_{\mu\nu}\varphi; B_\rho, F^{(1)}, F^{(2)}, \tilde{C}^{(1)}, C^{(2)}; B_{\rho\sigma}, D^{(2)}, \tilde{G}^{(2)}, G^{(3)}\right), \tag{44}$$

where

$$\tilde{C}^{(1)a}_{\alpha\beta\gamma} \stackrel{\mathrm{d}}{\equiv} C^{(1)a}_{\alpha\beta\gamma} - D^{(1)a}_{\alpha\beta\gamma}, \tag{45}$$

$$\tilde{G}^{(2)a}_{\alpha\beta\gamma\delta} \stackrel{\mathrm{d}}{\equiv} G^{(2)a}_{\alpha\beta\gamma\delta} - f^a_{bc}\Xi^{(2)\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} D^{(1)b}_{\mu\nu\rho} B^c_\sigma. \tag{46}$$

The Lagrangian $\mathcal{L}_4$ changes the form of the identities (29) into

$$\begin{aligned}&\frac{\partial \mathcal{L}_4}{\partial\partial_{\alpha\beta}\varphi_A} M_{aA}{}^B\varphi_B \\ &+ \frac{\partial \mathcal{L}_4}{\partial C^{(2)b}_{\mu\nu\rho}} f^b_{ac}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho} B^c_\gamma - \frac{1}{2}\frac{\partial \mathcal{L}_4}{\partial D^{(2)b}_{\mu\nu\rho}} f^b_{ac}\Delta^{(2)\gamma\alpha\beta}_{\mu\nu\rho} B^c_\gamma \\ &+ \frac{\partial \mathcal{L}_4}{\partial \tilde{G}^{(2)b}_{\mu\nu\rho\sigma}}\left\{f^b_{ac}\left[\Xi^{(2)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} B^c_{\gamma\delta} + \left(\Xi^{(2)\alpha\gamma\beta\delta}_{\mu\nu\rho\sigma} + \Xi^{(2)\beta\gamma\alpha\delta}_{\mu\nu\rho\sigma}\right) F^{(2)c}_{\gamma\delta}\right]\right. \\ &\qquad\qquad\qquad\qquad \left. + f^d_{ea} f^b_{dc}\Xi^{(2)\xi\eta\zeta\upsilon}_{\mu\nu\rho\sigma}\Delta^{(1)\alpha\beta\gamma}_{\xi\eta\zeta} B^c_\upsilon B^e_\gamma\right\} \\ &+ \frac{\partial \mathcal{L}_4}{\partial G^{(3)b}_{\mu\nu\rho\sigma}} f^b_{ac}\Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\left(-F^{(1)c}_{\gamma\delta} + B^c_{\gamma\delta}\right) \\ &+ \frac{\partial \mathcal{L}_4}{\partial F^{(1)a}_{\alpha\beta}} + \frac{\partial \mathcal{L}_4}{\partial B^a_{\alpha\beta}} \equiv 0.\end{aligned} \tag{47}$$

They require that the gauge fields $B^a_{\mu\nu}$ be incorporated into $\mathcal{L}_4$ in the form of

$$\mathcal{L}_4 \equiv \mathcal{L}_5\left(\varphi, \partial_\mu\varphi, \nabla_{\mu\nu}\varphi; B_\rho, \tilde{F}^{(1)}, F^{(2)}, \tilde{C}^{(1)}, \tilde{C}^{(2)}; \tilde{D}^{(2)}, \tilde{\tilde{G}}^{(2)}, \tilde{G}^{(3)}\right), \tag{48}$$

where

$$\nabla_{\mu\nu}\varphi_A \stackrel{\rm d}{\equiv} \partial_{\mu\nu}\varphi_A - B^a_{\mu\nu}M_{aA}{}^B\varphi_B, \tag{49}$$

$$\tilde{F}^{(1)a}_{\alpha\beta} \stackrel{\rm d}{\equiv} F^{(1)a}_{\alpha\beta} - B^a_{\alpha\beta}, \tag{50}$$

$$\tilde{C}^{(2)a}_{\alpha\beta\gamma} \stackrel{\rm d}{\equiv} C^{(2)a}_{\alpha\beta\gamma} - f^a_{bc}\Gamma^{(2)\mu\nu\rho}_{\alpha\beta\gamma}B^b_{\mu\nu}B^c_\rho, \tag{51}$$

$$\tilde{D}^{(2)a}_{\alpha\beta\gamma} \stackrel{\rm d}{\equiv} D^{(2)a}_{\alpha\beta\gamma} + \frac{1}{2}f^a_{bc}\Gamma^{(2)\mu\nu\rho}_{\alpha\beta\gamma}B^b_{\mu\nu}B^c_\rho, \tag{52}$$

$$\begin{aligned}\tilde{\tilde{G}}^{(2)a}_{\alpha\beta\gamma\delta} &\stackrel{\rm d}{\equiv} \tilde{G}^{(2)a}_{\alpha\beta\gamma\delta} - f^a_{bc}\left(\frac{1}{2}\Xi^{(2)\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}B^b_{\mu\nu}B^c_{\rho\sigma} + 2\Xi^{(2)\mu\rho\nu\sigma}_{\alpha\beta\gamma\delta}B^b_{\mu\nu}F^{(2)c}_{\rho\sigma}\right)\\ &\quad + f^e_{bc}f^a_{ed}\Xi^{(2)\xi\eta\zeta\sigma}_{\alpha\beta\gamma\delta}\Gamma^{(1)\mu\nu\rho}_{\xi\eta\zeta}B^b_{\mu\nu}B^c_\rho B^d_\sigma,\end{aligned} \tag{53}$$

$$\tilde{G}^{(3)a}_{\alpha\beta\gamma\delta} \stackrel{\rm d}{\equiv} G^{(3)a}_{\alpha\beta\gamma\delta} + f^a_{bc}\Xi^{(3)\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}B^b_{\mu\nu}F^{(1)c}_{\rho\sigma}. \tag{54}$$

The Lagrangian $\mathcal{L}_5$ gives the identities (28) the following form:

$$\begin{aligned}&\frac{\partial\mathcal{L}_5}{\partial\partial_\alpha\varphi_A}M_{aA}{}^B\varphi_B + 2\frac{\partial\mathcal{L}_5}{\partial\nabla_{\alpha\mu}\varphi_A}\left(M_{aA}{}^B\partial_\mu\varphi_B - \frac{1}{2}f^b_{ac}M_{bA}{}^B B^c_\mu\varphi_B\right)\\ &+ \frac{\partial\mathcal{L}_5}{\partial F^{(2)b}_{\mu\nu}}f^b_{ac}\Phi^{(2)\alpha\beta}_{\mu\nu}B^c_\beta\\ &+ \frac{\partial\mathcal{L}_5}{\partial\tilde{C}^{(1)b}_{\mu\nu\rho}}f^b_{ac}\Gamma^{(1)\alpha\beta\gamma}_{\mu\nu\rho}\tilde{F}^{(1)c}_{\alpha\beta}\\ &+ 2\frac{\partial\mathcal{L}_5}{\partial\tilde{C}^{(2)b}_{\mu\nu\rho}}\left\{f^b_{ac}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho}\left(\tilde{F}^{(1)c}_{\beta\gamma} + F^{(2)c}_{\beta\gamma}\right) - \frac{1}{2}f^e_{ac}f^b_{ed}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho}B^c_\beta B^d_\gamma\right\}\\ &+ \frac{\partial\mathcal{L}_5}{\partial\tilde{D}^{(2)b}_{\mu\nu\rho}}\left\{f^b_{ac}\Delta^{(2)\beta\gamma\alpha}_{\mu\nu\rho}\left(\tilde{F}^{(1)c}_{\beta\gamma} + F^{(2)c}_{\beta\gamma}\right) + \frac{1}{2}f^e_{ac}f^b_{ed}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho}B^c_\beta B^d_\gamma\right\}\\ &+ \frac{\partial\mathcal{L}_5}{\partial\tilde{\tilde{G}}^{(2)b}_{\mu\nu\rho\sigma}}\left\{f^b_{ac}\Xi^{(2)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\left(-\tilde{C}^{(1)c}_{\beta\gamma\delta} + \frac{8}{3}\tilde{C}^{(2)c}_{\beta\gamma\delta} + \frac{8}{3}\tilde{D}^{(2)c}_{\gamma\delta\beta}\right)\right.\\ &\qquad + f^e_{da}f^b_{ec}\Xi^{(2)\xi\eta\zeta\beta}_{\mu\nu\rho\sigma}\Gamma^{(1)\alpha\gamma\delta}_{\xi\eta\zeta}B^c_\beta\tilde{F}^{(1)d}_{\gamma\delta} - 2f^e_{ac}f^b_{ed}\Xi^{(2)\alpha\gamma\beta\delta}_{\mu\nu\rho\sigma}B^c_\beta F^{(2)d}_{\gamma\delta}\\ &\qquad \left. + f^f_{ac}f^g_{fd}f^b_{ge}\Xi^{(2)\xi\eta\zeta\delta}_{\mu\nu\rho\sigma}\Gamma^{(1)\alpha\beta\gamma}_{\xi\eta\zeta}B^c_\beta B^d_\gamma B^e_\delta\right\}\\ &+ \frac{\partial\mathcal{L}_5}{\partial\tilde{G}^{(3)b}_{\mu\nu\rho\sigma}}\left\{-\frac{8}{3}f^b_{ac}\Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\left(\tilde{C}^{(2)c}_{\beta\gamma\delta} - \tilde{D}^{(2)c}_{\gamma\delta\beta}\right) + f^e_{ac}f^b_{ed}\Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}B^c_\beta\tilde{F}^{(1)d}_{\gamma\delta}\right\}\\ &+ \frac{\partial\mathcal{L}_5}{\partial B^a_\alpha} \equiv 0.\end{aligned} \tag{55}$$

From them we find that the gauge fields B^a_μ should be contained as follows:

$$\mathcal{L}_5 \equiv \mathcal{L}_6\left(\varphi, \nabla_\mu\varphi, \tilde{\nabla}_{\mu\nu}\varphi; \tilde{F}^{(1)}, \tilde{F}^{(2)}, \tilde{\tilde{C}}^{(1)}, \tilde{\tilde{C}}^{(2)}; \tilde{\tilde{D}}^{(2)}, \tilde{\tilde{\tilde{G}}}^{(2)}, \tilde{\tilde{G}}^{(3)}\right), \tag{56}$$

where

$$\nabla_\mu\varphi_A \stackrel{\rm d}{\equiv} \partial_\mu\varphi_A - B^a_\mu M_{aA}{}^B\varphi_B, \tag{57}$$

$$\tilde{\nabla}_{\mu\nu}\varphi_A \stackrel{\rm d}{\equiv} \nabla_{\mu\nu}\varphi_A - \left(B^a_\mu M_{aA}{}^B\partial_\nu\varphi_B + B^a_\nu M_{aA}{}^B\partial_\mu\varphi_B\right)$$

$$+\frac{1}{2}B^a_\mu B^b_\nu\{M_a, M_b\}_A{}^B\varphi_B, \tag{58}$$

$$\tilde{F}^{(2)a}_{\mu\nu} \stackrel{\rm d}{\equiv} F^{(2)a}_{\mu\nu} - \frac{1}{2}f^a_{bc}B^b_\mu B^c_\nu, \tag{59}$$

$$\tilde{\tilde{C}}^{(1)a}_{\mu\nu\rho} \stackrel{\rm d}{\equiv} \tilde{C}^{(1)a}_{\mu\nu\rho} - f^a_{bc}\Gamma^{(1)\alpha\beta\gamma}_{\mu\nu\rho}B^b_\alpha\tilde{F}^{(1)c}_{\beta\gamma}, \tag{60}$$

$$\tilde{\tilde{C}}^{(2)a}_{\mu\nu\rho} \stackrel{\rm d}{\equiv} \tilde{C}^{(2)a}_{\mu\nu\rho} - 2f^a_{bc}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho}B^b_\alpha\left(\tilde{F}^{(1)c}_{\beta\gamma} + F^{(2)c}_{\beta\gamma}\right) + \frac{2}{3}f^e_{bc}f^a_{ed}\Gamma^{(2)\beta\gamma\alpha}_{\mu\nu\rho}B^b_\alpha B^c_\beta B^d_\gamma, \tag{61}$$

$$\tilde{\tilde{D}}^{(2)a}_{\mu\nu\rho} \stackrel{\rm d}{\equiv} \tilde{D}^{(2)a}_{\mu\nu\rho} - f^a_{bc}\Delta^{(2)\beta\gamma\alpha}_{\mu\nu\rho}B^b_\alpha\left(\tilde{F}^{(1)c}_{\beta\gamma} + F^{(2)c}_{\beta\gamma}\right) - \frac{1}{3}f^e_{bc}f^a_{ed}\Gamma^{(2)\beta\gamma\alpha}_{\mu\nu\rho}B^b_\alpha B^c_\beta B^d_\gamma, \tag{62}$$

$$\begin{aligned}\tilde{\tilde{G}}^{(2)a}_{\mu\nu\rho\sigma} \stackrel{\rm d}{\equiv}\; & \tilde{G}^{(2)a}_{\mu\nu\rho\sigma} - f^a_{bc}\Xi^{(2)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}B^b_\alpha\left(-\tilde{C}^{(1)c}_{\beta\gamma\delta} + \frac{8}{3}\tilde{C}^{(2)c}_{\beta\gamma\delta} + \frac{8}{3}\tilde{D}^{(2)c}_{\gamma\delta\beta}\right) \\ & - 2f^e_{bd}f^a_{ec}\Xi^{(2)\alpha\gamma\beta\delta}_{\mu\nu\rho\sigma}B^b_\alpha B^c_\beta F^{(2)d}_{\gamma\delta} \\ & - \frac{1}{2}f^f_{bd}f^g_{fc}f^a_{ge}\Xi^{(2)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}B^b_\alpha B^c_\beta B^d_\gamma B^e_\delta,\end{aligned} \tag{63}$$

$$\begin{aligned}\tilde{\tilde{G}}^{(3)a}_{\mu\nu\rho\sigma} \stackrel{\rm d}{\equiv}\; & \tilde{G}^{(3)a}_{\mu\nu\rho\sigma} + \frac{8}{3}f^a_{bc}\Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}B^b_\alpha\left(\tilde{C}^{(2)c}_{\beta\gamma\delta} - \tilde{D}^{(2)c}_{\gamma\delta\beta}\right) \\ & + f^e_{bd}f^a_{ec}\Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}B^b_\alpha B^c_\beta\tilde{F}^{(1)d}_{\gamma\delta}.\end{aligned} \tag{64}$$

The Lagrangian $\mathcal{L}_6$ should satisfy the last identities (27) rewritten as

$$\begin{aligned}&\frac{\partial\mathcal{L}_6}{\partial\varphi_A}M_{aA}{}^B\varphi_B + \frac{\partial\mathcal{L}_6}{\partial\nabla_\mu\varphi_A}M_{aA}{}^B\nabla_\mu\varphi_B + \frac{\partial\mathcal{L}_6}{\partial\tilde{\nabla}_{\mu\nu}\varphi_A}M_{aA}{}^B\tilde{\nabla}_{\mu\nu}\varphi_B \\ &+ \frac{\partial\mathcal{L}_6}{\tilde{F}^{(1)b}_{\mu\nu}}f^b_{ac}\tilde{F}^{(1)c}_{\mu\nu} + \frac{\partial\mathcal{L}_6}{\tilde{F}^{(2)b}_{\mu\nu}}f^b_{ac}\tilde{F}^{(2)c}_{\mu\nu} + \frac{\partial\mathcal{L}_6}{\tilde{\tilde{C}}^{(1)b}_{\mu\nu\rho}}f^b_{ac}\tilde{\tilde{C}}^{(1)c}_{\mu\nu\rho} + \frac{\partial\mathcal{L}_6}{\tilde{\tilde{C}}^{(2)b}_{\mu\nu\rho}}f^b_{ac}\tilde{\tilde{C}}^{(2)c}_{\mu\nu\rho} \\ &+ \frac{\partial\mathcal{L}_6}{\tilde{\tilde{D}}^{(2)b}_{\mu\nu\rho}}f^b_{ac}\tilde{\tilde{D}}^{(2)c}_{\mu\nu\rho} + \frac{\partial\mathcal{L}_6}{\tilde{\tilde{G}}^{(2)b}_{\mu\nu\rho\sigma}}f^b_{ac}\tilde{\tilde{G}}^{(2)c}_{\mu\nu\rho\sigma} + \frac{\partial\mathcal{L}_6}{\tilde{\tilde{G}}^{(3)b}_{\mu\nu\rho\sigma}}f^b_{ac}\tilde{\tilde{G}}^{(3)c}_{\mu\nu\rho\sigma} \equiv 0.\end{aligned} \tag{65}$$

They show the Lagrangian $\mathcal{L}_6$ should be an arbitrary G-invariant function of the arguments. This is our final result.

In the above we have determined the covariant derivatives and gauge-field strengths step by step. For completeness we express them by using the original gauge fields and the irreducible components of their derivatives:

$$\nabla_\mu\varphi_A = \partial_\mu\varphi_A - B^a_\mu M_{aA}{}^B\varphi_B, \tag{66}$$

$$\begin{aligned}\tilde{\nabla}_{\mu\nu}\varphi_A =\; & \partial_{\mu\nu}\varphi_A - B^a_{\mu\nu}M_{aA}{}^B - \left(B^a_\mu M_{aA}{}^B\partial_\nu\varphi_B + B^a_\nu M_{aA}{}^B\partial_\mu\varphi_B\right) \\ & + \frac{1}{2}B^a_\mu B^b_\nu\{M_a, M_b\}_A{}^B\varphi_B,\end{aligned} \tag{67}$$

$$\tilde{F}^{(1)a}_{\mu\nu} = F^{(1)a}_{\mu\nu} - B^a_{\mu\nu}, \tag{68}$$

$$\tilde{F}^{(2)a}_{\mu\nu} = F^{(2)a}_{\mu\nu} - \frac{1}{2}f^a_{bc}B^b_\mu B^c_\nu, \tag{69}$$

$$\tilde{\tilde{C}}^{(1)a}_{\mu\nu\rho} = C^{(1)a}_{\mu\nu\rho} - D^{(1)a}_{\mu\nu\rho} - f^a_{bc}\Gamma^{(1)\alpha\beta\gamma}_{\mu\nu\rho}B^b_\alpha F^{(1)c}_{\beta\gamma} + f^a_{bc}\Gamma^{(1)\alpha\beta\gamma}_{\mu\nu\rho}B^b_\alpha B^c_{\beta\gamma}, \tag{70}$$

$$\tilde{\tilde{C}}^{(2)a}_{\mu\nu\rho} = C^{(2)a}_{\mu\nu\rho} - 2f^a_{bc}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho}B^b_\alpha\left(F^{(1)c}_{\beta\gamma} + F^{(2)c}_{\beta\gamma}\right) + \frac{2}{3}f^a_{be}f^e_{cd}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho}B^b_\alpha B^c_\beta B^d_\gamma, \tag{71}$$

$$\tilde{\tilde{D}}^{(2)a}_{\mu\nu\rho} = D^{(2)a}_{\mu\nu\rho} - f^a_{bc}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho} B^b_\alpha \left(F^{(1)c}_{\beta\gamma} - F^{(2)c}_{\beta\gamma}\right) + 2 f^a_{bc}\Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho} B^b_\alpha B^c_{\beta\gamma} - \frac{1}{3} f^a_{be} f^e_{cd} \Gamma^{(2)\alpha\beta\gamma}_{\mu\nu\rho} B^b_\alpha B^c_\beta B^d_\gamma, \quad (72)$$

$$\begin{aligned}\tilde{\tilde{G}}^{(2)a}_{\mu\nu\rho\sigma} = {} & G^{(2)a}_{\mu\nu\rho\sigma} - 2 f^a_{bc}\Xi^{(2)\alpha\gamma\beta\delta}_{\mu\nu\rho\sigma} B^b_{\alpha\beta} F^{(2)c}_{\gamma\delta} + 2 f^a_{be} f^e_{cd} \Xi^{(2)\alpha\gamma\beta\delta}_{\mu\nu\rho\sigma} B^b_\alpha B^c_\beta F^{(2)d}_{\gamma\delta} \\ & + f^a_{bc}\Xi^{(2)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} B^b_\alpha \left(C^{(1)c}_{\beta\gamma\delta} - \frac{8}{3} C^{(2)c}_{\beta\gamma\delta} - 2 D^{(1)c}_{\beta\gamma\delta} + \frac{16}{3} D^{(2)c}_{\beta\gamma\delta}\right) \\ & - \frac{1}{2} f^a_{bc}\Xi^{(2)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} B^b_{\alpha\beta} B^c_{\gamma\delta} + f^a_{be} f^e_{cd} \left(\Xi^{(2)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} - 2\Xi^{(2)\alpha\gamma\beta\delta}_{\mu\nu\rho\sigma}\right) B^b_\alpha B^c_\beta B^d_{\gamma\delta} \\ & - \frac{1}{2} f^a_{bg} f^g_{cf} f^f_{de} \Xi^{(2)\alpha\gamma\beta\delta}_{\mu\nu\rho\sigma} B^b_\alpha B^c_\beta B^d_\gamma B^e_\delta, \end{aligned} \quad (73)$$

$$\begin{aligned}\tilde{\tilde{G}}^{(3)a}_{\mu\nu\rho\sigma} = {} & G^{(3)a}_{\mu\nu\rho\sigma} + f^a_{bc}\Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} B^b_{\alpha\beta} F^{(1)c}_{\gamma\delta} - f^a_{be} f^e_{cd} \Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} B^b_\alpha B^c_\beta F^{(1)d}_{\gamma\delta} \\ & + \frac{8}{3} f^a_{bc}\Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} B^b_\alpha \left(C^{(2)c}_{\beta\gamma\delta} + 2 D^{(2)c}_{\beta\gamma\delta}\right) \\ & + f^a_{be} f^e_{cd} \Xi^{(3)\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} B^b_\alpha B^c_\beta B^d_{\gamma\delta}. \end{aligned} \quad (74)$$

6 Summary and Discussion

In the present paper we have generalized the usual prescription for constructing gauge-invariant Lagrangian to the case of including second derivatives of fields as well as first derivatives. By solving a series of identities which follow from generalized Noether's theorems, we have found the covariant derivatives and the gauge-field strengths.

Many problems remain to be solved:
—physical implications of the tensor gauge fiels;
—geometrical meanings of gauging second derivatives;
—extension to space-time symmetries;
—higher-derivative gravity in the light of this new type of gauge theories.

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