# Deformation of Properly Discontinuous Actions of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$

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**Abstract**

We consider the deformation of a discontinuous group acting on the Euclidean space by affine transformations. A distinguished feature here is that even a 'small' deformation of a discrete subgroup may destroy proper discontinuity of its action. In order to understand the local structure of the deformation space of discontinuous groups, we introduce the concepts from a group theoretic perspective, and focus on 'stability' and 'local rigidity' of discontinuous groups. As a test case, we give an explicit description of the deformation space of $\mathbb{Z}^k$ acting properly discontinuously on $\mathbb{R}^{k+1}$ by affine nilpotent transformations. Our method uses an idea of 'continuous analogue' and relies on the criterion of proper actions on nilmanifolds.

# 1 Local rigidity and stability

Our concern in this article is with the deformation of discontinuous groups for non-Riemannian homogeneous spaces.

## 1.1 Deformation of discontinuous groups — the non-Riemannian case

In contrast to the traditional case of discontinuous groups acting on Riemannian manifolds as isometries, our problem in the non-Riemannian case includes the following subproblem: if a discrete subgroup can be deformed, determine the range of the deformation parameters that does not destroy the proper discontinuity of its action.

As a clue to understanding the local structure of the 'deformation space', we consider a manifold $X$ acted on by a Lie group $G$. Suppose $\Gamma$ is a discontinuous group for $X$, that is, $\Gamma$ is a discrete subgroup of $G$ acting properly discontinuously and freely on $X$. Let $\Gamma'$ be another discrete subgroup of $G$ which is 'sufficiently close' to $\Gamma$. Now, our basic question is if the following statements hold or not:

**(R)′** : (*Local Rigidity*) $\Gamma'$ is conjugate to $\Gamma$ by an inner automorphism of $G$.

**(S)′** : (*Stability*) The $\Gamma'$-action on $X$ is properly discontinuous and free.

For a homogeneous space $X = G/H$ ($H$ being a closed subgroup of $G$), obvious remarks are:

1) If $H$ is compact, then (S)′ automatically holds.

2) If (R)′ holds, so does (S)′.

This article studies the deformation of discontinuous groups for $G/H$ in the case that (S)′ does not hold. This implies particularly that $H$ is noncompact and that (R)′ does not hold.

Let us now formalize the above two statements (R)′ and (S)′ more rigorously. We begin with an (abstract) finitely generated group $\Gamma$, and denote by $\mathrm{Hom}(\Gamma, G)$ the set of all group homomorphisms of $\Gamma$ into a Lie group $G$. Taking generators $\gamma_1, \dots, \gamma_k$ of $\Gamma$, we use the injective map

$$\mathrm{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \quad \varphi \mapsto (\varphi(\gamma_1), \dots, \varphi(\gamma_k))$$

to endow $\mathrm{Hom}(\Gamma, G)$ with the induced topology from the direct product $G \times \cdots \times G$. This topology is independent of the choice of generators. Now, suppose $G$ acts continuously on a manifold $X$. We recall from [6] the following definition:

$$
\begin{aligned}
(1.1) \quad R(\Gamma, G; X) := \{\varphi \in \mathrm{Hom}(\Gamma, G) \ : \ &\varphi \text{ is injective, and} \\
&\varphi(\Gamma) \text{ acts properly discontinuously and freely on } X\}.
\end{aligned}
$$

We remark that our notation here is slightly different from that in [6, 9]: for a homogeneous space $X = G/H$ our notation $R(\Gamma, G; G/H)$ here coincides with $R(\Gamma, G, H)$ loc. cit.

For each $\varphi \in R(\Gamma, G; X)$, the quotient space $\varphi(\Gamma)\backslash X \simeq \varphi(\Gamma)\backslash G/H$ (a *Clifford–Klein form* of $X$) becomes a Hausdorff topological space, and can be given a unique manifold structure for which the natural quotient map $X \to \varphi(\Gamma)\backslash X$ is a local diffeomorphism. Therefore, the Clifford–Klein form $\varphi(\Gamma)\backslash X$ enjoys all $G$-invariant local geometric structures on $X$. Thus, $R(\Gamma, G; X)$ may be regarded as a parameter space of Clifford–Klein forms $\varphi(\Gamma)\backslash X$ with parameter $\varphi$.

To be more precise on 'parameter', we should take into account 'unessential' deformation arising from inner automorphisms. If two homomorphisms $\varphi_1$ and $\varphi_2$ belonging to $R(\Gamma, G; X)$ satisfy $\varphi_2 = g \circ \varphi_1 \circ g^{-1}$ for some $g \in G$, then the corresponding Clifford–Klein forms are isomorphic to each other by the natural diffeomorphism $\varphi_1(\Gamma)\backslash X \xrightarrow{\sim} \varphi_2(\Gamma)\backslash X$, $\varphi_1(\Gamma)xH \mapsto \varphi_2(\Gamma)gxH$. In light of this observation, we define the *deformation space* as the quotient set

$$
(1.2) \qquad \mathcal{T}(\Gamma, G; X) := R(\Gamma, G; X)/G.
$$

For example, if $G = PSL(2, \mathbb{R})$ and $X$ is the upper half plane, and if $\Gamma$ is the fundamental group of a closed Riemann surface $M_g$ of genus $g \geq 2$, then $\mathcal{T}(\Gamma, G; X)$ is nothing other than the Teichmüller space of $M_g$. We refer the reader to an expository paper [9] for some elementary examples of the deformation space for non-Riemannian $X$.

Suppose now that $\varphi_0 : \Gamma \to G$ belongs to $R(\Gamma, G; X)$, and we reformalize (R)′ and (S)′ as follows:

**(R):** (*Local rigidity*) $G \cdot \varphi_0$ is open in $\mathrm{Hom}(\Gamma, G)$.

**(S):** (*Stability*) There is an open subset $V$ of $\mathrm{Hom}(\Gamma, G)$ such that $\varphi_0 \in V \subset R(\Gamma, G; X)$.

We say $\varphi_0 \in R(\Gamma, G; X)$ is *locally rigid as a discontinuous group* for $X$ if (R) holds. For a Riemannian symmetric space $X$, our terminology here is consistent with Weil's terminology used in [16].

A celebrated Selberg–Weil rigidity [16] for an irreducible **Riemannian** symmetric space $X$ asserts that (R) holds for any torsion free uniform lattice $\varphi_0(\Gamma)$ of $G$ unless $G$ is locally isomorphic to $SL(2, \mathbb{R})$, whereas $R(\Gamma, G; X)$ is always open in $\mathrm{Hom}(\Gamma, G)$ and thus (S) holds. In contrast, there is an example that (R) fails for an irreducible **non-Riemannian** symmetric space $X$ of an arbitrarily high dimension (see [6]).

The failure of (R) arouses our interest in the deformation space $\mathcal{T}(\Gamma, G; X)$ like the classical Teichmüller theory of Riemann surfaces. Besides, the concept of the stability (S) may be regarded as a first step to understand the local structure of the deformation space $\mathcal{T}(\Gamma, G; X)$ in the setting where (R) fails, in particular, where $H$ is non-compact.

Such a viewpoint traces back to the paper [4] on three dimensional Lorentz space forms, where Goldman discovered a discontinuous group $\Gamma$ for which (R) fails, and raised a question if (S) still holds (not exactly in the way formulated here). His case concerns with a semisimple Lie group $G$ which is locally isomorphic to $SO(2, 1) \times SO(2, 1)$. This question was solved affirmatively, namely, there exists a cocompact discontinuous group $\Gamma$ for which (R) fails but (S) holds in the generality that $G$ is a semisimple Lie group which is locally isomorphic to the direct product of two copies of $SO(n, 1)$ or $SU(n, 1)$ (see [8, 15]). Its proof relies on the criterion for properly discontinuous actions [3, 7] on homogeneous spaces of reductive groups.

For a more general $(\Gamma, G, X)$ such as the affine transformation group $G$, both (R) and (S) can fail, as is seen by the following one dimensional example:

**Example 1.1.** Let $\Gamma := \mathbb{Z}$, and $G$ be the $ax + b$ group, that is, $G = \{(a, b) : a > 0, b \in \mathbb{R}\}$ with the multiplication given by $(a, b) \cdot (a', b') = (aa', ab' + b)$. Consider the affine transformation of $G$ on $X := \mathbb{R}$. Then, $\mathrm{Hom}(\Gamma, G) \simeq G$, whereas $R(\Gamma, G; X) \simeq \{(1, b) : b \neq 0\}$. Hence, neither (R) nor (S) holds.

The above example deals with a homogeneous space $X$ of a solvable group $G$ and with a cocompact discontinuous group $\Gamma$.

## 1.2 Summary of this article

This article analyses the failure of (R) and (S) for homogeneous spaces of **nilpotent** Lie groups $G$. For a simply-connected nilpotent Lie group $G$, $R(\Gamma, G; X)$ becomes open in $\mathrm{Hom}(\Gamma, G)$ and thus (S) always holds if $\Gamma\backslash X$ is compact ([19]). Thus, our interest here is in the case when $\Gamma\backslash X$ is non-compact. As a test case, we initiate a detailed analysis on the deformation space in the following setting:

$$\begin{aligned} \Gamma &:= \mathbb{Z}^k && \text{(free abelian group of rank } k\text{)}, \\ X &:= \mathbb{R}^{k+1} && \text{(nilmanifold)}, \\ G &\subset \mathrm{Aff}(\mathbb{R}^{k+1}) && \text{(a two-step nilpotent subgroup)}, \end{aligned}$$

where $\Gamma$ acts on $X$ as nilpotent affine transformations via $G$. Then, we propose a method of giving a concrete description of $R(\Gamma, G; X)$ and the deformation space $\mathcal{T}(\Gamma, G; X)$ for a specific choice of $G$. Our main results are Theorems 2.3 and 5.1.

Besides, we shall see in Corollary 5.1.1 that the deformation space contains a smooth manifold $\mathcal{T}'(\Gamma, G; X)$ as its open dense subset such that

$$\dim \mathcal{T}'(\Gamma, G; X) = \begin{cases} 2k^2 - 1 & (k : \text{even}), \\ 2k^2 - 2 & (k : \text{odd}, \ \geq 3), \\ \quad 2 & (k = 1). \end{cases}$$

Thus, local rigidity (R) fails for any $k$ because the dimension of the deformation space is positive. In the above formula, one sees that the dimension of the deformation space $\mathcal{T}(\Gamma, G; X)$ has a different feature according to whether $k$ is even or odd. This will be explained by the criterion of properly discontinuous actions which involves the existence of a non-zero real eigenvalue of a certain $k \times k$ matrix, whence the parity of $k$ counts. Moreover, it follows from the complete description of $R(\Gamma, G; X)$ that we can determine for which $\varphi_0 \in R(\Gamma, G; X)$ the stability (S) fails.

Our specific choice of $G$ was motivated by Lipsman's classification [11] of maximal nilpotent affine transformation groups on $\mathbb{R}^3$, in which the two-step nilpotent group $G$ for $k = 2$ played a crucial role. It is noteworthy that for any subgroup $\widetilde{G}$ of the affine transformation group $\mathrm{Aff}(\mathbb{R}^{k+1})$ containing our specific $G$, $R(\Gamma, \widetilde{G}; X)$ is not open in $\mathrm{Hom}(\Gamma, \widetilde{G})$ by Theorem 2.3, and consequently both (R) and (S) fail.

The key idea of our proof is to take a connected subgroup $L$ that contains $\Gamma$ as a cocompact discrete subgroup, and then to show that every injective homomorphism from $\Gamma$ into $G$ extends uniquely to a continuous homomorphism from $L$ into $G$ (an idea of *syndetic hull*). A second step is to find explicitly $\mathrm{Hom}(L, G)$ in place of $\mathrm{Hom}(\Gamma, G)$, and to determine which homomorphism yields a properly discontinuous action. Unlike the reductive case [3, 5, 7], properly discontinuous actions for affine transformation groups on $\mathbb{R}^{k+1}$ are still far from being fully understood in general, as one sees from the current status of the long-standing Auslander conjecture (see [1] and references therein). However, fortunately in our special setting, we can use the criterion [13] of proper actions for two-step nilpotent Lie groups, which was obtained as an affirmative solution to Lipsman's conjecture [11]. Then, the final step to the proof of Theorem 2.3 (the description of $R(\Gamma, G; X)$) is reduced to a certain problem of Lie algebras, which we can solve explicitly.

# 2 Description of deformation parameter

This section gives a complete description of the parameter space $R(\mathbb{Z}^k, G; \mathbb{R}^{k+1})$ of properly discontinuous $\mathbb{Z}^k$-actions on $\mathbb{R}^{k+1}$ through a certain nilpotent affine transformation group $G$. This is the first of the main results of this paper, and is stated in Theorem 2.3. Building on it, we shall determine the deformation space $\mathcal{T}(\Gamma, G; \mathbb{R}^{k+1}) \simeq R(\Gamma, G; \mathbb{R}^{k+1})/G$ in Section 5 (see Theorem 5.1).

## 2.1 Nilpotent affine transformation group

We fix a positive integer $k$. Our basic setting in this paper is:

$$\begin{aligned}
&\Gamma := \mathbb{Z}^k, \\
&G := \left\{ \begin{pmatrix} I_k & \vec{x} & \vec{y} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : \vec{x}, \vec{y} \in \mathbb{R}^k, z \in \mathbb{R} \right\}, \\
&H := \left\{ \begin{pmatrix} I_k & \vec{x} & \vec{0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \vec{x} \in \mathbb{R}^k \right\}.
\end{aligned} \tag{2.1}$$

Then $G$ is a simply connected two-step nilpotent Lie group, and the homogeneous space $G/H$ is diffeomorphic to $\mathbb{R}^{k+1}$. We shall first give a description of

$\mathrm{Hom}(\Gamma, G)$ in Proposition 2.2.1, and then determine explicitly $R(\Gamma, G; \mathbb{R}^{k+1})$ as its subset in Theorem 2.3.

Geometrically, this means that we determine all possible properly discontinuous affine actions of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$ preserving differential forms $d\xi_{k+1}$ and $d\xi_i \wedge d\xi_{k+1}$ $(1 \leq i \leq k)$, where $(\xi_1, \ldots, \xi_{k+1})$ is the coordinate of $\mathbb{R}^{k+1}$.

## 2.2 Description of $\mathrm{Hom}(\Gamma, G)$

Any group homomorphism $\varphi : \Gamma \to G$ is determined by its evaluation at generators of $\Gamma$. Taking a standard basis $\{e_1, \ldots, e_k\}$ of the abelian group $\Gamma = \mathbb{Z}^k$, we regard $\mathrm{Hom}(\Gamma, G)$ as a subset of the direct product $G \times \cdots \times G$ by the evaluation map:

$$\mathrm{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \quad \varphi \mapsto (\varphi(e_1), \ldots, \varphi(e_k)). \tag{2.2}$$

Let us describe the image of the injective map (2.2). For this, first we set

$$M_1 := \{(\vec{x}, Y, \vec{z}) \in \mathbb{R}^k \oplus M(k, \mathbb{R}) \oplus \mathbb{R}^k : \vec{z} \neq \vec{0}\} \subset M(k, k+2; \mathbb{R}), \tag{2.3}$$

$$M_2 := M(k, 2k; \mathbb{R}). \tag{2.4}$$

Then, $\dim M_1 = k(k+2)$ and $\dim M_2 = 2k^2$. Second, for $\vec{x}, \vec{y} \in \mathbb{R}^k$ and $z \in \mathbb{R}$, we define a $(k+2) \times (k+2)$ matrix by

$$g(\vec{x}, \vec{y}, z) := \exp \begin{pmatrix} 0_k & \vec{x} & \vec{y} \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_k & \vec{x} & \vec{y} + \frac{1}{2} z \vec{x} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

With this expression, our groups $G$ and $H$ (see Section 2.1) are expressed as

$$G = \{g(\vec{x}, \vec{y}, z) : \vec{x}, \vec{y} \in \mathbb{R}^k, z \in \mathbb{R}\},$$
$$H = \{g(\vec{x}, \vec{0}, 0) : \vec{x} \in \mathbb{R}^k\}.$$

Now we define maps

$$\Psi_i : M_i \longrightarrow G \times \cdots \times G \quad (i = 1, 2)$$

such that their $j$th $(1 \leq j \leq k)$ components are respectively given by

(2.6)

$$\Psi_1(\vec{x}, Y, \vec{z})_j := g(z_j \vec{x}, \vec{y_j}, z_j) \quad \text{for } Y = (\vec{y_1}, \cdots, \vec{y_k}),\ {}^t\vec{z} = (z_1, \cdots, z_k),$$

$$\Psi_2(X, Y)_j := g(\vec{x_j}, \vec{y_j}, 0) \qquad \text{for } X = (\vec{x_1}, \cdots, \vec{x_k}),\ Y = (\vec{y_1}, \cdots, \vec{y_k}). \tag{2.7}$$

Then, the topological space $\mathrm{Hom}(\Gamma, G)$ is described via (2.2) as follows:

**Proposition 2.2.1 (Description of $\mathrm{Hom}(\Gamma, G)$).**
1) *The maps $\Psi_1$ and $\Psi_2$ induce a bijection*

$$\Psi_1 \cup \Psi_2 : M_1 \cup M_2 \xrightarrow{\sim} \mathrm{Hom}(\Gamma, G).$$

*In particular, $\Psi_1$ and $\Psi_2$ are injective, and their images $\Psi_1(M_1)$ and $\Psi_2(M_2)$ are disjoint subsets contained in $\mathrm{Hom}(\Gamma, G)$.*
2) (closure relation) *$\Psi_2(M_2)$ is closed in $\mathrm{Hom}(\Gamma, G)$, whereas the closure of $\Psi_1(M_1)$ is given as*

$$\overline{\Psi_1(M_1)} = \Psi_1(M_1) \cup \Psi_2(M_2^d). \tag{2.8}$$

*Here, $M_2^d$ is a subset of $M_2$ defined by*

$$M_2^d := \{(X, Y) : X,\ Y \in M(k, \mathbb{R}),\ \operatorname{rank} X \leq 1\}. \tag{2.9}$$

We shall give a proof of this proposition in Section 3.

## 2.3 Description of $R(\Gamma, G; \mathbb{R}^{k+1})$

Let us introduce the following subsets of $M_1$ and $M_2$:

$$M_1^r := \{(\vec{x}, Y, \vec{z}) \in M(k, k+2; \mathbb{R}) : \vec{z} \neq \vec{0},\ \operatorname{rank}({}^tY, \vec{z}) = k\} \tag{2.10}$$

$$M_2^r := \{(X, Y) \in M(k, 2k; \mathbb{R}) : \det(Y - \lambda X) \neq 0 \text{ for any } \lambda \in \mathbb{R}\}. \tag{2.11}$$

We are now ready to characterize $R(\Gamma, G; \mathbb{R}^{k+1})$ as a subset of $\mathrm{Hom}(\Gamma, G)$. Here is the first of the main results in this paper:

**Theorem 2.3 (Description of $R(\Gamma, G; \mathbb{R}^{k+1})$).** *Let $G$ be a nilpotent Lie group defined as* (2.1) *and $\Gamma = \mathbb{Z}^k$. Then, the maps $\Psi_1$ and $\Psi_2$ (see* (2.6) *and* (2.7)*) induce the bijection*

$$\Psi_1 \cup \Psi_2 : M_1^r \cup M_2^r \xrightarrow{\sim} R(\Gamma, G; \mathbb{R}^{k+1}).$$

We shall give a proof of this theorem in Section 4.

## 2.4 Generic points of $R(\Gamma, G; \mathbb{R}^{k+1})$

This subsection studies a generic part of $R(\Gamma, G; \mathbb{R}^{k+1})$ by analysing the sets $M_1^r$ and $M_2^r$ in detail.

For $X$, $Y \in M(k,\mathbb{R})$, we define a polynomial of $\lambda$ by

$$f(X,Y;\lambda) := \det(Y-\lambda X) = \sum_{l=0}^{k} a_l(X,Y)\lambda^l. \tag{2.12}$$

Here, we note

$$\begin{aligned} a_0(X,Y) &= \det Y, \\ a_{k-1}(X,Y) &= (-1)^{k-1}\sum_{i,j=1}^{k}(-1)^{i+j}Y_{ij}\det \widehat{X}_{ij}, \\ a_k(X,Y) &= (-1)^k \det X, \end{aligned}$$

where $\widehat{X}_{ij}$ denotes the submatrix obtained by deleting row $i$ and column $j$ from $X$.

Let us recall from (2.11) and (2.12) that

$$M_2^r = \{(X,Y) \in M(k,2k;\mathbb{R}) : f(X,Y;\lambda) \neq 0 \text{ for any } \lambda \in \mathbb{R}\}.$$

In order to give a 'generic' part of $R(\Gamma, G; \mathbb{R}^{k+1})$ by means of Theorem 2.3, we set

$$M_2^{\mathrm{ro}} := \begin{cases} M_2^r \cap \{(X,Y) : \det X \neq 0\} & (k\text{: even}), \\ M_2^r \cap \{(X,Y) : \operatorname{rank} X = k-1, a_{k-1}(X,Y) \neq 0\} & (k\text{: odd}). \end{cases}$$

**Proposition 2.4.1.**
1) *$M_1^r$ is open dense in $M_1$. In particular, it has dimension $k(k+2)$.*
2) *$M_2^{\mathrm{ro}}$ is open in $M_2^r$, and the complement of $M_2^{\mathrm{ro}}$ in $M_2^r$ has a smaller dimension than that of $M_2^{\mathrm{ro}}$. The dimension of $M_2^{\mathrm{ro}}$ is given by*

$$\dim M_2^{\mathrm{ro}} = \begin{cases} 2k^2 & (k\text{: even}), \\ 2k^2-1 & (k\text{: odd}). \end{cases}$$

*Proof.* 1) Clear.

2) For an even integer $l$, we consider a monic polynomial of the real variable $x$:

$$g(x) = x^l + b_{l-1}x^{l-1} + \cdots + b_1 x + b_0.$$

Then, it attains its minimum, denoted by $m(b_0, b_1, \cdots, b_{l-1})$, which is a continuous function of the real coefficients $b_0, b_1, \cdots, b_{l-1}$.

**Case 1** **($k$ : even)** Suppose $(X,Y) \in M_2^{\mathrm{ro}}$. Then, the monic polynomial $\frac{f(X,Y;\lambda)}{a_k(X,Y)}$ must be positive for all $\lambda \in \mathbb{R}$. Thus we have

$$M_2^{\mathrm{ro}} = \left\{ (X,Y) \in M_2 : \det X \neq 0,\ m\left(\frac{a_0(X,Y)}{a_k(X,Y)}, \cdots, \frac{a_{k-1}(X,Y)}{a_k(X,Y)}\right) > 0 \right\}.$$

Hence, $M_2^{\mathrm{ro}}$ is open in $M_2 = M(k, 2k; \mathbb{R})$.

To see $M_2^{\mathrm{ro}} \neq \emptyset$, we set $J_k := \begin{pmatrix} 0 & -I_{\frac{k}{2}} \\ I_{\frac{k}{2}} & 0 \end{pmatrix} \in M(k, \mathbb{R})$. Then $(J_k,\ I_k) \in M_2^{\mathrm{ro}}$ because

$$f(J_k, I_k; \lambda) = \left(\det \begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix}\right)^{\frac{k}{2}} = (1+\lambda^2)^{\frac{k}{2}} > 0.$$

Hence, $M_2^{\mathrm{ro}} \neq \emptyset$ and $\dim M_2^{\mathrm{ro}} = \dim M_2 = 2k^2$.

Next, suppose $(X,Y) \in M_2^r \setminus M_2^{\mathrm{ro}}$. Then $\det X = 0$ by definition. Furthermore, it follows from $f(X,Y;\lambda) \neq 0$ for any $\lambda \in \mathbb{R}$ that the coefficient $a_{k-1}(X,Y)$ of $\lambda^{k-1}$ must vanish because $f(X,Y;\lambda) = a_{k-1}(X,Y)\lambda^{k-1} + \cdots + a_0(X,Y)$ and $k-1$ is odd. Thus we have seen that

$$M_2^r \setminus M_2^{\mathrm{ro}} \subset \{ (X,Y) \in M_2 : \det X = a_{k-1}(X,Y) = 0 \}.$$

Hence the complement $M_2^r \setminus M_2^{\mathrm{ro}}$ has at least codimension two in $M_2^r$.

**Case 2** **($k$: odd)** First, we claim

$$M_2^r \subset \{ (X,Y) \in M(k, 2k; \mathbb{R}) : \det X = 0 \}.$$

In fact, since $k$ is odd, the polynomial $f(X,Y;\lambda)$ of the real variable $\lambda$ has zeros unless the top term $a_k(X,Y)\lambda^k$ vanishes. Therefore, $a_k(X,Y)$ $(= (-1)^k \det X) = 0$ if $(X,Y) \in M_2^r$.

Next, let us prove that $M_2^{\mathrm{ro}}$ is open in the set

$$\begin{aligned} S &:= \{ (X,Y) \in M(k, 2k; \mathbb{R}) : \det X = 0,\ \operatorname{grad}\det X \neq 0 \} \\ &= \{ (X,Y) \in M(k, 2k; \mathbb{R}) : \operatorname{rank} X = k-1 \}. \end{aligned}$$

Suppose $(X,Y) \in M(k, 2k; \mathbb{R})$ satisfies

$$\det X = 0 \text{ and } a_{k-1}(X,Y) \neq 0.$$

Then $f(X,Y;\lambda) \neq 0$ for any $\lambda \in \mathbb{R}$ if and only if the monic polynomial

$$\frac{f(X,Y;\lambda)}{a_{k-1}(X,Y)} = \lambda^{k-1} + \sum_{i=0}^{k-2} \frac{a_i(X,Y)}{a_{k-1}(X,Y)} \lambda^i$$

is positive for all $\lambda$. Thus, we have seen

$$M_2^{\mathrm{ro}} = \left\{ (X,Y) \in M(k,2k;\mathbb{R}) : \begin{array}{l} \operatorname{rank} X = k-1,\ a_{k-1}(X,Y) \neq 0, \\ m\left( \frac{a_0(X,Y)}{a_{k-1}(X,Y)}, \cdots, \frac{a_{k-2}(X,Y)}{a_{k-1}(X,Y)} \right) > 0. \end{array} \right\}.$$

It is now clear that $M_2^{\mathrm{ro}}$ is open in $S$. To see $M_2^{\mathrm{ro}} \neq \emptyset$, we set $J_k' := \begin{pmatrix} J_{k-1} & 0 \\ 0 & 0 \end{pmatrix} \in M(k;\mathbb{R})$. Then, $(J_k', I_k) \in M_2^{\mathrm{ro}}$ because

$$f\left(J_k', I_k; \lambda\right) = (1+\lambda^2)^{\frac{k-1}{2}} > 0.$$

Hence, $M_2^{\mathrm{ro}}$ is a non-empty open subset of $S$. Since $S$ is a non-singular manifold of dimension $2k^2-1$, so is $M_2^{\mathrm{ro}}$.

Finally, it follows from the definition of $M_2^{\mathrm{ro}}$ that $M_2^r \setminus M_2^{\mathrm{ro}}$ is contained in the algebraic variety:

$$\{(X,Y) : \det X = 0,\ a_{k-1}(X,Y) \|\operatorname{grad} \det X\|^2 = 0\},$$

which is of dimension $2k^2-2$. Thus, Proposition 2.4.1 has been proved. $\square$

**Remark 2.4.2.** Proposition 2.4.1 implies that $M_2^r$ contains an open subset of $M_2$ if and only if $k$ is even. However, $M_2^r$ itself is not open in $M_2$ even if $k$ is even because $(O, I_k) \in M_2^r$ is not an inner point, as we shall see in the following example:

**Example 2.4.3.** Take a half dimensional affine subspace

$$V = \{(X, I_k) : X \in M(k,\mathbb{R})\}$$

of $M(k,2k;\mathbb{R})$. Then we have

$$V \cap M_2^r = \{(X, I_k) : \text{any eigenvalue of } X \text{ is in } \mathbb{C} \setminus \mathbb{R}^\times\}.$$

This gives a partial information on $R(\Gamma, G; \mathbb{R}^{k+1})$, and was proved in Lipsman [11, Theorem 4.4] for $k=2$ as a crucial step to the classification of maximal nilpotent affine subgroups that act properly on $\mathbb{R}^3$, and was generalized in [12] for $k \geq 3$. Our proof here is different and simpler.

# 3 Description of $\mathrm{Hom}(\Gamma, G)$

This section determines $\mathrm{Hom}(\Gamma, G)$ explicitly, and gives a proof of Proposition 2.2.1.

## 3.1 Parametrization of $\mathrm{Hom}(\Gamma, G)$

Recall from (2.2) that any $\varphi \in \mathrm{Hom}(\Gamma, G)$ is determined by $\varphi(e_j)$ $(1 \leq j \leq k)$, which we write as $\varphi(e_j) = g(\vec{x_j}, \vec{y_j}, z_j)$ for some $\vec{x_j}, \vec{y_j} \in \mathbb{R}^k$ and $z_j \in \mathbb{R}$ according to (2.5). Collecting these data $\vec{x_j}, \vec{y_j}$ $(1 \leq j \leq k)$ and $\vec{z} := {}^t(z_1, \ldots, z_k)$, we obtain an injective map defined by

$$\mathrm{Hom}(\Gamma, G) \to M(k, 2k+1; \mathbb{R}), \ \ \varphi \mapsto \left(\vec{x_1}, \ldots, \vec{x_k}; \vec{y_1}, \ldots, \vec{y_k}; \vec{z}\right). \tag{3.1}$$

Let us determine the image of (3.1). Since $\varphi \in \mathrm{Hom}(\Gamma, G)$ satisfies

$$\varphi(e_i)\varphi(e_j) = \varphi(e_i + e_j) = \varphi(e_j)\varphi(e_i)$$

for any $i, j$ $(1 \leq i, j \leq k)$, we have

$$z_i \vec{x_j} = z_j \vec{x_i} \quad \text{for any } i, j \ (1 \leq i, j \leq k). \tag{3.2}$$

Conversely, given $k$ elements $g_1, \cdots, g_k$ in $G$ that mutually commute, we can define a group homomorphism $\varphi : \Gamma \to G$ by $\varphi(\sum_{j=1}^k m_j e_j) := g_1^{m_1} \cdots g_k^{m_k}$. Therefore, the image of (3.1) is characterized by the condition (3.2), that is, we have a bijection:

$$\mathrm{Hom}(\Gamma, G) \simeq \{\left(\vec{x_1}, \ldots, \vec{x_k}; Y; \vec{z}\right) : Y \in M(k, \mathbb{R}); \vec{x_1}, \ldots, \vec{x_k}, \vec{z} \text{ satisfies } (3.2)\}.$$

We shall find all solutions of (3.2), according to the following two cases: (a) $\vec{z} \neq \vec{0}$ and (b) $\vec{z} = \vec{0}$.

In the case (a), there exists uniquely an element $\vec{r} \in \mathbb{R}^k$ such that $\vec{x_j} = z_j \vec{r}$ for all $j$ $(1 \leq j \leq k)$. This amounts to $\Psi_1(M_1)$ (see (2.6) for the definition of $\Psi_1$).

In the case (b), any $\vec{x_1}, \ldots, \vec{x_k}$ solves (3.2). This amounts to $\Psi_2(M_2)$ (see (2.7) for the definition of $\Psi_2$).

Hence, $\mathrm{Hom}(\Gamma, G)$ is the disjoint union of $\Psi_1(M_1)$ and $\Psi_2(M_2)$. Thus we have completed the proof of Proposition 2.2.1 (1).

□

### 3.2 Closure relation in $\mathrm{Hom}(\Gamma, G)$

This subsection gives a proof of Proposition 2.2.1 (2). It is clear that $\Psi_2(M_2)$ is a closed set. Let us consider the closure of $\Psi_1(M_1)$, and find its boundary. What we need is to prove:

$$\overline{\Psi_1(M_1)} \cap \Psi_2(M_2) = \Psi_2(M_2^d).$$

**Proof of the inclusion $\supset$ :** Suppose $(X, Y) \in \Psi_2(M_2^d)$. Since $\mathrm{rank}\, X \leq 1$, we find $\vec{x} \in \mathbb{R}^k$ and $\vec{a} \in \mathbb{R}^k \setminus \{0\}$ such that $X = \vec{x}\,{}^t\vec{a}$. In light of the obvious formula

$$g(a_j \vec{x}, \vec{y_j}, 0) = \lim_{l \to \infty} g\big(\frac{a_j}{l} l\vec{x}, \vec{y_j}, \frac{a_j}{l}\big) \quad (1 \leq j \leq k),$$

we conclude from the definitions (2.6) and (2.7) of $\Psi_1$ and $\Psi_2$ that

$$\Psi_2(X, Y) = \lim_{l \to \infty} \Psi_1\big(l\vec{x}, Y, \frac{\vec{a}}{l}\big).$$

As $(l\vec{x}, Y, \frac{\vec{a}}{l})$ $(l = 1, 2, \cdots)$ is a sequence of $M_1$, we have proved the inclusion $\supset$.

**Proof of the inclusion $\subset$ :** Take any sequence $(\vec{x^{(l)}}, Y^{(l)}, \vec{z^{(l)}})$ in $M_1$ such that $\Psi_1(\vec{x^{(l)}}, Y^{(l)}, \vec{z^{(l)}})$ converges to an element of $\Psi_2(M_2)$, say, $\Psi_2(X, Y)$ for some $X$, $Y \in M(k, \mathbb{R})$. Then the formula

$$\lim_{l \to \infty} \Psi_1(\vec{x^{(l)}}, Y^{(l)}, \vec{z^{(l)}}) = \Psi_2(X, Y)$$

implies that $X$ is the limit of $X^{(l)} := (z_1^{(l)}\vec{x^{(l)}}, \cdots, z_k^{(l)}\vec{x^{(l)}})$ as $l$ tends to infinity. Since $\mathrm{rank}\, X^{(l)} \leq 1$, its limit also satisfies $\mathrm{rank}\, X \leq 1$. Thus we have proved the inclusion $\subset$.

Thus, Proposition 2.2.1 (2) is proved. □

# 4 Proof of Theorem 2.3

This section gives a proof of Theorem 2.3. Our strategy here is to rewrite the condition of $R(\Gamma, G; \mathbb{R}^{k+1})$, in particular, the condition for properly discontinuous actions in the following scheme:

| | | | |
|---|---|---|---|
| | $\Gamma$ | discrete subgroup | (see (1.1)) |
| $\Rightarrow$ | $L = \overline{\Gamma}$ | its syndetic hull | (see Proposition 4.3.2) |
| $\Rightarrow$ | $\mathfrak{l}$ | its Lie algebra. | (see Section 4.4) |

## 4.1 Proper actions and properly discontinuous actions

In dealing with properly discontinuous actions of a discrete group, a more general notion "proper action" is sometimes useful. We recall:

**Definition 4.1.1 (Palais [14]).** Suppose that a locally compact topological group $L$ acts continuously on a Hausdorff, locally compact space $X$. For a subset $S$ of $X$, we define a subset of $L$ by $L_S = \{\gamma \in L : \gamma S \cap S \neq \emptyset\}$. The $L$-action on $X$ is said to be *proper* if $L_S$ is compact for every compact subset $S$ of $X$.

We note that the $L$-action is *properly discontinuous* if $L$ is a discrete group and if the $L$-action is proper.

The following elementary observation is a bridge between the action of a discrete group and that of a connected group.

**Observation 4.1.2 ([5, Lemma 2.3]).** *Suppose a locally compact group $L$ acts on a Hausdorff, locally compact space $X$. Let $\Gamma$ be a cocompact discrete subgroup of $L$. Then*

1) *The $L$-action on $X$ is proper if and only if the $\Gamma$-action is properly discontinuous.*

2) *$L\backslash X$ is compact if and only if $\Gamma\backslash X$ is compact.*

## 4.2 Extension from a discrete subgroup

Suppose we are in the setting of Section 2.1. We set

$$L := \mathbb{R}^k$$

and regard $\Gamma = \mathbb{Z}^k$ as a cocompact discrete subgroup of $L$. We write $\mathrm{Hom}(L, G)$ for the set of continuous group homomorphisms from $L$ into $G$. In our setting (2.1), every homomorphism from $\Gamma$ into $G$ extends uniquely to a continuous homomorphism from $L$ to $G$. That is, we have:

**Lemma 4.2.1.** *The restriction map* $\mathrm{Hom}(L, G) \to \mathrm{Hom}(\Gamma, G), \psi \mapsto \psi|_\Gamma$ *is bijective.*

*Proof.* As $G$ is a simply connected nilpotent group, the exponential map, $\exp : \mathfrak{g} \to G$ is bijective. We write log for its inverse. Then, $\psi \in \operatorname{Hom}(L, G)$ satisfies

$$\psi(\sum_{j=1}^{k} a_j e_j) = \exp\big(\sum_{j=1}^{k} a_j \log \psi(e_j)\big) \text{ for any } a_1, \cdots, a_k \in \mathbb{R}. \tag{4.1}$$

This shows that the homomorphism $\psi$ is determined by its restriction $\psi|_\Gamma$. Conversely, the formula (4.1) also indicates how to extend a homomorphism from $\Gamma$ to $L$. Thus we have proved Lemma 4.2.1. □

In light of Lemma 4.2.1, any property of $\psi$ should be expressed in terms of the restriction $\psi|_\Gamma$ in principle. We show:

**Lemma 4.2.2.** *The following two conditions on $\psi \in \operatorname{Hom}(L, G)$ are equivalent:*

(i) *$\psi$ is injective.*

(ii) *$\psi|_\Gamma$ is injective and $\psi(\Gamma)$ is discrete in $G$.*

*Proof.* Since $G$ is a simply connected nilpotent Lie group, any connected subgroup of $G$ is closed. Therefore, $\psi :\ L/\operatorname{Ker}\psi \to G$ is a homeomorphism onto a closed subgroup of $G$. In particular, $\psi(\Gamma)$ is discrete in $G$ if and only if $\Gamma/\Gamma \cap \operatorname{Ker}\psi$ is discrete in $L/\operatorname{Ker}\psi$. Now, it is clear that (i) implies (ii).

Conversely, if $\psi$ is not injective and if $\psi|_\Gamma$ is injective, then the composition map $\Gamma \subset L \to L/\operatorname{Ker}\psi$ is injective with non-discrete image because $\operatorname{rank}\Gamma < \dim(L/\operatorname{Ker}\psi)$. Hence, $\psi(\Gamma)$ is not discrete in $G$, too. Thus, (ii) also implies (i). □

## 4.3 A continuous analogue of properly discontinuous actions

Following Observation 4.1.2, we amplify Lemma 4.2.2 with the condition of proper actions on the homogeneous space $G/H$:

**Lemma 4.3.1.** *Let $\psi \in \operatorname{Hom}(L, G)$ and $\varphi = \psi|_\Gamma$ (see Lemma 4.2.1). Then the following two conditions are equivalent:*

(i) *$\psi : L \to G$ is injective and $\psi(L)$ acts properly on $G/H$.*

(ii) $\varphi : \Gamma \to G$ *is injective and* $\varphi(\Gamma)$ *acts properly discontinuously and freely on* $G/H$.

*Proof.* (i) $\Rightarrow$ (ii): Since $\psi$ is injective, it follows from Lemma 4.2.2 that $\varphi(\Gamma)$ is discrete in a closed subgroup $\psi(L)$. Therefore, $\varphi(\Gamma)$ acts properly discontinuously on $G/H$ because $\psi(L)$ acts properly on $G/H$. Furthermore, any properly discontinuous action of $\varphi(\Gamma)$ is automatically free because $\varphi(\Gamma) \simeq \Gamma$ is torsion-free. Hence (ii) is proved.

(ii) $\Rightarrow$ (i): If $\varphi(\Gamma)$ acts properly discontinuously on $G/H$ then $\varphi(\Gamma)$ is discrete in $G$. Hence, $\psi : L \to G$ is injective by Lemma 4.2.2. Furthermore, $\psi(L)$ with its relative topology contains $\varphi(\Gamma)$ as a cocompact discrete subgroup. Therefore, $\psi(L)$ acts properly on $G/H$ by Observation 4.1.2 (1). Thus, we have proved the implication (ii) $\Rightarrow$ (i). □

We are ready to characterize $R(\Gamma, G; \mathbb{R}^{k+1})$ by means of the connected subgroup $L$:

**Proposition 4.3.2.** *Under the isomorphism* $\mathrm{Hom}(L, G) \xrightarrow{\sim} \mathrm{Hom}(\Gamma, G)$ *in Lemma 4.2.1, we have*

$$R(\Gamma, G; \mathbb{R}^{k+1}) \simeq \{\psi \in \mathrm{Hom}(L, G) : \text{i) } \psi \textit{ is injective}, \quad \text{ii) } \psi(L) \textit{ acts properly on } G/H\}.$$

## 4.4 Reformulation of $R(\Gamma, G; \mathbb{R}^{k+1})$

So far, we have transferred proper discontinuity and freeness of discrete group actions into a certain property of connected group actions. Now, let us rewrite the latter condition in terms of Lie algebras. We use the German lower case letters $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ to denote the Lie algebras of $G$, $H$ and $L$ respectively. We write $d\psi$ for the differential of $\psi \in \mathrm{Hom}(L, G)$. Consider the following conditions on $d\psi$:

$$d\psi : \mathfrak{l} \to \mathfrak{g} \text{ is injective,} \tag{4.2}$$

$$d\psi(\mathfrak{l}) \cap \bigcup_{g \in G} \mathrm{Ad}(g)\mathfrak{h} = \{0\}. \tag{4.3}$$

Now we can restate Proposition 4.3.2 as

**Proposition 4.4.1.** *Under the isomorphism* $\mathrm{Hom}(L,G) \xrightarrow{\sim} \mathrm{Hom}(\Gamma,G)$ *(see Lemma 4.2.1), we have*

$$R(\Gamma, G; \mathbb{R}^{k+1}) \simeq \{\psi \in \mathrm{Hom}(L,G) \,:\, d\psi \textit{ satisfies } (4.2) \textit{ and } (4.3)\,\}.$$

*Proof.* Any connected subgroup of a simply connected nilpotent Lie group is simply connected. Hence, $d\psi$ is injective if and only if $\psi$ is injective. Now use the criterion of proper actions for a homogeneous space of a two-step nilpotent Lie group $G$ as follows. ☐

**Lemma 4.4.2 ([13, Theorem 2.11]).** *Let $G$ be a simply connected Lie group, and $H, L$ its closed subgroups. Suppose $G$ is a two-step nilpotent Lie group, which means that the commutator subgroup of $G$ is contained in the centre of $G$. Then, the following three conditions on $\psi$ are equivalent.*

(i) *$\psi(L)$ acts on $G/H$ properly.*

(ii) *$\psi(L) \cap gHg^{-1} = \{e\}$ for all $g \in G$.*

(iii) *$d\psi(\mathfrak{l}) \cap \bigcup_{g\in G} \mathrm{Ad}(g)\mathfrak{h} = \{0\}$.*

**Remark 4.4.3.** Lemma 4.4.2 gives an affirmative solution to Lipsman's conjecture [11] for two-step nilpotent Lie groups. Recently, Baklouti–Khlif [2] and Yoshino [18] proved independently that Lipsman's conjecture is still true for three-step nilpotent Lie groups.

## 4.5 Completion of the proof of Theorem 2.3

Now let us complete the proof of Theorem 2.3. We have already reduced it to a problem of Lie algebras. Now, we use the following:

**Lemma 4.5.1.** *We define the variety in $\mathfrak{g}$ by*

$$\mathcal{V} = \bigcup_{g\in G} \mathrm{Ad}(g)\mathfrak{h}.$$

*Then we have*

$$\begin{aligned} \mathcal{V} &= \{W - [W,V] : W \in \mathfrak{h}, V \in \mathfrak{g}\} \\ &= \{\begin{pmatrix} 0 & \vec{x} & b\vec{x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \vec{x} \in \mathbb{R}^k, b \in \mathbb{R}\}. \end{aligned} \tag{4.4}$$

*Proof.* Elementary computation. $\square$

According to the parametrization

$$\Psi_1 \cup \Psi_2 : M_1 \cup M_2 \xrightarrow{\sim} \mathrm{Hom}(L, G) \xrightarrow{\sim} \mathrm{Hom}(\Gamma, G)$$

given in Proposition 2.2.1 and Lemma 4.2.1, we examine if $d\psi$ satisfies (4.2) and (4.3) for $\psi \in \Psi_1(M_1)$ and $\psi \in \Psi_2(M_2)$, respectively. The following proposition gives criteria for (4.2) and (4.3):

**Proposition 4.5.2.**
1) *Let* $\psi := \Psi_1(\vec{x}, Y, \vec{z})$ *for* $(\vec{x}, Y, \vec{z}) \in M_1$. *Then, we have the following equivalence:*

$$\psi \textit{ satisfies } (4.2) \iff \mathrm{rank}({}^tY, \vec{z}) = k.$$

*In this case,* $\psi$ *satisfies* (4.3)*, too.*
2) *Let* $\psi := \Psi_2(X, Y)$ *for* $(X, Y) \in M_2$. *Then*

$$\psi \textit{ satisfies } (4.2) \iff \mathrm{rank}\begin{pmatrix} X \\ Y \end{pmatrix} = k.$$

*In this case, we have the following equivalence:*

$$\psi \textit{ satisfies } (4.3) \iff \det(Y - bX) \neq 0 \textit{ for any } b \in \mathbb{R}.$$

*Proof.* 1) It follows from the definition of $\Psi_1$ (see (2.6)) that

$$d\psi(\vec{a}) = \begin{pmatrix} 0 & \langle \vec{a}, \vec{z} \rangle \vec{x} & Y\vec{a} \\ 0 & 0 & \langle \vec{a}, \vec{z} \rangle \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

Here, $\langle \ , \ \rangle$ denotes the standard inner product on $\mathbb{R}^k$. Then

$$\begin{aligned} \psi \text{ satisfies } (4.2) &\iff \{\vec{a} \in \mathfrak{l} : \langle \vec{z}, \vec{a} \rangle = 0, \, Y\vec{a} = \vec{0}\} = \{0\} \\ &\iff \mathrm{rank}({}^tY, \vec{z}) = k. \end{aligned}$$

Furthermore, by (4.4) and (4.5)

$$d\psi(\vec{a}) \in \mathcal{V} \iff d\psi(\vec{a}) = \vec{0}.$$

Therefore, (4.2) implies (4.3).

2) It follows from the definition of $\Psi_2$ (see (2.7)) that

$$d\psi(\vec{a}) = \begin{pmatrix} 0 & X\vec{a} & Y\vec{a} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

Then,

$$\begin{aligned} \psi \text{ satisfies (4.2)} &\iff \{\vec{a} \in \mathfrak{l} : X\vec{a} = \vec{0}, Y\vec{a} = \vec{0}\} = \{\vec{0}\} \\ &\iff \operatorname{rank} \begin{pmatrix} X \\ Y \end{pmatrix} = k. \end{aligned}$$

Suppose (4.2) is satisfied. It follows from (4.4) and (4.6) that

$$\begin{aligned} &\psi \text{ satisfies (4.3)} \\ \iff &\text{there is no } b \in \mathbb{R} \text{ and } \vec{a} \in \mathbb{R}^k \text{ such that } Y\vec{a} = bX\vec{a} \neq \vec{0} \\ \iff &\det(Y - bX) = 0 \text{ has no real solution for any } b \in \mathbb{R}. \end{aligned}$$

Hence, Proposition is proved. □

We note that if $\det(Y - bX) \neq 0$ for any $b \in \mathbb{R}$ then $\operatorname{rank} \begin{pmatrix} X \\ Y \end{pmatrix} = k$ because $\det Y \neq 0$. Then, Theorem 2.3 follows from Propositions 4.4.1 and 4.5.2. Hence, we have completed the proof of Theorem 2.3. □

# 5 Deformation space

Building on the description of the parameter space $R(\Gamma, G; \mathbb{R}^{k+1})$ of discontinuous groups given in Theorem 2.3, we determine explicitly the deformation space $\mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$. This is stated in Theorem 5.1 and Corollary 5.1.1, and is the second of the main results of this paper.

## 5.1 Description of the deformation space $\mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$

We define subsets of $M_1^r$ and $M_2^r$, respectively, by

(5.1)

$$D_1 := \left\{ \left(\vec{x}, {}^t(\vec{\eta_1}, \dots, \vec{\eta_k}), \vec{z}\right) \in M(k, k+2; \mathbb{R}) : \begin{array}{ll} 1) & \vec{z} \neq \vec{0}, \\ 2) & \vec{\eta_j} \perp \vec{z} \ (1 \le j \le k), \\ 3) & \operatorname{rank}(\vec{\eta_1}, \cdots, \vec{\eta_k}) = k-1. \end{array} \right\},$$

(5.2)

$$D_2 := \left\{ (X, Y) \in M(k, \mathbb{R}) \oplus M(k, \mathbb{R}) : \begin{array}{ll} 1) & \operatorname{Trace}(X\,{}^tY) = 0, \\ 2) & \det(Y - \lambda X) \neq 0 \text{ for any } \lambda \in \mathbb{R}. \end{array} \right\}.$$

We note that the third condition in (5.1) asserts that $\operatorname{rank}(\vec{\eta_1}, \cdots, \vec{\eta_k})$ attains its maximum because all the vectors $\vec{\eta_j}$ $(1 \le j \le k)$ are orthogonal to $\vec{z}$.

We retain the setting as in Section 2.1. In particular, $\Gamma = \mathbb{Z}^k$ and $G$ is a nilpotent affine transformation group defined in (2.1). For $i = 1, 2$, we denote by

$$\overline{\Psi}_i : M_i^r \to \mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$$

the composition of $\Psi_i : M_i^r \to R(\Gamma, G; \mathbb{R}^{k+1})$ (see (2.6) and (2.7)) and the natural quotient map $R(\Gamma, G; \mathbb{R}^{k+1}) \to \mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$.

Here is an explicit description of the deformation space:

**Theorem 5.1.** *The maps $\overline{\Psi}_1$ and $\overline{\Psi}_2$ induce the following bijection:*

$$\overline{\Psi}_1 \cup \overline{\Psi}_2 : D_1 \cup D_2 \xrightarrow{\sim} \mathcal{T}(\Gamma, G; \mathbb{R}^{k+1}).$$

In particular, we find the dimension of the deformation space:

**Corollary 5.1.1.** *The deformation space $\mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$ contains a smooth manifold $\mathcal{T}'$ as its open dense subset, where the dimension of $\mathcal{T}'$ is given by*

$$\dim \mathcal{T}' = \begin{cases} 2k^2 - 1 & (k : \textit{even}), \\ 2k^2 - 2 & (k : \textit{odd}, \ \ge 3), \\ 2 & (k = 1). \end{cases}$$

## 5.2 Proof of Theorem 5.1 and Corollary 5.1.1

We let $G$ act on $M(k, k+2; \mathbb{R})$ and $M(k, 2k; \mathbb{R})$, respectively, as follows: for $h = \begin{pmatrix} I_k & \vec{a} & \vec{b} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in G$, the actions of $h$ are given by

$$(5.3) \qquad (\vec{x}, Y, \vec{z}) \mapsto (\vec{x}; Y + (\vec{a} - c\vec{x})^t\vec{z}, \vec{z}) \qquad \text{on} \quad M(k, k+2; \mathbb{R}),$$

$$(5.4) \qquad (X, Y) \mapsto (X, Y - cX) \qquad \text{on} \quad M(k, 2k; \mathbb{R}).$$

**Proposition 5.2.1.**
1) *Both $M_1$ and $M_1^r$ are $G$-stable subsets of $M(k, k+2; \mathbb{R})$.*
2) *$M_2^r$ is a $G$-stable subset of $M_2 = M(k, 2k; \mathbb{R})$.*
3) *For $i = 1, 2$, the maps $\Psi_i : M_i^r \to \mathrm{Hom}(\Gamma, G)$ respect $G$-actions.*
4) *For $i = 1, 2$, $D_i$ are complete representatives of the $G$-orbit on $M_i^r$.*

*Proof.* 1) Clear from the definitions (2.3) and (2.10) of $M_1$ and $M_1^r$.
2) Clear from the definition (2.11) of $M_2^r$.
3) We first note that via (2.2) the $G$-action on $\mathrm{Hom}(\Gamma, G)$ is compatible with the diagonal $G$-action on $G \times \cdots \times G$:

$$(g_1, \cdots, g_k) \ \mapsto \ (hg_1h^{-1}, \cdots, hg_kh^{-1}).$$

Now, we compute the $j$th components of the image of $\Psi_1$ (see (2.6) for the definition) and $\Psi_2$ (see (2.7) for the definition), respectively, as follows:
For $h = \begin{pmatrix} I_k & \vec{a} & \vec{b} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in G$, we have

$$h \exp \begin{pmatrix} 0 & z_j\vec{x} & \vec{y_j} \\ 0 & 0 & z_j \\ 0 & 0 & 0 \end{pmatrix} h^{-1} = \exp \begin{pmatrix} 0 & z_j\vec{x} & \vec{y_j} + z_j(\vec{a} - c\vec{x}) \\ 0 & 0 & z_j \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$h \exp \begin{pmatrix} 0 & \vec{x_j} & \vec{y_j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} h^{-1} = \exp \begin{pmatrix} 0 & \vec{x_j} & \vec{y_j} - c\vec{x_j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is what we wanted to prove.
4) This is an elementary linear algebra. ☐

Then, Theorem 5.1 is an immediate consequence of Theorem 2.3 and Proposition 5.2.1 (4). Corollary 5.1.1 now follows from Proposition 2.4.1, and from the $G$-action on $R(\Gamma, G; \mathbb{R}^{k+1})$ described in the above proof.

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