Casimir-Polder interaction between an atom and a conducting wall in cosmic string spacetime

E. R. Bezerra de Mello[1]*, V. B. Bezerra[1]†, H. F. Mota[1,2], A. A. Saharian[3]‡

[1] *Departamento de Física, Universidade Federal da Paraíba 58.059-970, Caixa Postal 5.008, João Pessoa, PB, Brazil*

[2] *Department of Physics and Astronomy, Tufts University 212 College Avenue, Medford, MA 02155, USA*

[3] *Department of Physics, Yerevan State University Alex Manoogian Street, 0025 Yerevan, Armenia*

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Abstract

The Casimir-Polder interaction potential is evaluated for a polarizable microparticle and a conducting wall in the geometry of a cosmic string perpendicular to the wall. The general case of the anisotropic polarizability tensor for the microparticle is considered. The corresponding force is a function of the wall-microparticle and cosmic string-microparticle distances. Depending on the orientation of the polarizability tensor principal axes the force can be either attractive or repulsive. The asymptotic behavior of the Casimir-Polder potential is investigated at large and small separations compared to the wavelength of the dominant atomic transitions. We show that the conical defect may be used to control the strength and the sign of the Casimir-Polder force.

1 Introduction

Casimir-Polder (CP) interactions between atoms and surfaces are among the most interesting manifestations of the electromagnetic quantum fluctuations (for a review see [1]). The wide applications of the corresponding forces in many areas of science and technology motivate the investigations of various mechanisms to control their strength and sign. Recently large efforts have been focused on the investigation of the nature of CP forces and its dependence on the geometry of boundaries. In particular, there has been extensive interest in geometries with repulsive CP forces (for a recent discussion see [2] and references therein). In [3] we have shown that the presence of topological defects can serve as an additional tool for the control of the forces.

The formation of topological defects is predicted in some field-theoretical and condensed matter systems as a result of symmetry breaking phase transitions. In particular, within the

*E-mail: emello@fisica.ufpb.br
†E-mail: valdir@fisica.ufpb.br
‡E-mail: saharian@ysu.am

framework of grand unified theories, different types of topological defects could be produced in the early universe [4, 5]. Among them, the cosmic strings have attracted considerable attention. Although the recent observational data on the cosmic microwave background radiation have ruled out cosmic strings as the primary source for primordial density perturbations, they are still candidates for the generation of a number of interesting astrophysical effects such as the generation of gravitational waves, high-energy cosmic rays and gamma ray bursts.

In the simplest model for a straight cosmic string the geometry outside the string core is flat with a planar angle deficit. The corresponding non-trivial topology leads to the modification of the zero-point fluctuations of quantum fields. The change of vacuum properties due to this modification has been discussed in a large number of papers (see, for instance, the references given in [6]). The vacuum polarization effect by a cosmic string in the background of de Sitter and anti-de Sitter spacetimes is investigated in [7, 8]. The distortion of the electromagnetic field vacuum fluctuations by a cosmic string also gives rise to a CP force acting on a polarizable microparticle. This force is investigated in [9, 10]. It is shown that this force depends on the eigenvalues for the polarizability tensor and on the orientation of its principal axes. The CP force can be either repulsive or attractive with respect to the string. For an isotropic polarizability tensor the force is always repulsive. In [3] it was considered the influence of the cosmic string on the CP force between a microparticle and a conducting cylindrical shell coaxial with the string. The combined effects arising from the topology of the cosmic string and boundaries on the vacuum energy and stresses have been discussed in [6], [11]-[18].

In the present paper we investigate the influence of a conical defect (cosmic string) on the CP force between a polarizable microparticle and a conducting plate perpendicular to the defect. The paper is organized as follows. In the next section we evaluate the retarded Green tensor for the electromagnetic field in the geometry of a cosmic string with a conducting plate. This tensor is used in section 3 to evaluate the CP force acting on a polarizable microparticle for the general case of anisotropic polarizability tensor. The CP force in the oscillator model for the polarizability tensor is discussed in section 4. Section 5 summarizes the main results.

2 Retarded Green tensor

For a long straight cosmic string, at distances much larger than the core radius, the corresponding spacetime geometry is flat with a planar angle deficit $2\pi - \phi_0$. Considering the string situated along the z-axis, the line element, in cylindrical coordinates $(x^1, x^2, x^3) = (r, \phi, z)$, can be written as

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \tag{1}$$

where $0 \leqslant \phi \leqslant \phi_0 = 2\pi/q$, with q being a parameter associated with the cosmic string. This line element has been derived in [19] in the weak-field and thin-string approximations. In this case the angle deficit is related to the mass per unit length μ of the string by the formula $2\pi - \phi_0 = \mu/m_{\mathrm{Pl}}^2$, where m_{Pl} is the Planck mass. In the standard scenario for the cosmic string formation in the early universe one has $\mu \sim \eta^2$, where η is the energy scale of the phase transition at which the string is formed. For GUT scale strings $\mu \ll m_{\mathrm{Pl}}^2$ and the weak-field approximation is well justified. However, the validity of the line element (1) has been extended beyond the linear perturbation theory by several authors [20] (see also [4]). In this case the parameter q need not to be close to 1. Note that the conical defects appear as an effective geometry in a number of condensed matter systems such as crystals, liquid crystals and quantum liquids (see, for example, [21]).

Our main interest in the present paper is the CP force acting on a polarizable microparticle (atom, molecule or any small object described by an electric-dipole polarizability tensor) near a conducting plate perpendicular to the cosmic string and located at $z = 0$. For a microparticle situated at a point $\mathbf{r}$, the CP interaction energy can be expressed as

$$U(\mathbf{r}) = \frac{1}{2\pi}\int_0^\infty d\xi\, \alpha_{jl}(i\xi)G_{jl}^{(s)}(\mathbf{r},\mathbf{r};i\xi), \tag{2}$$

where the summation is understood over $j,l = 1,2,3$, $\alpha_{jl}(\omega)$ is the polarizability tensor of the particle and

$$G_{jl}^{(s)}(\mathbf{r},\mathbf{r}';\omega) = \int_{-\infty}^{+\infty} d\tau G_{jl}^{(s)}(x,x')e^{i\omega\tau}, \tag{3}$$

with $x = (t,\mathbf{r})$, $x' = (t',\mathbf{r}')$, $\tau = t - t'$. In (3), $G_{jl}^{(s)}(x,x')$ is given by

$$G_{jl}^{(s)}(x,x') = G_{jl}(x,x') - G_{jl}^{(\mathrm{M})}(x,x'), \tag{4}$$

where $G_{jl}(x,x')$ is the retarded Green tensor for the electromagnetic field in the geometry under consideration and $G_{jl}^{(\mathrm{M})}(x,x')$ is the retarded Green tensor in the boundary-free Minkowski spacetime. The local geometry induced by the cosmic string is flat and the subtracted Green tensor $G_{jl}^{(s)}(x,x')$ is finite in the coincidence limit for points $z \neq 0$ and $r \neq 0$.

The retarded Green tensor for the electromagnetic field is given by the expression

$$G_{jl}(x,x') = -i\theta(\tau)\left\langle E_j(x)E_l(x') - E_l(x')E_j(x)\right\rangle, \tag{5}$$

where $\theta(x)$ is the unit-step function, $E_j(x)$ is the operator of the j-th component of the electric field, and the angular brackets mean the vacuum expectation value. If $\{E_{\alpha j}(x), E_{\alpha j}^*(x)\}$ is a complete set of mode functions for the electric field, with the collective index α specifying the modes, then the Green tensor can be expressed as the sum over the modes:

$$G_{jl}(x,x') = -i\theta(\tau)\sum_\alpha [E_{\alpha j}(x)E_{\alpha l}^*(x') - E_{\alpha l}(x')E_{\alpha j}^*(x)], \tag{6}$$

where the asterisk stands for complex conjugate. In (6), $E_{\alpha j}(x)$ is the j-th physical component of the electric field vector in cylindrical coordinates and the values $j = 1,2,3$ correspond to the r,ϕ,z coordinates, respectively.

In the geometry under consideration we have two classes of mode functions corresponding to the waves of the transverse magnetic (TM) and transverse electric (TE) types. The corresponding mode functions for the electric field, obeying the boundary condition $\mathbf{n}\times\mathbf{E} = 0$ on the conducting plate at $z = 0$, with $\mathbf{n}$ being the normal to the plate, are given by the expressions

$$E_{\alpha l}^{(\lambda)}(x) = \beta_\alpha E_{\alpha l}^{(\lambda)}(r,z)e^{i(qm\phi-\omega t)}, \tag{7}$$

where $\lambda = 0,1$ correspond to TM and TE modes, respectively, and

$$\omega^2 = \gamma^2 + k^2,\ m = 0,\pm1,\pm2,\ldots, \tag{8}$$

with $0 \leqslant \gamma < \infty$, $0 \leqslant k < \infty$. The components of the electric field in (7), are given by

$$\begin{aligned} E_{\alpha 1}^{(0)}(r,z) &= -k\gamma J'_{q|m|}(\gamma r)\sin(kz), \\ E_{\alpha 2}^{(0)}(r,z) &= -ik\frac{qm}{r}J_{q|m|}(\gamma r)\sin(kz), \\ E_{\alpha 3}^{(0)}(r,z) &= \gamma^2 J_{q|m|}(\gamma r)\cos(kz), \end{aligned} \tag{9}$$

for the TM modes and by the functions

$$\begin{aligned} E^{(1)}_{\alpha 1}(r,z) &= -\omega \frac{qm}{r} J_{q|m|}(\gamma r)\sin(kz), \\ E^{(1)}_{\alpha 2}(r,z) &= -i\omega\gamma J'_{q|m|}(\gamma r)\sin(kz), \\ E^{(1)}_{\alpha 3}(r,z) &= 0. \end{aligned} \tag{10}$$

for the TE modes. In these expressions $J_\nu(x)$ is the Bessel function and the prime means derivative with respect to the argument of the function. As it is seen from the formulas for the mode functions, they are specified by the set $\alpha = (\lambda, \gamma, m, k)$.

The problem under consideration is symmetric under the reflection with respect to the plate, $z \to -z$. Here we consider the region $z > 0$. In this region the mode functions (7) are normalized by the condition

$$\int_0^\infty dr r \int_0^{\phi_0} d\phi \int_0^\infty dz \, \mathbf{E}^{(\lambda)}_\alpha \cdot \mathbf{E}^{(\lambda')*}_{\alpha'} = 2\pi\omega\delta_{\alpha\alpha'}, \tag{11}$$

where $\delta_{\alpha\alpha'}$ is understood as the Dirac delta function for continuous components of the collective index α and as the Kronecker delta for discrete ones. Substituting the expressions for the mode functions, it can be seen that the normalization coefficient is given by the expression

$$\beta^2_\alpha = \frac{2q}{\pi\gamma\omega}, \tag{12}$$

for both TM and TE modes.

Substituting the mode functions into the mode sum (6), for the Green tensor one finds

$$\begin{aligned} G_{jl}(x,x') &= -2i\theta(\tau)\frac{q}{\pi}\sum_{m=-\infty}^{+\infty}\sum_{\lambda=0,1}\int_0^\infty dk \int_0^\infty d\gamma \frac{1}{\gamma\omega} \\ &\times \left[e^{iqm\Delta\phi - i\omega\tau} E^{(\lambda)}_{\alpha j}(r,z) E^{(\lambda)*}_{\alpha l}(r',z') - e^{-iqm\Delta\phi + i\omega\tau} E^{(\lambda)}_{\alpha l}(r',z') E^{(\lambda)*}_{\alpha j}(r,z) \right], \end{aligned} \tag{13}$$

where $\Delta\phi = \phi - \phi'$. The spectral components of the Green tensor are presented in the form:

$$\begin{aligned} G_{jl}(\mathbf{r},\mathbf{r}';i\xi) &= -\frac{2q}{\pi}\sum_{m=-\infty}^{+\infty}\sum_{\lambda=0,1}\int_0^\infty dk \int_0^\infty d\gamma \frac{1}{\gamma\omega} \\ &\times \left[E^{(\lambda)}_{\alpha j}(r,z) E^{(\lambda)*}_{\alpha l}(r',z') \frac{e^{iqm\Delta\phi}}{\omega - i\xi} + E^{(\lambda)}_{\alpha l}(r',z') E^{(\lambda)*}_{\alpha j}(r,z) \frac{e^{-iqm\Delta\phi}}{\omega + i\xi} \right]. \end{aligned} \tag{14}$$

By taking into account the expressions (9) and (10), the Green tensor may be decomposed as

$$G_{jl}(\mathbf{r},\mathbf{r}';i\xi) = G^{(0)}_{jl}(\mathbf{r},\mathbf{r}';i\xi) + G^{(b)}_{jl}(\mathbf{r},\mathbf{r}';i\xi), \tag{15}$$

where $G^{(0)}_{jl}(\mathbf{r},\mathbf{r}';i\xi)$ is the corresponding function for the boundary-free cosmic string geometry and the part $G^{(b)}_{jl}(\mathbf{r},\mathbf{r}';i\xi)$ is induced by the presence of the conducting plate at $z = 0$. The CP interaction in the boundary-free cosmic string geometry has been discussed in [9, 10] and here we will be mainly concerned with the boundary-induced part.

The components of the tensor $G^{(b)}_{jl}(\mathbf{r},\mathbf{r}';i\xi)$ can be expressed in terms of the functions

$$\begin{aligned} A(r,r',\Delta\phi,y,\xi) &= {\sum_{m=0}^{\infty}}' \cos(qm\Delta\phi) \int_0^\infty dk \cos(ky) \\ &\times \int_0^\infty d\gamma \frac{\gamma J_{qm}(\gamma r) J_{qm}(\gamma r')}{\omega^2 + \xi^2}, \end{aligned} \tag{16}$$

and

$$
\begin{aligned}
B(r,r',\Delta\phi,y,\xi) &= \sum_{j=\pm1}\sum_{m=0}^{\infty}{}' \cos(qm\Delta\phi)\int_0^\infty dk\cos(ky) \\
&\times\int_0^\infty d\gamma \frac{\gamma J_{qm-j}(\gamma r)J_{qm-j}(\gamma r')}{\omega^2+\xi^2},
\end{aligned} \tag{17}
$$

where the prime on the sign of the sum means that the term $m=0$ should be taken with weight $1/2$. For the diagonal components one has the expressions

$$
\begin{aligned}
G_{11}^{(b)}(\mathbf{r},\mathbf{r}';i\xi) &= -\frac{2q}{\pi}\left[\partial_z^2 B(r,r',\Delta\phi,z+z')+\frac{2}{rr'}\partial_{\Delta\phi}^2 A(r,r',\Delta\phi,z+z',\xi)\right], \\
G_{22}^{(b)}(\mathbf{r},\mathbf{r}';i\xi) &= -\frac{2q}{\pi}\left[\partial_z^2 B(r,r',\Delta\phi,z+z')-2\partial_r\partial_{r'}A(r,r',\Delta\phi,z+z',\xi)\right], \\
G_{33}^{(b)}(\mathbf{r},\mathbf{r}';i\xi) &= \frac{4q}{\pi}\left(-\partial_z^2+\xi^2\right)A(r,r',\Delta\phi,z+z',\xi).
\end{aligned} \tag{18}
$$

The off-diagonal components are presented as

$$
\begin{aligned}
G_{12}^{(b)}(\mathbf{r},\mathbf{r}';i\xi) &= -\frac{4q}{\pi rr'}\left[2\partial_{\xi^2}\partial_z^2\partial_{\Delta\phi}+r'\partial_{r'}\partial_{\Delta\phi}\right]A(r,r',\Delta\phi,z+z',\xi), \\
G_{13}^{(b)}(\mathbf{r},\mathbf{r}';i\xi) &= -\frac{4q}{\pi}\partial_z\partial_r A(r,r',\Delta\phi,z+z',\xi), \\
G_{23}^{(b)}(\mathbf{r},\mathbf{r}';i\xi) &= -\frac{4q}{\pi r}\partial_{\Delta\phi}\partial_z A(r,r',\Delta\phi,z+z',\xi).
\end{aligned} \tag{19}
$$

The remained off-diagonal components of the Green tensor are obtained from those in (19) by using the relation

$$
G_{lj}(\mathbf{r}',\mathbf{r};-i\xi)=G_{jl}(\mathbf{r},\mathbf{r}';i\xi). \tag{20}
$$

With formulas (18) and (19), the evaluation of the Green tensor is reduced to the evaluation of the functions (16) and (17). For the function (16) one has the following representation [10]:

$$
A(r,r',\Delta\phi,z+z',\xi)=\frac{\pi}{4q}\left[\sum_k\frac{e^{-\xi u_k}}{u_k}-\frac{q}{2\pi}\sum_{j=\pm1}\int_0^\infty dx\frac{\sin(q\pi+jq\Delta\phi)e^{-\xi v(x)}/v(x)}{\cosh(qx)-\cos(q\pi+jq\Delta\phi)}\right], \tag{21}
$$

where we have defined

$$
\begin{aligned}
u_k &= \sqrt{r^2+r'^2+(z+z')^2-2rr'\cos(2\pi k/q-\Delta\phi)}, \\
v(x) &= \sqrt{r^2+r'^2+(z+z')^2+2rr'\cosh x}.
\end{aligned} \tag{22}
$$

In the first term on the right-hand side of (21) the summation goes under the condition

$$
-q/2+q\Delta\phi/(2\pi)\leqslant k\leqslant q/2+q\Delta\phi/(2\pi). \tag{23}
$$

A similar representation takes place for the function (17) [10]:

$$
\begin{aligned}
B(r,r',\Delta\phi,z+z',\xi) &= \frac{2r}{r'}A(r,r',\Delta\phi,z+z',\xi)+\frac{\pi}{2q\xi}\frac{1}{r'}\partial_r\left[\sum_k e^{-\xi u_k}\right. \\
&\left.-\frac{q}{2\pi}\sum_{j=\pm1}\int_0^\infty dx\frac{\sin(q\pi+jq\Delta\phi)e^{-\xi v(x)}}{\cosh(qx)-\cos(q\pi+jq\Delta\phi)}\right].
\end{aligned} \tag{24}
$$

For the evaluation of the CP potential, we need the components of the boundary-induced part of the Green tensor $G_{jl}^{(b)}(\mathbf{r},\mathbf{r}';i\xi)$ in the coincidence limit: $\mathbf{r}'\to\mathbf{r}$. In order to evaluate $\partial_{\Delta\phi}^2 A(r,r',\Delta\phi,z+z',\xi)$, in the coincidence limit, it is convenient to use the relation

$$\lim_{\mathbf{r}'\to\mathbf{r}}\partial_{\Delta\phi}^2 A(r,r',\Delta\phi,\Delta' z,\xi)=-\lim_{\mathbf{r}'\to\mathbf{r}}[r\partial_r(r\partial_r)-4r^2\partial_{z^2}(r\partial_r+1)]A(r,r',\Delta\phi,z+z',\xi).$$

For the diagonal components one finds the following expression:

$$\begin{aligned}G_{ll}^{(b)}(\mathbf{r},\mathbf{r};i\xi) &= -2\xi^3\left[\sum_{k=0}^{[q/2]}{}' f_l(2\xi\sqrt{r^2s_k^2+z^2},s_k,z)-\frac{q}{\pi}\sin(q\pi)\right.\\ &\quad\left.\times\int_0^\infty dy\frac{f_l(2\xi\sqrt{r^2\cosh^2 y+z^2},\cosh y,z)}{\cosh(2qy)-\cos(q\pi)}\right],\end{aligned}\tag{25}$$

where $[q/2]$ means the integer part of $q/2$ and we have introduced the notation

$$s_k=\sin(\pi k/q).\tag{26}$$

As before, the prime on the sign of sum in (25) means that the term $k=0$ is taken with weight 1/2. In (25) we have defined the function

$$f_l(u,v,z)=e^{-u}\sum_{p=1}^{3}[b_{lp}(v)u^{p-4}+4z^2\xi^2c_{lp}(v)u^{p-6}],\tag{27}$$

with

$$\begin{aligned}b_{lp}(v) &= b_{lp}^{(0)}+b_{lp}^{(1)}v^2,\\ c_{lp}(v) &= c_{lp}^{(0)}+c_{lp}^{(1)}v^2.\end{aligned}\tag{28}$$

The coefficients in (28) are given by the matrices

$$b_{lp}^{(0)}=\begin{pmatrix}1&1&1\\-2&-2&0\\-1&-1&-1\end{pmatrix},\quad b_{lp}^{(1)}=\begin{pmatrix}1&1&-1\\1&1&-1\\0&0&0\end{pmatrix},\tag{29}$$

and

$$c_{lp}^{(0)}=\begin{pmatrix}0&0&0\\3&3&1\\3&3&1\end{pmatrix},\quad c_{lp}^{(1)}=\begin{pmatrix}-3&-3&-1\\-3&-3&-1\\0&0&0\end{pmatrix},\tag{30}$$

where the rows and columns are numbered by l and p, respectively.

In the coincidence limit $\mathbf{r}'\to\mathbf{r}$, the only nonzero off-diagonal component of the boundary-induced part of the Green tensor is $G_{13}^{(b)}(\mathbf{r},\mathbf{r};i\xi)$. From (19) one finds that this component is given by

$$\begin{aligned}G_{13}^{(b)}(\mathbf{r},\mathbf{r};i\xi) &= -8rz\xi^5\left[\sum_{k=0}^{[q/2]}{}' s_k^2 f_{13}(2\xi\sqrt{r^2s_k^2+z^2})-\frac{q}{\pi}\sin(q\pi)\right.\\ &\quad\left.\times\int_0^\infty dy\frac{f_{13}(2\xi\sqrt{r^2\cosh^2 y+z^2})}{\cosh(2qy)-\cos(q\pi)}\cosh^2 y\right],\end{aligned}\tag{31}$$

with the notation

$$f_{13}(u) = u^{-5}e^{-u}(u^2+3u+3). \tag{32}$$

Note that the problem under consideration has less symmetry than the one for a conducting cylindrical boundary coaxial with the string, considered in [3], and in the coincidence limit the Green tensor is non-diagonal.

For integer values of the parameter q, formulas (25) and (31) are reduced to

$$\begin{aligned} G_{ll}^{(b)}(\mathbf{r},\mathbf{r};i\xi) &= -\xi^3 \sum_{k=0}^{q-1} f_l(2\xi\sqrt{r^2s_k^2+z^2}, s_k, z), \\ G_{13}^{(b)}(\mathbf{r},\mathbf{r};i\xi) &= -4rz\xi^5 \sum_{k=0}^{q-1} s_k^2 f_{13}(2\xi\sqrt{r^2s_k^2+z^2}). \end{aligned} \tag{33}$$

The $k=0$ terms in these expressions correspond to the boundary-induced part of the Green tensor for a conducting plate in Minkowski spacetime. Note that the corresponding off-diagonal component vanishes.

3 Casimir-Polder potential

Having the components of the retarded Green tensor, we can evaluate the CP potential using formula (2). Taking into account Eq. (15), the potential may be decomposed as

$$U(\mathbf{r}) = U_0(\mathbf{r}) + U_b(\mathbf{r}), \tag{34}$$

where

$$U_0(\mathbf{r}) = \frac{1}{2\pi}\int_0^\infty d\xi\, \alpha_{jl}(i\xi)[G_{jl}^{(0)}(\mathbf{r},\mathbf{r};i\xi) - G_{jl}^{(\mathrm{M})}(\mathbf{r},\mathbf{r};i\xi)] \tag{35}$$

is the potential in a boundary-free cosmic string geometry and the part

$$U_b(\mathbf{r}) = \frac{1}{2\pi}\int_0^\infty d\xi\, \alpha_{jl}(i\xi) G_{jl}^{(b)}(\mathbf{r},\mathbf{r};i\xi) \tag{36}$$

is induced by the plate at $z=0$.

Substituting expressions (25) and (31), for the components of the Green tensor, into (36), we get the following result

$$U_b(\mathbf{r}) = -\frac{1}{16\pi}\left[\sideset{}{'}\sum_{k=0}^{[q/2]} f(r,z,s_k) - \frac{q}{\pi}\int_0^\infty dy \frac{\sin(q\pi) f(r,z,\cosh y)}{\cosh(2qy)-\cos(q\pi)}\right]. \tag{37}$$

Here we have introduced the notation

$$f(r,z,x) = \sum_{l,p=1}^{3}\left[(r^2x^2+z^2)b_{lp}(x) + z^2c_{lp}(x)\right]\frac{h_{lp}(2\sqrt{r^2x^2+z^2})}{(r^2x^2+z^2)^3} + 2rzx^2\frac{h(2\sqrt{r^2x^2+z^2})}{(r^2x^2+z^2)^3}, \tag{38}$$

with the functions

$$\begin{aligned} h_{lp}(y) &= \int_0^\infty du\, u^{p-1}e^{-u}\alpha_{ll}(iu/y), \\ h(y) &= \int_0^\infty du\, e^{-u}(u^2+3u+3)\alpha_{13}(iu/y). \end{aligned} \tag{39}$$

Assuming that $r \gg z$, the dominant contribution to the CP potential comes from the $k = 0$ term and, to the leading order, the potential coincides with the corresponding potential for a plate in Minkowski spacetime: $U_b(\mathbf{r}) \approx U_b^{(\mathrm{M})}(\mathbf{r})$ (see Eq. (46) below). In the opposite limit, when $r \ll z$, the potential is dominated by the pure string part $U_0(\mathbf{r})$.

In (39), $\alpha_{jl}(i\xi)$ are the physical components of the polarizability tensor in the cylindrical coordinates corresponding to line element (1). These components depend on the orientation of the polarizability tensor principal axes. As a consequence, the CP potential depends on the distance of the microparticle from the string, on the distance from the plate and on the angles determining the orientation of the principal axes. Let $x'^l = (x', y', z')$ be the Cartesian coordinates with the origin at the location of the microparticle and with the axes directed along the principal axes of the polarizability tensor (see Figure 1). We also introduce the intermediate Cartesian coordinates $x''^l = (x'', y'', z'')$ with the same origin and with the z'' axis parallel to the string and with the string coordinate $x'' = -r$. Let β_{ln} be the cosine of the angle between the axes x''^l and x'^n. One has $\sum_{n=1}^{3} \beta_{ln}^2 = 1$. The coefficients β_{ln} can be expressed in terms of the Euler angles (α, β, γ) (see Figure 1) determining the orientation of the principal axes with respect to the coordinate system x''^l (see, for example, [22]). The corresponding matrix $\hat{R}$, with the elements β_{ln}, is given by the expression

$$\hat{R} = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}, \tag{40}$$

where β is the angle between the axes z' and z'', γ (α) is the angle between the axis y' (y'') and the line of nodes (line of the intersection of the $x'y'$ and the $x''y''$ coordinate planes, the line N in Figure 1). For the diagonal components of the polarizability tensor appearing in (39) we have

$$\alpha_{ll}(\omega) = \sum_{n=1}^{3} \beta_{ln}^2 \alpha_n(\omega), \tag{41}$$

where $\alpha_n(\omega)$ are the principal values of the polarizability tensor. The off-diagonal component can be written as $\alpha_{13}(\omega) = \sum_{n=1}^{3} \beta_{1n}\beta_{3n}\alpha_n(\omega)$, or by taking into account (40):

$$\begin{aligned} \alpha_{13}(\omega) &= \sin\beta[(\alpha_1(\omega) - \alpha_2(\omega))\sin\gamma\,(\sin\alpha\cos\gamma + \cos\alpha\sin\gamma\cos\beta) \\ &\quad + (\alpha_3(\omega) - \alpha_1(\omega))\cos\alpha\cos\beta]. \end{aligned} \tag{42}$$

In the isotropic case $\alpha_n(\omega) \equiv \alpha(\omega)$ and we have $\alpha_{jl}(\omega) = \alpha(\omega)\delta_{jl}$. When

$$\alpha_1(\omega) = \alpha_2(\omega), \tag{43}$$

from the general expressions one has simpler relations

$$\begin{aligned} \alpha_{ll}(\omega) &= \alpha_1(\omega) + [\alpha_3(\omega) - \alpha_1(\omega)]\,\beta_{l3}^2, \\ \alpha_{13}(\omega) &= \frac{1}{2}\,[\alpha_3(\omega) - \alpha_1(\omega)]\cos\alpha\sin(2\beta). \end{aligned} \tag{44}$$

In this special case the CP potential does not depend on the angle γ.

For integer values of the parameter q, the general formula (37) is further simplified to

$$U_b(\mathbf{r}) = -\frac{1}{32\pi}\sum_{k=0}^{q-1} f(r, z, s_k). \tag{45}$$

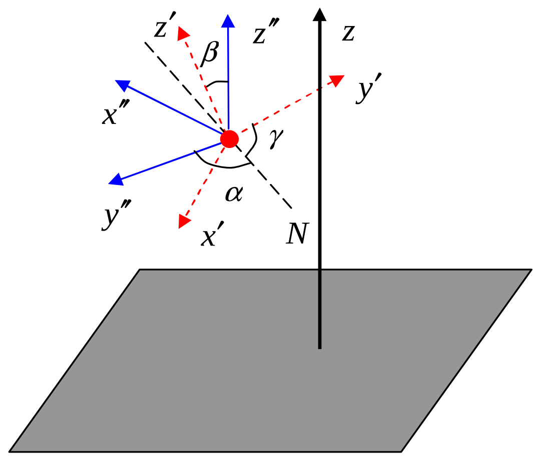

Figure 1: Microparticle near a conducting plate. The cosmic string is directed along the z-axis.

The $k = 0$ term in this expression (and also in (37)) coincides with the CP potential for the geometry of a plate in Minkowski spacetime, $U_b^{(\mathrm{M})}(\mathbf{r})$. Taking into account the expression (38), we find

$$U_b^{(\mathrm{M})}(\mathbf{r}) = -\frac{z^{-4}}{32\pi}\int_0^\infty dx e^{-x}[\left(1+x+x^2\right)\sum_{n=1}^{3}\alpha_n(ix/2z) + \left(1+x-x^2\right)\alpha_{33}(ix/2z)], \tag{46}$$

where

$$\alpha_{33}(\omega) = [\alpha_1(\omega)\cos^2\gamma + \alpha_2(\omega)\sin^2\gamma]\sin^2\beta + \alpha_3(\omega)\cos^2\beta. \tag{47}$$

For $\alpha_n(ix/2z) > 0$, the corresponding CP force is always attractive. This force does not depend on the angle α. In the special case (43), we have no dependence on the angle γ as well.

Now let us consider the asymptotic of the CP potential (37) at large distances from the string and from the boundary compared to the wavelength of the main atomic absorption lines. In this case, the expression for the function $f(r, z, x)$ takes the form

$$f(r,z,x) \approx 4\sum_{l=1}^{3}\alpha_{ll}(0)\frac{\left(b_l r^2 + c_l z^2\right)x^2 + z^2}{(r^2x^2+z^2)^3} + \frac{16rzx^2\alpha_{13}(0)}{(r^2x^2+z^2)^3}, \tag{48}$$

with the coefficients

$$b_l = (1,-1,-1),\ c_l = (-2,-2,0). \tag{49}$$

If, in addition, $z \gg r$, the total CP potential (34) is dominated by the pure string part $U_0(\mathbf{r})$ and to the leading order one gets

$$U(\mathbf{r}) \approx U_0(\mathbf{r}) \approx \frac{\left(q^2-1\right)\left(q^2+11\right)}{360\pi r^4}\left[\alpha_{11}(0) - \alpha_{22}(0) + \alpha_{33}(0)\right]. \tag{50}$$

In the special case (43) the explicit dependence of the potential on the orientation of the principal axes is given by taking into account the relation

$$\alpha_{11}(0) - \alpha_{22}(0) + \alpha_{33}(0) = \alpha_3(0) + 2\left[\alpha_1(0) - \alpha_3(0)\right]\sin^2\alpha\sin^2\beta. \tag{51}$$

In this case, for a fixed value of r, the equilibrium orientation corresponds to $\alpha = \beta = 0$ for $\alpha_1(0) > \alpha_3(0)$ and to $\alpha = \beta = \pi/2$ for $\alpha_1(0) < \alpha_3(0)$.

The z-projection of the force is determined by the boundary-induced part of the CP. At distances from the boundary much larger than the relevant transition wavelengths, assuming $z \gg r$, the function $f(r,z,x)$ given by (48) can be written, approximately, as

$$f(r,z,x) \approx \frac{4}{z^4}\sum_{m=1}^{3}\alpha_m(0)\left[1-2x^2\left(1-\beta_{3m}^2\right)\right]. \tag{52}$$

The corresponding force is attractive with respect to the plate. In particular, for integer $q \geqslant 2$, after the summation over k in (45), one finds

$$U_b(\mathbf{r}) \approx -\frac{q}{8\pi z^4}\alpha_{33}(0). \tag{53}$$

Note from (46) that for a plate in Minkowski spacetime, at large distances, one has

$$U_b^{(\mathrm{M})}(\mathbf{r}) \approx -\frac{1}{8\pi z^4}\sum_{m=1}^{3}\alpha_m(0). \tag{54}$$

The latter does not depend on the orientation of the principal axes for the polarizability tensor. Comparing with (53), we see that this is not the case when the string is present.

At distances smaller than the wavelength of the main atomic absorption lines the dominant contribution to the CP potential comes from the $p=1$ term and from the last term in (38). Up to the leading order, the CP potential is given by (37) with

$$\begin{aligned} f(r,z,x) \quad &\approx \quad 2\sum_{l}^{3}\frac{(r^2x^2+z^2)b_{l1}(x)+z^2c_{l1}(x)}{(r^2x^2+z^2)^{5/2}}\int_0^\infty du\,\alpha_{ll}(iu) \\ &\quad +\frac{12rzx^2}{(r^2x^2+z^2)^{5/2}}\int_0^\infty du\,\alpha_{13}(iu). \end{aligned} \tag{55}$$

If in addition, $r \ll z$, one finds

$$f(r,z,x) \approx \frac{2}{z^3}[\left(1-2x^2\right)\sum_{l=1}^{3}\int_0^\infty du\,\alpha_l(iu)+\left(1+2x^2\right)\int_0^\infty du\,\alpha_{33}(iu)], \tag{56}$$

where $\alpha_{33}(\omega)$ is given by the expression (47). For integer values $q \geqslant 2$, the CP potential has the asymptotic form

$$U_b(\mathbf{r}) \approx -\frac{q}{8\pi z^3}\int_0^\infty du\,\alpha_{33}(iu). \tag{57}$$

Note that for the CP potential in the Minkowski spacetime at small distances we have

$$U_b^{(\mathrm{M})}(\mathbf{r}) \approx -\frac{z^{-3}}{16\pi}\int_0^\infty du\,[\sum_{l=1}^{3}\alpha_l(iu)+\alpha_{33}(iu)]. \tag{58}$$

In the isotropic case one has $\alpha_{jl}(\omega)=\alpha(\omega)\delta_{jl}$ and the expression for the function (38) takes the form

$$\begin{aligned} f(r,z,x) &= 2(r^2x^2+z^2)^{-3}\int_0^\infty du e^{-u}\alpha(iu/2\sqrt{r^2x^2+z^2}) \\ &\quad \times\left\{(x^2-1)(1+u)\left(r^2x^2-2z^2\right)+u^2\left[z^2(1-2x^2)-r^2x^4\right]\right\}. \end{aligned} \tag{59}$$

At large distances we find

$$U_b(\mathbf{r}) \approx \frac{\alpha(0)}{4\pi z^4}\left[\sum_{k=0}^{[q/2]}{}' g_1(r/z, s_k) - \frac{q}{\pi}\int_0^\infty dy \frac{\sin(q\pi) g_1(r/z, \cosh y)}{\cosh(2qy) - \cos(q\pi)}\right], \tag{60}$$

where

$$g_1(y, x) = \frac{y^2x^2 + 4x^2 - 3}{(y^2x^2+1)^3}. \tag{61}$$

In particular, for integer values of q one has

$$U_b(\mathbf{r}) \approx \frac{\alpha(0)}{8\pi}\sum_{k=0}^{q-1}\frac{s_k^2 r^2 + \left(4s_k^2 - 3\right) z^2}{(s_k^2 r^2 + z^2)^3}. \tag{62}$$

Assuming $z \gg r$, for $q \geqslant 2$, we find from (62) that up to the leading order $U_b(\mathbf{r}) \approx -q\alpha(0)/(8\pi z^4)$. For a plate in Minkowski spacetime the corresponding asymptotic is given by the formula $U_b^{(\mathrm{M})}(\mathbf{r}) \approx -3\alpha(0)/(8\pi z^4)$.

For the isotropic polarizability and at distances smaller than the wavelength of the dominant atomic transitions wavelength the asymptotic of the CP potential has the form

$$\begin{aligned} U_b(\mathbf{r}) \;\; &\approx \;\; -\frac{1}{4\pi z^3}\int_0^\infty du\, \alpha(iu)\left[\sum_{k=0}^{[q/2]}{}' g_2(r/z, s_k)\right. \\ &\quad \left. -\frac{q}{\pi}\int_0^\infty dy \frac{\sin(q\pi) g_2(r/z, \cosh y)}{\cosh(2qy) - \cos(q\pi)}\right], \end{aligned} \tag{63}$$

where

$$g_2(y, x) = (x^2 - 1)\frac{y^2x^2 - 2}{(y^2x^2+1)^{5/2}}. \tag{64}$$

For integer values q this gives

$$U_b(\mathbf{r}) \approx \frac{1}{8\pi}\sum_{k=0}^{q-1}(1 - s_k^2)\frac{r^2 s_k^2 - 2z^2}{(r^2 s_k^2 + z^2)^{5/2}}\int_0^\infty du\, \alpha(iu). \tag{65}$$

In particular, for $r \ll z$ and $q \geqslant 2$, from (65), we find $U_b(\mathbf{r}) \approx -q\int_0^\infty du\, \alpha(iu)/(8\pi z^3)$. For the CP potential in the Minkowski spacetime at small distances from the conducting plate one has $U_b^{(\mathrm{M})}(\mathbf{r}) \approx -\int_0^\infty du\, \alpha(iu)/(4\pi z^3)$.

4 Oscillator model

For further transformation of the general formula (37), the frequency dependence of the polarizability tensor appearing in (39) should be specified. For the eigenvalues of the polarizability tensor we use the anisotropic oscillator model. In this model,

$$\alpha_n(i\xi) = \sum_j \frac{g_j^{(n)}}{\omega_j^{(n)2} + \xi^2}, \tag{66}$$

where $\omega_j^{(n)}$ and $g_j^{(n)}$ are the oscillator frequencies and strengths, respectively. For the functions (39) we find the expressions

$$\begin{aligned} h_{lp}(y) &= y^2 \sum_{n=1}^{3} \beta_{ln}^2 \sum_j g_j^{(n)} B_p(y\omega_j^{(n)}), \\ h(y) &= y^2 \sum_{n=1}^{3} \beta_{1n}\beta_{3n} \sum_j g_j^{(n)} \sum_{p=1}^{3} h_p B_p(y\omega_j^{(n)}), \end{aligned} \tag{67}$$

with $h_1 = h_2 = 3$, $h_3 = 1$, and

$$B_p(x) = \int_0^\infty du \frac{u^{p-1}e^{-u}}{u^2 + x^2}. \tag{68}$$

For the first two functions in (68) one has

$$\begin{aligned} B_1(x) &= x^{-1} \left[\sin(x)\mathrm{Ci}(x) - \cos(x)\mathrm{si}(x)\right], \\ B_2(x) &= -\cos(x)\mathrm{Ci}(x) - \sin(x)\mathrm{si}(x), \end{aligned} \tag{69}$$

where the functions $\mathrm{Ci}(x)$ and $\mathrm{si}(x)$ are defined in [23]. The functions $B_p(x)$ for $p \geqslant 3$ are obtained by using the recurrence formula

$$B_{p+2}(x) = \Gamma(p) - x^2 B_p(x). \tag{70}$$

Now the expression for the CP potential is given by (37) where

$$\begin{aligned} f(r,z,x) = 4 &\sum_{n=1}^{3} \sum_j g_j^{(n)} \sum_{p=1}^{3} \frac{B_p(2\omega_j^{(n)}\sqrt{r^2x^2+z^2})}{(r^2x^2+z^2)^2} \\ &\times \left\{ \sum_{l=1}^{3} \left[(r^2x^2+z^2)b_{lp}(x) + z^2 c_{lp}(x)\right]\beta_{ln}^2 + 2h_p rzx^2 \beta_{1n}\beta_{3n} \right\}. \end{aligned} \tag{71}$$

Note that for a plate in the Minkowski spacetime the CP potential is given by the expression

$$U_b^{(\mathrm{M})}(\mathbf{r}) = -\frac{1}{8\pi z^2} \sum_{n=1}^{3} \sum_j g_j^{(n)} \left\{ \left[B_1(2\omega_j^{(n)}z) + B_2(2\omega_j^{(n)}z)\right]\left(1+\beta_{3n}^2\right) + B_3(2\omega_j^{(n)}z)\left(1-\beta_{3n}^2\right) \right\}. \tag{72}$$

This potential is a monotonic increasing function of z and the corresponding force is attractive for all distances.

At small distances, $\omega_j^{(n)}\sqrt{r^2+z^2} \ll 1$, the dominant contribution comes from the term with $p = 1$ by using the asymptotic expression $B_1(y) \approx \pi/(2y)$, valid for $y \ll 1$. If in addition $r \ll z$ one finds

$$f(r,z,x) \approx \frac{\pi}{z^3} \sum_{n=1}^{3} \sum_j \frac{g_j^{(n)}}{\omega_j^{(n)}} \left[1 + \beta_{3n}^2 - 2x^2\left(1-\beta_{3n}^2\right)\right]. \tag{73}$$

For integer $q \geqslant 2$, by using this expression, for the plate-induced part in the CP potential we obtain the asymptotic expression

$$U_b(\mathbf{r}) \approx -\frac{q}{16z^3} \sum_{n=1}^{3} \sum_j \frac{g_j^{(n)}}{\omega_j^{(n)}} \beta_{3n}^2. \tag{74}$$

For a plate in Minkowski spacetime, at distances smaller than the wavelength of the main atomic absorption lines, in the leading order one has

$$U_b^{(\mathrm{M})}(\mathbf{r}) \approx -\frac{z^{-3}}{32}\sum_{n=1}^{3}\sum_j \frac{g_j^{(n)}}{\omega_j^{(n)}}\left(1+\beta_{3n}^2\right). \tag{75}$$

In the opposite limit of large distances, $\omega_j^{(n)}\sqrt{r^2+z^2} \gg 1$, we use $B_p(z) \approx \Gamma(p)/z^2$, $z \gg 1$ and the result (48) is recovered with $\alpha_n(0) = \sum_j g_j^{(n)}/\omega_j^{(n)2}$.

In the isotropic case $g_j^{(n)} = g_j$, $\omega_j^{(n)} = \omega_j$ and the expression (71) reduces to

$$\begin{aligned} f(r,z,x) &= 8\sum_j g_j \left\{ B_3(y_j)\frac{z^2(1-2x^2)-r^2x^4}{(r^2x^2+z^2)^2} \right. \\ &\quad \left. +(x^2-1)\left[B_1(y_j)+B_2(y_j)\right]\frac{r^2x^2-2z^2}{(r^2x^2+z^2)^2}\right\}, \end{aligned} \tag{76}$$

with the notation

$$y_j = 2\omega_j\sqrt{r^2x^2+z^2}. \tag{77}$$

For $y_j \gg 1$ we obtain the result (60) with $\alpha_n(0) = \sum_j g_j\omega_j^{-2}$. An asymptotic expression for the CP potential at small distances, corresponding to $y_j \ll 1$, is obtained from (63) with the substitution $\int_0^\infty du\,\alpha(iu) = (\pi/2)\sum_j g_j/\omega_j$.

Note that in the isotropic case for the pure string part one has [10]

$$U_0(\mathbf{r}) = \frac{1}{2\pi}\left[\sum_{k=1}^{[q/2]} f_0(r,s_k) - \frac{q}{\pi}\int_0^\infty dy\frac{\sin(q\pi)f_0(r,\cosh y)}{\cosh(2qy)-\cos(q\pi)}\right], \tag{78}$$

where

$$f_0(r,v) = \sum_j \frac{g_j}{r^2v^2}\left\{v^2\left[B_1(2rv\omega_j)+B_2(2rv\omega_j)\right]+\left(1-v^2\right)B_3(2rv\omega_j)\right\}. \tag{79}$$

As a numerical example, in figure 2 we plot the dependence of the CP potential of the microparticle with an isotropic polarizability tensor on the distances from the wall and from the string. Single-oscillator model is used for the polarizability. For the parameter q describing the conical space we have taken the value $q = 3$. As it is seen from the plot, the Casimir-Polder force is repulsive with respect to the string and attractive with respect to the wall.

As we have mentioned before, for an anisotropic polarizability tensor the CP potential, in addition to the coordinates r and z of the polarizable particle, depends also on the orientation of the principal axes for the polarizability tensor. As a result of this dependence a moment of force acts on the microparticle. In the numerical example below we use the single oscillator model with $\alpha_n(i\xi) = g^{(n)}/[\omega^{(n)2}+\xi^2]$ and $g^{(1)} = g^{(2)}$, $\omega^{(1)} = \omega^{(2)}$. In this case the CP potential depends on the angles α and β only. In figure 3 we display the dependence of the CP potential, $r^2U(\mathbf{r})/g^{(1)}$, as a function of the angles α and β for $q = 3$, $\omega^{(1)}r = 1$, $\omega^{(1)}z = 1$, $\omega^{(3)}/\omega^{(1)} = 1.5$, and $g^{(3)}/g^{(1)} = 1.25$. The values for the angles α and β corresponding to the minimum of the potential determine the equilibrium orientation of the principal axes for the polarizability tensor.

5 Conclusions

In this paper we have investigated the CP interaction between a polarizable microparticle and a conducting plate in a conical space. The corresponding potential is expressed in terms of

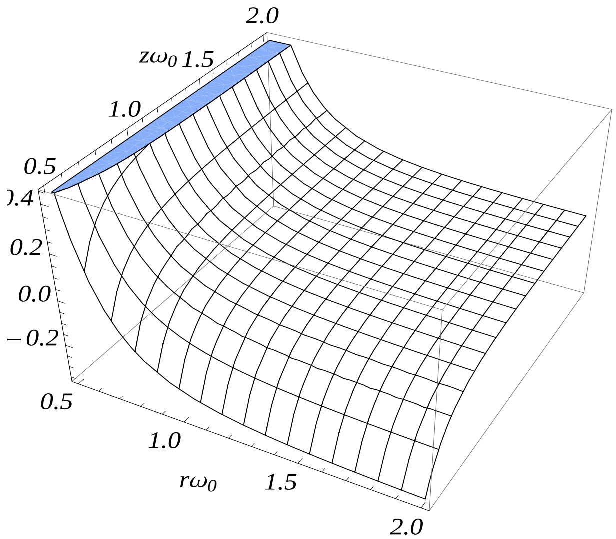

Figure 2: CP potential, $U(\mathbf{r})/(g_0\omega_0^2)$, as a function of the rescaled distances from the wall and from the string in the conical space with the parameter $q = 3$.

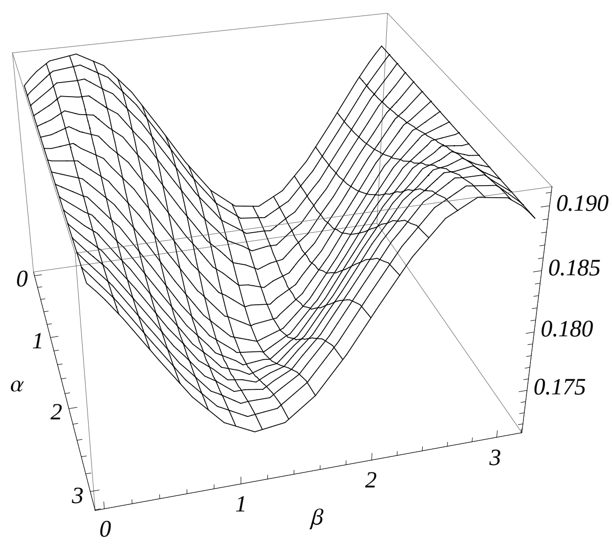

Figure 3: CP potential, $r^2U(\mathbf{r})/g^{(1)}$, as a function of the angles α and β for an anisotropic polarizability tensor. The values of the parameters used in the numerical evaluation are given in the text.

the retarded Green tensor for the electromagnetic field by formula (2). This tensor contains the information about the physical and geometrical properties of vacuum fluctuations. For the evaluation of the Green tensor we have used the direct mode-summation method. In this way the Green tensor is decomposed into the boundary-free and plate-induced parts. The CP interaction in the boundary-free conical space has been discussed previously and here we were mainly concerned with the effects related to the presence of the conducting plate. The corresponding contribution to the Green tensor is given by the expressions (25) and (31) for the diagonal and off-diagonal components, respectively.

Similarly to the Green tensor, the CP potential is decomposed as (34) with the boundary-induced part given by the expression (37). The corresponding force depends on the distance of the microparticle from the string, on the distance from the plate and on the orientation of the polarizability tensor principal axes. The dependence on the orientation enters into the potential through the dependence of the components of the polarizability tensor on the Euler angles. The latter is given by the formulas (41) and (42) with the matrix $\hat{R}$ given by (40). With dependence of the polarizability tensor eigenvalues and the orientation of the principal axes, the CP force can be either attractive or repulsive. The general formula is simplified in the special case with integer values of the parameter $q = 2\pi/\phi_0$ [see (45)]. At distances much larger than the relevant transition wavelengths, the expression for the function $f(r, z, x)$ appearing in the expression for the CP potential takes the form (48). If in addition $z \gg r$, one has the asymptotic (52). In this case the potential varies inversely with the fourth power of the distance from the conducting plate and the corresponding force is attractive with respect to the plate. For integer values of $q \geqslant 2$, the asymptotic behavior of the potential is given by the expression (53) which depends on the orientation of the polarizability tensor principal axes. For a plate in Minkowski spacetime the corresponding asymptotic expression is given by (54) and in the leading order the force does not depend on the orientation. For the isotropic polarizability tensor the plate-induced part in the CP potential is given by the expression (37) with the function $f(r, z, x)$ given by (59). At large distances and for integer values of q, the corresponding asymptotic expression is given by (62). If in addition $z \gg r$, for $q \geqslant 2$ one has $U_b/U_b^{(\mathrm{M})} \approx q/3$.

It is important to call attention to the fact that U_b is divergent at $z = 0$. Otherwise, near the string, its behavior is well defined. For the frequency dependence of the polarizability tensor we have used the anisotropic oscillator model with the eigenvalues given by (66). With this model, the function $f(r, z, x)$ in the expression (37) for the CP potential takes the form (71) for the general case and the form (76) in the case of isotropic polarizability tensor. In the case of anisotropic polarizability, the dependence of the CP potential on the orientation of the polarizability tensor principal axes also leads to a moment of force acting on the particle. This results in the macroscopic polarization of a system of particles induced by combined effects of the string and the boundary.

In the discussion above we have assumed that the electromagnetic field is prepared in the vacuum state. If the field is prepared in a thermal state with temperature T a new length scale appears, $\lambda_T = (k_B T)^{-1}$, with k_B being the Boltzmann constant. At nonzero temperature the results obtained in this paper remain valid in the region $r, z \ll \lambda_T$.

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