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A view on extending morphisms from ample divisors

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Dedicated to Andrew J. Sommese on his 60th birthday

Abstract. The philosophy that "a projective manifold is more special than any of its smooth hyperplane sections" was one of the classical principles of projective geometry. Lefschetz type results and related vanishing theorems were among the typically used techniques. We shall survey most of the problems, results and conjectures in this area, using the modern setting of ample divisors, and (some aspects of) Mori theory.

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1. Introduction

In the context of classical algebraic geometry, consider a given embedded complex projective manifold $X \subset \mathbb{P}^N$. One of the typically used techniques was to

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replace X by some of its smooth hyperplane sections, $Y \subset X$. Thus, the dimension of X is decreased and classification results may be obtained inductively. The efficiency of the method depends on the possibility of transferring some known special properties from Y to X. In general, given $Y \subset \mathbb{P}^{N-1}$, there is no *smooth* $X \subset \mathbb{P}^N$ such that Y is one of its hyperplane sections. One can say that X *is more special then* Y.

The present paper is a survey of contemporary aspects of the hyperplane section technique. A first important "modern" incarnation of the above principle is given by Lefschetz's theorem, showing that the topology of Y strongly reflects that of X (see [**5**]). From a geometrical point of view, we are usually given some regular map, say $p : Y \to Z$, making Y *special*; e.g., a Fano fibration. We would like to extend this map to X. It was discovered by Sommese, in his innovative early paper [**67**], that the extension is always possible, if one only assumes that the general fiber of p has dimension at least two. His proof is based on Lefschetz's theorem and on (very much related) vanishing results of Kodaira type. In the same paper, Sommese showed that when p is smooth and extends, the dimension of Z cannot be too large. It soon became clear that the extension problem is much harder when $\dim Y \leq \dim Z + 1$. Fujita [**25**] further refined some of the techniques and considered new applications e.g., when p is a $\mathbb{P}^d$-bundle or a blowing-up. In the case of three folds, fine results were found by Bădescu [**6, 7, 8**], when Y is a $\mathbb{P}^1$-bundle over a curve and by Sommese [**69, 70**], when Y is not relatively minimal. It is worth pointing out that the classical context of hyperplane sections was gradually replaced by the more general situation when $Y \subset X$ is merely an ample divisor, and no projective embedding of X is given. This is a substantial generalization, since in the new setting the normal bundle of Y in X is *not* specified.

The appearance of Mori theory made possible a change of both the point of view and the techniques (see [**52, 38**]). The isomorphism between the Picard groups of X and Y given by the Lefschetz theorem leads to an inclusion between the Kleiman–Mori cones $\overline{NE}(Y)$ and $\overline{NE}(X)$. As is well known, faces of these cones describe non-trivial morphisms defined on Y and X, respectively. So, the original question of extending maps from Y to X translates into a comparison problem between these cones. Ideally, when the two cones are equal, all morphisms from Y extend to X (see e.g., [**40, 74, 13, 2**] and Section 8 for results in this direction, usually when X or Y are Fano manifolds). In the general case, what we can hope for is to extend the contraction of an extremal ray of Y (cf. [**35, 57**]). This is not always possible, but very few counterexamples are known (see Section 4 for this intriguing aspect). The techniques used in this setting are the cone theorem, due to Mori, and the contraction theorem, due to Kawamata–Reid–Shokurov, combined with the well behaved deformation theory of families of rational curves [**41, 20, 38**]. See also [**1, 34, 40**] and [**74**] for some useful facts about special families of rational curves, coming from extremal rays.

General results on extending morphisms are discussed in Section 3; in Section 5 we concentrate on the special situation when p is a $\mathbb{P}^d$-bundle or a blowing-up. We pay special attention to the case of $\mathbb{P}^1$-bundles, which is the most difficult. A (still open) main conjecture on the subject is stated and various related facts are proved in Section 7. The afore mentioned results by Bădescu and Sommese on three folds are recovered in Section 6, using the Mori theory point of view (cf. [**34**]). In the

last section we discuss the ascent of some good properties from Y to X: e.g., being uniruled, or rationally connected, or rational, etc.

We have tried to write a complete and coherent exposition, also accessible to the nonspecialist. We included several new proofs and sometimes substantial simplifications of the original arguments. Several possible generalizations are mentioned at the end of the paper, together with appropriate references to the existing literature.

2. Background material

We work over the complex field $\mathbb{C}$. Throughout the paper we deal with irreducible, reduced, projective varieties X. We use the term manifold if X is moreover assumed to be smooth. We denote by $\mathcal{O}_X$ the structure sheaf of X. For any coherent sheaf $\mathcal{F}$ on X, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(X, \mathcal{F})$. If $p : X \to Y$ is a morphism, we write $p_{(i)}$ for its i-th direct image.

Let L be a line bundle on X. L is said to be *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on X. We say that L is *strictly nef* (or *numerically positive*) if $L \cdot C > 0$ for all effective curves C on X. L is said to be *big* if $\kappa(L) = \dim X$, where $\kappa(L)$ denotes the Iitaka dimension of L. If L is nef then this is equivalent to $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L and $n = \dim X$. The pull-back $\iota^* L$ of a line bundle L on X by an embedding $\iota : Y \hookrightarrow X$ is denoted by L_Y. We denote by $N_{Y/X}$ the normal bundle of Y in X and by K_X the canonical bundle of a smooth variety X.

We use standard notation from algebraic geometry, among which we recall the following ones:

- $\approx$, the linear equivalence of line bundles; $\sim$, the numerical equivalence of line bundles;
- $|L|$, the complete linear system associated to a line bundle L;
- $\kappa(D)$, the Iitaka dimension of the line bundle associated to a $\mathbb{Q}$-Cartier divisor D on X; and $\kappa(X) := \kappa(K_X)$, the Kodaira dimension of X, for X smooth.
- $\pi_i(X)$, the i-th homotopy group, omitting the base point when its choice is irrelevant.

$\mathbb{P}^n$ denotes the projective n-space, $\mathcal{Q}^n \subset \mathbb{P}^{n+1}$ denotes the n-dimensional hyperquadric. For a vector bundle $\mathcal{E}$, we write $\mathbb{P}(\mathcal{E})$ for the associated projective bundle and $\xi_{\mathbb{P}}$, or ξ_X when $X = \mathbb{P}(\mathcal{E})$, for the tautological line bundle, using the Grothendieck convention.

Line bundles and divisors are used with little (or no) distinction. We almost always use the additive notation. We say that a line bundle L is *spanned* if it is spanned, i.e., globally generated, at all points of X by $H^0(X, L)$.

2.1. Setting up and motivation.

Let X be a projective manifold and let $Y \subset X$ be a smooth ample divisor. It is a natural classical question to try to understand how the structure of Y determines the one of X.

More precisely, given a surjective morphism $p = p_{|D|} : Y \to Z$ associated to a linear system $|D|$, we look for a linear system $|\overline{D}|$ on X defining a regular map

$\overline{p} = p_{|\overline{D}|} : X \to W$ onto a projective variety W, such that the following diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{p}} \\ Z & \xrightarrow[\alpha]{} & W \end{array} \tag{2.1}$$

commutes. If the morphism $\alpha : Z \to \alpha(Z)$ is finite we say that $\overline{p}$ is a *lifting* of p. If $\overline{p}_{|Y} = p$, that is if $\alpha : Z \to \alpha(Z) \subset W$ is an isomorphism onto its image, we say that $\overline{p}$ is a *strict lifting* of p, or that p is *extendable* to $\overline{p}$. Note that this is always the case whenever the restriction map $H^0(X, \overline{D}) \to H^0(Y, D)$ is surjective. Note also that this further condition will be a posteriori satisfied in our setting (see the proof of Theorem 3.8).

Assume that the morphism p has a lifting $\overline{p}$. Up to taking the Remmert–Stein factorization, we can always assume that p has connected fibers and Z is normal. Therefore, by using the ampleness of Y in X, it is a standard fact that one of the following holds:

(1) $\dim Y - \dim Z \geq 1$ and $\alpha : Z \xrightarrow{\sim} W$ (in particular p is extendable);
(2) $p, \overline{p}$ and $\alpha : Z \to \alpha(Z)$ are birational; so, α is the normalization morphism;
(3) p is birational and $\dim X - \dim W = 1$; in this case $\alpha : Z \to \alpha(Z)$ may be of degree ≥ 2.

A simple example is obtained as follows. Consider $X := \mathbb{P}^1 \times \mathbb{P}^{n-1}$ embedded in $\mathbb{P}^N$ by $\mathcal{O}(2,1)$, $n \geq 4$. By Bertini's theorem, we can choose a hyperplane H in $\mathbb{P}^N$ such that the restriction Y of H to X is a smooth ample divisor. Then we get a commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X = \mathbb{P}^1 \times \mathbb{P}^{n-1} \\ {\scriptstyle p}\downarrow & \searrow & \downarrow{\scriptstyle \overline{p}} \\ Z & \xrightarrow[\alpha]{} & \mathbb{P}^{n-1} \end{array}$$

where p and α are given by the Remmert–Stein factorization of the restriction $\overline{p}_{|Y} : Y \to \mathbb{P}^{n-1}$ and $\overline{p}$ is the natural projection. Note that the morphism α is finite of degree two. Moreover, p is not an isomorphism. Indeed, assume otherwise. Then $Y \to \mathbb{P}^{n-1}$ is a two-to-one finite covering, so that it induces an isomorphism $\mathrm{Pic}(Y) \cong \mathbb{Z}$ (see [**49**, II, 7.1.20] for details and complete references). On the other hand, $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ by the Lefschetz theorem; a contradiction.

If the morphism p is extendable, our aim is to describe X by using the structure morphism $\overline{p}$. The occurrence that p is not extendable forces X to satisfy geometric constraints which, in turn, make X special enough to be completely classified.

As a typical example, consider the following natural question, formulated in the classical context:

Question 2.1. Let X be an n-dimensional manifold embedded in a projective space $\mathbb{P}^N$. Assume that a smooth hyperplane section, $Y = X \cap H$, of X is a $\mathbb{P}^d$-bundle over some manifold Z, such that the fibers are linearly embedded. Does it follow that the bundle projection $p : Y \to Z$ extends to X giving a $\mathbb{P}^{d+1}$-bundle projection $\overline{p} : X \to Z$?

As soon as $n \geq 4$, the (positive) answer to this question relies on some non-trivial results from the deformation theory of rational curves. It turns out that the

key-fact is the condition $H \cdot f = 1$, where f is a line in a fiber $\mathbb{P}^d$ of $p : Y \to Z$, i.e., f is a linear $\mathbb{P}^1$ with respect to the embedding of X in $\mathbb{P}^N$ given by H. Moreover, the above can happen only if $2 \dim Z \leq \dim X$ (Proposition 5.9).

2.2. Special varieties. Let X be a projective manifold of dimension n. We say that X is a *Fano manifold* if $-K_X$ is ample; its *index*, i, is the largest positive integer such that $K_X \approx -i\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on X. Let L be a given ample line bundle on X. We say that (X, L) is a *del Pezzo variety* (respectively a *Mukai variety*) in the adjunction theoretic sense if $K_X \approx -(n-1)L$ (respectively $K_X \approx -(n-2)L$). Note that del Pezzo manifolds are completely described by Fujita [**27**, I, Section 8]. We refer to Mukai [**54**] for results on Mukai varieties.

We say that (X, L) is a *scroll* over a normal variety Z of dimension m if there exists a surjective morphism with connected fibers $p : X \to Z$, such that $K_X + (n - m + 1)L \approx p^*\mathcal{L}$ for some *ample* line bundle $\mathcal{L}$ on Z.

We refer to [**15**] and [**16**, Sections 14.1, 14.2] for relations between the adjunction theoretic and the classical definition of scrolls.

Let X be a projective manifold and let $p : X \to Z$ be a surjective morphism onto another manifold, Z. We say that X, $p : X \to Z$ is a $\mathbb{P}^d$*-bundle* if each closed fiber of p is isomorphic to the projective space $\mathbb{P}^d$. We also say that X, $p : X \to Z$ is a *linear* $\mathbb{P}^d$*-bundle* if $X = \mathbb{P}(\mathcal{E})$ for some rank $d+1$ vector bundle $\mathcal{E}$ on Z.

We say that X, $p : X \to Z$ is a *conic fibration* over a normal projective variety Z if every fiber of the morphism p is a conic, i.e., it is isomorphic to the zero scheme of a non-trivial section of $\mathcal{O}_{\mathbb{P}^2}(2)$. Note that the above definition is equivalent to saying that there exists a rank 3 vector bundle $\mathcal{E}$ over Z such that its projectivization $\widetilde{p} : \mathbb{P}(\mathcal{E}) \to Z$ contains X embedded over Z as a divisor whose restriction to any fiber of $\widetilde{p}$ is an element of $|\mathcal{O}_{\mathbb{P}^2}(2)|$. The push-forward $p_*(-K_X)$ can be taken as the above $\mathcal{E}$. It is a standard fact to show that $p : X \to Z$ is a flat morphism; since X is smooth, it follows that the base Z is smooth, too.

2.3. Lefschetz-type and vanishing results. A basic tool for dealing with the problems discussed above are Lefschetz's theorems, which, in turn, are very much related (in fact, almost equivalent) to vanishing results of Kodaira type (see [**49**, I, Chapters 3, 4] for a nice general presentation and complete references). See [**5**] for the classical statement of Lefschetz's theorem.

Theorem 2.2. (Hamm–Lefschetz theorem) *Let L be an ample line bundle on a projective manifold, X, and let $D \in |L|$. Then given any point $x \in D$ it follows that the j-th relative homotopy group, $\pi_j(X, D, x)$, vanishes for $j \leq \dim X - 1$. In particular, the restriction mapping, $H^j(X, \mathbb{Z}) \to H^j(D, \mathbb{Z})$ is an isomorphism for $j \leq \dim X - 2$, and is injective with torsion free cokernel for $j = \dim X - 1$.*

Theorem 2.3. (Barth–Lefschetz theorem) *Let Y be a connected submanifold of a projective manifold, X. Let $n = \dim X$, $m = \dim Y$. Assume that $N_{Y/X}$ is ample. Then for any $x \in Y$, we have $\pi_j(X, Y, x) = 0$ for $j \leq 2m - n + 1$. In particular, under the natural map we have $\pi_1(Y, x) \cong \pi_1(X, x)$ if $2m - n \geq 1$. Moreover:*

(i) *If $2m - n = 1$, the restriction map $r : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective with torsion free cokernel; and*
(ii) *If $2m - n \geq 2$, then $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$ via r.*

Kawamata and Viehweg showed that the Kodaira vanishing theorem holds for any nef and big line bundle (see e.g., [**38**, Sections 1–2]).

Theorem 2.4. (Kawamata–Viehweg vanishing theorem) *Let X be a projective manifold of dimension n, and let D be a nef and big divisor on X. Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0 \quad \text{for } i > 0.$$

2.4. Basic facts from Mori theory. Let us recall some definitions and a few facts from Mori theory we need. Basic references for details are [**52, 53**], and [**38**]. Let X be a connected normal projective variety of dimension $n(\geq 2)$.

- $\mathrm{Num}(X) = \mathrm{Pic}(X)/\sim$;
- $N^1(X) = \mathrm{Num}(X) \otimes \mathbb{R}$;
- $N_1(X) = (\{1\text{-cycles}\}/\sim) \otimes \mathbb{R}$;
- $NE(X)$, the convex cone in $N_1(X)$ generated by the effective 1-cycles;
- $\overline{NE}(X)$, the closure of $NE(X)$ in $N_1(X)$ with respect to the Euclidean topology;
- $\varrho(X) = \dim_{\mathbb{R}} N_1(X)$, the *Picard number* of X;
- $\overline{NE}(X)_{D\geq 0} = \{\zeta \in \overline{NE}(X) \mid \zeta \cdot D \geq 0\}$ for given $D \in \mathrm{Pic}(X) \otimes \mathbb{Q}$;
- $\overline{NE}(X)_{D<0} = \{\zeta \in \overline{NE}(X) \mid \zeta \cdot D < 0\}$ for given $D \in \mathrm{Pic}(X) \otimes \mathbb{Q}$;
- $\overline{NE}(X)_{D\leq 0} = \{\zeta \in \overline{NE}(X) \mid \zeta \cdot D \leq 0\}$ for given $D \in \mathrm{Pic}(X) \otimes \mathbb{Q}$;
- $\mathrm{Nef}(X)$, the dual cone of $\overline{NE}(X)$, namely, the cone in $N^1(X)$ spanned by classes of nef divisors.

If γ is a 1-dimensional cycle in X we denote by $[\gamma]$ its class in $\overline{NE}(X)$. Note that the vector spaces $N^1(X)$ and $N_1(X)$ are dual to each other via the usual intersection of cycles "$\cdot$".

Assume that X is smooth. We say that a half line $R = \mathbb{R}_+[\zeta]$ in $\overline{NE}(X)$, where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, is an *extremal ray* if $K_X \cdot \zeta < 0$ and $\zeta_1, \zeta_2 \in R$ for every $\zeta_1, \zeta_2 \in \overline{NE}(X)$ such that $\zeta_1 + \zeta_2 \in R$.

An extremal ray $R = \mathbb{R}_+[\zeta]$ is *nef* if $D \cdot \zeta \geq 0$ for every effective divisor D on X. An extremal ray which is not nef is said to be *non-nef*.

Let $D \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ be a nef $\mathbb{Q}$-divisor, $D \not\sim 0$. Let

$$F_D := D^{\perp} \cap (\overline{NE}(X) \setminus \{0\}),$$

where "$\perp$" means the orthogonal complement of D in $N_1(X)$. Then F_D is called a *good extremal face* of $\overline{NE}(X)$ and D is the *supporting hyperplane* of F_D, if F_D is entirely contained in the set $\{\zeta \in N_1(X) \mid K_X \cdot \zeta < 0\}$. An extremal ray is a 1-dimensional good extremal face. Indeed, for any extremal ray R there exists a nef $D \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ such that $R = D^{\perp} \cap (\overline{NE}(X) \setminus \{0\})$.

Theorem 2.5. (Mori cone theorem) *Let X be a projective manifold of dimension n. Then there exists a countable set of curves $C_i, i \in I$, with $K_X \cdot C_i < 0$, such that one has the decomposition*

$$\overline{NE}(X) = \sum_{i \in I} \mathbb{R}_+[C_i] + \overline{NE}(X)_{K_X \geq 0}.$$

The decomposition has the properties:

(i) *the set of curves C_i is minimal, no smaller set is sufficient to generate the cone;*

(ii) *given any neighborhood U of $\overline{NE}(X)_{K_X \geq 0}$, only finitely many $[C_i]$'s do not belong to U.*

The semi-lines $\mathbb{R}_+[C_i]$ are the extremal rays of X. Moreover, the curves C_i are (possibly singular) reduced irreducible rational curves which satisfy the condition $1 \leq -K_X \cdot C_i \leq n+1$.

Theorem 2.6. (Kawamata–Reid–Shokurov base point free theorem) *Let X be a projective manifold of dimension $n \geq 2$. Let D be a nef Cartier divisor such that $aD - K_X$ is nef and big for some positive integer a. Then $|mD|$ has no base points for $m \gg 0$.*

It is a standard fact that, for a good extremal face F_D, the line bundle $mD-K_X$ is ample for $m \gg 0$. Therefore, by Theorem 2.6, the linear system $|mD|$ is base point free for $m \gg 0$, so that it defines a morphism, say $\varphi : X \to W$. By taking m big enough, we may further assume that W is normal and the fibers of φ are connected. Note that $\varphi_*\mathcal{O}_X \cong \mathcal{O}_W$, the pair (W, φ) is unique up to isomorphism and $D \in \varphi^* \operatorname{Pic}(W)$. If C is an irreducible curve on X, then $[C] \in F_D$ if and only if $D \cdot C = 0$, which means $\dim \varphi(C) = 0$, i.e., φ contracts the good extremal face F_D. We will call such a contraction, φ, the *contraction of F_D*. If $F_D = R$, R an extremal ray, we will denote $\operatorname{cont}_R : X \to W$ the contraction morphism. Let

$$E := \{x \in X \mid \operatorname{cont}_R \text{ is not an isomorphism at } x\}.$$

Note that E is the locus of curves whose numerical class is in R. We will refer to E simply as the *locus* of R.

If X is smooth we define the *length of an extremal ray*,

$$\operatorname{length}(R) = \min\{-K_X \cdot C \mid C \text{ rational curve}, [C] \in R\}.$$

Note that the cone theorem yields the bound $0 < \operatorname{length}(R) \leq n+1$. We will also use the notation $\operatorname{length}(R) = \ell(R)$. We say that a rational curve C generating an extremal ray $R = \mathbb{R}_+[C]$ is a *minimal curve* if $\ell(R) = -K_X \cdot C$.

The following useful inequality is inspired by Mori's bend and break (cf. [**34**, Theorem 0.4], and also [**74**, Theorem 1.1], [**41**, Corollary IV.2.6]).

Theorem 2.7. *Let X be a projective manifold of dimension n. Assume that K_X is not nef and let R be an extremal ray on X of length $\ell(R)$. Let ρ be the contraction of R and let E be any irreducible component of the locus of R. Let Δ be any irreducible component of any fiber of the restriction, ρ_E, of ρ to E. Then*

$$\dim E + \dim \Delta \geq n + \ell(R) - 1.$$

By combining the theorem above with a result due to Ando [**1**], Wiśniewski [**74**, Theorem (1.2)] showed the following, which plays an important role in the sequel.

Theorem 2.8. (Ando [**1**], Wiśniewski [**74**]) *Let X be a projective manifold of dimension $n \geq 3$. Assume that K_X is not nef. Let $\varphi : X \to Z$ be the contraction morphism of an extremal ray R. If every fiber of φ has dimension at most one, then Z is smooth and either φ is the blowing-up of a smooth codimension two subvariety of Z, or φ is a conic fibration.*

Ando [**1**, (3.10), (2.3)] proved the theorem above assuming that the locus E of R satisfies the condition $\dim E \geq n-1$. From the inequality of Theorem 2.7 it follows that this is the case. Indeed, if $\dim E \leq n-2$, for any irreducible component Δ of any fiber of the restriction of φ to E, we would have

$$\dim \Delta \geq \ell(R) + 1 \geq 2,$$

contradicting the fibers dimension assumption.

2.5. Families of rational curves. We follow the notation in [**41**], to which we refer for details; see also [**20**]. Let X be a projective manifold. By $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ we denote the scheme parameterizing morphisms from $\mathbb{P}^1$ to X which are birational onto their image. We will denote by $[f]$ the point of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ determined by such a morphism $f : \mathbb{P}^1 \to X$.

A reduced, irreducible subvariety $V \subset \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ determines a *family of rational curves* on X. We let $\mathcal{F}$ be the universal family, restricted to V, with $p : \mathcal{F} \to V$ and $q : \mathcal{F} \to X$ the natural projections. We call the image of q the *locus of the family*, denoted by $\mathrm{Locus}(V)$. A *covering family* is a family satisfying $\mathrm{Locus}(V) = X$. We say that a family V, closed under the action of $\mathrm{Aut}(\mathbb{P}^1)$, is *unsplit* if the image of V in $\mathrm{Chow}(X)$ under the natural morphism $[f] \to [f(\mathbb{P}^1)]$ is closed. In general, the closure of the image of V in $\mathrm{Chow}(X)$ determines a *family of rational 1-cycles* on X. If $x \in X$ is a fixed (closed) point, we denote by V_x the closed subfamily of V consisting of morphisms sending a fixed point $O \in \mathbb{P}^1$ to x. We say that V is *locally unsplit* if, for $x \in \mathrm{Locus}(V)$ a general point, the family V_x is unsplit. A family V of rational 1-cycles on X is *quasi-unsplit* if any two irreducible components of cycles in V are numerically proportional. Such families typically arise from cycles belonging to an extremal ray.

3. General results

We discuss throughout this section some general results on extending morphisms $p : Y \to Z$ from ample (smooth) divisors Y of a manifold X.

To begin with, let us prove two early theorems due to Sommese [**67**] (see also [**16**, (5.2.1), (5.2.5)]) that marked the starting point of the subject. The first one shows that the morphism p is always extendable whenever $\dim Y - \dim Z \geq 2$. The second one gives the restriction that $\dim X \geq 2 \dim Z$ for a smooth $p : Y \to Z$ to extend.

Theorem 3.1. (Sommese [**67**]) *Let Y be a smooth ample divisor on a projective manifold X. Let $p : Y \to Z$ be a surjective morphism. If*

$$\dim Y - \dim Z \geq 2,$$

then p extends to a surjective morphism $\overline{p} : X \to Z$.

Proof. Let $\dim X =: n$. Without loss of generality it can be assumed that $\dim Z \geq 1$. Thus we have that $n \geq 4$ and therefore by Lefschetz theorem we see that the restriction map gives an isomorphism, $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$. Moreover, by Remmert–Stein factorizing p it can be assumed that Z is normal and p has connected fibers.

Let $\mathcal{L}$ be a very ample line bundle on Z. Since $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$ there exists an $H \in \mathrm{Pic}(X)$ whose restriction, H_Y, to Y is isomorphic to $p^*\mathcal{L}$. Let $L := \mathcal{O}_X(Y)$. We claim that

$$H^1(Y, (H - tL)_Y) = 0 \quad \text{for } t \geq 1. \tag{3.1}$$

By Serre duality we are reduced to showing $H^{n-2}(Y, K_Y + (tL - H)_Y) = 0$ for $t \geq 1$.

From the relative form of the Kodaira vanishing theorem (see e.g., [**38, 22, 66**]) we see that

$$p_{(j)}(K_Y + (tL - H)_Y) = p_{(j)}(K_Y + tL_Y) \otimes (-\mathcal{L}) = 0$$

for $j \geq 1$. Using the Leray spectral sequence we deduce that

$$H^{n-2}(Y, K_Y + (tL - H)_Y) = H^{n-2}(Z, p_*(K_Y + (tL - H)_Y)).$$

The last group is zero, since $n - 2 = \dim Y - 1 > \dim Z$ by our assumption. This shows (3.1).

Now consider the exact sequence

$$0 \to K_X \otimes (L-H) \otimes (tL) \to K_X \otimes (L-H) \otimes (t+1)L \to K_Y \otimes (L-H)_Y \otimes (tL_Y) \to 0.$$

By (3.1) we have $H^1(Y, (H-L)_Y - tL_Y) = 0$ for $t \geq 0$. Therefore, by Serre duality, $H^{n-2}(Y, K_Y \otimes (L-H)_Y \otimes (tL_Y)) = 0$ for $t \geq 0$. Thus the exact sequence above gives an injection of $H^{n-1}(X, K_X \otimes (L-H) \otimes (tL))$ into $H^{n-1}(X, K_X \otimes (L-H) \otimes (t+1)L)$. By Serre's vanishing theorem, $H^{n-1}(X, K_X \otimes (L - H) \otimes (t + 1)L) = 0$ for $t \gg 0$. Therefore we conclude that

$$H^{n-1}(X, K_X \otimes (L - H) \otimes (tL)) = H^1(X, H - (t + 1)L) = 0 \quad \text{for } t \geq 0.$$

Hence in particular $H^1(X, H - L) = 0$, so that we have a surjection

$$H^0(X, H) \to H^0(Y, H_Y) \to 0.$$

Since $H_Y \cong p^*\mathcal{L}$ with $\mathcal{L}$ very ample on Z, we infer that there exist $\dim Z + 1$ divisors $D_1, \ldots, D_{\dim Z+1}$ in $|H|$ such that $D_1 \cap \cdots \cap D_{\dim Z+1} \cap Y = \emptyset$. Since Y is ample it thus follows that $\dim(\cap_{i=1}^{\dim Z+1} D_i) \leq 0$. We claim that H is spanned by its global sections. If $\cap_{i=1}^{\dim Z+1} D_i = \emptyset$, then $\mathrm{Bs}|H| = \emptyset$. If $\cap_{i=1}^{\dim Z+1} D_i \neq \emptyset$, then

$$\dim(D_1 \cap \cdots \cap D_{\dim Z+1}) \geq \dim X - \dim Z - 1 \geq 2.$$

This contradicts the above inequality.

Let $\overline{p} : X \to \mathbb{P}^N$ be the map associated to $H^0(X, H)$. Ampleness of Y yields that $\overline{p}(X) = p(Y)$, so $\overline{p}|_Y = p$ and we are done. Q.E.D.

Theorem 3.2. (Sommese [**67**]) *Let Y be a smooth ample divisor on a projective manifold, X. Let $p : Y \to Z$ be a morphism of maximal rank onto a normal variety, Z. If p extends to a morphism $\overline{p} : X \to Z$, then $\dim X \geq 2 \dim Z$.*

Proof. We follow the topological argument from [**67**, Proposition V]. Let S be the image of the set of points where $\overline{p}$ is not of maximal rank. By ampleness of Y the set S is finite. Let $f = p^{-1}(z)$, $F = \overline{p}^{-1}(z)$ be the fibers of $p, \overline{p}$ over $z \in Z \setminus S$ respectively. Let $\dim Z = b$ and let $r = \dim Y - \dim Z$. From standard results in topology we deduce: $\pi_j(Y, f) \cong \pi_j(Z)$ for all j, $\pi_j(X, F) \cong \pi_j(Z)$ for $j \leq 2b - 2$ and $\pi_{2b-1}(X, F) \to \pi_{2b-1}(Z)$ is onto. It follows that $\pi_j(Y, f) \to \pi_j(X, F)$ is an isomorphism for $j \leq 2b - 2$ and it is onto for $j = 2b - 1$. From Theorem 2.2, we have that $\pi_j(Y) \to \pi_j(X)$ is an isomorphism if $j < \dim Y$ and is onto for $j = \dim Y$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
\cdots & \to & \pi_j(f) & \to & \pi_j(Y) & \to & \pi_j(Y, f) & \to & \pi_{j-1}(f) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \pi_j(F) & \to & \pi_j(X) & \to & \pi_j(X, F) & \to & \pi_{j-1}(F) & \to & \cdots
\end{array}$$

Arguing by contradiction, let us assume $\dim X < 2 \dim Z$, or $r < b - 1$. It follows that $2r + 2 \leq r + b = \dim Y$ and $2r + 2 < 2b - 1$. This, the above, and the

five lemma show that $\pi_j(f) \cong \pi_j(F)$ for $j < 2r+2$ and $\pi_{2r+2}(f) \to \pi_{2r+2}(F)$ is onto.

By Whitehead's generalization of Hurewicz's theorem [**71**, Theorem 9, p. 399] we get $H_j(f,\mathbb{Z}) \cong H_j(F,\mathbb{Z})$ for $j < 2r+2$ and a surjection

$$H_{2r+2}(f,\mathbb{Z}) \to H_{2r+2}(F,\mathbb{Z}) \to 0.$$

By noting that $2r+2 = 2(\dim f + 1) = \dim_{\mathbb{R}} F$, this leads to the contradiction $H_{2r+2}(F,\mathbb{Z}) = 0$. Q.E.D.

Remark 3.3. In the situation of Theorem 3.2, if $\dim X = 2\dim Z$, a more refined argument based on results of Lanteri and Struppa [**48**] shows that the general fiber $(F, \mathcal{O}_F(Y))$ of $\overline{p}$ is isomorphic to $(\mathbb{P}^{\dim F}, \mathcal{O}_{\mathbb{P}^{\dim F}}(1))$. We refer to [**16**, (5.2.6), (2.3.9)] for more details and complete references.

From now on,

- *we are reduced to consider the problem of extending morphisms $Y \to Z$ from ample divisors Y of a manifold X in the hardest case when* $\dim Y - \dim Z \leq 1$.

In this setting, we shall use (part of) Mori theory; we compare the cone of curves of Y and X and give results on extending contractions of extremal rays (cf. [**40, 74, 35**]).

As noted in [**2**, Section 3], the following useful fact holds true. It is an easy consequence of Theorem 2.8 and a lemma due to Kollár [**40**].

Proposition 3.4. *Let X be a projective manifold of dimension ≥ 4. Assume that K_X is not nef and let $R = \mathbb{R}_+[C]$ be an extremal ray on X. Let Y be a smooth ample divisor on X. If $(K_X + Y)\cdot C \leq 0$, then $R \subset \overline{NE}(Y)$.*

Proof. By the Lefschetz theorem, the embedding $i : Y \hookrightarrow X$ gives an isomorphism $N_1(Y) \cong N_1(X)$, under which we get a natural inclusion $i_* : \overline{NE}(Y) \hookrightarrow \overline{NE}(X)$.

Let $\varphi : X \to Z$ be the contraction of the extremal ray R and let E be the locus where φ is not an isomorphism, i.e., the locus of curves whose numerical class is in R. If there is a fiber $F \subset X$ of φ whose dimension is at least two, then $Y \cap F$ contains a curve γ which generates R in $\overline{NE}(X)$, and hence $R \subset \overline{NE}(Y)$. Thus we can assume that every fiber of φ has dimension at most one, so that Theorem 2.8 applies. Therefore we are done after showing that in each case of 2.8 the divisor Y contains a fiber of φ.

In the birational case, E is a $\mathbb{P}^1$-bundle over $\varphi(E)$. Let $F \cong \mathbb{P}^1$ be a fiber of the bundle projection $E \to \varphi(E)$. Then $-K_X \cdot F = 1$, so that $(K_X + Y)\cdot F \leq 0$ and the ampleness of Y give $Y \cdot F = 1$. Therefore Lemma 3.6(i) below leads to the contradiction $\dim \varphi(E) \leq 1$, so $\dim X \leq 3$. In the conic fibration case, for any fiber F of φ, we have $-F \cdot K_X \leq 2$, and hence we get $1 \leq Y \cdot F \leq 2$. Thus Lemma 3.6(ii) gives the contradiction $\dim Z \leq 2$. Q.E.D.

Let us point out the following consequence of Proposition 3.4 (cf. Section 8).

Proposition 3.5. *Let X be a projective manifold of dimension $n \geq 4$, let H be an ample line bundle on X, and let Y be an effective smooth divisor in $|H|$. Assume that $-(K_X + H)$ is nef. Then X is a Fano manifold and $\overline{NE}(X) \cong \overline{NE}(Y)$.*

PROOF. Let $D := -(K_X + H)$, which is nef. Then $-K_X = H + D$ is ample, so that X is a Fano manifold. Let $R = \mathbb{R}_+[C]$ be an extremal ray in the polyhedral cone $\overline{NE}(X)$. By assumption, $(K_X + H) \cdot C \leq 0$. Then Proposition 3.4 applies to give that R is contained in $\overline{NE}(Y)$. So $\overline{NE}(X) = \overline{NE}(Y)$. Q.E.D.

Lemma 3.6. (Kollár [**40**]) *Let X be a projective manifold.*

(i) *Let $p : X \to Z$ be a $\mathbb{P}^1$-bundle over a normal projective variety Z. Let $Y \subset X$ be a divisor such that the restriction $p : Y \to Z$ is finite of degree one. If Y is ample then* $\dim Z \leq 1$.

(ii) *Let $p : X \to Z$ be a conic fibration over a normal projective variety Z. Let $Y \subset X$ be a divisor such that the restriction $p : Y \to Z$ is finite of degree two (or one). If Y is ample then* $\dim Z \leq 2$.

PROOF. (i) Note that $p_*\mathcal{O}_X(Y)$ is an ample rank 2 vector bundle since Y is ample. On the other hand, the section $\mathcal{O}_X \to \mathcal{O}_X(Y)$ gives an extension

$$0 \to \mathcal{O}_Z \to p_*\mathcal{O}_X(Y) \to L \to 0,$$

where L is a line bundle. Thus $c_2(p_*\mathcal{O}_X(Y)) = 0$, which contradicts ampleness for $\dim Y \geq 2$.

(ii) If $p : X \to Z$ has only smooth fibers, then, after a finite base change $Z' \to Z$, we get a $\mathbb{P}^1$-bundle $p' : X' \to Z'$. Now, $p'_*\mathcal{O}_{X'}(Y')$ has rank 3 and we get an extension

$$0 \to \mathcal{O}_{Z'} \to p'_*\mathcal{O}_{X'}(Y') \to \mathcal{E} \to 0,$$

where $\mathcal{E}$ is a rank 2 vector bundle. Thus $c_3(p'_*\mathcal{O}_{X'}(Y')) = 0$, which contradicts ampleness for $\dim Z \geq 3$.

If $p : X \to Z$ has singular fibers, then let $p' : X' \to Z'$ be the universal family of lines in the fibers. The pull-back of Y to X' intersects every line once and it is ample. Moreover, $\dim Z' = \dim Z - 1$, thus we are done by (i). Q.E.D.

Corollary 3.7. *Let X be a projective manifold of dimension ≥ 4. Let Y be a smooth ample divisor on X. Assume that K_Y is not nef and let R_Y be an extremal ray on Y. Further assume that there exists a nef divisor D on X such that $D \cdot R_Y = 0$ and such that $D^\perp$ is contained in the set $\{\zeta \in \overline{NE}(X) \mid (K_X + Y) \cdot \zeta \leq 0\}$. Then there exists an extremal ray $R \subset \overline{NE}(X)$ which induces R_Y.*

PROOF. Note that $D^\perp$ is a locally polyhedral face of $\overline{NE}(X)$ since $\{\zeta \in \overline{NE}(X) \mid (K_X + Y) \cdot \zeta \leq 0\} \subset \overline{NE}(X)_{K_X<0} \cup \{0\}$. Let $D^\perp = \langle R_1, \ldots, R_s \rangle$, R_i extremal rays on X, $i = 1, \ldots, s$. We can assume $R_Y \neq R_i$ for each i since otherwise we are done. Since $D \cdot R_Y = 0$, it follows that, for some $s' \leq s$,

$$R_Y \subset \langle R_1, \ldots, R_{s'} \rangle \subseteq \{\zeta \in \overline{NE}(X) \mid (K_X + Y) \cdot \zeta \leq 0\}.$$

Therefore $R_i \subset \overline{NE}(Y)$, $i = 1, \ldots, s'$, by Proposition 3.4. Recalling that $R_Y \neq R_i$, we thus conclude that R_Y is not extremal on Y, a contradiction. Q.E.D.

The following numerical invariant was introduced in [**35**]. Let X be a projective manifold of dimension $n \geq 3$. Let Y be a smooth ample divisor on X. Let R be an extremal ray on Y and let H be an ample divisor on X. Then define

$$\alpha_H(R) := \frac{H \cdot C}{\ell(R)}, \quad C \text{ minimal rational curve generating } R,$$

and

$$\alpha_H(Y) := \min\{\alpha_H(R) \mid R \text{ extremal ray on } Y\}.$$

The following slightly improves the main result of [**35**].

Theorem 3.8. *Let X be a projective manifold of dimension $n \geq 4$. Let Y be a smooth ample divisor on X. Assume that K_Y is not nef and let R be an extremal ray on Y. Let $p := \mathrm{cont}_R : Y \to Z$ be the contraction of R. The following conditions are equivalent:*

(i) *There exists an extremal ray $\mathcal{R}$ on X which induces R on Y;*
(ii) *There exists a nef divisor $\mathcal{D}$ on X such that $R = \mathcal{D}_Y^{\perp} \cap (\overline{NE}(Y) \setminus \{0\})$;*
(iii) *p is extendable;*
(iv) *p has a lifting;*
(v) *There exists an ample line bundle H on X such that $\alpha_H(R) = \alpha_H(Y)$.*

Proof. (i) $\Longrightarrow$ (ii) We have $\mathcal{R} = \mathcal{D}^{\perp} \cap (\overline{NE}(X) \setminus \{0\})$ for some nef divisor $\mathcal{D}$ on X. Restricting to Y, gives $R = \mathcal{D}_Y^{\perp} \cap (\overline{NE}(Y) \setminus \{0\})$.

(ii) $\Longrightarrow$ (iii) There exists an extremal ray $\mathcal{R} \subseteq \mathcal{D}^{\perp} \cap \overline{NE}(X)_{K_X+Y<0}$. Then by Proposition 3.4, $\mathcal{R} \subset \overline{NE}(Y)$, that is, $\mathcal{R}$ induces R on Y. Replacing $\mathcal{D}$ by some other nef divisor on X, we may assume that $\mathcal{R} = \mathcal{D}^{\perp} \cap (\overline{NE}(X) \setminus \{0\})$. It follows that $m\mathcal{D} - (K_X + Y)$ is ample for $m \gg 0$ and $\mathrm{cont}_{\mathcal{R}}$ is given by $|m\mathcal{D}|$. By Kodaira vanishing we have $H^1(X, m\mathcal{D} - Y) = 0$, showing that $\mathrm{cont}_{\mathcal{R}}$ extends cont_R.

(iii) $\Longrightarrow$ (iv) is obvious.

(iv) $\Longrightarrow$ (i) The morphism $\overline{p} : X \to W$ which lifts p is associated to a complete linear system $|L|$, for some nef line bundle L on X. Clearly, $R \subset L^{\perp}$ and $R \cdot (K_X + Y) = R \cdot K_Y < 0$. Hence, in particular, $L^{\perp} \cap \{\zeta \in \overline{NE}(X) \mid (K_X + Y) \cdot \zeta \leq 0\} \neq (0)$. Therefore, since $\{\zeta \in \overline{NE}(X) \mid (K_X + Y) \cdot \zeta \leq 0\}$ is locally polyhedral in $\overline{NE}(X)$ (see also the proof of 3.7), there exists an extremal ray $\mathcal{R}$ on X such that $\mathcal{R} \subset L^{\perp} \cap \{\zeta \in \overline{NE}(X) \mid (K_X + Y) \cdot \zeta \leq 0\}$. We also know by Proposition 3.4 that $\mathcal{R} \subset L^{\perp} \cap (\overline{NE}(Y) \setminus \{0\})$. It thus follows that p contracts $\mathcal{R}$. Since p is the contraction of an extremal ray, R, we conclude that $R = \mathcal{R}$ in $\overline{NE}(Y)$.

(i) $\Longrightarrow$ (v) As noted in the proof of (ii)$\Longrightarrow$ (iii), if $\mathcal{R} = \mathcal{D}^{\perp} \cap (\overline{NE}(X) \setminus \{0\})$, we have that $m\mathcal{D} - (K_X + Y) := H$ is ample for $m \gg 0$. Nefness of $\mathcal{D}$ yields for each extremal ray $R' = \mathbb{R}_+[C']$ on Y, C' minimal curve, $(K_Y + H_Y) \cdot C' \geq 0$. Moreover, since $R \subset \mathcal{D}^{\perp}$, we get $1 = \alpha_H(R) = \frac{H \cdot C}{\ell(R)} \leq \frac{H \cdot C'}{\ell(R')} = \alpha_H(R')$. This means that $\alpha_H(R) = \alpha_H(Y)$ is minimal.

Finally, to show (v) $\Longrightarrow$ (i), let $R := \mathbb{R}_+[C]$, C minimal rational curve. Let $a = (H \cdot C)$ and let $b = \ell(R)$. Assume that $\alpha_H(R) = \alpha_H(Y)$. Then $aK_Y + bH_Y$ is nef on Y, since otherwise $(aK_Y + bH_Y) \cdot C' < 0$ for some extremal ray $R' = \mathbb{R}_+[C']$ on Y. This would give $\frac{a}{b} = \frac{(H \cdot C)}{\ell(R)} > \frac{(H \cdot C')}{\ell(R')}$, contradicting the minimality of $\alpha_H(R)$.

We claim that $D := a(K_X + Y) + bH$ is nef on X. Assuming the contrary, by Mori's cone theorem there exists an extremal ray $R' = \mathbb{R}_+[C']$, such that $(a(K_X + Y) + bH) \cdot C' < 0$. Let ρ be the contraction of R'. If there is a fiber F of ρ of dimension ≥ 2, then some curve $\gamma \subset F$ is contained in Y by the ampleness of Y. Since $(a(K_X + Y) + bH) \cdot \gamma < 0$, this contradicts the nefness of the restriction of $a(K_X + Y) + bH$ to Y. Thus we conclude that all fibers of ρ are of dimension ≤ 1. By Theorem 2.8 we deduce that X is either a conic fibration, or the blowing-up of a smooth codimension two subvariety. In this latter case one has $K_X \cdot C' = -1$, which contradicts $(a(K_X + Y) + bH) \cdot C' < 0$. In the first case one has either $K_X \cdot C' = -2$

or $K_X \cdot C' = -1$ according to whether C' is a conic or a line. If $K_X \cdot C' = -1$ we have the same numerical contradiction. If $K_X \cdot C' = -2$ we get $Y \cdot C' = 1$ by using again the above inequality. It thus follows that X is a $\mathbb{P}^1$-bundle over a smooth variety W and Y is a smooth section. By pushing forward the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Y) \to \mathcal{O}_Y(Y) \to 0,$$

we get an exact sequence

$$0 \to \mathcal{O}_W \to \mathcal{E} \to L \to 0,$$

where $\mathcal{E}$ is an ample rank 2 vector bundle such that $X = \mathbb{P}(\mathcal{E})$ and $L \cong N_{Y/X}$ is an ample line bundle. By the Kodaira vanishing theorem we have $H^1(W, -L) = 0$, so that the sequence splits. This contradicts the ampleness of $\mathcal{E}$. Thus we conclude that $a(K_X + Y) + bH = D$ is nef.

Now, note that $D \cdot R = 0$ and that $D^\perp$ is strictly contained in $\overline{NE}(X)_{K_X+Y<0}$. Thus Corollary 3.7 applies to say that there exists an extremal ray $\mathcal{R} \subset \overline{NE}(X)$ which induces R. Q.E.D.

Corollary 3.9. *Let X be a projective manifold of dimension $n \geq 4$. Let Y be a smooth ample divisor on X. Assume that K_Y is not nef. Then there exists an extremal ray on Y which extends to an extremal ray on X. In particular, when Y has only one extremal ray, it always extends.*

Proof. For instance, take any extremal ray R of Y attaining the minimal value of the invariant $\alpha_Y(R)$ (i.e., $\alpha_Y(R) = \alpha_Y(Y)$). Q.E.D.

The following theorem is a version of a result of Occhetta [**57**, Proposition 5] (who states it in the more general case when Y is the zero locus of a section of an ample vector bundle $\mathcal{E}$ on X of the expected dimension $\dim X - \mathrm{rank}\mathcal{E}$). His argument contains an unclear critical point. With notation as in the proof below, the conclusion in [**57**] uses in an essential way the fact that D is an adjoint divisor, i.e., $D = aK_X + \mathcal{L}$ for some integer $a > 0$ and some *ample* line bundle $\mathcal{L}$ on X (implying that $D^\perp$ is strictly contained in $\overline{NE}(X)_{K_X<0}$). However, we only know that $\mathcal{L}_Y$ is ample! See also Remark 3.12 below.

Theorem 3.10. (cf. Occhetta [**57**]) *Let X be a projective manifold of dimension ≥ 4. Let Y be a smooth ample divisor on X. Assume that K_Y is not nef and let R be an extremal ray on Y. Let $C \subset Y$ be a rational curve whose numerical class is in R. Assume that the deformations of C in X yield a covering and quasi-unsplit family of rational cycles. Then R is an extremal ray of X, too. In particular, the conclusion holds if the deformations of C in Y cover Y and $H \cdot C = 1$ for some ample divisor H on X.*

Proof. We show first the last claim. Let $\nu : \mathbb{P}^1 \to C$ be the normalization of C and let $g : \mathbb{P}^1 \to Y$ and $f : \mathbb{P}^1 \to X$ be the induced morphisms to Y and X respectively. If C yields a covering family of Y, its deformations in X cover X, too. Indeed, consider the tangent bundle sequence

$$0 \to T_Y \to T_{X|Y} \to \mathcal{O}_Y(Y) \to 0.$$

By pulling back to $\mathbb{P}^1$, we get the exact sequence

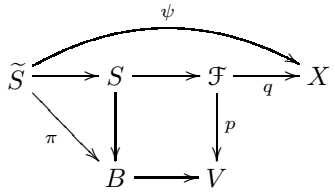
$$0 \to g^*T_Y \to g^*(T_{X|Y}) = f^*T_X \to g^*\mathcal{O}_Y(Y) \to 0.$$

Since both $g^*\mathcal{O}_Y(Y)$ and g^*T_Y are nef and hence spanned, we conclude that f^*T_X is spanned, and therefore that C induces a covering family on X, see [**41**, II, Section 3, IV, (1.9)]. Moreover, the condition $H \cdot C = 1$ ensures that the deformations of C in X yield an unsplit, hence also quasi-unsplit, family.

Let $D_Y \in \mathrm{Pic}(Y)$ be some (nef) supporting divisor of R, i.e., $R = D_Y^{\perp} \cap (\overline{NE}(Y) \setminus \{0\})$. By the Lefschetz theorem, there exists a divisor $D \in \mathrm{Pic}(X)$ which restricts to D_Y. Let V be a covering and quasi-unsplit family of rational cycles on X, containing C.

Claim 3.11. *D is nef.*

Proof. Assume that $D \cdot \Gamma < 0$ for some irreducible curve Γ on X. Clearly, we can assume that Γ is not contained in Y since the restriction of D to Y is nef. Since V is a covering family and Y is ample, we can find a curve B' in V parameterizing curves meeting both Γ and Y. Let B be the normalization of B'. Consider the base-change diagram



where $\mathcal{F}$ is the universal family and $\widetilde{S}$ is a desingularization of S, an irreducible component of $p^{-1}(B)$, whose locus contains Γ. Note that $\widetilde{S}$ is a ruled surface over the curve B. Let $A := \psi(\widetilde{S}) \cap Y$ be the trace on Y of the image in X of the surface $\widetilde{S}$. Since A is an ample divisor on $\psi(\widetilde{S})$, there exists at least one irreducible component, say $\mathcal{C}$, of A which is not contracted by $\mathrm{cont}_R : Y \to Z$. Let $\widetilde{\Gamma}$, $\widetilde{\mathcal{C}}$ be two irreducible curves on $\widetilde{S}$ such that $\psi(\widetilde{\Gamma}) = \Gamma$, $\psi(\widetilde{\mathcal{C}}) = \mathcal{C}$. By the above and the hypothesis that V is quasi-unsplit, $\widetilde{\mathcal{C}}$ is not a fiber of $\pi : \widetilde{S} \to B$.

We can write, for some integers $\varepsilon, \delta_i, \varepsilon > 0, \widetilde{\mathcal{C}} \sim \varepsilon C_0 + \sum_i \delta_i F_i$, where C_0 is a section of π and each F_i is contained in a fiber of π. We also have $\widetilde{\Gamma} \sim \alpha C_0 + \sum_i \beta_i F_i$, for some integers α, β_i, $\alpha > 0$. Thus

$$\varepsilon\widetilde{\Gamma} \sim \alpha\varepsilon C_0 + \varepsilon \sum_i \beta_i F_i \sim \alpha\widetilde{\mathcal{C}} - \sum_i (\alpha\delta_i - \varepsilon\beta_i) F_i,$$

that is, in $\mathrm{Pic}(\widetilde{S}) \otimes \mathbb{Q}$, one has $\widetilde{\Gamma} \sim a\widetilde{\mathcal{C}} + \sum_i b_i F_i$, with $a = \frac{\alpha}{\varepsilon} > 0$.

Let $\widetilde{D} := \psi^* D$. Since V is quasi-unsplit, $\widetilde{D} \cdot F_i = D \cdot R = 0$, so that

$$D \cdot \Gamma = \widetilde{D} \cdot \widetilde{\Gamma} = \widetilde{D} \cdot \left(a\widetilde{\mathcal{C}} + \sum_i b_i F_i \right) = a(\widetilde{D} \cdot \widetilde{\mathcal{C}})_{\widetilde{S}} = a(D \cdot \mathcal{C}) \geq 0$$

since D_Y is nef. This shows the claim.

To conclude we have to show that R is an extremal ray on X; see also the proof of [**11**, Theorem (5.1)]. By Lefschetz's theorem, the embedding $i : Y \hookrightarrow X$ gives a natural inclusion $i_* : \overline{NE}(Y) \hookrightarrow \overline{NE}(X)$. Clearly, $\mathcal{R} := i_*(R)$ is K_X-negative by the adjunction formula.

Since R is an extremal ray of $\overline{NE}(Y)$, by duality it corresponds to it an extremal face of maximal dimension $\varrho(Y) - 1$ of the cone of nef divisors $\mathrm{Nef}(Y)$. Therefore we can find $\varrho(Y) - 1$ good supporting divisors of R whose numerical classes are linearly

independent in $N^1(Y)$. By Claim 3.11, this implies that such good supporting divisors extend to divisors on X that are nef, trivial on $\mathcal{R}$, and whose numerical classes are linearly independent in $N^1(X)$. Since there are $\varrho(Y)-1$ of them, and $\varrho(Y) = \varrho(X)$ by the isomorphism $N^1(X) \cong N^1(Y)$, this implies that $\mathcal{R}$ is an extremal ray of $\overline{NE}(X)$. Q.E.D.

Remark 3.12. In [**57**, Proposition 5], the author states the result assuming that R is nef. However, the theorem also applies to non-nef extremal rays of Y, see Proposition 5.13 below. Note that, even when $R = \mathbb{R}_+[C]$ is nef, in general C does not define a covering family of Y. E.g., take cont_R to be a conic fibration, C being a line in a degenerate fiber.

4. Some convex geometry speculations

First, we recall the following simple observations, due to Bădescu.

Lemma 4.1. ([**6**, Remark 1), p. 170]) *On the projective line $\mathbb{P}^1$, consider a line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$, for some integer $a \geq 2$. Write $a = b + c$, with $b, c > 0$. Then there exists a surjective map*

$$\mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c) \to \mathcal{O}_{\mathbb{P}^1}(a) \to 0.$$

Proof. Consider the global sections

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) = \langle u^a, u^{a-1}v, \ldots, v^a \rangle$$

as homogeneous polynomials in two variables u, v. We have natural inclusions

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)) \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \quad \text{and} \quad H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(c)) \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)),$$

given by multiplication by u^{a-b} and v^{a-c} respectively. Then we get a surjective map

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(c)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \to 0,$$

and injections

$$0 \to \mathcal{O}_{\mathbb{P}^1}(b) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^1}(a), \quad 0 \to \mathcal{O}_{\mathbb{P}^1}(c) \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^1}(a).$$

Thus $\beta \oplus \gamma : \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c) \to \mathcal{O}_{\mathbb{P}^1}(a)$ gives the requested map. The surjectivity follows from the commutative square

$$\begin{array}{ccc} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(c)) & \longrightarrow & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \\ \downarrow{\scriptstyle ev_x} & & \downarrow{\scriptstyle ev_x} \\ (\mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))_x & \longrightarrow & (\mathcal{O}_{\mathbb{P}^1}(a))_x \end{array}$$

where the vertical arrows are the evaluation maps in a given point $x \in \mathbb{P}^1$ and $ev_x : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \to (\mathcal{O}_{\mathbb{P}^1}(a))_x$ is onto by spannedness of $\mathcal{O}_{\mathbb{P}^1}(a)$. Q.E.D.

Proposition 4.2. ([**6**]) *Given a vector bundle $\mathcal{E} = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$, with $a_1 \geq 2$, $a_1 = b + c$, and $b, c > 0$, there exists an exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F} := \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c) \oplus \Big(\bigoplus_{i=2}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a_i) \Big) \to \mathcal{E} \to 0. \tag{4.1}$$

PROOF. Lemma 4.1 yields a surjective map

$$\mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c) \oplus \Big(\bigoplus_{i=2}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a_i)\Big) \to \mathcal{E} \to 0,$$

whose kernel is the trivial bundle since $\det(\mathcal{E}) = \det(\mathcal{F})$. Q.E.D.

Remark 4.3. Note that Proposition 4.2 gives rise to a method to construct ample divisors which are projective bundles over $\mathbb{P}^1$. Indeed, let $Y := \mathbb{P}(\mathcal{E})$ and $X := \mathbb{P}(\mathcal{F})$. As soon as $a_i > 0$ for each index $i = 2, \ldots, n-1$, the exact sequence (4.1) expresses Y as a smooth ample divisor of X; it is recovered by pushing forward the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Y) \to \mathcal{O}_Y(Y) \to 0.$$

The following fact is well known. We include the proof for reader's convenience.

Lemma 4.4. *Let $V = \mathbb{P}(\mathcal{E})$ be a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$, for some rank n vector bundle $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$. Assume that V is a Fano manifold, of index $i(V)$. Then, for some integer a, either*

(i) *$V = \mathbb{P}^{n-1} \times \mathbb{P}^1$, $i(V) \leq 2$ and $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a)$; or*
(ii) *V is the blowing-up, $\sigma : V \to \mathbb{P}^n$, along a codimension two linear subspace of $\mathbb{P}^n$, $i(V) = 1$ and $\mathcal{E} = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)$.*

PROOF. After normalization of the integers a_i, write $0 = a_1 \leq a_2 \leq \cdots \leq a_n$ and consider the section Γ of $p : V \to \mathbb{P}^1$ corresponding to the quotient

$$\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

onto the first summand. Note that the morphism $\varphi_{|\xi_V|} : V \to \mathbb{P}^m$ maps the curve Γ to a point. On the other hand, since V is a Fano manifold with $\mathrm{Pic}(V) \cong \mathbb{Z} \oplus \mathbb{Z}$, there are two extremal rays, R_1, corresponding to p, and R_2, generating the cone $\overline{NE}(V)$. Since the morphism $\varphi_{|\xi_V|}$ is not finite, it must coincide with the contraction of R_2. Now, setting $d := \sum_{i=1}^n a_i$, the canonical bundle formula yields

$$-K_V \cong p^*\mathcal{O}_{\mathbb{P}^1}(2-d) \otimes \mathcal{O}_V(n).$$

Therefore, dotting with Γ, we get $d < 2$, so that either $\mathcal{E} \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}$, or $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, leading to the two cases as in the statement. Note that by the canonical bundle formula, in the first case the index of V is $i(V) \leq 2$, while, in the second case, $i(V) = 1$. Q.E.D.

Examples 4.5. (Only known examples of non-extendable extremal rays). Let X be a projective manifold of dimension $n \geq 4$. Let Y be a smooth ample divisor on X. The only known examples of extremal rays R of Y which do not extend to X are the following:

(1) $Y = \mathbb{P}^1 \times \mathbb{P}^{n-2}$, and R is the nef extremal ray corresponding to the $\mathbb{P}^1$-bundle projection $q : Y \to \mathbb{P}^{n-2}$. The manifold X is constructed as in Remark 4.3;
(2) $Y = \mathbb{P}\big(\bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)\big)$ for some integer $a \geq 2$, R is the non-nef extremal ray corresponding to the blowing-up, $\sigma : Y \to \mathbb{P}^{n-1}$, along a codimension two linear subspace of $\mathbb{P}^{n-1}$. Again, X is constructed as in Remark 4.3, $X \neq \mathbb{P}^1 \times \mathbb{P}^{n-1}$.

In case (1), the $\mathbb{P}^{n-2}$-bundle projection $p : Y \to \mathbb{P}^1$ on the first factor extends by construction. Then, if $q : Y \to \mathbb{P}^{n-2}$ extends too, we would have a surjective map $\mathbb{P}^{n-1} \to \mathbb{P}^{n-2}$, where $\mathbb{P}^{n-1}$ is a fiber of the extension of p; a contradiction.

Let Y be as in case (2). By Proposition 4.2, Y embeds as a smooth ample divisor of $X := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a-1) \oplus \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus(n-3)} \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)\big)$. Note that $\overline{NE}(Y) = \langle R_1, R_2\rangle$, where R_1, R_2 are the extremal rays corresponding to the bundle projection $Y \to \mathbb{P}^1$, and to the blowing-up $\sigma : Y \to \mathbb{P}^{n-1}$ respectively. Moreover, Lemma 4.4 applies to say that X is not a Fano manifold. Therefore $\overline{NE}(Y)$ is strictly contained in $\overline{NE}(X)$. Since $\varrho(Y) = \varrho(X)$ by the Lefschetz theorem, and the projection $Y \to \mathbb{P}^1$ extends by construction, we thus conclude that the extremal ray R_2 does not extend to X. Clearly $X \neq \mathbb{P}^1 \times \mathbb{P}^{n-1}$ in the above example. Note that by taking as Y a hyperplane section of the Segre embeding X of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, the restriction to Y of the bundle projection $X \to \mathbb{P}^{n-1}$ is in fact the blowing-up $\sigma : Y \to \mathbb{P}^{n-1}$ along a codimension two linear subspace. Of course, in this case, the extremal ray defining σ extends to X. In terms of Proposition 4.2, this situation corresponds to the case when $a = 1$, that is $Y = \mathbb{P}\big(\bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)\big)$ is an ample divisor of $X = \mathbb{P}\big(\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(1)\big)$.

That those in 4.5 are the only known examples of non-extendable extremal rays looks quite surprising to us. We propose the following speculations with the hope they may eventually lead to an explanation of this fact.

Let X be a projective manifold of dimension $n \geq 4$. Let Y be a smooth ample divisor on X. Consider the following property, for an extremal ray, R, of Y.

($\star$) *For any nef divisor H on Y, such that $H^\perp \cap (\overline{NE}(Y) \setminus \{0\}) = R$, it follows that $\overline{H}$ is nef (here $\overline{H}$ is the unique divisor class on X such that $\overline{H}_Y = H$).*

First, note that ($\star$) implies that R is a ray of X (cf. [**11**, Theorem 4.1] and end of proof of Theorem 3.10).

Proposition 4.6. *Assume that property ($\star$) holds for all extremal rays of Y such that $R \subset \mathcal{D}^\perp$ for some nef divisor $\mathcal{D}$ on X, $\mathcal{D} \not\sim 0$. Assume, also, that $\varrho(X) \geq 3$. Then every extremal ray of Y extends to an extremal ray of X.*

Proof. Assume that we have some extremal ray of Y, say R_0, which is not a ray of X. We may assume that $R_0 \subset \overline{NE}(X)_{(K_X+(1+\varepsilon)Y)<0}$ for some $\varepsilon > 0$.

Observe that, by our hypothesis, any ray R of Y which is contained in some face of $\overline{NE}(X)$ satisfies property ($\star$), and hence, as noted above, R is a ray of X.

Let $\mathcal{C}_\circ := (K_X + (1+\varepsilon)Y)^\perp \cap (\overline{NE}(Y) \setminus \{0\})$ and let $R_0, R_1, \ldots, R_s$ be all the extremal rays of $\overline{NE}(Y)$ such that $R_i \subset \overline{NE}(X)_{(K_X+(1+\varepsilon)Y)<0}$, $i = 0, 1, \ldots, s$. Next consider the non-degenerate convex cone $\mathcal{C} \subset \mathbb{R}^{\varrho(Y)}$ defined by

$$\mathcal{C} = \langle \mathcal{C}_\circ, R_0, R_1, \ldots, R_s\rangle = \overline{NE}(Y) \cap \overline{NE}(X)_{(K_X+(1+\varepsilon)Y)\leq 0}.$$

Then $\mathcal{C}_\circ$ is a face of the cone $\mathcal{C}$ and, for each index i,

$$R_i \not\subset \langle R_0, R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_s\rangle.$$

Claim 4.7. *There exists a face F of the cone $\mathcal{C}$ such that F contains two extremal rays of Y, say R', R'', such that R' is a ray of X and R'' is not.*

Proof. Recall that by Proposition 3.4 all extremal rays of X contained in $\overline{NE}(X)_{(K_X+Y)\leq 0}$ are also extremal rays of Y. Therefore, some of the extremal

rays R_j, $j \geq 1$, are rays of X. Take one of them, say R_1. Then there exists a face F_1 of $\mathcal{C}$ containing R_1 and some of the other rays $R_{j'}$, for some $j' \in \{2, \ldots, s\}$. If one of the rays $R_{j'}$ (say R_2), is not a ray of X, then take $R' = R_1$, $R'' = R_2$ and $F = F_1$. If all the extremal rays $R_{j'}$, $j' \in \{2, \ldots, s\}$, are rays of X, we apply the same argument to conclude that either we prove the claim, or every extremal ray of Y contained in $\mathcal{C}$ lifts to an extremal ray of X, contradicting the assumption that R_0 does not.

Thus we may assume to be in the situation described in Claim 4.7. Take a nef divisor D on Y such that $F = D^{\perp} \cap (\overline{NE}(Y) \setminus \{0\})$. If the unique divisor class $\overline{D}$ on X which restricts to D is nef, then by our assumption R'' is a ray of $\overline{NE}(X)$. This contradicts the claim, so $\overline{D}$ is not nef on X.

Now, take a nef divisor $\mathcal{H}$ on X such that $R' = \mathcal{H}^{\perp} \cap (\overline{NE}(X) \setminus \{0\})$. For $0 \leq \alpha \leq 1$, consider $D_\alpha = \alpha\overline{D} + (1-\alpha)\mathcal{H}$ and let $\lambda := \sup\{\alpha \mid D_\alpha \text{ is nef}\}$. Then $0 \leq \lambda < 1$.

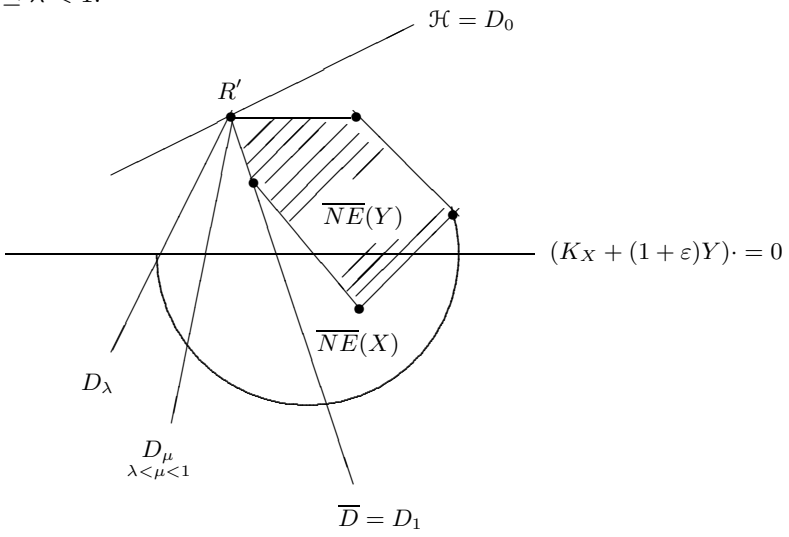


FIGURE 1. The two cones of curves.

Claim 4.8. *The extremal ray R' does not satisfy property $(\star)$.*

PROOF. For some $\lambda < \mu < 1$, the divisor D_μ is not nef and its restriction $D_{\mu|Y} = \mu D + (1-\mu)\mathcal{H}_Y$ is nef on Y. In particular one has

$$R' \subseteq (D_{\mu|Y})^{\perp} \cap (\overline{NE}(Y) \setminus \{0\}) \subseteq \mathcal{H}_Y^{\perp} \cap (\overline{NE}(Y) \setminus \{0\}).$$

Thus, since $R' = \mathcal{H}^{\perp} \cap (\overline{NE}(X) \setminus \{0\})$, equalities hold above, so that $R' = (D_{\mu|Y})^{\perp} \cap (\overline{NE}(Y) \setminus \{0\})$. Thus R' does not satisfy property $(\star)$; see Figure 1.

The claim above leads to a contradiction, and hence concludes the proof of the proposition. Q.E.D.

Remark 4.9. (1) The examples of non-extending extremal rays in Example 4.5 were known to the experts since the early 80's. The resulting observation that, in general, $\overline{NE}(Y) \subsetneq \overline{NE}(X)$ when $Y \subset X$ is ample, was rediscovered in [**31**].

(2) The (open) problem of deciding whether or not the property $(\star)$ holds for all extremal rays R of Y such that $R \subset D^{\perp}$ for some nef divisor D on X, $D \not\sim 0$, looks hard but very interesting. A positive answer to it would imply, via Proposition 4.6, that non-extendable rays may only occur when $\varrho(X) = 2$. Moreover, a positive answer would give a proof for the claim made in [**31**, Theorem 4.3]. See also Remark 7.7 for further applications.

5. Applications to $\mathbb{P}^d$-bundles and blowing-ups

Let us start by recalling some useful preliminary facts.

The following result goes back to Goren [**28**] and Kobayashi–Ochiai [**39**]. We refer also to Fujita [**27**, Chapter I, (1.1), (1.2)] where the Cohen–Macaulay assumption was removed.

Theorem 5.1. *Let L be an ample line bundle on an n-dimensional irreducible projective variety X. If $L^n = 1$ and $h^0(L) \geq n+1$, then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

Corollary 5.2. *If X is a Fano manifold of dimension n and index i, we have $i \leq n+1$ and $X \cong \mathbb{P}^n$ if equality holds.*

Proof. Use the Hilbert polynomial and Kodaira vanishing to check the hypothesis of Theorem 5.1. See [**39**] for details. Q.E.D.

Remark 5.3. In a completely similar way one proves that if X is as above and $i = n$, then $X \cong \mathcal{Q}^n$, see again [**39**].

The smooth version of the following fact was proved by Ramanujam [**59**] and Sommese [**67**]. The general case is due to Bădescu [**9**]. We also refer to [**16**, Section 2.6 and (5.4.10)] for a different argument based on Rossi's extension theorem.

Theorem 5.4. *Let $Y \cong \mathbb{P}^{n-1}$ be an ample Cartier divisor on a normal projective variety X of dimension $n \geq 3$. Then X is the cone $\mathcal{C}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(s))$, where $\mathcal{O}_Y(Y) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$. If $s = 1$, the assertion is true for $n = 2$ as well.*

Proof. Let us recall the argument in the smooth case.

First, note that, by Lefschetz theorem, $i^* : \mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y)$ via the embedding $i : Y \hookrightarrow X$. This is clear if $n \geq 4$. If $n = 3$, then $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective with torsion free cokernel. Therefore i^* is an isomorphism, since $\mathrm{Pic}(Y) \cong \mathbb{Z}$.

Let L be the ample generator of $\mathrm{Pic}(X)$ and assume $Y \in |aL|$ for some $a \geq 1$. Note that $L_Y = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. It then follows that

$$1 = (L_Y)_Y^{n-1} = (L^{n-1} \cdot Y)_X = a(L^n).$$

Therefore $a = 1$ and $L^n = 1$.

Next, Kodaira vanishing and the exact sequence

$$0 \to \mathcal{O}_X(-L) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

give $H^1(X, \mathcal{O}_X) = 0$. Hence the exact sequence

$$0 \to \mathcal{O}_X \to L \to L_Y \to 0$$

yields $h^0(L) = h^0(L_Y) + 1 = \dim X + 1$. Then Theorem 5.1 applies to give $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Q.E.D.

For the first statement of the theorem below, see Fujita [**26**, (2.12)] or Ionescu [**34**, p. 467]; the second point follows from Theorem 5.4 and some of Bădescu's arguments in [**6, 7**]; the third point was noticed in [**15**, Section 2] and [**17**].

Theorem 5.5. *Let X be an n-dimensional projective manifold.*

(i) *Let $\pi : X \to Z$ be a surjective morphism from X onto a normal variety Z. Let L be an ample line bundle on X. Assume that $(F, L_F) \cong (\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ for a general fiber, F, of π and that all fibers of π are d-dimensional. Then $\pi : X \to Z$ is a linear $\mathbb{P}^d$-bundle with $X = \mathbb{P}(\pi_* L)$.*

(ii) *Let $p : Y \to Z$ be a $\mathbb{P}^d$-bundle over a projective manifold Z. Assume that Y is an ample divisor on X. Furthermore assume that p extends to a morphism $\overline{p} : X \to Z$. Then there exists a non-splitting exact sequence*

$$0 \to \mathcal{O}_Z \to \overline{\mathcal{E}} \xrightarrow{u} \mathcal{E} \to 0,$$

where $\overline{\mathcal{E}}$, $\mathcal{E}$ are ample vector bundles on Z such that $X = \mathbb{P}(\overline{\mathcal{E}})$, $Y = \mathbb{P}(\mathcal{E})$, p, $\overline{p}$ are the bundle projections on Z, and the inclusion $Y \subset X$ is induced by u.

(iii) *Let $\pi : X \to Z$ be a linear $\mathbb{P}^{d+1}$-bundle over a projective manifold Z. Assume that $\dim Z < d + 1$ and the tautological line bundle of X, say L, is ample. Then $K_X + (d + 2)L$ is nef. Moreover, the bundle projection π is associated to the linear system $|m(K_X + (d + 2)L)|$ for $m \gg 0$ (i.e., (X, L) is a scroll over Z).*

Proof. (i) Following the argument as in [**34**, p. 467], let us first show that Z is smooth. Indeed, let $z \in Z$ be a closed point and denote by Δ the fiber over z. Consider the embedding of X given by $|mL|$ for $m \gg 0$. Let $\widetilde{Z}$ be the smooth $(n - d)$-fold got by intersecting d general members $H_1, \ldots, H_d$ of $|mL|$. Since $(F, L_F) \cong (\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ for a general fiber F, the restriction $\widetilde{p}$ of p to $\widetilde{Z}$ has degree m^d. Since p has equidimensional fibers, $\widetilde{Z} \cap \Delta$ is a 0-dimensional scheme, and its length is given by $\ell(\widetilde{Z} \cap \Delta) = m^d(L_\Delta^d) \geq m^d$. Since Z is normal, it follows by a well known criterion (see e.g., [**65**, Chapter II, Theorem 6]) that the above inequality is in fact an equality. Hence in particular $L_\Delta^d = 1$, so that Δ is irreducible and generically reduced. Therefore, by the generality of $H_1, \ldots, H_d$, we may assume that $\widetilde{Z} \cap \Delta$ is a reduced 0-cycle consisting of $\#(\widetilde{Z} \cap \Delta) = \ell(\widetilde{Z} \cap \Delta) = \deg(\widetilde{p})$ distinct points. It thus follows that $\widetilde{p}$ is étale over z. Therefore Z is smooth at z since $\widetilde{Z}$ is smooth.

Since all fibers of π are equidimensional and X and Z are smooth, the morphism π is flat. Let now Δ be any fiber of π. We have seen above that Δ is irreducible and generically reduced. Moreover, Δ is Cohen–Macaulay since every fiber is defined by exactly $n - d$ coordinate functions. It thus follows that Δ is in fact reduced. By the semicontinuity theorem for dimensions of spaces of sections on fibers of a flat morphism [**29**, Chapter III, Theorem 12.8], $h^0(L_\Delta) \geq h^0(L_F) = d + 1$. Then by Theorem 5.1 we conclude that $(\Delta, L_\Delta) \cong (\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ for every fiber Δ of π. Then $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} := \pi_* L$.

(ii) Note that, by ampleness of Y, p equidimensional implies $\overline{p}$ equidimensional. Let F be a general fiber of $\overline{p}$ and let $f = F \cap Y$ be the corresponding fiber of p. By (i), it is enough to show that $F \cong \mathbb{P}^{d+1}$ and $L_F \cong \mathcal{O}_{\mathbb{P}^{d+1}}(1)$, where $L = \mathcal{O}_X(Y)$. If $d \geq 2$ we conclude by Theorem 5.4.

Thus we can assume $d = 1$. By taking general hyperplane sections of Z and by base change, we can also assume that Z is a smooth curve (and hence F is a divisor). Here we follow Bădescu's argument. By the Lefschetz theorem, we have an exact sequence

$$0 \to \mathrm{Num}(X) \xrightarrow{i^*} \mathrm{Num}(Y) \to \mathrm{Coker}(i^*) \to 0,$$

where i^* is the morphism induced by the embedding $i : Y \hookrightarrow X$ and $\mathrm{Coker}(i^*)$ is torsion free. First note that $\mathrm{Num}(X) \ncong \mathbb{Z}$, since otherwise F would be an ample divisor. Since $\mathrm{Num}(Y) \cong \mathbb{Z} \oplus \mathbb{Z}$, we thus conclude that $\mathrm{Num}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, and therefore that $\mathrm{Coker}(i^*) = (0)$ since it is torsion free. Let h be a section of the bundle $p : Y \to Z$. Then $\mathrm{Num}(Y) = \mathbb{Z}\langle f \rangle \oplus \mathbb{Z}\langle h \rangle$ and $\mathrm{Num}(X) = \mathbb{Z}\langle F \rangle \oplus \mathbb{Z}\langle H \rangle$ for some line bundle H on X inducing h on Y. Write $Y \sim aF + bH$ for some integers a, b. Since $h \cdot f = H \cdot Y \cdot F = 1$, we get $1 = b(H^2 \cdot F)$ and therefore $b = \pm 1$. Then $Y \cdot f = (aF \pm H) \cdot f = \pm(H \cdot f) = \pm 1$. By ampleness of Y, it must be $b = 1$. Thus $f \cong \mathbb{P}^1$ has self-intersection $f^2 = Y^2 \cdot F = Y \cdot f = 1$ on F. This implies $F \cong \mathbb{P}^2$ and $L_F \cong \mathcal{O}_{\mathbb{P}^2}(1)$ by using Theorem 5.4. Applying the first part of the statement and pushing forward under $\overline{p}$ the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Y) \to \mathcal{O}_Y(Y) \to 0$$

we get the desired conclusion.

(iii) If $K_X + (d+2)L$ is not nef, there exists an extremal ray R of X such that $(K_X + (d+2)L) \cdot R < 0$. It follows that $\ell(R) > d + 2$. Therefore, if Δ is a positive dimensional fiber of the contraction cont_R, we have $\dim \Delta \geq \ell(R) - 1 \geq d + 2$ (see Theorem 2.7). Then, for a fiber F of π, we have

$$\dim F + \dim \Delta \geq 2d + 3 > n + 1$$

(where the last inequality follows from the assumption $d + 1 > \dim Z$, which is equivalent to saying that $2d + 2 > n$). Hence $\dim(F \cap \Delta) \geq 2$. Thus there exists a curve $C \subset F$ such that $(K_X + (d+2)L) \cdot C < 0$, contradicting $(K_X + (d+2)L)_{|F} \approx 0$. Therefore we conclude that $K_X + (d+2)L$ is nef, and hence, by Theorem 2.6, the linear system $|m(K_X + (d+2)L)|$ defines a morphism, say φ, for $m \gg 0$.

Let now $R \subset (K_X + (d+2)L)^{\perp} \cap (\overline{NE}(X) \setminus \{0\})$ be an extremal ray. We have $(K_X + (d+2)L) \cdot R = 0$, so that $\ell(R) \geq d + 2$, and hence, as above, $\dim \Delta \geq \ell(R) - 1 \geq d + 1$.

Since $2d + 2 > n$, it follows, again by Theorem 2.7, that $\dim F + \dim \Delta \geq 2d + 2 \geq n + 1$. Then $\dim(F \cap \Delta) \geq 1$. This implies that π is the contraction cont_R of the extremal ray R. Since this is true for each extremal ray as above, we conclude that the face $(K_X + (d+2)L)^{\perp} \cap (\overline{NE}(X) \setminus \{0\})$ is in fact 1-dimensional and that p coincides with the morphism φ. Q.E.D.

Remark 5.6. (1) In the boundary case $d+1 = \dim Z$ of Theorem 5.5(iii), the same argument gives the nefness of $K_X + (d+2)L$; moreover, further considerations show that the bundle projection π is associated to $|m(K_X + (d+2)L)|$ for $m \gg 0$ unless $X \cong \mathbb{P}^{d+1} \times \mathbb{P}^{d+1}$. We refer for this to [**17**, (3.1)].

(2) Note that by Theorem 3.2 or Lemma 5.7 below, statement (iii) of Theorem 5.5 applies under the conditions in 5.5(ii).

In the case of $\mathbb{P}^d$-bundles, Sommese's theorem 3.2 admits the following simple alternative proof.

Lemma 5.7. *Let Y be a smooth ample divisor on a projective manifold, X. Assume that Y is a $\mathbb{P}^d$-bundle over a manifold Z. Further assume that p has an extension $\overline{p} : X \to Z$. Then* $\dim Z \leq d+1$.

PROOF. By Theorem 5.5 we get the exact sequence

$$0 \to \mathcal{O}_Z \to \overline{\mathcal{E}} \to \mathcal{E} \to 0, \tag{5.1}$$

where $\overline{\mathcal{E}}$, $\mathcal{E}$ are ample vector bundles on Z. Arguing by contradiction, assume $\dim Z > \mathrm{rk}\mathcal{E} = d+1$, that is $1 \leq \dim Z - \mathrm{rk}\mathcal{E}$. By le Potier's vanishing theorem [**50**] we have $H^i(Z, \mathcal{E}^*) = 0$ for $i \leq \dim Z - \mathrm{rk}\mathcal{E}$. Therefore $H^1(Z, \mathcal{E}^*) = 0$, so we conclude that (5.1) splits, contradicting ampleness of $\overline{\mathcal{E}}$. Q.E.D.

Lemma 5.8. *Let L be an ample line bundle on a projective manifold, X, of dimension $n \geq 4$. Assume that there is a smooth $Y \in |L|$ such that Y is a $\mathbb{P}^d$-bundle, $p : Y \to Z$, over a manifold Z. Let ℓ be a line in a fiber $\mathbb{P}^d$ of p. Further assume that $H \cdot \ell = 1$ for some ample line bundle H on X. Then $(X, L) \cong (\mathbb{P}(\mathcal{E}), \xi_{\mathbb{P}})$ for an ample rank $d+2$ vector bundle, $\mathcal{E}$, on Z with p equal to the restriction to Y of the induced projection $\mathbb{P}(\mathcal{E}) \to Z$.*

PROOF. Lines in the fibers of p define a covering family of Y. By our assumptions, the induced family on X is unsplit. Therefore Theorem 3.10 applies to give that p extends. We conclude by Theorem 5.5(ii). Q.E.D.

The following gives a precise answer to Question 2.1.

Proposition 5.9. *Let X be an n-dimensional projective manifold embedded in $\mathbb{P}^N$, $n \geq 4$.*

(i) *Assume that X has a smooth hyperplane section $Y = X \cap H$ which is a $\mathbb{P}^d$-bundle over a manifold Z, say $p : Y \to Z$, such that the fibers of p are linear subspaces of $\mathbb{P}^N$. Then p lifts to a linear $\mathbb{P}^{d+1}$-bundle $\overline{p} : X \to Z$. Moreover, this is possible only if $d+1 \geq \dim Z$.*

(ii) *Conversely, assume that $\pi : X \to Z$ is a $\mathbb{P}^{d+1}$-bundle with linear fibers and $d+1 \geq \dim Z$. Then there exists a smooth hyperplane section $Y = X \cap H$ which is a $\mathbb{P}^d$-bundle.*

PROOF. (i) If ℓ is a line contained in some fiber of p, we have $H \cdot \ell = 1$. Thus the first assertion follows from Lemma 5.8 and the second from Theorem 3.2 (or Lemma 5.7).

(ii) Consider the incidence relation

$$W := \{(z, h) \mid H \supseteq F_z\} \subseteq Z \times (\mathbb{P}^N)^\vee,$$

where $F_z = \pi^{-1}(z)$ is the fiber $\mathbb{P}^{d+1}$ over a point $z \in Z$ and $h \in (\mathbb{P}^N)^\vee$ is the point corresponding to the hyperplane H in $\mathbb{P}^N$. Then $\dim W = \dim Z + N - (d+1) - 1$, so that $\dim Z \leq d+1$ gives $\dim W \leq N-1$. Therefore there exists a hyperplane H in $\mathbb{P}^N$ not containing fibers of π and we are done. Q.E.D.

Consider the setting as in diagram (2.1) from Section 2 with $\dim X \geq 4$. Let us discuss here some applications under the assumption that the canonical bundle K_Z of Z is nef. We follow the exposition in [**35**, Section 4], where the results are proved for a strictly nef and big divisor Y on X.

As a first application, we consider the case when the morphism $p : Y \to Z$ is a $\mathbb{P}^d$-bundle. An analogous result, assuming $\kappa(Z) \geq 0$ instead of K_Z to be nef, was proved in [**24**] in a completely different way.

The following result is essentially due to Wiśniewski [**74**, (3.3)].

Lemma 5.10. *Let Y be a $\mathbb{P}^d$-bundle over a smooth projective variety and let $p : Y \to Z$ be the bundle projection. If K_Z is nef, then Y admits a unique extremal ray, and p is its contraction.*

Proof. Assume by contradiction that there exists an extremal rational curve, C, which is not contracted by p. Let $\Gamma \cong \mathbb{P}^1$ be the normalization of $p(C)$. Let $f : \Gamma \to Z$ be the induced morphism and consider the base change diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \Gamma & \xrightarrow[f]{} & Z. \end{array}$$

Since Γ is a smooth curve, we have a vector bundle $\mathcal{E}$ on Γ, of rank $r := d+1$, such that $Y' = \mathbb{P}(\mathcal{E})$. Let F' be the fiber of the bundle projection p' and let $C' \subset Y'$ be a curve mapped onto C under g. Let $\mathcal{T}'$ be the tautological line bundle on Y'. Then we get

$$0 > (C \cdot K_Y) = (C' \cdot g^* K_Y) = -r(C' \cdot \mathcal{T}') + (C' \cdot (f \circ p')^* K_Z) + (C' \cdot p'^* \det(\mathcal{E})).$$

Since K_Z is nef, it thus follows $r(C' \cdot \mathcal{T}') > (C' \cdot p'^* \det(\mathcal{E}))$. By the Grothendieck theorem, we have $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$, where $a_1 \geq \cdots \geq a_r$. Then the inequality above yields $r(C' \cdot \mathcal{T}') > r a_r (C' \cdot F')$. Thus

$$(C' \cdot \mathcal{T}') > a_r (C' \cdot F'). \tag{5.2}$$

Let $R = \mathbb{R}_+[C]$ be the extremal ray generated by C. The composition $\varphi := \mathrm{cont}_R \circ g$ is a morphism defined by a linear sub-system of $|\alpha \mathcal{T}' + \beta F'|$, for some $\alpha > 0$. Since $C' \cdot (\alpha \mathcal{T}' + \beta F') = 0$, we get $\alpha (C' \cdot \mathcal{T}') = -\beta (C' \cdot F')$. Therefore (5.2) gives

$$-\beta > \alpha a_r. \tag{5.3}$$

Let C_r be the section of p' corresponding to the surjection $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(a_r) \to 0$. Then $C_r \cdot (\alpha \mathcal{T}' + \beta F') \geq 0$, which contradicts (5.3). Q.E.D.

Proposition 5.11. *Let $p : Y \to Z$ be a $\mathbb{P}^d$-bundle over a projective manifold Z. Assume that K_Z is nef. If Y is an ample divisor on a manifold X, then p extends to a morphism $\overline{p} : X \to Z$. Furthermore there exists a non-splitting exact sequence*

$$0 \to \mathcal{O}_Z \to \overline{\mathcal{E}} \to \mathcal{E} \to 0$$

such that $X = \mathbb{P}(\overline{\mathcal{E}})$, $Y = \mathbb{P}(\mathcal{E})$, and p, $\overline{p}$ are the bundle projections on Z.

Proof. By Lemma 5.10, the bundle projection p is the contraction of the unique extremal ray on Y. Then, by Corollary 3.9, p extends. Thus Theorem 5.5(ii) applies to give the result. Q.E.D.

Next, we consider the case when the morphism $p : Y \to Z$ is a blowing-up. The following general fact is a direct consequence of Lemma 5.10.

Lemma 5.12. *Let Z be a projective manifold. Let $p : Y \to Z$ be the blowing-up along a smooth subvariety T of codimension ≥ 2. Assume that the canonical bundles K_T and K_Z are nef. Then Y has only one extremal ray.*

PROOF. Let c be the codimension of T in Z, and let E be the exceptional divisor of p. Let $R = \mathbb{R}_+[C]$ be any extremal ray on Y. Since $K_Y \approx p^*K_Z \otimes \mathcal{O}_Y((c-1)E)$, we conclude that $E \cdot C < 0$. Therefore C is contained in E and $K_E \cdot C < 0$. Then apply the proof of Lemma 5.10 to the $\mathbb{P}^{c-1}$-bundle $E \to T$. Q.E.D.

The following generalizes a result due to Sommese concerning the reduction map in the case of threefolds (see [**69**], [**70**, Theorem I], and also [**33, 34**]) and is closely related to Fujita's results in [**25**].

Proposition 5.13. *Let Z be a projective manifold. Assume that K_Z is nef. Let $p : Y \to Z$ be the blowing-up along a smooth subvariety T of codimension $c \geq 2$, such that K_T is nef. If Y is an ample divisor on a manifold X, then there exists a commutative diagram* (2.1)*, where W is a smooth projective variety and either*

(i) *$\overline{p} : X \to W$ is the blowing-up of W along the image of T. Moreover, $\overline{p}(Y)$ is an ample divisor on W whenever T is 0-dimensional; or*
(ii) *X is generically a $\mathbb{P}^1$-bundle over Z and Y is a rational section of it. Moreover, $\dim \overline{p}^{-1}(T) = n-2$, $c = 2$ and fibers of $\overline{p}$ are at most two-dimensional.*

PROOF. By Lemma 5.12, Y contains a unique extremal ray and p is its contraction. Then, by Corollary 3.9, p extends to a contraction $\overline{p} : X \to W$ of an extremal ray on X, which gives rise to a commutative diagram (2.1). Assume first that $\overline{p}$ is birational. Let E, $\overline{E}$ be the exceptional loci of p, $\overline{p}$ respectively, so that $E = \overline{E} \cap Y$. As the restriction $p_{|E} : E \to T$ of p to E is a $\mathbb{P}^{c-1}$-bundle, it follows from Theorem 5.5(ii) that $\overline{p}_{|\overline{E}} : \overline{E} \to T$ is a $\mathbb{P}^c$-bundle. It is now standard to see that W is smooth, Z is contained in W as a divisor and $\overline{p}$ is the blowing-up of W along T, cf. also [**25**, Section 5]. In [**25**, Section 5] it is also proved that $\overline{p}(Y) \cong Z$ is an ample divisor on W under the extra assumption that the restriction to T of the line bundle $\mathcal{O}_W(Z)$ is ample. In particular, $\overline{p}(Y)$ is ample on W if T is 0-dimensional.

Now, assume that $\overline{p}$ is not birational. Then $\alpha : Z \to W$ from (2.1) is an isomorphism and Y is a rational section for $\overline{p}$, which is generically a $\mathbb{P}^1$-bundle. Let $t \in T$ be a general point and let $l \subset F := p^{-1}(t) \cong \mathbb{P}^{c-1}$ be a line. We put $a := Y \cdot l$ and we denote by f a general fiber of $\overline{p}$. Since l is contracted by $\overline{p}$, it is numerically proportional to f. It follows easily that $l \sim af$ as 1-cycles. As $K_X \cdot f = -2$, we get $-2a = K_X \cdot l = K_Y \cdot l - Y \cdot l = 1 - c - a$. So $a = c - 1$. Assume that $\dim \overline{p}^{-1}(T) = n-1$. We obtain that $0 = a(\overline{p}^{-1}(T) \cdot f) = \overline{p}^{-1}(T) \cdot l = -1$, a contradiction. So $\dim \overline{p}^{-1}(T) = n-2$. Now, denote by V the family of all deformations of l in X. We claim that $\dim \mathrm{Locus}(V) \geq n-1$. Assuming the contrary, we would have $\dim \mathrm{Locus}(V) \leq n-2$. From the exact sequence

$$0 \to N_{l/Y} \to N_{l/X} \to \mathcal{O}_l(a) \to 0,$$

using standard facts from deformation theory of rational curves, we find that $\dim V = h^0(N_{l/X}) = n + c + a - 4$. Thus, if x is a point on l,

$$\dim V_x \geq \dim V + 1 - (n-2) = c + a - 1.$$

But the same exact sequence gives

$$\dim V_x \leq h^0(N_{l/X}(-1)) \leq c + a - 2.$$

This is a contradiction and the claim is proved. Since $\dim \overline{p}^{-1}(T) = n - 2$, the claim implies that some deformation of l equals a fiber of $\overline{p}$. In particular, $a = 1$, so $c = 2$ and we are done. Q.E.D.

Remark 5.14. Let us explicitly point out that an analogous result was proved by Fujita [**25**, Section 5], under the assumption that $\operatorname{codim}_Z T \geq 3$, but with no nefness condition on K_T and K_Z. However, the contraction morphism $X \to W$ obtained in [**25**] is in general analytic, not necessarily projective.

Next, let us consider the case when Y admits a pluricanonical fibration.

Recall that a *Calabi–Yau manifold* Y is a projective variety with trivial canonical bundle and $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, \ldots, \dim Y - 1$.

Proposition 5.15. *Let X be a projective manifold of dimension ≥ 3. Let Y be a smooth ample divisor on X. Assume that K_Y is nef. Then the linear system $|m(K_X + Y)|$ is base points free for $m \gg 0$. If K_Y is numerically trivial, then Y is a Calabi–Yau variety and X is a Fano manifold. If $(K_Y)^{k+1}$ is a trivial cycle and $(K_Y)^k$ is non-trivial for $0 < k < n - 1$, then X is a Fano fibration over Z and $k = \dim Z$.*

Proof. The proof runs parallel to that of Theorem 3.8. Since K_Y is nef and Y is ample, we conclude that $K_X + Y$ is nef. Thus $m(K_X + Y)$ is spanned for $m \gg 0$ by the Kawamata–Reid–Shokurov base point free theorem, and it defines a morphism $\pi : X \to W$. By restricting to Y we find that $|mK_Y|$ is base points free for $m \gg 0$. Then Y admits a pluricanonical map, say $\varphi := \varphi_{|mK_Y|}$.

If $(K_Y)^{n-1} = 0$, the morphism φ is a fibration. If K_Y is numerically trivial, then $K_X + Y$ is also, and thus X is a Fano manifold. Hence in particular $H^i(X, \mathcal{O}_X) = H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < \dim Y$, so that Y is a Calabi–Yau manifold. The remaining part of the statement is clear. Q.E.D.

We say that a projective manifold Y is *extendable* if there exists a projective manifold X such that $Y \subset X$ is an ample divisor. Proposition 5.15 shows that a manifold Y such that K_Y is numerically trivial and either K_Y is not linearly trivial or $h^1(\mathcal{O}_Y) > 0$, e.g., Y an abelian variety, is not extendable.

It is worth noting that Proposition 5.15 shows that the Abundance conjecture [**38**] holds true for extendable manifolds. Let us recall what the conjecture says in the smooth case. *Let Y be a projective manifold with K_Y nef. Then mK_Y is spanned by its global sections for $m \gg 0$.*

We refer to [**35**] for a further discussion in the case when Y is a strictly nef divisor on X.

6. Complete results in the three dimensional case

Throughout this section we assume that X is a smooth projective three fold and $Y \subset X$ is a smooth ample divisor. The following theorem implies a number of results from [**6, 7, 8, 69, 70**] and [**33**]. We follow the arguments in [**34**]. Note that [**52**] contains a precise description of all types of extremal rays of X.

Definition 6.1. (cf. [**69**]) A *reduction* of the pair (X, Y) is another pair (X', Y'), with $Y' \subset X'$ a smooth ample divisor, such that X' is got by contracting all (-1) planes $E \cong \mathbb{P}^2$ contained in X such that $Y_E \in |\mathcal{O}_E(1)|$ and Y' is just the image of Y in X'.

Theorem 6.2. *Let X, Y be as above and assume that K_Y is not nef. Then one of the following holds.*

(i) *$\varrho(X) = 1$, X is Fano, of index ≥ 2 and either:*
  (a) *$X \cong \mathbb{P}^3$, $Y \in |\mathcal{O}_{\mathbb{P}^3}(a)|$, $a = 1, 2, 3$; or*
  (b) *$X \cong \mathcal{Q}^3$, $Y \in |\mathcal{O}_{\mathcal{Q}^3}(a)|$, $a = 1, 2$; or*
  (c) *X is a del Pezzo three fold, $Y \in |\mathcal{O}_X(1)|$, cf.* [**27**] *or* [**37**] *for a complete list.*

(ii) *X is a linear $\mathbb{P}^2$-bundle over a curve and, for each fiber F, either $Y_F \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ or $Y_F \in |\mathcal{O}_{\mathbb{P}^2}(2)|$;*

(iii) *X admits a contraction of an extremal ray, $\varphi : X \to W$, such that W is a (smooth) curve, we have $F \cong \mathcal{Q}^2$ for a general fiber of φ and $Y_F \in |\mathcal{O}_{\mathcal{Q}^2}(1)|$ (we call φ a* quadric fibration*);*

(iv) *X is a linear $\mathbb{P}^1$-bundle over a surface and Y is a rational section;*

(v) *A reduction (X', Y') of (X, Y) exists.*

Proof. Since $K_X + Y$ is not nef, there exists an extremal ray $R = \mathbb{R}_+[C]$ of X such that $(K_X + Y) \cdot C < 0$. Consider the length $\ell(R)$ of R and observe that we have $\ell(R) \geq 2$. Let $\varphi = \mathrm{cont}_R : X \to W$ be the contraction of R and let F be a general fiber of φ. If $\dim W = 0$, we fall in case (i). So, from now on, we may assume $\dim W > 0$. If $\ell(R) = 4$, by Theorem 2.7, $\dim W = 0$. If $\ell(R) = 3$, by Theorem 2.7, W is a curve. Moreover, $Y \cdot C = 1$ or 2. By Corollary 5.2, $F \cong \mathbb{P}^2$. If $Y \cdot C = 1$, we get case (ii), $Y_F \in |\mathcal{O}_F(1)|$ by Theorem 5.5. Assume now that $Y \cdot C = 2$ (and $\ell(R) = 3$). Let $L := K_X + 2Y$ and let F_0 be an arbitrary fiber of φ. Remark that $L \cdot R > 0$, therefore $L \cdot C_0 > 0$ for any curve $C_0 \subset F_0$. We have that $(L_{F_0})^2 = 1$ and L_{F_0} is ample by the Nakai–Moishezon criterion. By Theorem 5.1, φ makes X a $\mathbb{P}^2$-bundle and $Y_F \in |\mathcal{O}_{\mathbb{P}^2}(2)|$. Thus, when $\ell(R) = 3$ and W is a curve, we get case (ii). Next, suppose that $\ell(R) = 2$, so $Y \cdot C = 1$. If W is a curve we get $K_F + 2Y_F \sim 0$ and we deduce from Remark 5.3 that $F \cong \mathcal{Q}^2$, leading to case (iii). If W is a surface, we get case (iv). Indeed, φ is generically a $\mathbb{P}^1$-bundle, Y being a rational section. So it is enough to see that all fibers of φ are one-dimensional. Let S be an irreducible surface contracted by φ and let $D := S \cap Y$. We obtain $S^2 \cdot Y = (D_Y)^2 < 0$ and $S^2 \cdot Y = D \cdot S = 0$ since S is contracted by φ. This contradiction shows that φ is a $\mathbb{P}^1$-bundle. Finally, assume that φ is birational. For such a ray, it follows from Theorem 2.7 that E, the locus of R, is an irreducible surface, contracted to a point. Moreover, $E \cdot C := -c < 0$ since R is not nef. We get $K_E + (c+2)Y_E \sim 0$; as above, using suitable vanishings, (see [**34**] for details) we deduce that $E \cong \mathbb{P}^2$, $E_E \in |\mathcal{O}_{\mathbb{P}^2}(-1)|$ and $Y_E \in |\mathcal{O}_{\mathbb{P}^2}(1)|$. This leads to the reduction from case (v). Q.E.D.

Corollary 6.3. ([**6, 7, 8**]) *Let (X, Y) be as above and assume that $p : Y \to B$ is a $\mathbb{P}^1$-bundle. Then p extends to a linear $\mathbb{P}^2$-bundle $\overline{p} : X \to B$, unless either $X \cong \mathbb{P}^3$, $Y \in |\mathcal{O}_{\mathbb{P}^3}(2)|$, or $X \cong \mathcal{Q}^3$, $Y \in |\mathcal{O}_{\mathcal{Q}^3}(1)|$, or $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$, p is one of the projections and the other projection extends.*

PROOF. Assume first that $\varrho(X) = 1$. The conclusion follows by looking at the list in Theorem 6.2(i), using the classification of del Pezzo three folds. Next, suppose $\varrho(X) > 1$. By Lefschetz's theorem, we get an isomorphism $\mathrm{Num}(X) \cong \mathrm{Num}(Y)$. Then Corollary 3.9 applies to give that some extremal ray of Y extends to X. If the genus $g(B) > 0$, such a ray is unique and its contraction, p, extends. Apply Theorem 5.5 to conclude. Assume $g(B) = 0$. Unless either $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $Y \cong \mathbb{F}_1$, Y has only one extremal ray, so the previous argument applies. To conclude, we only have to examine the case when the contraction of the (-1) curve of $\mathbb{F}_1$, say $\pi : \mathbb{F}_1 \to \mathbb{P}^2$, extends to a morphism $\overline{\pi} : X \to W$.

Case 1. $\overline{\pi} : X \to W$ is the contraction of a (-1) plane E. The diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\overline{\pi}} \\ \mathbb{P}^2 \cong Z & \hookrightarrow & W \end{array}$$

shows that $W \cong \mathbb{P}^3$, $Z \in |\mathcal{O}_{\mathbb{P}^3}(1)|$ (see Theorem 5.4). Let $L := \overline{\pi}^*\mathcal{O}_{\mathbb{P}^3}(1)$. We get $Y^3 = (L - E)^3 = 0$, contradicting ampleness of Y.

Case 2. $\overline{\pi} : X \to \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle, Y is a rational section (see the argument from the proof of Theorem 6.2, case (iv)). Let f be a fiber of p and let C_0 be the (-1) curve contracted by π. Write $Y_Y \approx aC_0 + bf$, for some $a > 0$. From $Y \cdot C_0 = 1$ it follows $b = a + 1$. Let $L := \overline{\pi}^*(l)$, $l \subset \mathbb{P}^2$ a line. Since $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$, there is some $F \in \mathrm{Pic}(X)$ such that $\mathcal{O}_Y(F) \cong \mathcal{O}_Y(f)$. We find easily that $Y \approx F + aL$. Now consider the exact sequence

$$0 \to -aL \to \mathcal{O}_X(F) \to \mathcal{O}_Y(F) \to 0.$$

We have $H^1(X, -aL) = H^1(\mathbb{P}^2, -al) = 0$. Therefore, using also the ampleness of Y, it follows that the linear system $|F|$ gives a morphism $\overline{p} : X \to \mathbb{P}^1$ which extends p. Clearly, $\overline{p}$ is a $\mathbb{P}^2$-bundle and, in fact, $X \cong \mathbb{P}^1 \times \mathbb{P}^2$. Q.E.D.

A classification of all cases when Y is birationally ruled also follows from Theorem 6.2.

Corollary 6.4. (cf. [**69, 70**]) *Let (X, Y) be as above and assume that Y is not birationally ruled. Then, either X is a (linear) $\mathbb{P}^1$-bundle and Y is a rational section, or there is a reduction (X_0, Y_0) such that K_{Y_0} is nef.*

PROOF. Looking over the cases (i)–(v) in Theorem 6.2 and using the hypothesis that Y is not ruled, we see that only cases (iv) and (v) are possible. Q.E.D.

Corollary 6.5. ([**34**]) *Let (X, Y) be as above. Assume that X is not a $\mathbb{P}^1$-bundle, Y being a rational section.*

(i) *If $\kappa(Y) = 0$, there is a reduction (X_0, Y_0) of (X, Y) such that Y_0 is a $K3$ surface and X_0 is Fano.*
(ii) *If $\kappa(Y) = 1$, there is a reduction (X_0, Y_0) of (X, Y) such that X_0 fibers over a curve, with general fiber a del Pezzo surface.*

PROOF. Use the preceding corollary and Proposition 5.15. Q.E.D.

7. Extending $\mathbb{P}^1$-bundles

We start with the following proposition.

Proposition 7.1. *Let X be a projective manifold of dimension $n \geq 4$. Let Y be a smooth ample divisor on X. Assume that Y is a conic fibration, with general fiber f. Let V be the family of rational curves induced by f on X. Then the following conditions are equivalent:*

(i) $Y \cdot f = 1$;
(ii) *V is unsplit;*
(iii) *V is locally unsplit.*

If (i)–(iii) *hold, then p is a $\mathbb{P}^1$-bundle which extends to $\overline{p} : X \to Z$. Moreover,* $\dim Z = 2$ *and $\overline{p}$ is a $\mathbb{P}^2$-bundle. Conversely, if p is smooth and extends, conditions* (i)–(iii) *hold.*

Proof. Since (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are clear, it is enough to show (iii) $\Longrightarrow$ (i). Let $a := Y \cdot f$, let $y \in Y$ be a fixed general point and consider the standard exact sequence

$$0 \to \bigoplus^{n-2} \mathcal{O}_f(-1) \to N_{f/X}(-1) \to \mathcal{O}_f(a-1) \to 0.$$

Since $h^1(N_{f/X}(-1)) = 0$, general facts from deformation theory of rational curves say that $\dim V_y = h^0(N_{f/X}(-1)) = a$, and hence $\dim \mathcal{F}_y = a + 1$, see 2.5. By semicontinuity, the same holds at a general point $x \in X$. Fix such a general point $x \in X$ and take another point $t \in \mathrm{Locus}(V_x)$. Since V is locally unsplit, we know that each curve from V_x is irreducible. By the non-breaking lemma, we thus conclude that there is a finite number of curves in V_x passing through t. That is the projection $q : \mathcal{F}_x \to \mathrm{Locus}(V_x)$ is a finite map. Therefore $\dim \mathrm{Locus}(V_x) = a + 1$. Thus we obtain $\dim Y \cap \mathrm{Locus}(V_x) \geq a$. Assume by contradiction that $a \geq 2$. Then there exists a curve $C \subset Y \cap \mathrm{Locus}(V_x)$ such that $p(C)$ is a curve in Z. In this case, a variant of the non-breaking lemma (see [**73**, (1.14)] and also [**17**, (1.4.5)]) implies that the curve C is numerically equivalent in X to λf for some positive rational number λ. Now, take a hyperplane section H_Z of Z and let $\mathcal{L} \in \mathrm{Pic}(X)$ be the extension of $p^*(H_Z)$ on X via the isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$. In particular, $\mathcal{L} \cdot f = 0$, this leading to the numerical contradiction $0 < \mathcal{L} \cdot C = \lambda(\mathcal{L} \cdot f) = 0$.

If (i)–(iii) hold, p extends to a $\mathbb{P}^2$-bundle by Lemma 5.8. Moreover, $\dim Z = 2$ by Lemma 5.7. Conversely, if p extends, we have (i) by Theorem 5.5. Q.E.D.

We consider now the extension problem for $\mathbb{P}^1$-bundles. For perspective we also recall the (much easier) case of $\mathbb{P}^d$-bundles, for $d \geq 2$.

The following major conjecture on the topic [**16**, Section 5.5] describes all known examples. We refer to [**16**, Section 5.5] for the more general case when X is a local complete intersection.

Conjecture 7.2. *Let L be an ample line bundle on a projective manifold, X, of dimension $n \geq 3$. Assume that there is a smooth $Y \in |L|$ such that Y is a $\mathbb{P}^d$-bundle, $p : Y \to Z$, over a manifold, Z, of dimension b. Then $d \geq b - 1$ and $(X, L) \cong (\mathbb{P}(\mathcal{E}), \xi_{\mathbb{P}})$ for an ample vector bundle, $\mathcal{E}$, on Z with p equal to the restriction to Y of the induced projection $\mathbb{P}(\mathcal{E}) \to Z$, except if either:*

(i) $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$; *or*
(ii) $(X, L) \cong (\mathcal{Q}^3, \mathcal{O}_{\mathcal{Q}^3}(1))$; *or*
(iii) $Y \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$, *$p$ is the product projection onto the second factor,* $(X, L) \cong (\mathbb{P}(\mathcal{E}), \xi_{\mathbb{P}})$ *for an ample vector bundle, $\mathcal{E}$, on $\mathbb{P}^1$ with the product projection of Y onto the first factor equal to the induced projection $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$.*

Note. The inequality $d \geq b-1$ is a necessary condition for $p : Y \to Z$ to extend by Lemma 5.7.

The conjecture has been shown except when $d = 1$, $b \geq 3$, and the base Z does not map finite-to-one into its Albanese variety. The case when either $d \geq 2$ or Z is a submanifold of an abelian variety follows from Sommese's extension theorems [**67**] (see also Fujita [**25**]). This argument works also in the case when Z maps finite-to-one into its Albanese variety (see [**16**, (5.2.3)]).

Theorem 7.3. (Sommese) *Conjecture 7.2 is true for $d \geq 2$.*

Proof. Since the result is trivial if Z is a point we can assume without loss of generality that $\dim Z \geq 1$ and thus that $n \geq d+2 \geq 4$. From Theorem 3.1 we know that $p : Y \to Z$ extends to a morphism, $\overline{p} : X \to Z$. The result follows from Theorem 5.5(ii). Q.E.D.

The conjecture is also known when $d = 1$ and $b \leq 2$. If $b = 1$, the result is due to Bădescu [**6, 7, 8**]; we have seen a proof in Corollary 6.3. The case $b = 2$ is due to the contribution of several authors: Fania and Sommese [**24**], Fania, Sato and Sommese [**23**], Sato and Spindler [**62, 63**] and also [**61, 64**]. Below we propose a shorter proof. The basic ideas are those in [**23**] and [**64**], but we do not use the difficult papers [**24**] and [**61**].

Theorem 7.4. *Conjecture 7.2 is true when $d = 1$ and $b = 2$.*

Proof. Assume that p does not extend.

Step 1. Z is ruled. From Corollary 3.9 it follows that Y has some extremal ray, say R, which extends to an extremal ray $\overline{R}$ on X. Using [**52**], we consider the possible type of R. If R is nef, Z is covered by rational curves in the fibers of cont_R, so it is ruled. If R is not nef, E, the locus of R, covers Z (so again Z is ruled), unless $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $p(E) := C$ is a curve. Standard computations (cf. also Poposition 5.13) show that C is a (-1) curve and we may construct a commutative diagram

$$\begin{array}{ccccc} Z & \xleftarrow{p} & Y & \hookrightarrow & X \\ {\scriptstyle \mathrm{cont}_C}\downarrow & & \downarrow{\scriptstyle \mathrm{cont}_R} & & \downarrow{\scriptstyle \mathrm{cont}_{\overline{R}}} \\ Z' & \xleftarrow[p']{} & Y' & \hookrightarrow & X' \end{array}$$

where Y' is ample on X' (cf. [**25**]) and p' is again a $\mathbb{P}^1$-bundle. So, after finitely many steps, we conclude that Z is ruled.

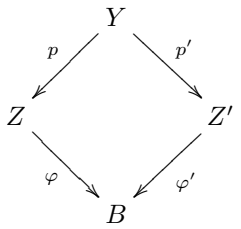
Step 2. $Z \cong \mathbb{P}^2$. Assume the contrary. As Z is ruled, there is a morphism $\varphi : Z \to B$ which is generically a $\mathbb{P}^1$-bundle. Apply Theorem 3.1 to the map $\pi := \varphi \circ p$ to get an extension $\overline{\pi} : X \to B$. Next we use Corollary 6.3 fiberwise. Let $F \cong \mathbb{F}_e$, $\overline{F}$ be the general fibers of $\pi, \overline{\pi}$ respectively. Denote by f, C_0 a fiber and a minimal section of F, respectively. Note that the classes of f and C_0 are not proportional in $N_1(Y)$. Then the diagram

$$\begin{array}{ccc} N_1(F) & \longrightarrow & N_1(\overline{F}) \\ \downarrow & & \downarrow \\ N_1(Y) & \xrightarrow{\sim} & N_1(X) \end{array}$$

shows that $\dim_{\mathbb{R}} N_1(\overline{F}) \geq 2$. So, from Corollary 6.3 we infer that either $Y \cdot f = 1$, or $F \cong \mathbb{P}^1 \times \mathbb{P}^1$. In the first case, p extends by Lemma 5.8. So we may assume that $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ from now on.

We refer to [**23**, pp. 7–11] for details concerning the next few arguments. Using standard properties of Hilbert schemes, one shows:

(a) φ is a $\mathbb{P}^1$-bundle;
(b) any fiber of π is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$;
(c) the family of curves on Y determined by minimal sections of the map $p_{|F} : F \to p(F)$ yields another $\mathbb{P}^1$-bundle $p' : Y \to Z'$. We deduce a cartesian diagram



where φ' is also a $\mathbb{P}^1$-bundle.

Next, from the above construction, Corollary 6.3 and Theorem 5.5, we find that p' extends to a linear $\mathbb{P}^2$-bundle $\overline{p}' : X \to Z'$.

(d) This yields an exact sequence

$$0 \to \mathcal{O}_{Z'} \to \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{F}, \mathcal{G}$ are ample vector bundles on Z'. If $Z' = \mathbb{P}(\mathcal{E}')$, one finds $\mathcal{G} \cong \varphi'^*(\mathcal{E}) \otimes \xi^a$, where $\mathcal{E}$ is a rank 2 vector bundle on B, $\xi = \xi_{Z'}$ and $a > 0$. Moreover, from (c) it follows that $Z \cong \mathbb{P}(\mathcal{E})$.
(e) If we assume $\mathcal{E}$ unstable, it is now standard to see that the exact sequence from (d) splits. This is a contradiction, since $\mathcal{F}$ is ample.

Finally, see [**62**] for a proof that the case $\mathcal{E}$ stable also leads to a contradiction.

Step 3. Conclusion. We know that $Z \cong \mathbb{P}^2$, so Proposition 7.5 below applies to give the result. Q.E.D.

Let us also explicitly note that in the relevant case $d = 1$, $b \geq 3$ (by Lemma 5.7 the bundle $p : Y \to Z$ does not extend in this case) Conjecture 7.2 is equivalent to saying that

- *A $\mathbb{P}^1$-bundle Y, $p : Y \to Z$, over a manifold Z cannot be an ample divisor in an n-dimensional manifold X unless $Y \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$, $Z \cong \mathbb{P}^{n-2}$, and X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ whose restriction to Y is the projection $Y \to \mathbb{P}^1$ on the first factor.*

The following proposition from [**23**] ensures that, to prove Conjecture 7.2, it is enough to show that $Z \cong \mathbb{P}^{n-2}$, assuming that p does not extend.

Proposition 7.5. ([**23**, Section 2]) *Let Y be a smooth ample divisor on an n-dimensional projective manifold X. Assume that $p : Y \to Z$ is a $\mathbb{P}^1$-bundle over $Z = \mathbb{P}^b$, $b \geq 2$. If p does not extend to X, then $Y \cong \mathbb{P}^1 \times \mathbb{P}^b$ and X is a $\mathbb{P}^{b+1}$-bundle over $\mathbb{P}^1$.*

PROOF. Let $F \in |p^*\mathcal{O}_{\mathbb{P}^b}(1)|$ and let $L := \mathcal{O}_X(Y)$. Looking over the proof of Theorem 3.1 we see that, once we have the vanishings in (3.1), the argument works also in the case $\dim Y - \dim Z = 1$. Therefore, the assumption that p does not extend translates into: there exists some $t > 0$ such that $H^1(Y, F - tL_Y) \neq 0$. Using Serre duality, Kodaira vanishing, and the exact sequence

$$0 \to K_Y + tL_Y - F \to K_Y + tL_Y \to K_F + tL_F - F \to 0$$

it follows that

$$H^{n-3}(F, K_F + tL_F - F) \neq 0 \quad \text{for some } t > 0. \tag{7.1}$$

Iterating this construction, we may assume that $b = 2$; in this case $F = \mathbb{F}_e$, for some $e \geq 0$. We write $Y = \mathbb{P}(\mathcal{E})$ for some rank 2 vector bundle on $\mathbb{P}^b$. We also may assume that, if $l \subset \mathbb{P}^b$ is a line, we have $\mathcal{E}_l \cong \mathcal{O}_l \oplus \mathcal{O}_l(-e)$. We shall prove that $e = 0$, so that $\mathcal{E}$ is trivial (see [**58**, Section 3.2]) and the conclusion follows.

So, assume that $b = 2$, $F = \mathbb{F}_e$ and write $L_F \sim aC_0 + bf$, using the notation from [**29**, Chapter V.2]. Since L is ample, $a > 0$ and $b > ae$. Hence, for $t > 0$, $bt > aet$. Therefore, either $bt - 1 > aet$ and $tL_F - F \sim atC_0 + (bt - 1)f$ is ample, or $bt = aet + 1$ and $tL_F - F \sim at(C_0 + ef)$. Now, if $e > 0$, $C_0 + ef$ is nef and big. Using Kawamata–Viehweg vanishing this contradicts (7.1). So $e = 0$ and we are done. Q.E.D.

Further evidence for Conjecture 7.2 is given by the following result (see [**14**] for a proof).

Proposition 7.6. ([**14**]) *Let $p : Y \to Z$ be a $\mathbb{P}^1$-bundle over a smooth projective three fold Z. Then Y cannot be a very ample divisor in any projective manifold X, unless $Z \cong \mathbb{P}^3$ and $Y \cong \mathbb{P}^1 \times \mathbb{P}^3$.*

Remark 7.7. The following discussion gives some further support to Conjecture 7.2, in relation to the content of Section 4. Let $p : Y \to Z$ be a smooth $\mathbb{P}^1$-bundle with Y ample divisor in a projective manifold X of dimension $n \geq 4$. If p does not extend to X, and we assume the hypothesis of Proposition 4.6 is fulfilled, then the following three conditions hold true.

(1) $\varrho(X) = \varrho(Y) = 2$;
(2) Y is a Fano manifold;
(3) Z is a Fano manifold (and $\varrho(Z) = 1$).

Condition (1) directly follows from Proposition 4.6.

To show (2), let R_1 be the extremal ray corresponding to the bundle projection p. By Corollary 3.9 we conclude that there exists an extremal ray R_2 on Y which extends to X. By (1), $\overline{NE}(Y) = \langle R_1, R_2 \rangle$ and Y is a Fano manifold.

Since Y is a Fano manifold and p is smooth, Z is also a Fano manifold, see [**41**, p. 244]. In our special case, we can give the following alternative argument. Assume that Z is not a Fano manifold. Then K_Z would be nef (since $\varrho(Z) = 1$). Therefore Proposition 5.11 applies to say that p extends; a contradiction. Whence (3) holds.

We already observed that, in view of Proposition 7.5, to prove Conjecture 7.2 it would be enough to show that $Z \cong \mathbb{P}^{n-2}$. This is in agreement with the fact that assuming the hypothesis of Proposition 4.6 to hold, we get condition (3) above. For instance, when $n = 4$, the only Fano surface Z with $\varrho(Z) = 1$ is $\mathbb{P}^2$, yielding a very short proof of Theorem 7.4.

8. Fano manifolds as ample divisors

Throughout this section, let X be a projective manifold of dimension $n \geq 4$, let H be an ample line bundle on X, and let Y be a smooth divisor in $|H|$. In this general setting, it is natural to ask the following questions.

Question 8.1. If Y is a Fano manifold, when is X a Fano manifold?

Question 8.2. If X is a Fano manifold, when do we have $\overline{NE}(X) \cong \overline{NE}(Y)$?

Question 8.2 has been solved in [**40**] and [**74**], by using Theorem 2.8, in the special case described in the following theorem.

Theorem 8.3. (Kollár, Wiśniewski) *Let X be a Fano manifold of dimension $n \geq 4$ and index $i \geq 1$, $-K_X \cong iL$, for some ample line bundle L on X. Assume that we are given a smooth member $Y \in |mL|$, for some integer m, $1 \leq m \leq i$. Then there is an isomorphism $\overline{NE}(Y) \cong \overline{NE}(X)$.*

Proof. The result follows from Proposition 3.5, since $-(K_X + Y) \approx (i-m)L$ is nef. Q.E.D.

Let us come back now to Question 8.1. This question is motivated by the problem of classifying polarized pairs (X, Y) as above, when Y is a Fano manifold of large index. Set $-K_Y \approx iL_Y$, where L_Y is an ample line bundle on Y.

The first cases to consider, cf. Corollary 5.2 and Remark 5.3, are $i = \dim Y + 1 = n$, so Y is a projective space, and $i = n-1$, so Y is a quadric; for a solution, see [**67**] and also [**9**], where X is only assumed to be normal. The case when (Y, L_Y) is a classical del Pezzo variety, i.e., $i = n-2$ with L_Y very ample, has been completely worked out by Lanteri, Palleschi and Sommese [**45, 46, 47**]. In [**12, 13**] the next case when (Y, L_Y) is a Mukai variety, i.e., $i = n-3$, is considered. The results of [**13**] have been refined and strengthened in [**2**] under the assumption that L_Y is merely ample, as a consequence of a comparing cones result which holds true in the range $i \geq \frac{\dim Y}{2}$. In [**56**] the classification is extended to the next case.

We will work under the *extra assumption* that the line bundle L_Y is spanned. Note that this is in fact the case when (Y, L_Y) is either a del Pezzo variety of degree at least two, or a Mukai variety. This follows from Fujita's classification [**27**] of del Pezzo manifolds and from Mukai's classification, see [**54**] and [**51**].

We have the following result (compare with [**2**, (4.2)]).

Theorem 8.4. *Let X be a projective manifold of dimension $n \geq 4$, let H be an ample line bundle on X, and let Y be an effective divisor in $|H|$. Assume that Y is a Fano manifold of index $i \geq 3$, $-K_Y \approx iL_Y$. Further assume that L_Y is spanned. Then either*

(i) *There exists an extremal ray R on X of length $\ell(R) \geq 2i+1$; or*
(ii) *Y contains an extremal ray of length $\geq 3i$; or*
(iii) *X is a Fano manifold and $\overline{NE}(X) \cong \overline{NE}(Y)$.*

Proof. By the Lefschetz theorem, there exists a unique line bundle L on X such that $L_{|Y} = L_Y$.

First, suppose that $K_Y + iH_Y$ is not nef. Then, by the cone theorem, there exists an extremal ray $R_Y = \mathbb{R}_+[C]$ on Y such that

$$(K_Y + iH_Y) \cdot C < 0.$$

We can also assume that C is a minimal curve, that is $\ell(R_Y) = -K_Y \cdot C$. Therefore $\ell(R_Y) > i(H \cdot C)$.

Set $a := L \cdot C$, $d := H \cdot C$. Since $-K_Y \approx iL_Y$, the last inequality yields $a > d \geq 1$. Then $a \geq 2$ with equality only if $d = 1$. Thus $\ell(R_Y) = ia \geq 2i$, so that either $\ell(R_Y) \geq 2i + 1$, or $a = 2$ and hence $d = H \cdot C = 1$.

From $-K_Y \approx iL_Y$ we get $\ell(R_Y) = i(L_Y \cdot C)$, that is, i divides $\ell(R_Y)$. Hence in the first case above it must be $\ell(R_Y) \geq 3i$, as in the case (ii) of the statement.

Thus we can assume that $a = 2$, $H \cdot C = 1$ and the adjunction formula gives $(K_X + H) \cdot C = K_Y \cdot C = -\ell(R_Y)$, or $-K_X \cdot C = \ell(R_Y) + 1 = 2i + 1$. Therefore $(K_X + 2iH) \cdot C < 0$, i.e., $K_X + 2iH$ is not nef. Then there exists a rational curve γ generating a ray $R = \mathbb{R}_+[\gamma]$ on X such that $(K_X + 2iH) \cdot \gamma < 0$. Since we can assume $\ell(R) = -K_X \cdot \gamma$, it follows that $\ell(R) > 2i$ and we are in case (i) of the statement.

Assume now that $K_Y + iH_Y$ is nef. Then by the ascent of nefness (see the proof of Theorem 3.8) we infer that $K_X + (i+1)H$ is nef and hence by Kawamata–Reid–Shokurov base point free theorem we conclude that $m(K_X + (i+1)H)$ is spanned for $m \gg 0$.

We proceed by cases, according to the Iitaka dimension of $K_X + (i+1)H$. Let $\psi : X \to W$ be the map with normal image and connected fibers associated to $|m(K_X + (i+1)H)|$ for $m \gg 0$.

If $\kappa(K_X + (i+1)H) = 0$, then $K_X + (i+1)H \approx 0$, so that X is a Fano manifold. From $iL_Y \approx -K_Y \approx iH_Y$ we conclude by Lefschetz that $H \approx L$. Therefore Theorem 8.3 applies to give $\overline{NE}(X) = \overline{NE}(Y)$.

Assume $\kappa(K_X + (i+1)H) = 1$. Since Y is ample, the restriction ψ_Y of ψ to Y maps onto W, so that $W \cong \mathbb{P}^1$ since Y is a Fano manifold. Note that ψ_Y is not the constant map by ampleness of Y. Recalling that $\overline{NE}(Y)$ is polyhedral, we conclude that there exists an extremal ray R on Y which is not contracted by ψ_Y. Let $\varphi : Y \to Z$ be the contraction of R. We claim that all fibers of φ are of dimension ≤ 1. Otherwise, let Δ be a fiber of dimension ≥ 2. Any fiber F of ψ_Y is a divisor on Y. Then we can find a curve $C \subset \Delta \cap F$. Therefore C generates R and $\dim \psi_Y(C) = 0$, contradicting the fact that R is not contracted by ψ_Y.

Thus by Theorem 2.8 we know that either φ is a blowing-up of a smooth codimension two subvariety of Z and $-K_Y \cdot C = 1$; or φ is a conic fibration and $-K_Y \cdot C \leq 2$. In each case, the equality $-K_Y \cdot C = i(L_Y \cdot C)$ contradicts the assumption that $i \geq 3$.

Assume now $\kappa(K_X + (i+1)H) \geq 2$. We follow here the argument from [**13**]. From $(K_X + H)_Y \approx K_Y \approx -iL_Y$ we get by Lefschetz

$$K_X + H + iL \approx 0, \tag{8.1}$$

that is $K_X + (i+1)H \approx i(H - L)$. Thus we conclude that $\kappa(H - L) \geq 2$ and that $m(H - L)$ is spanned for $m \gg 0$. Therefore the Mumford vanishing theorem [**55**] (see also [**66**, (7.65)]) applies to give

$$H^1(X, L - H) = 0. \tag{8.2}$$

Now consider the exact sequence

$$0 \to L - H \to L \to L_Y \to 0.$$

Since L_Y is spanned on Y, by (8.2) we see that sections of $H^0(Y, L_Y)$ lift to span L in a neighborhood of Y; but since Y is ample we conclude that L is spanned off

a finite set of points. Hence L is nef and therefore $-K_X$ is ample by (8.1), i.e., X is a Fano manifold.

We conclude that either $\overline{NE}(X) \cong \overline{NE}(Y)$ and hence we are done, or there exists an extremal ray $R = \mathbb{R}_+[C]$ on X, $R \subset \overline{NE}(X) \setminus \overline{NE}(Y)$ such that every fiber of the contraction $\varphi : X \to Z$ of R has dimension at most one. Then Theorem 2.8 applies again to say that either:

(1) φ is a blowing-up along a smooth codimension two center B and $K_X \cdot C = -1$, or
(2) φ is a conic fibration and either $K_X \cdot C = -2$ or $K_X \cdot C = -1$.

In case (1), from (8.1) and $K_X \cdot C = -1$ we get $1 = Y \cdot C + i(L \cdot C)$. Since $Y \cdot C > 0$ and $L \cdot C \geq 0$ it must be $Y \cdot C = 1$, $L \cdot C = 0$. Note that $(K_X + Y) \cdot C = 0$ and apply Proposition 3.4 to contradict our present assumption that $R \not\subset \overline{NE}(Y)$.

Let us consider case (2). If $K_X \cdot C = -2$ we have $2 = Y \cdot C + i(L \cdot C)$, giving $Y \cdot C = 2$, $L \cdot C = 0$. If $K_X \cdot C = -1$ we get $1 = Y \cdot C + i(L \cdot C)$, giving $Y \cdot C = 1$, $L \cdot C = 0$. In both cases, Proposition 3.4 applies again to give the same contradiction as above. Q.E.D.

Corollary 8.5. ([**2**, (4.2)]) *Let X be an n-dimensional projective manifold. Let H be an ample line bundle on X and let Y be a divisor in $|H|$. Assume that Y is a Fano manifold of index $i \geq \frac{\dim Y}{2} \geq 3$ (hence $n \geq 7$), $-K_Y \approx iL_Y$ and L_Y is spanned. Then X is a Fano manifold and $\overline{NE}(X) \cong \overline{NE}(Y)$.*

Proof. By the proof of Theorem 8.4, either we are done, or Y contains an extremal ray $R_Y = \mathbb{R}_+[C]$ such that either $\ell(R_Y) \geq 3i$, or $\ell(R_Y) \geq 2i$ and $H \cdot C = 1$. In the first case we get the numerical contradiction

$$\ell(R_Y) \geq \frac{3}{2}(n-1) \geq n + 1 = \dim Y + 2.$$

Thus we may assume $\ell(R_Y) \geq 2i$ and $H \cdot C = 1$. Therefore, if Δ is a positive dimensional fiber of the contraction $p = \mathrm{cont}_{R_Y} : Y \to W$, we have $\dim \Delta \geq \ell(R_Y) - 1 \geq \dim Y - 1$ (see Theorem 2.7). Thus, either $\Delta = Y$ or $\ell(R_Y) = \dim Y = n - 1$.

In the first case, the contraction p is the constant map, so that $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \cong \mathbb{Z}$ and the conclusion is clear.

In the latter case, since $\ell(R_Y) = \dim Y$, we know from Theorem 2.7 that $\dim W \leq 1$. If $\dim W = 0$ we conclude as above. Assume that W is a curve and let F be a general fiber of p. Since Y is a Fano manifold, $W \cong \mathbb{P}^1$. Moreover, since $H \cdot C = 1$, we get $K_F + (n-1)H_F \approx 0$. Corollary 5.2 applies to give $F \cong \mathbb{P}^{n-2}$, $H \in |\mathcal{O}_F(1)|$. Therefore, by Theorem 5.5(i), $Y \cong \mathbb{P}(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on $\mathbb{P}^1$. Using Lemma 4.4 and the assumption $i \geq 3$, we see that this case does not occur. Q.E.D.

9. Ascent properties

Let X be a projective n-dimensional manifold and let $Y \subset X$ be a smooth ample divisor. Here is a list of general facts concerning ascent properties from Y to X. E.g.,

• K_Y not ample $\Longrightarrow$ K_X not nef. It immediately follows from the adjunction formula.

• $\kappa(Y) < \dim Y \Longrightarrow \kappa(X) = -\infty$. Here is the argument from [**34**, Proposition 5]. Assume by contradiction that $|mK_X| \neq \emptyset$ for some $m > 0$ and let $E \in |mK_X|$. Write $E = aY + E'$, with $a \geq 0$ and $Y \not\subset \mathrm{Supp}(E')$. Since λY is very ample for $\lambda \gg 0$, we can find $\lambda \gg 0$ and $D \in |\lambda E|$ such that $Y \not\subset \mathrm{Supp}(D)$. Since we have $(D + \lambda mY)_{|Y} \approx \lambda mK_Y$, it follows that $|\lambda mK_Y|$ is very ample outside of $Y \cap \mathrm{Supp}(D)$, so that $\kappa(Y) = n - 1 = \dim Y$. This contradiction proves the assertion.

Uniruled manifolds are birationally Fano fibrations. This fact follows from Campana's construction (see e.g., the Preface and Chapters 3, 4 of Debarre's text [**20**]). Many results in our paper are concerned with the case in which Y carries a special Fano fibration structure.

• Y uniruled $\Longrightarrow$ X uniruled. It immediately follows from the uniruledness criterion [**41**, II, Section 3, IV, (1.9)]. Saying that Y is uniruled means that there is a morphism $f : \mathbb{P}^1 \to Y$ such that f^*T_Y is spanned. Consider the tangent bundle sequence

$$0 \to T_Y \to T_{X|Y} \to \mathcal{O}_Y(Y) \to 0.$$

Let $f' : \mathbb{P}^1 \to X$ be the induced morphism to X. By pulling back to $\mathbb{P}^1$, we get the exact sequence

$$0 \to f^*T_Y \to f^*(T_{X|Y}) = f'^*T_X \to f^*\mathcal{O}_Y(Y) \to 0.$$

Since both $f^*\mathcal{O}_Y(Y)$ and f^*T_Y are nef, we conclude that f'^*T_X is nef and hence spanned; this is equivalent to say that X is uniruled.

• Y rationally connected $\Longrightarrow$ X rationally connected. Saying that Y is rationally connected is equivalent to the existence of a curve $C \cong \mathbb{P}^1 \subset Y$ with ample normal bundle $N_{C/Y}$ (see e.g., [**41**]). Therefore the exact sequence of normal bundles

$$0 \to N_{C/Y} \to N_{C/X} \to \mathcal{O}_C(Y) \to 0$$

and the ampleness of Y give the ampleness of $N_{C/X}$.

• Y unirational $\Longrightarrow$ X unirational? This is a hard question and no answer is known. It is interesting to point out that, since unirationality implies rational connectedness, to find examples of Y unirational with X not unirational would give examples of rationally connected manifolds X which are not unirational. Quoting Kollár [**42**, Section 7, Problem 55], the latter is "one of the most vexing open problems" in the theory.

• In general, Y rational does not imply that X is rational. We present below a few results about this problem. In particular, we obtain a proof of the following classical statement ([**60**, Chapter IV]): *for a very ample smooth divisor on a three fold X, the ascent of rationality holds true with the only exception when X is the cubic hypersurface of $\mathbb{P}^4$.* The case of the cubic hypersurface is indeed an exception, see [**19**].

Theorem 9.1. (cf. also [**18**]) *Let L be an ample line bundle on a smooth projective three fold X. Assume that there is a smooth $Y \in |L|$ such that Y is rational. Then X is rational unless either:*

(i) *$L^3 = 1$ and (X, L) is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, 1, 1)$, $-K_X \approx 2L$; or*
(ii) *$L^3 = 2$ and (X, L) is the double covering of $\mathbb{P}^3$ branched along a smooth surface of degree 4, $-K_X \approx 2L$ and L is the pull-back of $\mathcal{O}_{\mathbb{P}^3}(1)$; or*
(iii) *X is the hypercubic in $\mathbb{P}^4$ and $L \approx \mathcal{O}_X(1)$.*

PROOF. Since Y is a rational surface, K_Y is not nef. We follow the cases (i)–(v) from Theorem 6.2. In case (i), we apply the well-known classification of Fano three folds of index ≥ 2 (and $\varrho(X) = 1$), see [**37**]. We either get one of the exceptional cases in the statement, or X is the complete intersection of two quadrics in $\mathbb{P}^5$, or $X \subset \mathbb{P}^6$ is a linear section of the Grassmannian of lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ by the Plücker embedding. In the last two cases X is rational (see e.g., [**37**]). A simple argument is given in Example 9.3 below.

Assume now that we are in case (ii) or (iii) from Theorem 6.2. For such a fibration the base curve is $\mathbb{P}^1$ and the general fiber is rational. Moreover, a section exists by Tsen's theorem, see e.g., [**41**, IV.6]. So X is rational, too.

If we are in case (iv), the base surface is birational to Y, so it is rational. We conclude that X is rational. Finally, case (v) leads to one of the previously discussed situations. Q.E.D.

The following result, contained in [**36**, Theorem 1.3], concerns the ascent of rationality from a suitable rational submanifold. The proof relies on Hironaka's desingularization theory [**32**] and on basic properties of rationally connected manifolds [**43**].

Theorem 9.2. ([**36**, Theorem 1.3]) *Let X be a projective variety and $|D|$ a complete linear system of Cartier divisors on it. Let $D_1, \ldots, D_s \in |D|$ and put $W_i := D_1 \cap \cdots \cap D_i$ for $1 \leq i \leq s$. Assume that W_i is smooth, irreducible of dimension $n - i$, for all i. Assume moreover that there is a divisor E on $W := W_s$ and a linear system $\Lambda \subset |E|$ such that:*

(i) *$\varphi_\Lambda : W \dashrightarrow \mathbb{P}^{n-s}$ is birational, and*
(ii) *$|D_W - E| \neq \emptyset$.*

Then X is rational.

PROOF. We proceed by induction on s. Let us explain the case $s = 1$, the general case being completely similar. So, let $W \in |D|$ be a smooth, irreducible Cartier divisor such that $\varphi_\Lambda : W \dashrightarrow \mathbb{P}^{n-1}$ is birational for $\Lambda \subset |E|$, $E \in \mathrm{Div}(W)$ and $|D_W - E| \neq \emptyset$. Replacing X by its desingularization, we may assume that X is smooth. As W is rational, it is rationally connected, so we may find some smooth rational curve $C \subset W$ with $N_{C/W}$ ample. We have $C \cdot E > 0$ and from (ii) we deduce $C \cdot D > 0$. From the exact sequence of normal bundles we get that $N_{C/X}$ is ample, so X is rationally connected. In particular, $H^1(X, \mathcal{O}_X) = 0$.

The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_W(D) \to 0,$$

shows that $\dim |D| = \dim |D_W| + 1 \geq \dim |E| + 1 \geq n$.

We may choose a pencil $(W, W') \subset |D|$, containing W, such that $W'_W = E_0 + E_1$, with $E_0 \geq 0$ and $E_1 \in \Lambda$. By Hironaka's theory [**32**], we may use blowing-ups with smooth centers contained in $W \cap W'$, such that after taking the proper transforms of the elements of our pencil, to get:

(a) $\mathrm{Supp}(E_0)$ has normal crossing;
(b) Λ is base points free (so $\varphi : W \to \mathbb{P}^{n-1}$ is a birational morphism).

Further blowing-up of the components of $\mathrm{Supp}(E_0)$ allows to assume $E_0 = 0$ so D_W is linearly equivalent to E. Using the previous exact sequence and the fact

that $H^1(X, \mathcal{O}_X) = 0$, it follows that $\mathrm{Bs}|D| = \emptyset$. Finally, $D^n = (D_W)_W^{n-1} = 1$, so φ is a birational morphism to $\mathbb{P}^n$. Q.E.D.

Example 9.3. ([**36**, Example 1.4]) Let $X \subset \mathbb{P}^{n+d-2}$ be a non-degenerate projective variety of dimension $n \geq 2$ and degree $d \geq 3$, which is not a cone. Then X is rational, unless it is a smooth cubic hypersurface, $n \geq 3$. If X is singular, by projecting from a singular point we get a variety of minimal degree, birational to X. So X is rational. If X is not linearly normal, X is isomorphic to a variety of minimal degree. Hence we may assume X to be smooth and linearly normal. One sees easily that such a linearly normal, non-degenerate manifold $X \subset \mathbb{P}^{n+d-2}$ has anticanonical divisor linearly equivalent to $n-1$ times the hyperplane section, i.e., they are exactly the so-called "classical del Pezzo manifolds". They were classified by Fujita in a series of papers; see [**27**] or [**37**]. Independently of their classification, the following simple argument shows that such manifolds are rational when $d \geq 4$. Consider the surface W obtained by intersecting X with $n-2$ general hyperplanes. Note that W is a non-degenerate, linearly normal surface of degree d in $\mathbb{P}^d$, so it is a del Pezzo surface. As such, W is known to admit a representation $\varphi : W \to \mathbb{P}^2$ as the blowing-up of $9-d$ points. Let $L \subset W$ be the pull-back via φ of a general line in $\mathbb{P}^2$. It is easy to see that L is a cubic rational curve in the embedding of W into $\mathbb{P}^d$. So, for $d \geq 4$, L is contained in a hyperplane of $\mathbb{P}^d$. This shows that the conditions of the Theorem 9.2 are fulfilled for X, $|D|$ being the system of hyperplane sections. We also see that Theorem 9.2 is sharp, as the previous argument fails exactly for the case of cubics.

Remark 9.4. In closing, we mention three possible generalizations of the problem of extending morphisms from ample divisors on X.

(1) The smoothness assumption on X may be relaxed by allowing normal singularities. Let Y be a smooth divisor in X (X is smooth), and let us only suppose that Y has ample normal bundle. Then a well-known result ([**30**]) shows that there is a birational map $\psi : X \to X'$, which is an isomorphism along Y, such that $\psi(Y) := Y' \subset X'$ is ample and X' is normal. See e.g., [**9, 10**] and [**18**] for results in this direction.
(2) Consider a smooth section $Y \subset X$ of the appropriate expected dimension $n - \mathrm{rk}\mathcal{E}$ of an ample vector bundle $\mathcal{E}$ on an n-fold X. Note that a Lefschetz type theorem for ample vector bundles, due to Sommese [**68**], implies that the restriction to Y gives an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$. See e.g., [**3, 21, 44, 4**] and [**2**] for results of this type.
(3) In the same spirit, let us consider a smooth subvariety Y of a manifold X such that $\mathrm{codim}_X Y \geq 2$, and Y has ample normal bundle. Further, let us add the Lefschetz type assumption that $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X)$. Then one can study extensions of rationally connected fibrations $p : Y \to Z$ onto a normal projective variety Z. See [**11**] and [**57**] for results in this direction.
(4) The very recent paper [**72**] classifies pairs (X, Y), when $Y \subset X$ is an ample divisor which is a homogeneous manifold.

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