Ground state solution for a problem with mean curvature operator in Minkowski space. *

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Abstract

In this paper we prove the existence of a radial ground state solution for a quasilinear problem involving the mean curvature operator in Minkowski space.

Introduction

In this paper we study the following quasilinear problem

$$\begin{cases} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right] + f(u) = 0, & x \in \mathbb{R}^N, \\ u(x) > 0, & \text{in } \mathbb{R}^N \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{1}$$

where $N \geqslant 2$ and $f : \mathbb{R} \to \mathbb{R}$.
The differential operator we are considering, known as the mean curvature operator in the Minkowski space, has been deeply studied in the recent years, in nonlinear equations on bounded domains with various type of boundary conditions (see [3, 4, 5, 6] and the references within) and in the whole $\mathbb{R}^N$ for nonlinearities f of the type u^p (see [7]).

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If we look for radial solutions, we can reduce equation (1) to the following ODE

$$\left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + \frac{N-1}{r}\frac{u'}{\sqrt{1-(u')^2}} + f(u) = 0 \tag{2}$$

where $u \in C^2([0,+\infty])$ is such that $u'(0) = 0$.
We will use the shooting method to establish the global existence of the solutions of the Cauchy problem

$$\begin{cases} \left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + \frac{N-1}{r}\frac{u'}{\sqrt{1-(u')^2}} + f(u) = 0 \\ u(0) = \xi, u'(0) = 0 \end{cases} \tag{3}$$

where ξ is allowed to vary in an interval which we will define later. As usual, in this type of problem the local existence is not difficult to prove, since standard fixed point theorems work fine.
What is really interesting is to find the conditions which permit to extend the solution to the whole $\mathbb{R}_+$ and to prove that the solution is a ground state, namely $\lim_{r\to\infty} u(r) = 0$.
The shooting argument has been used in the past to find ground state solutions to various types of equations. We recall two significant examples such as

$$\Delta u + f(u) = 0, \tag{4}$$

treated in [2] or the following prescribed mean curvature equation

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \lambda u + u^q = 0, \tag{5}$$

studied in [10]. The method consists in studying the profile of the solution of (3) as the initial value ξ varies into an interval. In particular, since we are interested in ground states, we aim to exclude the cases in which for a finite $R > 0$ either u or u' vanishes. Using the property of the intervals to be connected, if we proved that the values ξ corresponding to the *bad* cases constitute two open disjoint non empty subsets of an interval I, we should have found at least an initial value whose corresponding solution is a ground state.
We make the following assumptions over f

(f1) $f(0) = 0$,

(f2) f is locally Lipschitz in $[0,+\infty)$,

(f3) $\exists \alpha := \inf\{\xi > 0 \mid f(\xi) \geqslant 0\} > 0$,

(f4) (if $N \geqslant 3$) $\lim_{s \to \alpha^+} \frac{f(s)}{s-\alpha} > 0$,

(f5) $\exists \gamma > 0$ such that $F(\gamma) := \int_0^\gamma f(s)\, ds > 0$,

and, defining

$$\xi_0 := \inf\{\xi > 0 \mid F(\xi) > 0\}, \tag{6}$$

we assume

(f6) $f(\xi) > 0$ in $(\alpha, \xi_0]$.

In the sequel, we will suppose that f is extended in $\mathbb{R}_-$ by 0. Of course, since we are looking for positive solutions, this assumption does not involve the generality of the problem. The main result of the paper is the following

Theorem 0.1. *If*

- $N \geqslant 3$ *and* f *satisfies (***f1***-...-***f6***)*
- $N = 2$ *and* f *satisfies (***f1***), (***f2***), (***f3***), (***f5***) and (***f6***),*

then (1) *has a radially decreasing solution.*

Remark 0.2. *We do not treat the case* $N = 1$ *since it is definitely analogous to* $u'' + f(u) = 0$*. Then we refer to [1, Section 6] for sufficient and necessary condition for the existence of the unique solution of the problem*

$$\begin{cases} \left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + f(u) = 0 \\ u(x) > 0, \quad \textit{in } \mathbb{R} \\ u(x) \to 0, \quad \textit{as } |x| \to \infty. \end{cases}$$

Remark 0.3. *We exhibit some examples of functions* f *satisfying our assumptions*

1. $f(s) = -\lambda s + s^q$ *for* $\lambda > 0$ *and* $q > 1$ *is a nice function for* $N \geqslant 2$*,*
2. $f(x) = -s\sin(s)|\sin(s)|^{q-1}$ *is nice when* $N \geqslant 2$ *and* $q = 1$ *and when* $N = 2$ *and* $q \geqslant 1$*.*

Remark 0.4. *By comparing our main result with those in [2] and [10], some remarkable differences stand out. For example we point out that no assumption is required on the behaviour of f at infinity. On the contrary, when for instance f is as in example 1, a necessary condition both in [2] and in [10] is $q \in (1, \frac{N+2}{N-2})$, for $N \geqslant 3$.*
Moreover the existence result proved in [10] holds for λ sufficiently small. On the other hand a nonexistence result has been proved for (5) *in [9] when $\lambda > \left(2\frac{q+1}{q-1}\right)^{\frac{q-1}{q+1}}$. As shown in example 1, in our case λ is allowed to be any positive number.*

1 Proof of the existence result

Observe that the solution of (3) satisfies the equation

$$(r^{N-1}\phi'(u'))' = -r^{N-1}f(u), \tag{7}$$

where $\phi(s) := 1 - \sqrt{1-s^2}$ (for $s \in [-1,1]$).
It is easy to verify that $\phi' :]-1,1[\to \mathbb{R}$ is an increasing diffeomorphism. Set $\delta > 0$ (whose smallness will be later established) and denote by $C := C(\mathbb{R}_+, \mathbb{R})$ and by $C_\delta := C([0,\delta], \mathbb{R})$ respectively the set of the continuous functions defined in $\mathbb{R}_+$ and in the interval $[0,\delta]$. Define the following operators

$$S : C \to C, \quad S(u)(r) := \begin{cases} -\frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t)\, dt & \text{if } r > 0, \\ 0 & \text{if } r = 0, \end{cases}$$

and

$$K : C \to C, \quad K(u)(r) = \int_0^r u(t)\, dt.$$

For every $\xi \in \mathbb{R}$, define the translation operator $T_\xi : C \to C$ such that $T_\xi(u) = \xi + u$. Moreover, consider the Nemytskii operators associated to f and $(\phi')^{-1}$,

$$N_f : C \to C, \quad N_f(u)(r) = f(u(r)),$$
$$N_{(\phi')^{-1}} : C \to C, \quad N_{(\phi')^{-1}}(u)(r) = (\phi')^{-1}(u(r)).$$

Set $\rho > 0$ and denote with $B_\rho := \{u \in C_\delta \mid \|u\|_\infty \leqslant \rho\}$. We set the following fixed point problem: for any $\xi \in \mathbb{R}$ we want to find $u \in \xi + B_\rho$ such that

$$u = T_\xi \circ K \circ N_{(\phi')^{-1}} \circ S \circ N_f(u). \tag{8}$$

Since $(\phi')^{-1}$ and f are respectively Lipschitz and locally Lipschitz, Banach-Caccioppoli fixed point theorem guarantees the existence of a sufficiently small $\delta > 0$ such that the function $u := u(\xi, r) \in \xi + B_\rho$ is a solution of (8). It is easy to observe that u is a local solution of the Cauchy problem (3).

Now, let $R > 0$ be such that $[0, R)$ is the maximal interval where the function u is defined. Multiplying (2) by u' and integrating over $(0, r)$ we obtain the following equality for any $r \in (0, R)$

$$H(u'(r)) + (N-1) \int_0^r \frac{(u')^2(s)}{s\sqrt{1-(u')^2(s)}}\, ds = F(\xi) - F(u(r)) \tag{9}$$

where $H(t) = \frac{1-\sqrt{1-t^2}}{\sqrt{1-t^2}}$.
Denote by

$$\beta := \inf\{\xi > \xi_0 \mid f(\xi) = 0\}. \tag{10}$$

Of course $\alpha < \xi_0 < \beta \leqslant +\infty$. Denote by I the interval (α, β) and take $\xi \in I$. By **(f3)** and **(f6)**, for every $s \leqslant \beta$ we have $F(s) \geqslant F(\alpha)$, so from (9) we deduce that $H(u'(r))$ is bounded as far as $u(r) \leqslant \beta$.

Observe that, since $f(u(0)) = f(\xi) > 0$, from equation (2) we deduce that $u''(0) < 0$ and then there exists $\eta > 0$ such that $u'(r) < 0$ and $\xi > u(r) > 0$ for every $r \in (0, \eta)$. Set

$$\bar{R} := \begin{cases} \inf\{r \in (0,R) \mid u'(r) \geqslant 0\} & \text{if } u'(r) = 0 \text{ for some } r \in (0,R) \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

Remark 1.1. *According to the definition* (11) *we have that* $0 < \eta \leqslant \bar{R} \leqslant +\infty$ *and, since* $u(r) < \xi < \beta$ *for every* $r \in (0, \bar{R})$*, from* (9) *we have*

$$\exists \varepsilon > 0 \text{ such that, for any } r \in (0, \bar{R}), |u'(r)| \leqslant 1 - \varepsilon. \tag{12}$$

In particular we deduce that $\bar{R} = +\infty$ *implies* $R = +\infty$.

Define the following two intervals

$$I_+ := \left\{ \xi \in I \,\middle|\, \exists R' \leqslant R \text{ such that } \begin{array}{l} u(\xi,r) > 0, u'(\xi,r) < 0, \text{for } r < R' \\ u'(\xi,R') = 0 \end{array} \right\}$$

and

$$I_- := \left\{ \xi \in I \,\middle|\, \exists R' \leqslant R \text{ such that } \begin{array}{l} u(\xi,r) > 0, u'(\xi,r) < 0, \text{for } r < R' \\ u(\xi,R') = 0 \end{array} \right\}.$$

We will prove that I_+ and I_- do not cover I.

Lemma 1.2. *Suppose* $R = +\infty$ *and* $u'(r) < 0$, $u(r) > 0$ *for every* $r > 0$*. Then* $\lim_{r \to +\infty} u(r) = 0$.

Proof Of course by monotonicity there exists $l = \lim_{r\to+\infty} u(r) \geqslant 0$. By (2) and (12), we deduce that

$$\lim_{r\to+\infty} \left(\frac{u'(r)}{\sqrt{1-(u'(r))^2}}\right)' = -f(l). \tag{13}$$

Suppose that $f(l) \neq 0$, say $f(l) > 0$. By simple computations, from (12) and (13) we deduce that, definitively, $u''(r) < -\delta < 0$, for some $\delta > 0$. Of course this is not possible because of (12).
Since $f(l) = 0$, there are only two possibilities, either $l = 0$ or $l = \alpha$.

Suppose $N = 2$ and, by contradiction, $l = \alpha$. Since for any $r > 0$ $\beta > u(r) > \alpha$, from (7) we deduce that $r\phi'(u'(r))$ is decreasing in $\mathbb{R}_+$ and then, in particular, there exists $R_0 > 0$ and $\delta > 0$ such that for any $r > R_0$ we have $\phi'(u'(r)) < -\frac{\delta}{r}$. By (12) we infer that, for some $M > 0$, we have $Mu'(r) \leqslant \phi'(u'(r))$ and then

$$u'(r) \leqslant -\frac{\delta}{Mr} \quad \text{for any } r > R_0.$$

Integrating in (R_0, r) we obtain

$$u(r) \leqslant u(R_0) - \frac{\delta}{M}\log\left(\frac{r}{R_0}\right)$$

which contradicts $l = \alpha$.

Suppose $N \geqslant 3$. To prove $l \neq \alpha$, assume by contradiction that $l = \alpha$. Computing in (2), we have that the following equality holds in $(0, +\infty)$:

$$\frac{u''}{[1-(u')^2]^{\frac{3}{2}}} = -\frac{N-1}{r}\frac{u'}{\sqrt{1-(u')^2}} - f(u).$$

Taking into account (12), there exists $\delta > 0$ such that $\delta \leqslant \sqrt{1-(u')^2} \leqslant 1$. We deduce that

$$u'' = -\frac{N-1}{r}u'[1-(u')^2] - f(u)[1-(u')^2]^{\frac{3}{2}} \leqslant -\frac{N-1}{r}u' - \delta^3 f(u) \tag{14}$$

where we have used the fact that $u' < 0$ and $f(u) > 0$. Now we proceed as in [2], repeating the arguments for completeness. If we set $v = r^{\frac{N-1}{2}}u$, by (14) we get the following estimate

$$v'' \leqslant \left[\frac{(N-1)(N-3)}{r^2} - \delta^3\frac{f(u)}{u}\right]v \tag{15}$$

from which, in view of assumption (**f4**), we deduce that v'' is definitively negative. Now, since v' is definitively decreasing, certainly there exists $L = \lim_{r\to+\infty} v'(r) < +\infty$.
Of course L can not be negative, since otherwise $\lim_{r\to+\infty} v(r) = -\infty$.
On the other hand, if $L \geqslant 0$, then we deduce that v is definitively increasing and then there exists $R_0 > 0$ such that for any $r > R_0$ we have $v(r) > v(R_0)$. From (15) we infer that, for some positive constant C, $v''(r) \leqslant -C < 0$ definitively and this implies $L = \lim_{r\to+\infty} v'(r) = -\infty$: again a contradiction. □

Theorem 1.3. *I_+ is not empty.*

Proof Let $\xi \in (\alpha, \xi_0)$. By (6), $F(\xi) < 0$. By (9) we deduce that $F(u(r)) < F(\xi) < 0$ for any $r \in (0, R)$. As a consequence, by (**f6**) we have that there exists $m > 0$ such that

$$0 < m < u(r) < \xi, \tag{16}$$

and then, by Remark 1.1, $R = +\infty$. Now, assuming that $u'(r) < 0$ for any $r > 0$, by Lemma 1.2 we get a contradiction with (16). □

Now, to prove that I_- is not empty, we need some preliminary results. Consider the problem

$$\begin{cases} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right] + f(u) = 0, & \text{in } B_\rho, \\ u = 0, & \text{on } \partial B_\rho. \end{cases} \tag{17}$$

If $\beta < +\infty$ (we recall that β is defined in (10)), we replace f in (17) by

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \leqslant \beta, \\ 0 & \text{if } s > \beta. \end{cases} \tag{18}$$

As in [5], we use a variational approach to (17).
Set $W_\rho := W^{1,\infty}((0,\rho), \mathbb{R})$. It is well known that $W_\rho \hookrightarrow C_\rho$.
Define

$$K_0 := \{u \in W_\rho \mid \|u'\|_\infty \leqslant 1, u(\rho) = 0\}$$

and

$$\Psi(u) := \begin{cases} \displaystyle\int_0^\rho r^{N-1}(1 - \sqrt{1-(u')^2})\, dr & \text{if } u \in K_0 \\ +\infty & \text{if } u \in W_\rho \setminus K_0. \end{cases}$$

For any $u \in W_\rho$ we set

$$J(u) := \Psi(u) - \int_0^\rho r^{N-1} F(u)\, dr.$$

It is easy to verify that the functional J is a Szulkin's functional (see [12]) so that, by [12, Proposition 1.1], we have that if $u \in W_\rho$ is a local minimum of J, then it is a Szulkin critical point and for any $v \in K_0$ it solves the inequality

$$\int_0^\rho r^{N-1}(\phi(v') - \phi(u'))\, dr - \int_0^\rho r^{N-1} f(u)(v-u)\, dr \geqslant 0 \tag{19}$$

where we recall that ϕ is defined in (7).

Lemma 1.4. *If $u_0 \in K_0$ is a local minimum for J, then $u_0(|x|)$ is a classical solution of* (17).

Proof We will use an argument taken from [8].

Suppose $u_0 \in K_0$ is a minimum for J and consider the problem

$$(r^{N-1}\phi'(v'))' - r^{N-1}v = -r^{N-1}(f(u_0) + u_0), \quad v'(0) = 0, v(\rho) = 0. \tag{20}$$

By [3, Theorem 2.1], certainly (20) has a classical solution. As in [8, Lemma 3, Lemma 4] we deduce that the solution is unique, call it $\bar{v}$, and for any $w \in K_0$ it satisfies the following inequality

$$\int_0^\rho r^{N-1}(\phi(w') - \phi(\bar{v}'))\, dr + \int_0^\rho r^{N-1}(\bar{v} - f(u_0) - u_0)(w - \bar{v})\, dr \geqslant 0. \tag{21}$$

Now write (19) for $v = \bar{v}$ and (21) for $w = u_0$ and sum up the two inequalities. What we obtain is

$$-\int_0^\rho r^{N-1}(u_0 - \bar{v})^2\, dr \geqslant 0$$

which implies $u_0 = \bar{v}$ and then u_0 is the unique classical solution of (20). We conclude that $u_0(|x|)$ is a classical solution of (17). □

Theorem 1.5. *I_- is not empty.*

Proof As a first step, we show that

1. J is bounded below and achieves its infimum,

2. if $\rho > 0$ is sufficiently large, then $c_0 = \inf_{u \in W_\rho} J(u) < 0$.

Observe that
$$\forall u \in K_0 : \quad \|u\|_\infty \leqslant \rho.$$
As a consequence, it is easy to see that J is bounded below. Consider $(u_n)_n \in W_\rho$ a minimizing sequence. Of course we can assume $u_n \in K_0$ for any $n \geqslant 1$. By Ascoli Arzelà theorem, there exists a subsequence, relabeled $(u_n)_n$, and a continuous function u_0 such that
$$u_n \to u_0 \quad \text{uniformly in } [0, \rho]. \tag{22}$$
To prove that u_0 is in K_0, we just observe that, for any $x, y \in [0, \rho]$, with $x \neq y$, we have
$$\lim_n \frac{u_n(x) - u_n(y)}{x - y} = \frac{u_0(x) - u_0(y)}{x - y},$$
and then also u_0 has Lipschitz constant 1.
By (22) and [8, Lemma 1], $\Psi(u_0) \leqslant \liminf_n \Psi(u_n)$. Then, again by (22), we have
$$J(u_0) \leqslant c_0.$$
Now we prove our second claim. Consider the following function defined for $\rho > 2\gamma$
$$w_\rho(r) = \begin{cases} \gamma & \text{in } [0, \rho - 2\gamma] \\ \frac{\rho - r}{2} & \text{in } [\rho - 2\gamma, \rho]. \end{cases}$$
Of course $w_\rho \in K_0$. Moreover
$$\begin{aligned} J(w_\rho) \leqslant \frac{1}{2} \int_{\rho - 2\gamma}^{\rho} & (2 - \sqrt{3}) s^{N-1}\, ds \\ & - F(\gamma) \frac{(\rho - 2\gamma)^N}{N} + \frac{1}{N} \max_{0 \leqslant s \leqslant \gamma} |F(s)| (\rho^N - (\rho - 2\gamma)^N) \\ \leqslant{} & C_1 (\rho^N - (\rho - 2\gamma)^N) - \frac{F(\gamma)}{N} (\rho - 2\gamma)^N \\ \leqslant{} & C_2 \rho^{N-1} - C_3 \rho^N \end{aligned}$$
where C_1, C_2 and C_3 are suitable positive constant. The second claim is an obvious consequence of the previous chain of inequalities.

Now, suppose $\rho_0 > 0$ and $u_0 \in K_0$ are such that $I(u_0) = c_0 < 0$ and set $\bar{\xi} = u_0(0)$. The value $\bar{\xi} \in (\alpha, \beta)$. Indeed, by Lemma 1.4, $u_0(|\cdot|)$ is a classical solution of (17) and then u_0 is a local solution of (3), with $\xi = \bar{\xi}$ and $\tilde{f}$ instead of f if $\beta < +\infty$. If $\bar{\xi} \leqslant \alpha$, then $F(\bar{\xi}) \leqslant 0$ leads to an obvious contradiction to (9) computed in $r = \rho_0$. On the other hand, $\bar{\xi}$ can not be

greater than β, since in this case, by (18), the unique solution of the Cauchy problem (3) would be the constant function $u(r) = \bar{\xi}$.
By contradiction, suppose that $\bar{\xi} \notin I_-$. Since we can assume $u_0(r) > 0$ in $[0, \rho_0)$, otherwise we consider the function u_0 restricted to the interval $[0, R')$ where $R' := \inf\{r > 0 \mid u_0(r) = 0\}$, our contradiction assumption implies that $\bar{R} \in (0, \rho_0)$ (the definition of $\bar{R}$ is given in (11)).
Computing (9) for $r = \bar{R}$ and for $r = \rho_0$, we respectively have

$$(N-1)\int_0^{\bar{R}} \frac{(u')^2(s)}{s\sqrt{1-(u')^2(s)}}\, ds = F(\bar{\xi}) - F(u(\bar{R})), \tag{23}$$

$$H(u'(\rho_0)) + (N-1)\int_0^{\rho_0} \frac{(u')^2(s)}{s\sqrt{1-(u')^2(s)}}\, ds = F(\bar{\xi}). \tag{24}$$

Subtracting (23) from (24), we obtain

$$H(u'(\rho_0)) + (N-1)\int_{\bar{R}}^{\rho_0} \frac{(u')^2(s)}{s\sqrt{1-(u')^2(s)}}\, ds = F(u(\bar{R}))$$

that is $F(u(\bar{R})) > 0$.
Since $u'(r) < 0$ for any $r \in (0, \bar{R})$, we have that $u''(\bar{R}) \geqslant 0$ and then from (2) it follows that $f(u(\bar{R})) \leqslant 0$. Since f is positive in I and $0 < u(\bar{R}) < \bar{\xi} < \beta$, certainly $u(\bar{R}) \in (0, \alpha]$. From this we deduce that $F(u(\bar{R})) < 0$ and then the contradiction. □

Theorem 1.6. *I_+ and I_- are disjoint and open.*

Proof By contradiction, suppose $\bar{\xi} \in I_+ \cap I_-$. Then, since the solution of (3) with $\xi = \bar{\xi}$ is such that $u(R') = u'(R') = 0$, we can extend it by 0 in $(R', +\infty)$ and we get a compact support solution to the equation (2). Simple computations shows that this contradicts the strong maximum principle as it appears in [11, Theorem 1] (actually this theorem concerns a different class of operators, but the proof works also in our case), since $u(|x|)$ would be a compact support solution to the equation in (1).
An alternative (and simpler) proof consists in observing that, by uniqueness theorem, $u = 0$ is the unique solution of the Cauchy problem

$$\begin{cases} \left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + \frac{N-1}{r}\frac{u'}{\sqrt{1-(u')^2}} + f(u) = 0 \\ u(R') = 0, u'(R') = 0. \end{cases}$$

Finally, observe that, by continuous dependence on the initial datum, I_+ and I_- are open sets. □

By Theorem 1.3, 1.5 and 1.6, we can take $\xi \in I\backslash(I_+ \cup I_-)$. Since $\bar{R} = +\infty$, by Remark 1.1 $u(\xi, r)$ is defined in $\mathbb{R}_+$. By Lemma 1.2 $\lim_{r\to+\infty} u(\xi, r) = 0$. As a consequence $\bar{u}(x) = u(\xi, |x|)$ is a solution of (1).

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