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Electromagnetic Radiation and Motion of Real Particle

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Abstract. Relativistically covariant equation of motion for real dust particle under the action of electromagnetic radiation is derived. The particle is neutral in charge. Equation of motion is expressed in terms of particle's optical properties, standardly used in optics for stationary particles.

Key words: relativity theory, cosmic dust

1. Introduction

Relativistic equation of motion for perfectly absorbing spherical dust particle under the action of electromagnetic radiation was derived by Robertson (1937). Relativistic generalization for the case when scattered radiation is in the direction of the incident radiation – in proper reference frame of the particle – was presented by Klačka (1992, 2000).

However, real particles scatter radiation in a more complicated manner. The consequence of this reality may be a completely different orbital evolution of dust particles. Kocifaj and Klačka (1999) show this for really shaped stationary rapidly rotating particle.

Equation of motion for real (cosmic) dust particle was presented in Klačka (1994a) (and applied to real situation in Klačka 1994b). Paper by Klačka and Kocifaj (1994) expresses the equation in terms of optical properties used in optics. However, the last two papers express the result only to the first order in $\boldsymbol{v}/c$ (higher orders are neglected) where $\boldsymbol{v}$ is velocity of the particle, c is the speed of light. Such an accuracy may be sufficient in many applications in practice. However, to be sure that the equation of motion presented in the above two papers is correct, one has to derive relativistically correct equation of motion taking into account all orders in $\boldsymbol{v}/c$.

The aim of this paper is to derive relativistically covariant equation of motion for real dust particle under the action of electromagnetic radiation.

2. Proper reference frame of the particle – stationary particle

The term "stationary particle" will denote particle which does not move in a given inertial frame of reference, although we admit its rotational motion around an axis of rotation (with negligible rotational velocity). Primed quantities will denote quantities measured in the proper reference frame of the particle.

The flux density of photons scattered into an elementary solid angle $d\Omega' = \sin\vartheta' \, d\vartheta' \, d\varphi'$ is proportional to $p'(\vartheta', \varphi') \, d\Omega'$, where $p'(\vartheta', \varphi')$ is "phase function". Phase function depends on orientation of the particle with respect to the direction of the incident radiation and on the particle characteristics; angles ϑ', φ' correspond to the direction (and orientation) of travel of the scattered radiation, ϑ' is polar angle and it equals zero for the case of the travel of the ray in the orientation identical with the unit vector $\hat{\boldsymbol{S}}_i{}'$ of the incident radiation. The phase function fulfills the condition

$$\int_{4\pi} p'(\vartheta', \varphi') \, d\Omega' = 1 \; . \tag{1}$$

The momentum of the incident beam of photons which is lost in the process of its interaction with the particle is proportional to the cross-section C'_{ext} (extinction). The part proportional to C'_{abs} (absorption) is completely lost and the part proportional to $C'_{ext} - C'_{abs} = C'_{sca}$ (scattering) is again reemitted.

The momentum (per unit time) of the scattered photons into an elementary solid angle $d\Omega'$ is

$$d\boldsymbol{p}'_{sca} = \frac{1}{c} \, S' \, C'_{sca} \, p'(\vartheta', \varphi') \, \hat{\boldsymbol{K}}' \, d\Omega' \; , \tag{2}$$

where unit vector in the direction of scattering is

$$\hat{\boldsymbol{K}}' = \cos\vartheta' \, \hat{\boldsymbol{S}}_i{}' + \sin\vartheta' \, \cos\varphi' \, \hat{\boldsymbol{e}}_1{}' + \sin\vartheta' \, \sin\varphi' \, \hat{\boldsymbol{e}}_2{}' \; . \tag{3}$$

S' is the flux density of radiation energy. The system of unit vectors used on the RHS of the last equation forms an orthogonal basis. The total momentum (per unit time) of the scattered photons is

$$\boldsymbol{p}'_{sca} = \frac{1}{c} \, S' \, C'_{sca} \int_{4\pi} p'(\vartheta', \varphi') \, \hat{\boldsymbol{K}}' \, d\Omega' \; . \tag{4}$$

The momentum (per unit time) obtained by the particle due to the interaction with radiation is

$$\frac{d\,\boldsymbol{p}'}{d\,t'} = \frac{1}{c}\,S'\,\left\{C'_{ext}\,\hat{\boldsymbol{S}}_i{}' \;-\; C'_{sca}\,\int_{4\pi}\,p'(\vartheta', \varphi')\,\hat{\boldsymbol{K}}'\,d\Omega'\right\}\;. \tag{5}$$

As for the energy, we suppose that it is conserved: the energy (per unit time) of the incoming radiation E'_i, equals to the energy (per unit time) of the outgoing radiation (after interaction with the particle) E'_o. We will use the fact that time $t' = \tau$, where τ is proper time.

For the sake of brevity, we will use "effective factors" Q'_{xxx} instead of effective cross-sections C'_{xxx}: $C'_{xxx} = Q'_{xxx}\,A'$, where A' is geometrical cross-section of a sphere of volume equal to the volume of the particle. Equation (5) can be rewritten to the form

$$\begin{aligned}\frac{d\,\boldsymbol{p}'}{d\,\tau} \;=\; \frac{1}{c}\,S'\,A'\;\Big\{&[Q'_{ext} \;-\; <\cos\vartheta'>\;Q'_{sca}]\;\hat{\boldsymbol{S}}_i{}' \;+\\ &[-\;<\sin\vartheta'\;\cos\varphi'>\;Q'_{sca}]\;\hat{\boldsymbol{e}_1}{}' \;+\;[-\;<\sin\vartheta'\;\sin\varphi'>\;Q'_{sca}]\;\hat{\boldsymbol{e}_2}{}'\Big\}\;,\end{aligned} \tag{6}$$

or, in a short form

$$\frac{d\,\boldsymbol{p}'}{d\,\tau} = \frac{1}{c}\,S'\,A'\;\left\{Q'_R\,\hat{\boldsymbol{S}}_i{}' \;+\;Q'_1\,\hat{\boldsymbol{e}_1}{}' \;+\;Q'_2\,\hat{\boldsymbol{e}_2}{}'\right\}\;. \tag{7}$$

2.1. Summary of the important equations

Using the text concerning energy below Eq. (5) and the last Eq. (7), we may describe the total process of interaction in the form of the following equations (energies and momenta per unit time):

$$\begin{aligned}E'_o \;&=\; E'_i = A'\,S'\;,\\ \boldsymbol{p}'_o \;&=\;(1\;-\;Q'_R)\,\boldsymbol{p}'_i\;-\;(Q'_1\,\hat{\boldsymbol{e}_1}{}'\;+\;Q'_2\,\hat{\boldsymbol{e}_2}{}')\,E'_o/c\;,\\ \boldsymbol{p}'_i\;&=\;(E'_i/c)\,\hat{\boldsymbol{S}}_i{}'\;,\end{aligned} \tag{8}$$

The index "i" represents the incoming radiation, beam of photons, the index "o" represents the outgoing radiation.

The changes of energy and momentum of the particle due to the interaction with electromagnetic radiation are

$$\begin{aligned}\frac{d\,E'}{d\,\tau}\;&=\;E'_i\;-\;E'_o = 0\;,\\ \frac{d\,\boldsymbol{p}'}{d\,\tau}\;&=\;\boldsymbol{p}'_i\;-\;\boldsymbol{p}'_o\;.\end{aligned} \tag{9}$$

As for the condition for energy in our Eq. (8), it is equivalent to Eq. (121) in Klačka (1992), as for the condition for momentum $\boldsymbol{p}'_o$ in our Eq. (8), it is a generalization of Eq. (122) in Klačka (1992).

3. Stationary frame of reference

By the term “stationary frame of reference” we mean a frame of reference in which particle moves with a velocity vector $\boldsymbol{v} = \boldsymbol{v}(t)$. The physical quantities measured in the stationary frame of reference will be denoted by unprimed symbols.

Our aim is to derive equation of motion for the particle in the stationary frame of reference. We will use the fact that we know this equation in the proper frame of reference – see Eqs. (8) and (9). We have to use Lorentz transformation for the purpose of making transformation from proper frame of reference to stationary frame of reference.

If we have a four-vector $A^{\mu} = (A^0, \boldsymbol{A})$, where A^0 is its time component and $\boldsymbol{A}$ is its spatial component, generalized special Lorentz transformation yields

$$\begin{aligned} A^{0'} &= \gamma \left(A^0 - \boldsymbol{v} \cdot \boldsymbol{A}/c \right) , \\ \boldsymbol{A}' &= \boldsymbol{A} + [(\gamma - 1)\, \boldsymbol{v} \cdot \boldsymbol{A}/\boldsymbol{v}^2 - \gamma\, A^0/c]\, \boldsymbol{v} . \end{aligned} \tag{10}$$

The inverse generalized special Lorentz transformation is

$$\begin{aligned} A^{0} &= \gamma \left(A^{0'} + \boldsymbol{v} \cdot \boldsymbol{A}'/c \right) , \\ \boldsymbol{A} &= \boldsymbol{A}' + [(\gamma - 1)\, \boldsymbol{v} \cdot \boldsymbol{A}'/\boldsymbol{v}^2 + \gamma\, A^{0'}/c]\, \boldsymbol{v} . \end{aligned} \tag{11}$$

The γ factor is given by the well-known relation

$$\gamma = 1/\sqrt{1 - \boldsymbol{v}^2/c^2} . \tag{12}$$

As for four-vectors we immediately introduce four-momentum:

$$p^{\mu} = (p^0, \boldsymbol{p}) \equiv (E/c, \boldsymbol{p}) . \tag{13}$$

3.1. Incident radiation

Applying Eqs. (11) and (13) to quantity $(E_i'/c, \boldsymbol{p}_i')$ (four-momentum per unit time – proper time is a scalar quantity) and taking into account also Eqs. (8), we can write

$$\begin{aligned} E_i &= E_i'\, \gamma \left(1 + \boldsymbol{v} \cdot \hat{\boldsymbol{S}}_i{}'/c \right) , \\ \boldsymbol{p}_i &= \frac{E_i'}{c} \left\{ \hat{\boldsymbol{S}}_i{}' + \left[(\gamma - 1)\, \boldsymbol{v} \cdot \hat{\boldsymbol{S}}_i{}'/\boldsymbol{v}^2 + \gamma/c \right] \boldsymbol{v} \right\} . \end{aligned} \tag{14}$$

Using the fact that $p^{\mu} = (h\, \nu, h\, \nu\, \hat{\boldsymbol{S}}_i)$ for photon, Lorentz transformation yields

$$\begin{aligned} \nu' &= \nu\, w , \\ \hat{\boldsymbol{S}}_i{}' &= \frac{1}{w} \left\{ \hat{\boldsymbol{S}}_i + \left[(\gamma - 1)\, \boldsymbol{v} \cdot \hat{\boldsymbol{S}}_i/\boldsymbol{v}^2 - \gamma/c \right] \boldsymbol{v} \right\} , \end{aligned} \tag{15}$$

where abbreviation

$$w \equiv \gamma\,(1\ -\ \boldsymbol{v}\cdot\hat{\boldsymbol{S}}_i/c) \tag{16}$$

is used.

Inserting the second of Eqs. (15) into Eqs. (14), one obtains

$$\begin{aligned} E_i\ &=\ (1/w)\ E_i'\ , \\ \boldsymbol{p}_i\ &=\ (1/w)\ (E_i'/c)\ \hat{\boldsymbol{S}}_i\ . \end{aligned} \tag{17}$$

We have four-vector $p_i^\mu = (E_i/c, \boldsymbol{p}_i) = (1, \hat{\boldsymbol{S}}_i)E_i/c = (1/w, \hat{\boldsymbol{S}}_i/w)w\ E_i/c \equiv b_i^\mu\ w\ E_i/c$.

3.2. Outgoing radiation

The situation is analogous to the situation in the preceding subsection. It is only a little more algebraically complicated, since radiation may spread out also in directions given by unit vectors $\hat{\boldsymbol{e}_1}$, $\hat{\boldsymbol{e}_2}$. The relations between $\hat{\boldsymbol{e}_1}'$ and $\hat{\boldsymbol{e}_1}$, $\hat{\boldsymbol{e}_2}'$ and $\hat{\boldsymbol{e}_2}$, are analogous to that presented by the second of Eqs. (15):

$$\hat{\boldsymbol{e}_j}' = \frac{1}{w_j}\ \left\{\hat{\boldsymbol{e}_j}\ +\ \left[(\gamma\ -\ 1)\ \boldsymbol{v}\cdot\hat{\boldsymbol{e}_j}/\boldsymbol{v}^2\ -\ \gamma/c\right]\ \boldsymbol{v}\right\}\ ,\ \ j=1,2\ , \tag{18}$$

where

$$w_j \equiv \gamma\,(1\ -\ \boldsymbol{v}\cdot\hat{\boldsymbol{e}_j}/c)\ ,\ \ j=1,2\ . \tag{19}$$

Using Eqs. (11) and (13) to quantity $(E_o'/c, \boldsymbol{p}_o')$ (four-momentum per unit time – proper time is a scalar quantity), we can write

$$\begin{aligned} E_o\ &=\ \gamma\,(E_o'\ +\ \boldsymbol{v}\cdot\boldsymbol{p}_o')\ , \\ \boldsymbol{p}_o\ &=\ \boldsymbol{p}_o'\ +\ \left[(\gamma\ -\ 1)\ \boldsymbol{v}\cdot\boldsymbol{p}_o'/\boldsymbol{v}^2\ +\ \gamma\ \frac{E_o'}{c^2}\right]\ \boldsymbol{v}\ . \end{aligned} \tag{20}$$

Using also $\boldsymbol{p}_i' = E_i'\ \hat{\boldsymbol{S}}_i{}'/c$, Eqs. (8), (18), (20) and the second of Eqs. (15) yield

$$\begin{aligned} E_o\ &=\ Q_R'\ w\ E_i\ \gamma\ +\ (1\ -\ Q_R')\ E_i\ + \\ &\quad w\ E_i\ (Q_1'\ +\ Q_2')\ \gamma\ -\ w\ E_i\ (Q_1'/w_1\ +\ Q_2'/w_2)\ , \\ \boldsymbol{p}_o\ &=\ (1\ -\ Q_R')\ \frac{E_i}{c}\ \hat{\boldsymbol{S}}_i\ +\ Q_R'\ \frac{w\ E_i}{c^2}\ \gamma\ \boldsymbol{v}\ -\ \sum_{j=1}^{2} Q_j'\ \frac{w\ E_i}{c^2}\ (c\ \hat{\boldsymbol{e}}_j/w_j\ -\ \gamma\ \boldsymbol{v})\ . \end{aligned} \tag{21}$$

3.3. Equation of motion

In analogy with Eqs. (9), we have for the changes of energy and momentum of the particle due to the interaction with electromagnetic radiation

$$\begin{aligned} \frac{d\ E}{d\ \tau}\ &=\ E_i\ -\ E_o\ , \\ \frac{d\ \boldsymbol{p}}{d\ \tau}\ &=\ \boldsymbol{p}_i\ -\ \boldsymbol{p}_o\ . \end{aligned} \tag{22}$$

Putting Eqs. (21) into Eqs. (22), using also $\boldsymbol{p}_i = (E_i/c)\ \hat{\boldsymbol{S}}_i$, one easily obtains

$$\frac{d\ E}{d\ \tau} = Q'_R\ (E_i\ -\ w\ E_i\ \gamma)\ +\ \sum_{j=1}^{2} Q'_j\ w\ E_i\ (1/w_j\ -\ \gamma)\ ,$$

$$\frac{d\ \boldsymbol{p}}{d\ \tau} = Q'_R\ \left\{ \frac{E_i}{c}\ \hat{\boldsymbol{S}}_i\ -\ \frac{w\ E_i}{c^2}\ \gamma \boldsymbol{v} \right\}\ +\ \sum_{j=1}^{2} Q'_j\ \frac{w\ E_i}{c^2}\ (c\ \hat{\boldsymbol{e}}_j/w_j\ -\ \gamma \boldsymbol{v})\ . \qquad (23)$$

Equations (23) may be rewritten in terms of four-vectors:

$$\frac{d\ p^\mu}{d\ \tau} = Q'_R\ \left(p_i^\mu\ -\ \frac{w\ E_i}{c^2}\ u^\mu \right)\ +\ \sum_{j=1}^{2} Q'_j\ \frac{w\ E_i}{c^2}\ \left(c\ b_j^\mu\ -\ u^\mu \right)\ , \qquad (24)$$

where p^μ is four-vector of the particle of mass m

$$p^\mu = m\ u^\mu\ , \qquad (25)$$

four-vector of the world-velocity of the particle is

$$u^\mu = (\gamma\ c, \gamma\ \boldsymbol{v})\ . \qquad (26)$$

We have also found two new four-vectors

$$b_j^\mu = (1/w_j, \hat{\boldsymbol{e}}_j/w_j)\ ,\ \ j = 1, 2\ . \qquad (27)$$

It can be easily verified that:

i) the quantity $w\ E_i$ is a scalar quantity – see first of Eqs. (17);

ii) Eq. (24) reduces to Eq. (7) and to the first of Eqs. (9) for the case of proper inertial frame of reference of the particle;

iii) Eq. (24) yields $d\ m/d\ \tau = 0$.

4. Conclusion

We have derived equation of motion for real dust particle under the action of electromagnetic radiation. It is supposed that equation of motion is represented by Eqs. (8) and (9) in the proper frame of reference of the particle. The final covariant form is represented by Eq. (24), or, using the relations $E_i = w\ S\ A'$ (see Eq. (38) in Klačka 1992) and $\boldsymbol{p}_i = (E_i/c)\ \hat{\boldsymbol{S}}_i$

$$\frac{d\ p^\mu}{d\ \tau} = \frac{w^2\ S\ A'}{c^2}\ \left\{ Q'_R\ (c\ b_i^\mu\ -\ u^\mu)\ +\ \sum_{j=1}^{2} Q'_j\ \left(c\ b_j^\mu\ -\ u^\mu \right) \right\} \qquad (28)$$

(four-vector b_i^μ is defined below Eq. (17)).

Within the accuracy to the first order in $\boldsymbol{v}/c$, Eq. (28) yields

$$\frac{d\ \boldsymbol{v}}{d\ t} = \frac{S\ A'}{m\ c}\ \left\{ Q'_R\ \left[\left(1\ -\ \boldsymbol{v} \cdot \hat{\boldsymbol{S}}_i/c \right)\ \hat{\boldsymbol{S}}_i\ -\ \boldsymbol{v}/c \right]\ + \right.$$
$$\left. \sum_{j=1}^{2} Q'_j\ \left[\left(1\ -\ 2\ \boldsymbol{v} \cdot \hat{\boldsymbol{S}}_i/c\ +\ \boldsymbol{v} \cdot \hat{\boldsymbol{e}}_j/c \right)\ \hat{\boldsymbol{e}}_j\ -\ \boldsymbol{v}/c \right] \right\}\ . \qquad (29)$$

As for practical applications, the terms v/c standing at Q'_j are negligible for majority of real particles. (We want to stress that values of $Q'-$coefficients depend on particle's orientation with respect to the incident radiation – their values are time dependent.)

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