A dynamical time operator in Dirac's relativistic quantum mechanics[a]

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Abstract

A self-adjoint dynamical time operator is introduced in Dirac's relativistic formulation of quantum mechanics and shown to satisfy a commutation relation with the Hamiltonian analogous to that of the position and momentum operators. The ensuing time-energy uncertainty relation involves the uncertainty in the instant of time when the wave packet passes a particular spatial position and the energy uncertainty associated with the wave packet at the same time, as envisaged originally by Bohr. The instantaneous rate of change of the position expectation value with respect to the simultaneous expectation value of the dynamical time operator is shown to be the phase velocity, in agreement with de Broglie's hypothesis of a particle associated wave whose phase velocity is larger than c. Thus, these two elements of the original basis and interpretation of quantum mechanics are integrated into its formal mathematical structure. Pauli's objection is shown to be resolved or circumvented. Possible relevance to current developments in interference in time, in Zitterbewegung like effects in spintronics, grapheme and superconducting systems and in cosmology is noted

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I. INTRODUCTION

"One must be prepared to follow up the consequences of theory, and feel that one just has to accept the consequences no matter where they lead" P. A. M. Dirac[b].

In the formulation of quantum mechanics (QM) time appears as a parameter, not as a dynamical variable. It is a c-number, following Dirac's designation[1]. Thus QM fails to treat time and space coordinates on the almost equal footing accorded by the special theory of relativity, as it does with momentum and energy. As an explanation, it has interestingly been asserted that "time always arises in quantum mechanics as an externally defined classical parameter from the interaction with a classical environment". Indeed it is shown that the time independent Schr¨odinger or Dirac equations describing system and environment, give rise to the corresponding time dependent equations in a disentangling approximation where "the motion of the environment provides a time derivative which monitors the development of the quantum system". Consequently "time enters quantum mechanics only when an external force on the quantum system is considered classically"[2],[3].

The above then still leaves open the question of a "time operator" T satisfying a commutation relation

$$[T, H] = i\hbar, \tag{1}$$

as the one satisfied by the position and momentum operators. Its existence has had to deal with the fundamental objection pointed by Pauli, that the finite lower bound of the energy spectrum precludes mathematically the existence of a self adjoint operator ("i.e., as a function of q and p") canonically conjugate to the Hamiltonian.[4]

As a consequence, the role of time in quantum mechanics as well as the existence of time operators and the diverse formulations and interpretations of a time-energy uncertainty relation have been the subject of extensive investigations since[5]. To be noted in particular is the proposal of Aharonov and Bohm[6], in the framework of the non relativistic Schrödinger equation for the free particle Hamiltonian $H = p^2/2m$, of a maximally symmetric time operator $(1/2m)\{(1/p)x + x(1/p)\}$ associated with a "time-of-arrival" concept and needing the acceptance of Positive Operator Valued Measures (POVM), an extension of the standard von Neumann definition of "observables"[6].

Also to be noted is that, within the time quantities considered, such as parametric (clock) time, tunneling times, decay times, dwell times, delay times, arrival times or jump times, one finds both instantaneous values and intervals. To quote the introduction in Ref. 5, "In fact, the standard recipe to link the observables and the formalism does not seem to apply, at least in an obvious manner, to time observables".

In previous work[7] however, it was shown that Pauli's objection is overcome formally by enlarging the Hilbert space, obviously by continuing the energy spectrum to negative energies but also, equivalently, by introducing a spin-like quantum number so as to associate two states to each positive energy value. In both of these enlarged spaces, a unitary energy displacement operator can be introduced whose generator is a "time operator" that satisfies the above commutation relation. Its expectation value is equal to plus or minus the evolution parameter t, corresponding to the negative energy extension and to the positive energy extension, respectively.

In the present work, the relativistic free particle Dirac Hamiltonian is the starting point, instead of the non relativistic one. It suggests a particular form for a "dynamical time operator", to be denoted $T(t)$, i.e., dependent on the parametric time and introducing a new constant that would be a characteristic of the system. In Section II, the corresponding $[T, H]$ commutator is evaluated and the time evolution of the expectation value is derived from the dynamical postulate of QM. In Section III, the Heisenberg picture is used to establish its dependence on the parametric time. On the basis of its eigenvalues and eigenvectors derived in Appendix B, Section IV contains a clarification of the corresponding time-energy uncertainty relation. Section V addresses Pauli's objection. The behavior with respect to the discrete symmetries, parity, time reversal and charge conjugation, is considered next (Section VI), where the role of the new constant is clarified. Its value is suggested by the interpretation of the ensuing Zitterbewegung behavior, analysed in Section VII. Finally, Section VIII contains conclusions and possible relevance to current research areas.

II. A "TIME OPERATOR" IN RQM

In analogy with the free particle Dirac Hamiltonian[1],[4],[8],[9]

$$H = c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m_0 c^2 \tag{2}$$

where $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ and β are the Dirac matrices, the following expression for a self-adjoint time operator is considered:

$$T = \boldsymbol{\alpha} \cdot \boldsymbol{r}/c + \beta\tau_0, \tag{3}$$

where τ_0 is a real constant with dimensions of time, and in principle a property of the system. From the commutation relation between position and momentum $[x_i, p_j] = i\hbar\delta_{ij}$and the properties of the Dirac matrices, it follows:

$$[T, H] = i\hbar\{3I + 4\boldsymbol{s} \cdot \boldsymbol{l}/\hbar^2\} + 2\beta\{\tau_0(c\boldsymbol{\alpha} \cdot \boldsymbol{p}) - m_0c^2(\boldsymbol{\alpha} \cdot \boldsymbol{r}/c)\} \tag{4}$$

where I, $\boldsymbol{s}$ and $\boldsymbol{l}$ are the identity, spin and orbital angular momentum operators, respectively. Introducing the "spin-orbit" operator $K = \beta(2\boldsymbol{s} \cdot \boldsymbol{l}/\hbar^2 + 1)$, that commutes with $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$ and H , and is therefore a constant of motion[8],[9], one has

$$[T, H] = i\hbar\{I + 2\beta K\} + 2\beta\{\tau_0(c\boldsymbol{\alpha} \cdot \boldsymbol{p}) - m_0c^2(\boldsymbol{\alpha} \cdot \boldsymbol{r}/c)\} \tag{5}$$

or, equivalently

$$[T, H] = i\hbar\{I + 2\beta K\} + 2\beta\{\tau_0 H - m_0c^2T\}. \tag{6}$$

As such, this commutator is not entirely analogous to the position momentum commutation relation, but does contain a constant term whose expectation value is state independent, namely $i\hbar I$.

¿From the dynamical postulate of QM, the time evolution of the expectation value of the time operator is then given by:

$$d/dt < T >= (1/i\hbar) < [T, H] >=< \{I + 2\beta K\} + 2\beta\{\tau_0 H - m_0c^2T\} > . \tag{7}$$

In the presence of a potential$V(r)$ that depends only on position and thus commutes with the time operator, Eq. (5) is still valid, whereas the form (6) requires to substitute $H - V(r)$ for H in the last term (Appendix C examines the case of an electromagnetic field).

As a consequence of the other terms, the time evolution of the time operator includes the peculiar (Zitterbewegung) time dependence of the position operator in RQM, as can be seen more clearly in the Heisenberg picture.

III. THE FREE PARTICLE CASE IN THE HEISENBERG PICTURE

In the case of a free particle, where $\boldsymbol{p}$ and H are constants of motion, one has in the Heisenberg picture[1],[4],[8],[9]

$$\boldsymbol{\alpha}(t) = c\boldsymbol{p}/H + \{\boldsymbol{\alpha}(0) - c\boldsymbol{p}/H\}exp(-2iHt/\hbar), \tag{8}$$

$$\beta(t) = m_0c^2/H + \{\beta(0) - m_0c^2/H\}exp(-2iHt/\hbar) \tag{9}$$

and

$$\boldsymbol{r}(t) = \boldsymbol{r}(0) + (c^2\boldsymbol{p}/H)t + i(c\hbar/2H)\{\boldsymbol{\alpha}(0) - c\boldsymbol{p}/H\}[exp(-2iHt/\hbar) - 1] \tag{10}$$

As $(c^2p/H) = dE/dp$ represents the group velocity v_{gp}, $\boldsymbol{r}(t)$ is shown to follow the classical motion (Ehrenfest's theorem), albeit accompanied by oscillating terms (Zitterbewegung) that nevertheless vanish for only positive energy or only negative energy wave packets. Using Eqs. (8), (9) and (10), it follows that (see Appendix B for the full expression):

$$T(t) = (c\boldsymbol{p}/H).\{\boldsymbol{r}(0)/c^2 + (c\boldsymbol{p}/H)t\} + (m_0c^2/H)\tau_0 + oscillatingterms. \tag{11}$$

Leaving aside these oscillating terms and introducing explicitly the group velocity v_{gp} , one has, setting $\boldsymbol{r}(0) = 0$ for simplicity:

$$T(t) = (\boldsymbol{v}_{gp}/c)^2t + (m_0c^2/H)\tau_0, \tag{12}$$

$$\boldsymbol{r}(t) = \boldsymbol{v}_{gp}t. \tag{13}$$

It is seen that, although proportional to t, in general $T(t) < t$. Only in the limit $m_0c^2 = 0$, $\boldsymbol{v}_{gp}$ is equal to $\boldsymbol{c}$, $T(t) = t$ and $\boldsymbol{r}(t) = \boldsymbol{c}t = \boldsymbol{c}T(t)$. Non relativistic and ultra relativistic limits of the time operator are shown in Appendix C.

¿From Eqs. (10) and (11), it also follows that:

$$d\boldsymbol{r}(t)/dT(t) = \boldsymbol{v}_{gp}dt/(v_{gp}/c)^2dt = (\boldsymbol{v}_{gp}/v_{gp})(c/v_{gp}) = \boldsymbol{v}_{ph} \tag{14}$$

is the phase velocity $v_{ph} = E/p$, which is collinear with v_{gp}, and such that $v_{ph}v_{gp} = c^2$. Consequently $v_{ph} > c$. This agrees with the property that de Broglie derives for the wave he associates to a material particle[10]. And indeed it is shown in de Broglie's thesis that the phase velocity of the proposed wave satisfies $v_{ph}v_{gp} = c^2$, where v_{gp} is the speed of the "mobile" that is associated to the transport of energy, i.e., the group velocity of a superposition of waves with close-by frequencies.

IV. THE TIME-ENERGY UNCERTAINTY RELATION[11]

A time-energy uncertainty relation can now be derived in the usual way from the Schawrtz inequality, applied to the uncertainties $(\Delta T)^2 =< T^2 > - < T >^2$, and $(\Delta H)^2 =< H^2 > - < H >^2$ of the self-adjoint operators$T$ and H , namely:

$$(\Delta T)^2(\Delta H)^2 \geq (1/4)| < [T, H] > |^2 \geq (\hbar^2/4)| < (I + 2\beta K) > |^2 \tag{15}$$

As shown in Appendix A, in entire analogy with the eigenvectors and eigenvalues of the free particle relativistic Dirac Hamiltonian, the eigenvectors of the time operator are of the form

$$|\tau >= u_r|\boldsymbol{r} > \tag{16}$$

where $|\boldsymbol{r} >$ is the eigenvector of the position operator $\boldsymbol{r}$ and u_r is a four component spinor independent of the linear momentum $\boldsymbol{p}$. The corresponding doubly degenerate eigenvalues are

$$\tau = \pm\ \tau_r = \pm\ [(r/c)^2 + \tau_0^2]^{1/2} \tag{17}$$

Thus, a wave packet centered about $\tau_R = [(R/c)^2 + \tau_0^2]^{1/2}$ at a time t and of width ΔT is actually a wave packet centered at a point R of width Δr. Its Fourier transform yields a wave packet in momentum space of width Δp centered at a value P, which in turn represents a wave packet of width ΔE about $Ep = +[(pc)^2 + (m_0c^2)^2]^{1/2}$. Thus the position momentum uncertainty relation $(\Delta r)_t(\Delta p)_t \geq \hbar/2$ derives into a time energy uncertainty relation $(\Delta T)_t(\Delta E)_t \geq \hbar/2$, in agreement with the commutation relation, Eq.(6). To be emphasized is that the above expectation values and uncertainties correspond to instantaneous evaluations at time t, in agreement with Bohr's conception, as quoted by Pauli[4].

The dynamical time operator here proposed is the appropriate one to define the time of passage or arrival time at a specific point. In contrast, as pointed earlier, in many of the interpretations of a time-energy uncertainty relation the Δt corresponds to a time interval, not to an instantaneous value of the uncertainty. Dwell times, tunneling times, i.e., time intervals, should be expressed as differences of average values of the time operator taken at two different points of the trajectory, and consequently related to parametric time differences (Appendix C). Their time uncertainties would need to combine the instantaneous uncertainties of the end points. On the other hand, the often quoted uncertainty relation between line width and lifetimes of unstable states derives from the dynamics generated by the Schrödinger equation[12],[13].

V. WHAT ABOUT PAULI'S OBJECTION?

Considering that the position operator $\boldsymbol{r}$ in momentum space is the generator of momentum translations, that is,

$$exp(i\delta \boldsymbol{p} \cdot \boldsymbol{r}/\hbar)|\boldsymbol{p}>=|\boldsymbol{p}+\delta \boldsymbol{p}>, \tag{18}$$

the unitary operator

$$U(\epsilon) = (i\epsilon T/\hbar) = expi\epsilon\{\boldsymbol{\alpha} \cdot \boldsymbol{r}/c + \beta\tau_0\}/\hbar = (1 + i\epsilon\boldsymbol{\alpha} \cdot \boldsymbol{r}/c\hbar + ...)expi\beta(\epsilon\tau_0/\hbar) \tag{19}$$

where ϵ is a (positive or negative) infinitesimal energy, generates both a change in phase by the amount $\beta(\epsilon\tau_0/\hbar)$ and a momentum displacement by the amount $\boldsymbol{\delta p} = (\epsilon/c)\boldsymbol{\alpha}$ in the direction of the instantaneous velocity $\boldsymbol{c\alpha} = \boldsymbol{d}\boldsymbol{r}/dt$. Averaged over a wave packet, this can be seen as a "boost", that is, a change to a reference frame where the corresponding energy is shifted by $\delta E = (\epsilon/c)\boldsymbol{\alpha} \cdot \boldsymbol{v}_{gp}$where $\boldsymbol{v}_{gp}$ is the group velocity $c^2\boldsymbol{p}/H$.

Repeated applications can generate finite displacements over all the momentum space, and consequently finite energy shifts, without however leaving the positive (or negative) energy spectrum as the solutions for positive and negative energy transform separately under proper Lorentz transformations. Energy goes through a minimum (maximum) as the momentum goes through zero, remaining either above (or below) the $2m_0c^2$ energy gap. Both the positive and negative spectra eigenvalues $\pm[(pc)^2 + (m_0c^2)^2]^{1/2}$ of the Dirac Hamiltonian are degenerate with respect to $\boldsymbol{p}$ and $-\boldsymbol{p}$, providing the "pseudo spin" extension $|E; \sigma>$, with $\sigma = \pm 1$ being the sign of the momentum, needed for the formal introduction of a time operator as shown in Ref. 7. In this way Pauli's objection is resolved or circumvented.

VI. THE DYNAMICAL TIME OPERATOR AND THE DISCRETE SYMMETRIES SPACE INVERSION, CHARGE CONJUGATION AND TIME REVERSAL [8],[9]

a) **Space inversion (Parity $\boldsymbol{P}$)**: denoting by $<>_P$ the expectation value in the parity reversed state, one has:

$$<\boldsymbol{r}>_{\boldsymbol{P}}=-<\boldsymbol{r}>;<\boldsymbol{p}>_P=-<\boldsymbol{p}>;<\boldsymbol{\alpha}>_P=-<\boldsymbol{\alpha}>;<\beta>_{\boldsymbol{P}}=-<\beta> \quad (20)$$

Thus $[T,\boldsymbol{P}]=0$ and

$$<T>_{\boldsymbol{P}}=<T>\,. \quad (21)$$

b) **Charge conjugation**: Under charge conjugation $\boldsymbol{C}$, one has

$$<\boldsymbol{r}>_{\boldsymbol{C}}=<\boldsymbol{r}>;<\boldsymbol{p}>_{\boldsymbol{C}}=-<\boldsymbol{p}>;<\boldsymbol{\alpha}>_C=<\boldsymbol{\alpha}>;<\beta>_{\boldsymbol{C}}=-<\beta>\,(22) \quad (22)$$

Then:

$$<T>_{\boldsymbol{C}}=<\boldsymbol{\alpha}\cdot\boldsymbol{r}/c>_{\boldsymbol{C}}+<\beta\tau_0>_{\boldsymbol{C}}=<T>-2<\beta>\tau_0. \quad (23)$$

The expectation value in the charge conjugate state will only be equal to the expectation value in the original state if τ_0 is zero. Otherwise $[T,\boldsymbol{C}]\neq 0$.

c) **Time reversal**: Under time reversal $\boldsymbol{T}$ one has:

$$<r>_{\boldsymbol{T}}=<r>;<p>_{\boldsymbol{T}}=-<p>;<\alpha>_{\boldsymbol{T}}=-<\alpha>;<\beta>_{\boldsymbol{T}}=<\beta> \quad (24)$$

and therefore

$$<T>_{\boldsymbol{T}}=-<T>+2<\beta>\tau_0. \quad (25)$$

However it is seen that under the combined $\boldsymbol{C}$ and $\boldsymbol{T}$ symmetries one has:

$$<T>_{\boldsymbol{CT}}=-<\{\boldsymbol{\alpha}\cdot\boldsymbol{r}/c+\beta\tau_0\}>=-<T>, \quad (26)$$

or, as parity leaves invariant the dynamic time operator,

$$<T>_{\boldsymbol{CPT}}=-<T> \quad (27)$$

The plausible expectation that the dynamical time operator would reverse sign under time reversal occurs only if τ_0 is zero. On the other hand, if τ_0 is different from zero, then charge conjugation is needed in addition to produce the change in sign. This however brings it into agreement with Feyman's proposal of the equivalence of the negative energy electron states flowing backwards in time to positive energy positron states flowing forward in time, $< H(e) >_{\boldsymbol{CPT}} = - < H(-e) >$ when charge is taken into account. It is also in agreement with the positive energy extension of Ref. 7, necessary for the introduction of a time operator, where the needed degeneracy is provided by the $\boldsymbol{p}$ and $-\boldsymbol{p}$ degeneracy of the energy spectrum.

VII. ZITTERBEWEGUNG

Dirac's equation yields a position vector $\boldsymbol{r}(t)$ consisting of a term that follows the classical evolution to which is superimposed an oscillatory motion, the Zitterbewegung ("trembling motion"). This motion is characterized in the low energy range (see Appendix C) by an amplitude $\hbar/2m_0c$, the Compton wavelength divided by 4π and a frequency $2m_0c^2/\hbar$, thus an oscillation period $\hbar/2m_0c^2$. It is further demonstrated[8],[9] that this Zitterbewegung is not present in wave packets constructed with purely positive (negative) energy states. Alternatively[14],[15] it can also be shown that no finite space width wave packet of positive (negative) mean energy can be constructed without participation of negative (positive) energy states. Indeed, the narrowest packet that can be built of positive energy states alone has a width of the order $\hbar/m_0c$. Attempt to confine the packet within the spatial range $\hbar/2m_0c$ makes this participation considerable (to construct a δ function, negative and positive energy states must contribute with equal weight), this being interpreted as the onset of particle antiparticle pair creation. A similar situation arises with the system time operator. Its spectrum spans all positive and negative τ values except for a gap from τ_0 to $-\tau_0$. In this representation a wave packet of finite width with mean positive system time cannot be constructed without participation of negative system time states, and cannot be confined within a time span $2\tau_0$ without a considerable participation of these, that is, without the creation of particle antiparticle pairs. This leads to identify τ_0 with the Zitterbewegung period $\hbar/2m_0c^2$. A unified *spacetime "Compton scale"* $\hbar/2m_0c$ and $\hbar/2m_0c^2$ is thus established, that sets confinement limits in space and system time below which pair production becomes significantly present[16].

Zittervewegung occurs "naturally" in this formulation, as a result of the mixing of positive and negative energy eigenstates of the Dirac Hamiltonian. Its interpretation in the equation of motion $\boldsymbol{r}(t)$ is still subject to discussion. It is known that it can be eliminated by a redefinition of the position operator. Such is the so called Newton Wigner position operator, based on the Foldy-Wouthuysen representation, whose time derivative is just c^2p/H (the group velocity) instead of c, however at the price of an acausal propagation of initially localized particles. This is a common problem with all position operators commuting with the sign of energy[9].

VIII. CONCLUSIONS

Consideration of the free particle Dirac Hamiltonian leads to define a particular dynamical self adjoint "system time operator" (i.e. based on a dynamic observable, namely the position), and dependent on the parameter t . It is shown to satisfy a commutation relation with the Hamiltonian analogous to the one postulated for the position and momentum operators in the sense of containing a constant part independent of the particle state. The corresponding time energy uncertainty relation – now formally dependent on the position-momentum uncertainty relation - involves simultaneous definite time expectation values as envisaged originally by Bohr, i.e., the relation between the uncertainty in the instant of time when the wave packet passes a particular spatial position with the also instantaneous energy uncertainty associated with the wave packet. The daring de Broglie's hypothesis[10] of a particle associated wave whose phase velocity is larger than c is also derived as the instantaneous rate of change of the position expectation value with respect to the simultaneous expectation value of the time operator. Thus, these two elements of the original basis and interpretation of quantum mechanics are integrated into its formal mathematical structure.

The eigenvalue spectrum of the time operator contains a gap between positive and negative values, similar to the gap occurring in the energy spectrum. Its presence is needed to insure that charge conjugation has to be implemented in addition of time reversal to connect positive and negative time values, in agreement with Feyman's interpretation of the negative energy states. Associating this gap to the period of the Zitterbewgung allows setting a unified spacetime "Compton scale" that limits the width in space and time of the corresponding wave packets before the generation of particle antiparticle pairs occurs.

The introduction of a dynamical system time operator does not question nor invalidate the presence of the time parameter in the evolution postulate of quantum mechanics, whose validity has been justified extensively in experiments and applications (and whose presence may be explained by the interaction with the environment, as quoted in the Introduction). On the other hand, it may have relevance to current areas of research, such as:

i) Recently[22] it has been argued that the interpretation of a single particle experiment double slit interference in time[21] (investigation that "makes possible interferometry on the attosecond time scale") cannot be given in the non relativistic Schrödinger equation framework, as it requires "a wave function $\Psi(x,t) =< x|\Psi(t) >$ where x and t are the spectra of self-adjoint operators, to provide the possibility of coherence in time, and therefore, interference phenomena". This assertion is based correctly on the fact that the evolution of the state vector is given by a (external) parameter t and not by the eigenvalue of a self-adjoint operator canonically conjugate to the Hamiltonian, subject to the well known objections[4].

In the present case, the role of the ("external"[2],[3]) evolution parameter t (Schrödinger picture) is maintained but an additional "observable" represented by a self-adjoint operator T is introduced, with eigenvectors $|\tau\rangle$). Its spin-like eigenvector spectrum allows for the construction of an extended Hilbert space ($|\boldsymbol{r}\rangle \otimes |\tau\rangle$). As it commutes with the position operator, a "four dimensional" representation of the state vector $|\Psi(t)\rangle$ follows, namely $\langle \boldsymbol{r}, \tau\, |\Psi(t)\rangle = \Psi(\boldsymbol{r}, \tau; t)$. (Note that the free particle system is similarly represented by $\Phi(\boldsymbol{p}, E; t) = \langle \boldsymbol{p}, E\, |\Psi(t)\rangle$ in the energy momentum space). Furthermore, as the eigenvectors of the time operator are of the form$|\tau\rangle = u_r\, |\boldsymbol{r}\rangle$, one has that:

$$\langle \boldsymbol{r}, \tau\, |\Psi(t)\rangle = u_r\, \langle \boldsymbol{r}\, |\Psi(t)\rangle = u_r^{\dagger} \psi(\boldsymbol{r}; t) \tag{28}$$

where $\psi(\boldsymbol{r}; t)$ satisfies the time dependent Schrödinger equation. It then follows that

$$|\langle \boldsymbol{r}, \tau\, |\Psi(t)\rangle|^2 \approx N^2(\tau_r)\, |\psi(\boldsymbol{r}; t)|^2 \tag{29}$$

where $N(\tau_r)$ is the normalization coefficient of the spinor u_r (Appendix A). Finally, as shown numerically in Ref. 21, $|\psi(\boldsymbol{r}; t)|^2$ does exhibit a time interference pattern following a time double slit initial boundary condition.

ii) There is a current interest in the possibility of detecting Zitterbewegung like effects in spintronics, grapheme and superconducting systems, due to the similarity of their effective Hamiltonians with the Dirac Hamiltonian and the fact that their space time conditions are close to current experimental possibilities, these being in the spatial range of a few $\mathring{A}$ and of $1fs$ time pulses[16],[17],[18],[19],[20]. The possibility of defining an associated dynamical time operator and its relation to expected experimental observations may be addressed, based on the particular position operator $\boldsymbol{r}(t)$ appropiate in each case, as given by Eq. (4) in Ref. 17.

On the other hand, for electrons and heavier fermions the confinement limits associated with the Compton scale are far beyond the above experimental possibilities and no direct observation of Zitterbewegung can be expected. In the case of electron neutrinos or antineutrinos with masses of the order of 2.2 eV, $\hbar/2m_0c = 448\mathring{A}$ and $\hbar/2m_0c^2 = 0.15fs$. However, as they are found usually in an ultra relativistic range, the applicable characteristic amplitude and period are much reduced as they are attached to the de Broglie wave length (Appendix C).

iii) The dynamical system time operator may be helpful in resolving the so called time paradox in quantum gravity that concerns the incompatibility of the concepts of time in quantum mechanics – where "time continues to be treated as a background parameter" - and in general relativity – where "time is dynamical and local"[23].

The dynamical time operator here proposed, commutes with the position operator. However this does not lead to extend the normalization condition to the additional variable, as occurs when going from one to three space dimensions. Indeed, as pointed out in Ref.7, "a consistent definition of a probability density can include only points on a space-like surface, i.e., with no possible causal connection. In the non-relativistic limit ($c = \infty$) all such surfaces are reduced to $\tau = const$ planes, and the normalization applies only to the domain of space dimensions. Thus under no circumstances is the time variable on a complete equal footing as the space variables."

On the other hand the dynamical time operator, as defined, has a one to one correspondence with the timelike worldline $\boldsymbol{r}(t)$ and is monotonically linked to the time parameter t. Then to each point of the spectrum one can associate a spacelike surface that intersects the worldline at the corresponding point, thus providing a foliation of spacetime by spacelike surfaces over which one can define probability amplitudes. Consequently one can say that this operator yields an observable variable that "sets the conditions" for the other variables and defines a satisfactory notion of time, as required by the conditional probability interpretation of quantum gravity[24].

IX. APPENDIX A. EIGENVALUES AND EIGENVECTORS OF THE DYAMICAL TIME OPERATOR

Consider the eigenvalue equation of the self adjoint system time operator $T = \boldsymbol{\alpha} \cdot \boldsymbol{r}/c + \beta\tau_0$:

$$T|\tau >= \tau|\tau > \tag{A.1}$$

In complete analogy with the energy eigenvalue and eigenvector solution in the free particle case[1],[4],[8],[9] one has:

$$|\tau >= u_r|\boldsymbol{r} > \tag{A.2}$$

where $|\boldsymbol{r} >$ is the eigenvector of the position operator $\boldsymbol{r}$ and u_r is a four component spinor independent of the linear momentum $\boldsymbol{p}$. In the momentum representation the eigenfunction is

$$< \boldsymbol{p}|\tau >= u_r < \boldsymbol{p}|\boldsymbol{r} >= u_r(2\pi\hbar)^{-3/2} expi(\boldsymbol{p} \cdot \boldsymbol{r})/\hbar \tag{A.3}$$

Since from eq. (A.1) one has:

$$T^2|\tau >) = \tau^2|\tau > \tag{A.4}$$

and

$$T^2 = \{\boldsymbol{\alpha} \cdot \boldsymbol{r}/c + \beta\tau_0\}^2 = (r/c)^2 + \tau_0^2 \tag{A.5}$$

there are two (infinitely degenerate in the possible directions of $\boldsymbol{r}$) eigenvalues of the time operator, namely:

$$\tau = \pm\, \tau_r = \pm\, [(r/c)^2 + \tau_0^2]^{1/2} \tag{A.6}$$

Each of these eigenvalues is doubly degenerate with respect to the component $\boldsymbol{\sigma}\cdot\boldsymbol{r}/2r$ of the spin along the $\boldsymbol{r}$ direction which commutes with T. Thus one can find simultaneous eigenfunctions of $\boldsymbol{\sigma}\cdot\boldsymbol{r}/2r$ and T, giving rise to altogether four eigenvalue pairs:

$$(+\tau_r, +1/2); (+\tau_r, -1/2); (-\tau_r, +1/2); (-\tau_r, -1/2)$$

The four orthonormal spinors u_r are:

Sys.time	*Positive*	*Positive*	*Negative*	*Negative*
τ	$+\tau_r$	$+\tau_r$	$-\tau_r$	$-\tau_r$
Spin	$+1/2$	$-1/2$	$+1/2$	$-1/2$
u_1	1	0	$-r/d$	0
u_2	0	1	0	$+r/d$
u_3	$+r/d$	0	1	0
u_4	0	$-r/d$	0	1

with $d = [c(\tau_r + \tau_0)]$ and normalization coefficient $N(\tau_r) = [2\tau_r/(\tau_r + \tau_0)]^{-1/2}$ for normalization to unity. For a Lorentz covariant normalization, the normalization constant is

$[\tau_r/\tau_0]^{1/2}N(\tau_r) = [1 + (\tau_r/\tau_0)]^{1/2}$.

The term τ_0 is introduced solely by analogy with the rest mass term in the free particle Dirac Hamiltonian. As such it gives rise to $2\tau_0$ gap in the eigenvalue spectrum, separating positive and negative values in the same way as the $2m_0c^2$ gap in the energy spectrum. No interpretation as a property in analogy to the rest mass can be given at this stage, except perhaps by recalling that the starting hypothesis of de Broglie's thesis was to associate an oscillatory phenomenon of frequency $\nu_0 = m_0c^2/\hbar$ with the rest mass of the particle, measured in the rest frame of reference. The corresponding period would be $\hbar/m_0c^2$, that could be interpreted as a characteristic internal "system time" τ_0. This value is also related to the period of the Zittervewegung. Note that, in this formulation, τ_0 plays the role of an invariant quantity in the $(\boldsymbol{r}, \tau)$ space, i.e., $\tau_0^2 = \tau^2 - (\boldsymbol{r}/c)^2$, as m_0c^2 plays in the $(\boldsymbol{p}, E)$ space, namely $(m_0c^2)^2 = E^2 - (c\boldsymbol{p})^2$.

X. APPENDIX B. THE FULL DYNAMICAL TIME OPERATOR

Eqs. (8), (9) and (10) for the operators $\boldsymbol{\alpha}$, β and $\boldsymbol{r}$ in the Heisenberg representation, yield for the time operator $T(t)$ the following expression:

$$\begin{aligned}T(t) &= \boldsymbol{\alpha}(t)\cdot\boldsymbol{r}(t)/c + \beta(t)\tau_0 = \\ &= (1/c)(c\boldsymbol{p}/H)\cdot\boldsymbol{r}(0) + (c\boldsymbol{p}/H)^2 t + (m_0c^2/H)\tau_0 \\ &+ [exp(-2iHt/\hbar)]\{\boldsymbol{\alpha}(0) - c\boldsymbol{p}/H\}\cdot \\ &\{\boldsymbol{r}(0)/c + (c\boldsymbol{p}/H)t + (\hbar/H)sin(-Ht/\hbar)[(c\boldsymbol{p}/H)exp(iHt/\hbar) \\ &+ (\boldsymbol{\alpha}(0) - c\boldsymbol{p}/H)exp(-iHt/\hbar)]\} + \tau_0\{\beta(0) - m_0c^2/H\}exp(-2iHt/\hbar)\end{aligned} \quad \text{(B.1)}$$

For $t = 0$, this expression reduces correctly to $T(0) = \boldsymbol{\alpha}(0)\cdot\boldsymbol{r}(0)/c + \beta(0)\tau_0$.

XI. APPENDIX C - NON- AND ULTRA- RELATIVISTIC LIMITS

Setting $\boldsymbol{r}(0) = 0$ in Eq.(10) for simplicity, the non and ultra relativistic limits of the non oscillatory part of $T(t)$ are as follows:

1) Non relativistic limit $cp \ll m_0c^2$

$$T(t) \simeq \tau_0 + (cp/m_0c^2)^2 t + ...) \quad \text{(C.1)}$$

Then, a dwell time between two points of the trajectory is given by:

$$\delta T = T(t_2) - T(t_1) \simeq (cp/m_0c^2)^2(t_2 - t_1). \quad \text{(C.2)}$$

2) Ultra relativistic limit $cp \gg m_0c^2$

$$T(t) \simeq t + (m_0c^2/pc)\tau_0 + ... \qquad (C.3) \quad \text{(C.3)}$$

In this case, the dynamic time approaches the parametric (external) time t and:

$$\delta T = T(t_2) - T(t_1) \simeq t_2 - t_1 \quad \text{(C.4)}$$

As for the Zitterbewegung, whose amplitude and period are given by $c\hbar/H$ and H/h respectively, the non relativistic limit yields $\lambda C/2\pi$ and $\lambda C/c$, where λC is the Compton wave length h/m_0c, establishing a *spacetime "Compton scale"*. This amplitude restricts the localization of the particle in space to one half Compton wavelength. In a similar way, the period restricts the localization in time, in this case, to $\hbar/2m_0c^2$, suggesting the value $\hbar/2m_0c^2$ for the parameter τ_0, in direct relation to the rest mass. On the other hand, in the ultra relativistic limit the amplitude is $(1/2\pi)\lambda B$ where λB is the de Broglie wave length h/p , and the period is $\lambda B/c$, as noted in Ref. 18.

XII. APPENDIX D: CHARGED PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD

The "minimal coupling" Dirac Hamiltonian for a particle of charge q in an external electromagnetic field is:

$$H = \boldsymbol{\alpha} \cdot \boldsymbol{\pi}(\boldsymbol{r}, t) + \beta m_0 c^2 + q\Phi(\boldsymbol{r}, t), \tag{D.1}$$

where $\boldsymbol{\pi}(\boldsymbol{r}, t) = [\boldsymbol{p} - (q/c)\boldsymbol{A}(\boldsymbol{r}, t)]$, and $\boldsymbol{A}(\boldsymbol{r}, t)$ and $\Phi(\boldsymbol{r}, t)$ are the vector and scalar electromagnetic potentials, respectively. Now:

$$\begin{aligned}[\boldsymbol{\alpha} \cdot \boldsymbol{r}, \boldsymbol{\alpha} \cdot \boldsymbol{\pi}] &= \boldsymbol{r} \cdot \boldsymbol{\pi} + i\boldsymbol{\Sigma} \cdot (\boldsymbol{r} \times \ \boldsymbol{\pi}) - \boldsymbol{\pi} \cdot \boldsymbol{r} - i\boldsymbol{\Sigma} \cdot (\boldsymbol{\pi} \times \ \boldsymbol{r}) \\ &= 3I + (4/\hbar^2)\ \boldsymbol{s} \cdot \boldsymbol{r} \times \ [\boldsymbol{p} - (q/c)\boldsymbol{A}(\boldsymbol{r}, t)]. \end{aligned} \tag{D.2}$$

As $\quad [\boldsymbol{\alpha} \cdot \boldsymbol{r}, \Phi(\boldsymbol{r}, t)] = [\beta, \Phi(\boldsymbol{r}, t)] = 0,$

$$[T, H] = i\hbar \left[\begin{array}{c} I + 2\beta K - (4/\hbar^2)(q/c)\boldsymbol{s} \cdot \boldsymbol{r} \times \ \boldsymbol{A}(\boldsymbol{r},\boldsymbol{t}) \\ +2\beta \left\{ \tau_0(c\boldsymbol{\alpha} \cdot \boldsymbol{p}) - m_0 c^2 (\boldsymbol{\alpha} \cdot \boldsymbol{r}/c) \right\} \end{array} \right]. \tag{D.3}$$

Finally:

$$\begin{aligned} d/dt < T >= (1/i\hbar) < [T, H] >= 1 + 2 < \beta K > \\ -(4/\hbar^2)(q/c) < \boldsymbol{s} \cdot \boldsymbol{r} \times \ \boldsymbol{A}(\boldsymbol{r},\boldsymbol{t}) > -i(2/\hbar) < \beta\{\tau_0(H - \Phi(\boldsymbol{r}, t)) - m_0 c^2 T > \end{aligned} \tag{D.4}$$

The time rate of change of the expectation value of the time operator is modified by the electromagnetic interaction, in particular by the expectation value of the vector potential. In consequence, different trajectories through a non uniform electromagnetic field give rise to different time development of the expectation value of the system time operator and different associated phase velocities.

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b. quoted by J. Polchinsky of UCSB in ”23d Solvay Conference –The Quantum Structure of Space and Time”, World Scientific, 2007.

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