GRADED LIE ALGEBRAS OF MAXIMAL CLASS II

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Abstract. We describe the isomorphism classes of infinite-dimensional graded Lie algebras of maximal class over fields of odd characteristic.

1. Introduction

In this paper we describe a determination of all isomorphism types of infinite-dimensional graded Lie algebras with maximal class over fields having odd characteristic.

In our earlier paper with the same title [CMN97] we showed that there are $|\mathbf{F}|^{\aleph_0}$ infinite-dimensional graded Lie algebras of maximal class over a field $\mathbf{F}$ of positive characteristic (Corollary 9.4). This was done by describing the isomorphism type of such an algebra by its (normalized) sequence of two-step centralizers or, equivalently, a sequence of elements from the projective line over $\mathbf{F}$. We exhibited enough different sequences corresponding to algebras by using a known construction, which we called inflation, and combining it with taking limits. The starting point was some insoluble infinite-dimensional graded Lie algebras of maximal class over the field with p elements built by A. Shalev [Sha94] as positive parts of twisted loop algebras of some finite-dimensional simple algebras of Albert and Frank with a non-singular derivation on top.

These AFS-algebras (see Section 8) together with the process of inflation and taking limits in fact suffice to allow us here to describe *all* infinite-dimensional graded Lie algebras of maximal class up to isomorphism. This required us to refine our techniques for determining which infinite sequences of elements of the projective line can arise from sequences of two-step centralizers of infinite-dimensional graded Lie algebras of maximal class.

Carrara has shown [Car98, Car99] that each AFS-algebra L can be characterised by a finite quotient, in the sense that the sequence of two-step centralizers of AFS-algebras are characterized by a certain initial segment. In other words, each AFS-algebra has a finite quotient which is not a quotient of any other graded Lie

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algebra of maximal class. We call the smallest such quotient $Q(L)$. Our results show that for *odd* characteristic such a statement fails for all the other infinite-dimensional graded Lie algebras of maximal class. This perhaps explains the very distinguished role that AFS-algebras play in the present context.

The above distinction is the basis for the proof of our main result (Theorem 4.1) which states that an infinite-dimensional graded Lie algebra of maximal class arises either via a finite number of inflation steps from an AFS-algebra, or as the limit of an infinite number of inflation steps. (In the latter case the resulting algebra is independent of the starting algebra.)

A consequence of our result is that every infinite-dimensional graded Lie algebra of maximal class has at most three distinct constituent lengths (see Definition 3.4). This is not true in general though we expect that an arbitrary graded Lie algebra of maximal class has at most four constituent lengths.

Jurman [Jur98, Jur99] has shown that the characteristic two case is essentially (and not just technically) different by finding another family of uninflated graded Lie algebras of maximal class. Moreover he has shown that, with this family added, a result corresponding to Theorem 4.1 holds. His proof requires further technical refinements.

There are finite-dimensional graded Lie algebras of maximal class which do not arise as quotients of infinite-dimensional graded Lie algebras of maximal class. These algebras are quotients of only finitely many other graded Lie algebras of maximal class. Those which are not quotients of others are usually called *terminal*. Carrara has determined all the terminal graded Lie algebras of maximal class which have a $Q(L)$ as a quotient for some AFS-algebra L. It might be possible to refine her methods and our methods to obtain a description of all terminal graded Lie algebras of maximal class.

Computations with the ANU p-Quotient Program [HNO97] have been invaluable for understanding the structure of the algebras under consideration, and have constantly guided our proofs. We refer to [CMN97] for more details. We note here just that we made use of a more recent version of the program which allowed us occasionally to study algebras with dimensions in the thousands to help clarify matters.

We begin with some preliminaries: in Section 3 we review in particular the facts we need from [CMN97]. In Section 4 we then state our result, and give an outline of the proof. Although we use only elementary methods, the proof is technically involved, so that we try to state clearly the ideas behind the proof here. The actual proof is carried out in Sections 5-10.

We refer to the previous paper as I.

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2. Preliminaries

Throughout the paper, except where explicitly stated otherwise, p is an odd prime, $\mathbf{F}$ is a field of characteristic p, and $\mathbf{F}_p$ denotes the field with p elements.

We will use without further mention the Jacobi identity

$$[u[vw]] = [uvw] - [uwv];$$

unspecified brackettings are left-normed. We will use many times the generalized Jacobi identity

$$[v[y\underbrace{z\dots z}_{\lambda}]] = \sum_{i=0}^{\lambda} (-1)^i \binom{\lambda}{i} [v\underbrace{z\dots z}_{i}y\underbrace{z\dots z}_{\lambda-i}], \tag{2.1a}$$

also in its equivalent form where we "count from the back"

$$\begin{aligned}[v[y\underbrace{z\dots z}_{\lambda}]] &= (-1)^{\lambda} \sum_{i=0}^{\lambda} (-1)^i \binom{\lambda}{i} [v\underbrace{z\dots z}_{\lambda-i}y\underbrace{z\dots z}_{i}] \\ &\approx \sum_{i=0}^{\lambda} (-1)^i \binom{\lambda}{i} [v\underbrace{z\dots z}_{\lambda-i}y\underbrace{z\dots z}_{i}],\end{aligned} \tag{2.1b}$$

where we have written $a \approx b$ to mean $a = \pm b$. As we will be applying (2.1b) only when $[v[y\underbrace{z\dots z}_{\lambda}]]$ is known to be zero, the change of sign will be irrelevant.

Lucas' Theorem ([Luc78], but see for instance [KW89] for some recent developments on the subject), states that if λ and μ are arbitrary non-negative integers, and

$$\lambda = \lambda_0 + \lambda_1 p + \dots + \lambda_n p^n,$$
$$\mu = \mu_0 + \mu_1 p + \dots + \mu_n p^n,$$

with $0 \le \lambda_i, \mu_i < p$, are their p-adic representations, then

$$\binom{\lambda}{\mu} \equiv \prod_{i=0}^{n} \binom{\lambda_i}{\mu_i} \pmod{p}.$$

From a p-adic point of view, if $q = p^h$, and we write $\lambda = \lambda_0 + \lambda_1 q$ and $\mu = \mu_0 + \mu_1 q$, with $\lambda_0, \mu_0 < q$, then

$$\binom{\lambda}{\mu} \equiv \binom{\lambda_0}{\mu_0} \cdot \binom{\lambda_1}{\mu_1} \pmod{p}.$$

We use these results many times to evaluate binomial coefficients modulo p. For instance, if the remainder of λ modulo some power of p is less than the corresponding remainder of μ, then $\binom{\lambda}{\mu} \equiv 0 \pmod{p}$.

We also use the following elementary facts: if q is a power of p, then

$$(-1)^i \binom{q-1}{i} \equiv 1 \pmod{p},$$

for $0 \leq i \leq q-1$; and

$$0 = (1 + (-1))^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i}. \tag{2.2}$$

A typical application of this is

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} = -(-1)^n.$$

In several circumstances we will be using shorthands like

$$\lambda_1, (\lambda_2, \lambda_3^\mu)^\infty,$$

where the λ_i are arbitrary objects and μ is a non-negative integer, to denote the sequence

$$\lambda_1, \lambda_2, \underbrace{\lambda_3, \dots, \lambda_3}_{\mu}, \lambda_2, \underbrace{\lambda_3, \dots, \lambda_3}_{\mu}, \dots \ .$$

We will introduce more specific notation and known facts in Sections 3 and 4.

3. Background

This section recalls in a concise way all the material we need from I, and introduces some notation and conventions we use throughout the paper.

3.1. Definition. An *algebra of maximal class* is a graded Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i$$

over a field $\mathbf{F}$ of characteristic p, such that $\dim(L_1) = 2$, $\dim(L_i) \leq 1$, for $i \geq 2$, and L is generated by L_1. In other words, $[L_i L_1] = L_{i+1}$ for $i \geq 1$.

From now on, let L be an infinite-dimensional algebra of maximal class over the field $\mathbf{F}$; thus $\dim(L_i) = 1$ for all $i \geq 2$. The two-step centralizers

$$C_i = C_{L_1}(L_i),$$

for $i \geq 2$, are subspaces of dimension 1 of the two-dimensional space L_1. We choose an element $y \in L_1$ so that

$$C_2 = \mathbf{F} y \ .$$

It follows from I, Lemma 3.1, that if $C_i = \mathbf{F} y$ for all i, then L is isomorphic to the infinite-dimensional metabelian algebra of maximal class $\mathcal{M}(\mathbf{F})$ over $\mathbf{F}$. Suppose then that there is a second two-step centralizer and that C_n is its first occurrence. We say that the second two-step centralizer *occurs in weight* n. Write $C_n = \mathbf{F} x$ for some $x \in L_1$. From I, Theorem 5.5, we have

3.2. Proposition. *The second two-step centralizer first occurs in weight* $2p^h$*, for some* $h \geq 1$.

In other words, $n = 2p^h$, for some h. We will see in Section 4, Step 10, that a similar statement holds for all two-step centralizers. If there are other centralizers distinct from $\mathbf{F}y$ and $\mathbf{F}x$, then x and y can be redefined so that the third two-step centralizer in order of occurrence is $\mathbf{F} \cdot (y - x)$. All two-step centralizers can then be written uniquely as

$$C_i = \mathbf{F} \cdot (y - \alpha_i x),$$

for some $\alpha_i \in \mathbf{F} \cup \{\infty\}$. (See I, Section 3.) Here $y - \infty \cdot x = x$.

In the rest of the paper we use without further mention the following

3.3. Notation. Let L be an infinite-dimensional, non-metabelian algebra of maximal class. We write $\mathbf{F}y = C_2$ for the first two-step centralizer, and $\mathbf{F}x$ for the second one in order of occurrence. We also write $z = x + y$.

The sequence $(C_i)_{i \geq 2}$ determines L up to isomorphism (I, Theorem 3.2). Equivalently, L is determined by the sequence $(y - \alpha_i x)_{i \geq 2}$. We call either of these two equivalent sequences *the sequence of two-step centralizers* of L. We even mix usage. For instance, $\mathcal{M}(\mathbf{F})$ is determined, as recalled above, by the constant sequence of two-step centralizers $(y, y, y, \dots)$.

Suppose L is not isomorphic to $\mathcal{M}(\mathbf{F})$. Because of I, Lemma 3.3, the sequence of two-step centralizers of L consists of consecutive occurrences of y, interrupted by isolated occurrences of different two-step centralizers. Suppose that in the sequence of two-step centralizers we have a pattern of the form

$$C_i \neq \mathbf{F}y \ , \qquad C_{i+m} \neq \mathbf{F}y \ ,$$
$$C_{i+1} = C_{i+2} = \dots = C_{i+m-1} = \mathbf{F}y \ .$$

3.4. Definition (Constituents)**.** We call the subsequence

$$C_{i+1}, C_{i+2}, \dots, C_{i+m}$$

a *constituent* of L. The *length* of such a constituent is the number m. Note that this is a different definition from that of I, Section 3, where the final centralizer C_{i+m} was not included. Experience suggests this is a better usage; we apologise for the inconvenience the change causes. It follows that the lengths of constituents in this paper are increased by one with respect to I.

Also, the following special definition turns out to be useful.

3.5. Definition (First constituent and parameter)**.** Suppose the first occurrence of the second two-step centralizer is $C_{2q} = \mathbf{F}x$. It is convenient to regard the subsequence

$$C_2, C_3, \dots, C_{2q}$$

as a constituent (and thus the first one) *of length* $2q$. We call q the *parameter* of the algebra L. By Proposition 3.2, we have $q = p^h$ for some h.

It will be handy to have the following

3.6. Definition. A non-zero element v of L_{i+m} will be said to be *at the end* of the constituent

$$C_{i+1}, C_{i+1}, \dots, C_{i+m} \ .$$

Similarly, a non-zero element v of L_i will be said to be *at the beginning* of the constituent, or, equivalently, *at the end* of the previous constituent.

If L has only the two distinct two-step centralizers $\mathbf{F}y$ and $\mathbf{F}x$, then the sequence of constituent lengths determines the algebra up to isomorphism.

There are strong restrictions on the length of constituents, as proved in I, Proposition 5.6. In the notation of the present paper, this reads

3.7. Proposition. *Let L have parameter q. Then the constituent lengths take values in the set*

$$\{2q\} \cup \{2q - p^{\beta} : \textit{for } 0 \le \beta \le h\}.$$

In I, Section 6, a procedure called *inflation* is defined, which given an algebra L of maximal class, and a subspace $M = \mathbf{F}w$ of L_1 of dimension 1, yields another algebra ${}^{M}L$ of maximal class. There is a way of identifying L_1 and $({}^{M}L)_1$ (see I, Section 6). Once this is done, one can describe the sequence of two-step centralizers of ${}^{M}L$, starting with $C_{({}^{M}L)_1}\big(({}^{M}L)_2\big)$, as

$$\underbrace{w, \dots, w}_{2p-2}, C_2, \underbrace{w, \dots, w}_{p-1}, C_3, \dots, C_i, \underbrace{w, \dots, w}_{p-1}, C_{i+1}, \dots \ .$$

In other words, inflation can be described by inserting $p-1$ occurrences of w between every pair of consecutive terms of the sequence of two-step centralizers of L to obtain the corresponding sequence for ${}^{M}L$. The initial part of the sequence requires special treatment.

There is another procedure, called *deflation* (I, Section 7), that is a one-sided inverse of inflation. It associates to L another algebra of maximal class $L^{\downarrow}$. Here, too, there is a way of identifying L_1 with $(L^{\downarrow})_1$. With this identification, the sequence of two-step centralizers $(C_i')_{i \ge 2}$ of the deflation $L^{\downarrow}$ of L is described in I, Proposition 7.1, which we recall here as

3.8. Proposition. *If for $i \ge 2$ all the two-step centralizers*

$$C_{ip}, C_{ip+1}, \dots, C_{(i+1)p-1}$$

coincide with $C_2 = \mathbf{F}y$, then $C_i' = \mathbf{F}y$. If one of them equals $\mathbf{F}u \ne \mathbf{F}y$, then $C_i' = \mathbf{F}u$.

It should be noted that because of Proposition 3.7, at most one among the listed two-step centralizers can differ from $\mathbf{F}y$.

It is clear from the above that we always have

$$({}^{M}L)^{\downarrow} = L.$$

In other words if an algebra of maximal class is an inflation of another algebra, then the latter algebra is the deflation of the former.

3.9. Definition. An algebra L of maximal class is said to be *inflated* if

$$L = {}^{M}(L^{\downarrow})$$

for a suitable maximal subspace M of $(L^{\downarrow})_1$.

The Albert-Frank-Shalev algebras and their limits (Section 4 of I) are not inflated. This follows from the fact that inflated algebras can be explicitly characterized (I, Proposition 7.4). In the notation of the current paper, this can be formulated as

3.10. Proposition. *An algebra L is inflated if and only if all its constituents have length a multiple of p. In this case we have*

$${}^{M}(L^{\downarrow}) = L,$$

where $M = C_{L_1}(L_2)$ in the identification of L_1 with $(L^{\downarrow})_1$.

There is a particular case which is easy to describe.

3.11. Proposition. *Inflation with respect to the first two-step centralizer has the effect of multiplying all constituent lengths by p. The resulting algebra has parameter greater than p, and all constituents lengths are multiples of p.*

Conversely, suppose L has parameter q greater than p, and all constituents have length a multiple of p. Then the sequence of constituent lengths of $L^{\downarrow}$ is obtained from that of L by dividing all constituent lengths by p.

There is a special case of repeated deflation that is particularly useful. It follows immediately from Proposition 3.8.

3.12. Proposition. *Suppose L has parameter $q = p^h$. The result of deflating h times is an algebra in which x plays the role of the first centralizer. If all two-step centralizers*

$$C_{iq}, C_{iq+1}, \dots, C_{(i+1)q-1}$$

coincide with $C_2 = \mathbf{F}y$, then $C'_i = \mathbf{F}y$ in the algebra deflated h times. If one of them equals $\mathbf{F}u \neq \mathbf{F}y$, then $C'_i = \mathbf{F}u$ in the algebra deflated h times.

Here, too, Proposition 3.7 implies that at most one among the listed two-step centralizers can differ from $\mathbf{F}y$.

The following Lemma is proved in [CJ99]. We will give an independent proof in Section 10.

3.13. Lemma. *A two-step centralizer other than the first two in order of occurrence always occurs at the end of a short constituent, and it is followed by another short constituent.*

Once the characterization is obtained, one can read off from it a much more precise statement.

4. Leitfaden

In this section we state our main result, and describe the pattern of the proof.

Let p be a fixed odd prime. Let $\mathbf{F}$ be a field of characteristic p. Consider the following two classes of infinite-dimensional algebras of maximal class, defined over the field $\mathbf{F}_p$ with p elements:

- $\mathrm{AFS}(a, b, n, p)$, the insoluble Albert-Frank-Shalev algebras, for $0 < a < b \le n$, as described in [Sha94], and I, Section 4.
- $\mathrm{AFS}(a, b, \infty, p)$, the soluble limits associated to the Albert-Frank-Shalev algebras, for $0 < a < b$, as described in I, Section 9.

Note that all these algebras have two distinct two-step centralizers.

In I we made a different (equivalent) choice of parameters in the AFS-algebras as $0 \le a < b < n$; the current choice has the advantage of offering a more uniform description. (Compare also Proposition 7.3 of I.) What we called $\mathrm{AFS}(0, b, n, p)$ in I is now called $\mathrm{AFS}(b, n, n, p)$. In the notation of the present paper, the algebra $\mathrm{AFS}(a, b, n, p)$ has sequence of constituent lengths

$$2q, (q^{r-2}, 2q - 1, (q^{r-2}, 2q)^{s-1})^{\infty},$$

where $q = p^a$, $r = p^{b-a}$, and $s = p^{n-b}$. As mentioned in Section 2, the notation $(q^{r-2}, 2q)^{s-1}$ denotes $s - 1$ repetitions of the pattern $q^{r-2}, 2q$, and π^{∞}, where π is a pattern of constituent lengths, denotes periodic repetition of π. Here we understand of course that when $n = b$, so that $s - 1 = p^{n-b} - 1 = 0$, the pattern $(q^{r-2}, 2q)$ does not occur at all.

In the notation of the present paper, the algebra $\mathrm{AFS}(a, b, \infty, p)$ has sequence of constituent lengths

$$2q, q^{r-2}, 2q - 1, (q^{r-2}, 2q)^{\infty}.$$

This explains our notation.

We have the following

4.1. Theorem (Characterization)**.** *Let L be an infinite-dimensional algebra of maximal class over the field $\mathbf{F}$ of odd characteristic p.*

Then L is obtained

- *either via a finite number (possibly zero) of inflation steps from one of the algebras $\mathrm{AFS}(a, b, n, p) \otimes_{\mathbf{F}_p} \mathbf{F}$,*
- *or as the limit of an infinite number of inflation steps.*

We note that when one inflates an algebra infinitely many times, the result is independent of the algebra one began with, so that is not necessary to specify the latter in this case.

In the rest of the paper, we take L to be an infinite-dimensional algebra of maximal class over a fixed field $\mathbf{F}$ of odd characteristic p.

The key to our proof is that we show that *every non-inflated algebra of maximal class is an* AFS-*algebra.* Suppose we have done this, and let L be any algebra of maximal class. If L is not inflated, we are finished. If L is inflated, we note its first two-step centralizer κ_1, and consider its deflation $L^{\downarrow}$. If $L^{\downarrow}$ is not inflated, it has to be an AFS-algebra, and we recover L as ${}^{\kappa_1}L^{\downarrow}$. If $L^{\downarrow}$ is inflated, we note its first two-step centralizer κ_2, and consider its deflation $(L^{\downarrow})^{\downarrow}$. We keep repeating this procedure. If after a finite number m of deflation steps we reach an algebra N that is not inflated, then N will be an AFS-algebra, and we will recover L from N by inflating with respect to $\kappa_m, \dots, \kappa_2, \kappa_1$. If all the algebras we obtain in the process are inflated, then L can be obtained as the limit of an infinite number of inflation steps with respect to the sequence $(\kappa_i)_{i\geq 1}$.

Our proof of Theorem 4.1 is given in a number of steps. We have first, from Proposition 3.2 and Proposition 3.7,

Step 1. *If L is not metabelian, then the first constituent has length $2q$, for some power $q = p^h$ of p. All other constituents have length $2q$ or $2q - p^\beta$, for $0 \leq \beta \leq h$.*

From now on we will assume L is not metabelian, and we will write $q = p^h$ for its parameter.

We call a constituent of length $2q$ *long*, one of length q *short*. All others are *intermediate*.

From Proposition 3.10 we have

Step 2. *L is inflated if and only if all constituents have length a multiple of p.*

Step 2 clearly has the special case

Step 3. *If all constituents after the first one are short, then L is inflated.*

Because of Step 3, we may assume that there is another non-short constituent after the first long one. We first deal, roughly speaking, with the case when there are only two distinct two-step centralizers.

We begin with proving

Step 4. *Suppose L has two distinct two-step centralizers. If the second non-short constituent is long, then all constituents are short or long, and thus L is inflated.*

This follows from the results of Section 5 and 6.

We have to consider the case when the second non-short constituent is intermediate.

Step 5. *Suppose the second non-short constituent is an intermediate one, of length $2q - p^\beta$, for $0 \leq \beta < h$, and that no two-step centralizer higher than the second one occurs before it. Then the initial pattern of constituents is of the form*

$$2q, q^{r-2}, 2q - p^\beta,$$

for some power $r = p^k$ of p.

This is proved in Section 5.

Note that such a constituent pattern appears when one inflates an AFS-algebra β times with respect to the first two-step centralizer.

From now on we use the notation $r = p^k$.

Most of the next step has already been obtained by C. Carrara in [Car98, Car99]. We include a proof in Section 7 for completeness.

Step 6. *Suppose there is an initial segment of the sequence of two-step centralizers of L that involves only the first two two-step centralizers, and that the corresponding segment of the sequence of constituent lengths is*

$$2q, q^{r-2}, 2q - p^{\beta},$$

where $0 \leq \beta < h$. Then the algebra has two distinct two-step centralizers, and the sequence of constituent lengths consists of repetitions of the patterns

$$2q, q^{r-2} \qquad \text{or} \qquad 2q - p^{\beta}, q^{r-2}.$$

The following is clear in view of Proposition 3.11.

Step 7. *Suppose L has two distinct two-step centralizers. If the sequence of constituent lengths is*

$$2q, q^{r-2}, 2p - p^{\beta}, (q^{r-2}, 2q)^{\infty},$$

then L is obtained from an $\mathrm{AFS}(a, b, \infty, p) \otimes_{\mathbf{F}_p} \mathbf{F}$ via β inflation steps with respect to the first two-step centralizer.

We thus suppose that there is another intermediate constituent. In Section 8 we prove

Step 8. *Suppose there is an initial segment of the sequence of two-step centralizers of L that involves only the first two two-step centralizers, and that the corresponding segment of the sequence of constituent lengths is*

$$2q, q^{r-2}, 2q - p^{\beta}, (q^{r-2}, 2q)^{s-1}, q^{r-2}, 2q - p^{\beta}.$$

Then $s = p^l$ for some l.

Again, this is consistent with the structure of repeated inflations of the AFS-algebras with respect to the first two-step centralizer.

From now on we will use the notation $s = p^l$.

We now appeal to the main result of [Car98].

4.2. Theorem (C. Carrara)**.** *Let L be an infinite-dimensional algebra of maximal class over the field $\mathbf{F}_p$, where p is any prime.*

Suppose the sequence of two-step centralizers has an initial segment in which only two distinct two-step centralizers occur, and the corresponding initial segment of the sequence of constituent lengths is

$$2q, q^{r-2}, 2q - 1, (q^{r-2}, 2q)^{s-1}, q^{r-2}, 2q - 1,$$

with $q = p^h$, $r = p^k$ and $s = p^l$.

Then $L \cong \mathrm{AFS}(h, h + k, h + k + l, p)$. In particular, L has two distinct two-step centralizers.

Using this, we immediately obtain

Step 9. *Suppose the sequence of two-step centralizers of L has an initial segment involving only the first two two-step centralizers, and that the corresponding sequence of constituent lengths begins with*

$$2q, q^{r-2}, 2q - p^{\beta}, (q^{r-2}, 2q)^{s-1}, q^{r-2}, 2q - p^{\beta}.$$

Then L has two distinct two-step centralizers, and L is obtained from an AFS-*algebra via β inflation steps with respect to the first two-step centralizer.*

We now deal with the case of more than two distinct two-step centralizers. The following critical result is proved in Section 9.

4.3. Proposition (Specialization of two-step centralizers)**.** *Let L be an infinite-dimensional algebra of maximal class with three or more distinct two-step centralizers.*

Let y be the first two-step centralizer and w a two-step centralizer distinct from y.

Choose elements x_1 and x_2 in one of the following ways

$$x_1 = y \text{ and } x_2 = w \quad \text{or} \quad x_1 = w \text{ and } x_2 = y.$$

Consider the sequence of two-step centralizers of L. Leave x_1 fixed, and change all two-step centralizers other than y and w to x_2.

Then the resulting sequence is the sequence of two-step centralizers of an algebra of maximal class with two distinct two-step centralizers.

We use Proposition 4.3 to extend Proposition 3.2.

Step 10. *The i-th two-step centralizer in order of occurrence appears for the first time as*

$$C_{L_1}(L_{2p^n}),$$

for some $n \geq i - 1$.

This is proved in Section 10.
Finally we obtain

Step 11. *Suppose L has at least three distinct two-step centralizers. Then all constituents are either short or long, so that by Step 2 the algebra is inflated.*

This is proved in Section 10, and completes the proof.

5. Sequences of short constituents

In I we have proved that the second constituent is short, provided the characteristic of the field is odd. Here we will investigate more generally how many short constituents one gets between two non-short ones in odd characteristic. In doing this, we will introduce a few basic arguments that we use repeatedly in the course of the paper. These will be presented in some detail the first time they are used.

Because of Step 3, we may assume that not all constituents after the first one are short.

If the second non-short constituent is long, and no two-step centralizer other than the first two occurs before it, then deflation quickly tells us how many short constituents occur between the two long ones. In fact, suppose the sequence of constituent lengths of L begins as

$$2q, q^m, 2q.$$

The sequence of two-step centralizers therefore begins, starting with C_{2q}, as

$$x, \underbrace{y, \dots, y}_{q-1}, x, \underbrace{y, \dots, y}_{q-1}, x, \dots, \underbrace{y, \dots, y}_{q-1}, x, \underbrace{y, \dots, y}_{2q-1},$$

where the pattern $\underbrace{y, \dots, y}_{q-1}, x$ appears m times. If we deflate h times (remember that we use the standing notation $q = p^h$), according to Proposition 3.12, we obtain that the deflated algebra has sequence of two-step centralizers, beginning with C_2',

$$\underbrace{x, \dots, x}_{m+1}, y \ .$$

We obtain that the length of the first constituent in the deflated algebra is, according to Definition 3.5, $m + 3$. By Step 1, $m + 3 = 2r$, for some power $r = p^k$ of p, so that $m = 2r - 3$. We have obtained

5.1. Lemma. *Suppose the sequence of two-step centralizers of L has an initial segment involving only the first two two-step centralizers, and that the corresponding sequence of constituent lengths is*

$$2q, q^m, 2q.$$

Then $m = 2r - 3$ for some power $r = p^k$ of p.

If the second non-short constituent is intermediate, the situation is more complicated. Consider for instance the algebra $L = \mathrm{AFS}(1, 2, 2, 3)$, which has two distinct two-step centralizers, and sequence of constituent lengths $6, (3, 5)^\infty$, so that the parameter q is 3 here. We now write down the initial segment of the sequence of two-step centralizers of L, beginning with $C_{2q} = C_6$, isolating the two-step centralizers in groups of $q = 3$. We get

$$xyy|xyy|yyx|yyx|yyy|yxy|yxy|yyy|xyy|\dots \ .$$

The resulting sequence of two-step centralizers in $L^\downarrow$ is, according to Proposition 3.8,

$$x, x, x, x, y, x, x, y, x, \dots \ .$$

Here the intermediate constituent does not provide the end of the first constituent in the deflated algebra. We will need considerably more work in the rest of this section to deal with this case.

It might be worth noting that if all constituents are long or short, then all two-step centralizers other than the first one occur in the form C_i, where $i \equiv 0 \pmod{q}$. The occurrence of an intermediate constituent disturbs this pattern, and it takes further occurrences of intermediate constituents to bring it back into step. However, it takes an intermediate constituent of maximal length $2q - 1$ to disrupt the fact that all two-step centralizers other than the first one occur in the form C_i, where $i \equiv 0 \pmod{p}$.

We now introduce a simple but important

5.2. Calculation Device. Let $0 \neq v \in L_i$, for some $i > 1$. Suppose we know that all the two-step centralizers $C_i, C_{i+1}, \dots, C_{i+n-1}$ equal one of the first two, that is, they are either y or x. Consider, according to Definition 3.3, the element $z = x + y \in L_1$, a notation that we will use in the rest of the paper. We have that if $[vx_1x_2 \dots x_n]$ is a non-zero commutator in L, with $x_i \in \{x, y\}$, then

$$[vx_1x_2 \dots x_n] = [v\underbrace{z \dots z}_{n}] .$$

We begin with a simple generalization of Lemma 5.3 of I.

5.3. Lemma. *Suppose no two-step centralizer other than the first two occurs before the second non-short constituent. Then the number of short constituents between the first constituent and the second non-short one is odd.*

Proof. Suppose the sequence of constituent lengths starts with $2q, q^m$, where m is even. We will prove that this is followed by another short constituent.

We have to prove

$$[y\ \underbrace{z \dots z}_{q(m+3)-1}\ x] = 0. \tag{5.1}$$

Since $q(m+3) - 1$ is even, we can consider the integer

$$\lambda = \frac{q(m+3) - 1}{2} = q\frac{m+2}{2} + \frac{q-1}{2} .$$

We will obtain equation (5.1) from the expansion of the following expression with the generalized Jacobi identity (2.1b):

$$\begin{aligned} 0 &= [[y\underbrace{z \dots z}_{\lambda}][y\underbrace{z \dots z}_{\lambda}]] \\ &\approx \sum_{i=0}^{\lambda} (-1)^i \binom{\lambda}{i} [y\underbrace{z \dots z}_{2\lambda-i}y\underbrace{z \dots z}_{i}]. \end{aligned} \tag{5.2}$$

Note that the total weight of the commutators in (5.2) is that of the left-hand side of (5.1):

$$2\lambda + 2 = q(m+2) + q - 1 + 2 = q(m+3) + 1.$$

We now look at the resulting terms in (5.2). Since y is the first centralizer, the term $[y\underbrace{z \dots z}_{2\lambda-i}y]$ will vanish most of the time, except when $[y\underbrace{z \dots z}_{2\lambda-i}]$ occurs in a weight that marks the end of a constituent. Keeping in mind the information we already have on the previous constituents, the sum reduces to

$$\begin{aligned} 0 = {} & [y\underbrace{z \dots z}_{2\lambda}y] + \\ & + \left(\sum_{j=1}^{(m+2)/2} (-1)^{qj} \binom{q\frac{m+2}{2} + \frac{q-1}{2}}{qj} \right) \cdot [y\underbrace{z \dots z}_{2\lambda+1}]. \end{aligned} \tag{5.3}$$

Lucas' Theorem yields

$$(-1)^{qj}\binom{q\frac{m+2}{2}+\frac{q-1}{2}}{qj} = (-1)^{j}\binom{\frac{m+2}{2}}{j};$$

by (2.2)

$$\sum_{j=1}^{(m+2)/2} (-1)^{j}\binom{\frac{m+2}{2}}{j} = -1.$$

Therefore (5.3) yields

$$0 = [y \underbrace{z \dots z}_{2\lambda} z] - [y \underbrace{z \dots z}_{2\lambda} y] = [y \underbrace{z \dots z}_{2\lambda} x],$$

as $z = x + y$. This is precisely (5.1). □

5.4. Lemma. *Suppose L has an initial segment of the sequence of two-step centralizers involving only the first two two-step centralizers, and suppose the corresponding segment of the sequence of constituent lengths is*

$$2q, q^{m}.$$

Suppose further that an intermediate constituent in the sequence of two-step centralizers of L is preceded *by at least m short ones ending in x.*

Then that intermediate constituent is followed *by at least m short ones ending in x.*

We will make a corresponding statement later (Lemma 5.6).

Proof. Let the intermediate constituent have length $2q - p^{\beta}$. Suppose we have already proved that this intermediate constituent is followed by τ short ones ending in x, with $1 \le \tau < m$, so that there is a segment

$$\dots, q^{m}, 2q - p^{\beta}, q^{\tau}$$

in the sequence of constituent lengths of L. We will show that the next constituent is also short and ends in x. Let $\sigma = m - 1 - \tau \ge 0$.

According to Definition 3.6, take a homogeneous element v at the beginning of the σ-th constituent before the intermediate one. What has to be proved is

$$[v \underbrace{z \dots z}_{q(m+2)-p^{\beta}} x] = 0. \tag{5.4}$$

We assume this is not the case, and obtain a contradiction.

We expand

$$0 = [v[y \underbrace{z \dots z}_{q(m+2)-1} x]] \;;$$

here $[y \underbrace{z \dots z}_{q(m+2)-1} x] = 0$ because of our assumption on the initial segment of the sequence of two-step centralizers. Note that this will give us terms of weight p^{β} higher than the left-hand term of (5.4).

We get, using (2.1a) and the fact that $[vx] = 0$,

$$
\begin{aligned}
0 &= [v[y \underbrace{z \dots z}_{q(m+2)-1} x]] \\
&= [v[y \underbrace{z \dots z}_{q(m+2)-1}]x] \\
&= [v \underbrace{z \dots z}_{q(m+2)} x] \cdot \Bigg(\sum_{i=0}^{\sigma} (-1)^{qi} \binom{q(m+1)+q-1}{qi} \\
&\qquad + \sum_{i=\sigma+2}^{m+1} (-1)^{q(i-1)+q-p^\beta} \binom{q(m+1)+q-1}{q(i-1)+q-p^\beta} \Bigg) \\
&\quad + (-1)^{q(m+2)-p^\beta} \binom{q(m+2)-1}{q(m+2)-p^\beta} [v \underbrace{z \dots z}_{q(m+2)-p^\beta} y \underbrace{z \dots z}_{p^\beta - 1} x] \\
&= [v \underbrace{z \dots z}_{q(m+2)} x] \cdot \sum_{i=0}^{m} (-1)^i \binom{m+1}{i} \\
&\qquad + (-1)^{m+1} [v \underbrace{z \dots z}_{q(m+2)-p^\beta} y \underbrace{z \dots z}_{p^\beta - 1} x] \\
&= (-1)^{m+1} \cdot \big(- [v \underbrace{z \dots z}_{q(m+2)-p^\beta} z \underbrace{z \dots z}_{p^\beta - 1} x] \\
&\qquad + [v \underbrace{z \dots z}_{q(m+2)-p^\beta} y \underbrace{z \dots z}_{p^\beta - 1} x] \big) \\
&\approx [v \underbrace{z \dots z}_{q(m+2)-p^\beta} x \underbrace{z \dots z}_{p^\beta - 1} x].
\end{aligned}
$$

Here we have used several times Lucas' Theorem and its consequences.

Now if (5.4) does not hold we obtain a contradiction. In fact (5.4) can fail in two ways. It might be that $[v \underbrace{z \dots z}_{q(m+2)-p^\beta}]$ is centralized by y. This means that the $(\tau+1)$-th constituent after the intermediate one is not short, so by Proposition 3.7 it has length at least

$$2q - \frac{q}{p} = q + (p-1)\frac{q}{p} > q + \frac{q}{p},$$

as $p > 2$. However, we would have just obtained a constituent of length $q + p^\beta \leq q + q/p$, a contradiction.

As an alternative, it could be that $[v \underbrace{z \dots z}_{q(m+2)-p^\beta} w] = 0$ for some w, with $\mathbf{F}w \neq \mathbf{F}y, \mathbf{F}x$. We would then have a short constituent not ending in x; but we have shown that this constituent would be followed by a constituent of length $p^\beta < q$, and this is a final contradiction. $\square$

5.5. Lemma. *Suppose no two-step centralizer other than the first two occurs before the second non-short constituent. Suppose the sequence of constituent lengths of L begins as $2q, q^m, 2q - p^\beta$.*

Then $m = p^k - 2$ for some k.

This yields in particular Step 5.

Proof. Deflating β times, according to Proposition 3.11, we may assume $\beta = 0$.

Write $m+2 = rn$, where $r = p^k$ is a power of p (possibly 1), and $n \not\equiv 0 \pmod{p}$. By Lemma 5.4, we know that the first intermediate constituent is followed by at least m short ones ending in x. Let

$$\mu = q\bigl(m + 4 + r - 1\bigr) = q\bigl(r(n+1) + 1\bigr).$$

If $n > 1$, we have $m \geq r - 1$. Therefore we obtain

$$0 \neq [y \underbrace{z \dots z}_{\mu-2} y] = [y \underbrace{z \dots z}_{\mu-2} z] = [y \underbrace{z \dots z}_{\mu-1}].$$

We will prove

$$0 = [y \underbrace{z \dots z}_{\mu}] = [y \underbrace{z \dots z}_{\mu-2} yz], \tag{5.5}$$

so that the algebra is finite-dimensional, since constituents have length greater than 1. This contradiction will give that $n = 1$, so that $m = p^k - 2$ as claimed.

We consider

$$\lambda = \eta + q\vartheta, \text{ where } \eta = \frac{q-1}{2}, \text{ and } \vartheta = r\frac{n+1}{2}.$$

Here n is odd, because of Lemma 5.3. Note that

$$2\lambda + 2 = \mu + 1,$$

so that $[[y \underbrace{z \dots z}_{\lambda}][y \underbrace{z \dots z}_{\lambda}]]$ has the same weight as the commutators in (5.5).

We expand, using (2.1b) and the information about the constituents,

$$\begin{aligned} 0 &= [[y \underbrace{z \dots z}_{\lambda}][y \underbrace{z \dots z}_{\lambda}]] \\ &\approx [y \underbrace{z \dots z}_{\mu}] \cdot \left(\sum_{i=0}^{r-1} (-1)^{1+qi} \binom{\eta + q\vartheta}{1 + qi} \right. \\ &\qquad \left. + \sum_{i=r+1}^{\vartheta} (-1)^{qi} \binom{\eta + q\vartheta}{qi} \right). \end{aligned} \tag{5.6}$$

Now

$$\begin{aligned}\sum_{i=0}^{r-1}(-1)^{1+qi}\binom{\eta+q\vartheta}{1+qi} &= = -\eta\sum_{i=0}^{r-1}(-1)^{i}\binom{\vartheta}{i}\\ &= -\eta\binom{\vartheta}{0}\\ &= -\eta\\ &\equiv \frac{1}{2} \pmod p,\end{aligned}$$

as $\binom{r(n+1)/2}{i} \equiv 0 \pmod p$ for $0 < i < r$. As to the second summation, we have

$$\begin{aligned}\sum_{i=r+1}^{\vartheta}(-1)^{qi}\binom{\eta+q\vartheta}{qi} &= \sum_{i=r+1}^{\vartheta}(-1)^{i}\binom{\vartheta}{i}\\ &= \sum_{j=2}^{(n+1)/2}(-1)^{j}\binom{\frac{n+1}{2}}{j}\\ &= -\left(1-\frac{n+1}{2}\right).\end{aligned}$$

The total coefficient of $[y\underbrace{z\dots z}_{\mu}]$ is thus

$$\frac{1}{2}-1+\frac{n+1}{2}=\frac{n}{2}\not\equiv 0 \pmod p,$$

as claimed. □

5.6. Lemma. *Suppose the sequence of two-step centralizers of L has an initial segment involving only the first two two-step centralizers, and that the corresponding sequence of constituent lengths is*

$$2q, q^{r-2},$$

where r is a power of p. Then every long constituent in the sequence of two-step centralizers of L is followed by at least $r-2$ short ones ending in x.

Proof. By Lemma 5.4 and induction, we may assume that we have the subsequence $\dots, q^{r-2}, 2q, q^{\tau}$ in the sequence of two-step centralizers of L, for some $0 \le \tau < r-2$. Let $\sigma = r-2-\tau$, so that $\sigma \le r-2$, and let v be a homogeneous element at the end of the σ-th two-step centralizer before the long one, so that $[vx] = 0$. We have to prove

$$[v\underbrace{z\dots z}_{qr}x] = 0.$$

We have $[y \underbrace{z \dots z}_{qr-1} x] = 0$, from our assumption on the initial segment of the sequence of two-step centralizers of L. We compute with (2.1a)

$$\begin{aligned}
0 &= [v[y \underbrace{z \dots z}_{qr-1} x]] \\
&= [v[y \underbrace{z \dots z}_{qr-1}]x] \\
&= [v \underbrace{z \dots z}_{qr} x] \cdot \left(\sum_{i=0}^{\sigma-1} (-1)^{qi} \binom{qr-1}{qi} + \sum_{i=\sigma+1}^{r-1} (-1)^{qi} \binom{qr-1}{qi} \right) \\
&= [v \underbrace{z \dots z}_{qr} x] \cdot \left(-(-1)^{\sigma} \binom{r-1}{\sigma} \right) \\
&= -[v \underbrace{z \dots z}_{qr} x].
\end{aligned}$$

□

6. Algebras with long and short constituents

In this section we prove Step 4. We consider algebras of maximal class L with two distinct two-step centralizers in which the second non-short constituent is long, and show that all constituents are short and long. By Lemma 5.1, we know that the sequence of constituent lengths begins like

$$2q, q^{2r-3}, 2q,$$

where $q = p^h$ and $r = p^k$. The algebra we obtain by deflating h times has thus first constituent of length $2r$, as in the proof of Lemma 5.1.

By Lemma 5.6 and induction, we assume that at some point in the sequence of constituent lengths we have the pattern

$$\dots, q^{r-2}, 2q, q^{r-2}. \tag{6.1}$$

We first show that no intermediate constituent can now occur. Suppose there is a constituent of length $2q - p^{\beta}$, for some $0 \le \beta < h$, immediately following (6.1). Deflating β times, we may assume β to be zero.

We use the relation

$$[y \underbrace{z \dots z}_{q(r+1)-1} x] = 0,$$

commuted against a homogeneous element $v \ne 0$ of suitable weight, so that the resulting commutator

$$[v \underbrace{z \dots z}_{q(r+1)} y] \tag{6.2}$$

falls at the end of the final constituent in

$$\dots, q^{r-2}, 2q, q^{r-2}, 2q - 1.$$

Inspection shows that v falls within the long constituent in (6.1); in particular, $[vy] = 0$.

We have, using (2.1b),

$$\begin{aligned}
0 &= [v[y \underbrace{z \dots z}_{q(r+1)-1} x]] \\
&= [v[y \underbrace{z \dots z}_{q(r+1)-1}]x] - [vx[y \underbrace{z \dots z}_{q(r+1)-1}]] \\
&\approx [v \underbrace{z \dots z}_{q(r+1)} x] \cdot \left(\sum_{i=0}^{r-2} (-1)^{2q-2+qi} \binom{q(r+1)-1}{2q-2+qi} \right) \\
&\quad - [v \underbrace{z \dots z}_{q(r+1)} y] \\
&\quad - [v \underbrace{z \dots z}_{q(r+1)} z] \left(\sum_{i=0}^{r-2} (-1)^{2q-1+qi} \binom{q(r+1)-1}{2q-1+qi} \right).
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{i=0}^{r-2} (-1)^{2q-2+qi} \binom{q(r+1)-1}{2q-2+qi} &= \sum_{i=0}^{r-2} (-1)^{2q-2+qi} \binom{qr+q-1}{q(i+1)+q-2} \\
&= \sum_{i=0}^{r-2} (-1)^{i} \binom{r}{i+1} \\
&= -\sum_{i=1}^{r-1} (-1)^{i} \binom{r}{i} \\
&= -(1+(-1)^r) = 0 \ ,
\end{aligned}$$

and similarly

$$\sum_{i=0}^{r-2} (-1)^{2q-1+qi} \binom{q(r+1)-1}{2q-1+qi} = 0 \ .$$

We obtain that (6.2) vanishes, thereby providing a contradiction.

We now show that if there is a further short constituent at the end of (6.1), then this is followed by another short one. Using the appropriate v, which is again centralized by y, we compute with (2.1b)

$$\begin{aligned}
0 &= [v[y \underbrace{z \dots z}_{q(r+1)-1} x]]\\
&= [v[y \underbrace{z \dots z}_{q(r+1)-1}]x] - [vx[y \underbrace{z \dots z}_{q(r+1)-1}]]\\
&= [v \underbrace{z \dots z}_{q(r+1)} x] \cdot \left(\sum_{i=0}^{r-1} (-1)^{q-1+qi} \binom{q(r+1)-1}{q-1+qi} \right)\\
&\quad - [v \underbrace{z \dots z}_{q(r+1)} y]\\
&\quad - [v \underbrace{z \dots z}_{q(r+1)} z] \left(\sum_{i=1}^{r} (-1)^{qi} \binom{q(r+1)-1}{qi} \right).
\end{aligned}$$

We have easily

$$\begin{aligned}
\sum_{i=0}^{r-1} (-1)^{q-1+qi} \binom{q(r+1)-1}{q-1+qi} &= \sum_{i=0}^{r-1} (-1)^i \binom{r}{i}\\
&= -(-1)^r = 1,
\end{aligned}$$

and similarly

$$\sum_{i=1}^{r} (-1)^{qi} \binom{q(r+1)-1}{qi} = -1,$$

so that

$$0 = [v \underbrace{z \dots z}_{q(r+1)} x] - [v \underbrace{z \dots z}_{q(r+1)} y] + [v \underbrace{z \dots z}_{q(r+1)} z] = 2[v \underbrace{z \dots z}_{q(r+1)} x],$$

as required.

We finally show that no pattern

$$\dots, q^{r-2}, 2q, q^{\sigma}, 2q-1 \tag{6.3}$$

is allowed for $\sigma > r-1$. (Because of the deflation argument we have already employed at the beginning of the proof of Lemma 5.5, this takes care of any trailing intermediate constituent here.) In fact, this should be followed by Lemma 5.4 by at least q^{r-2}. Deflating h times, we see by direct inspection of the sequence of two-step centralizers, very much as we did after Lemma 5.1, that we obtain a constituent of length at least

$$\sigma + 2 + r - 2 + 1 > 2r.$$

We have noted at the beginning of this section that the deflated algebra has parameter $2r$; we have thus obtained a contradiction.

7. Proof of Step 6

We are assuming that the sequence of two-step centralizers of the algebra L has an initial segment involving only two distinct two-step centralizers, and that the corresponding sequence of constituent lengths is

$$2q, q^{r-2}, 2q - p^{\beta}, \tag{7.1}$$

for some $0 \le \beta < h$. We want to prove that the whole algebra L has two distinct two-step centralizers, and that the sequence of constituent lengths consists of repetitions of the patterns

$$2q, q^{r-2} \qquad \text{or} \qquad 2q - p^{\beta}, q^{r-2} \ .$$

As mentioned in the Introduction, most of the proofs of this section, except for minor technical differences, and for one point we will comment upon later, were obtained first by C. Carrara in her PhD Thesis [Car98].

From Lemmas 5.4 and 5.6, and induction, we know that under (7.1) there are at least $r-2$ short constituents ending in x after a non-short one. So we start by assuming, by induction, that we have a segment

$$\dots, q^{r-2}, 2q, q^{r-2}, \tag{7.2}$$

where all short constituents end in x, and show first that the next constituent is *not* short. This implies in particular, by Lemma 3.13, that no further two-step centralizer can occur. We use the relation $[y \underbrace{z \dots z}_{(r+1)q-1} y] = 0$, which says that in (7.1) the last constituent is *not* short. Let v be an element at the end of the last short constituent before the long one in (7.2). We have to show $[v \underbrace{z \dots z}_{(r+1)q} y] = 0$. We compute with (2.1b)

$$\begin{aligned} 0 &= [v[y \underbrace{z \dots z}_{(r+1)q-1} y]] \\ &= [v[y \underbrace{z \dots z}_{(r+1)q-1}]y] - [vy[y \underbrace{z \dots z}_{(r+1)q-1}]] \ . \end{aligned}$$

The first term expands, up to a sign, as

$$[v \underbrace{z \dots z}_{(r+1)q} y] \cdot \left(\sum_{i=0}^{r-2} (-1)^{q-1+qi} \binom{q-1+qr}{q-1+qi} + (-1)^{q-1+qr} \binom{q-1+qr}{q-1+qr} \right) - [v \underbrace{z \dots z}_{(r+1)q} y],$$

while the second one expands, up to the same sign, to

$$-[v \underbrace{z \dots z}_{(r+1)q} z] \cdot \left(\sum_{i=1}^{r-1} (-1)^i \binom{q-1+qr}{qi} \right).$$

Summing up, we obtain

$$\begin{aligned}0 &= [v\underbrace{z\dots z}_{(r+1)q}y]\cdot\big(-(-1)^r+(-1)^r-1\big)\\ &\quad+[v\underbrace{z\dots z}_{(r+1)q}z]\cdot\big(1+(-1)^r\big)\\ &= -[v\underbrace{z\dots z}_{(r+1)q}y].\end{aligned}$$

An entirely similar argument shows that no constituent of the form $2q-p^\gamma$, with $\gamma\notin\{\beta,h\}$ can occur at the end of (7.2). We use the same v, and the relation $[y\underbrace{z\dots z}_{(r+2)q-p^\gamma-1}y]=0$, which states that the last constituent in (7.1) is *not* of length $2q-p^\gamma$. We have, using (2.1b) as in the previous case,

$$\begin{aligned}0 &= [v[y\underbrace{z\dots z}_{(r+2)q-p^\gamma-1}y]]\\ &= [v[y\underbrace{z\dots z}_{(r+2)q-p^\gamma-1}]y]-[vy[y\underbrace{z\dots z}_{(r+2)q-p^\gamma-1}]]\\ &\approx [v\underbrace{z\dots z}_{(r+2)q-p^\gamma}y]\cdot\\ &\qquad\cdot\left(\sum_{i=1}^{r-1}(-1)^{qi+q-p^\gamma-1}\binom{q(r+1)+q-p^\gamma-1}{qi+q-p^\gamma-1}\right.\\ &\qquad\qquad\left.+(-1)^{q(r+1)+q-p^\gamma-1}\right)\\ &\quad-[v\underbrace{z\dots z}_{(r+2)q-p^\gamma}y]\\ &\quad-[v\underbrace{z\dots z}_{(r+2)q-p^\gamma}z]\cdot\\ &\qquad\cdot\left(\sum_{i=1}^{r}(-1)^{qi+q-p^\gamma}\binom{q(r+1)+q-p^\gamma-1}{qi+q-p^\gamma}\right)\\ &= \big(-(-1)^{q-p^\gamma-1}(1+(-1)^r)-1\big)[v\underbrace{z\dots z}_{(r+2)q-p^\gamma}y]\\ &= -[v\underbrace{z\dots z}_{(r+2)q-p^\gamma}y].\end{aligned}$$

Here we have used the fact that $\gamma\neq h$, so that $0<q-p^\gamma-1<q-p^\gamma<q$, and $\binom{q-p^\gamma-1}{q-p^\gamma}=0$.

We now assume, again proceeding by induction, that we have a segment

$$\dots,q^{r-2},2q-p^\beta,q^{r-2}, \tag{7.3}$$

and show that the next constituent can be either long or of length $2q - p^{\beta}$. In both cases, by Lemma 3.13, no further two-step centralizer can thus occur. We distinguish three cases.

Suppose first, by way of contradiction, that in (7.3) a short constituent follows. We can deflate β times, according to Proposition 3.11, and suppose that (7.3) takes the form

$$\dots, q^{r-2}, 2q-1, q^{r-1}. \tag{7.4}$$

Take v to be an element at the end of the last short constituent before the intermediate one. In particular, $[vy] \ne 0$. The configuration (7.4) implies that $[v \underbrace{z \dots z}_{\lambda} y] \ne 0$, where

$$\lambda = q(r+1) - 1 = qr + q - 1.$$

We will prove $[v \underbrace{z \dots z}_{\lambda} yz] = [v \underbrace{z \dots z}_{\lambda+2}] = 0$, a contradiction, as we also have $[v \underbrace{z \dots z}_{\lambda} yy] = [v \underbrace{z \dots z}_{\lambda+1} y] = 0$, since constituents have positive length.

We use the relation $[y \underbrace{z \dots z}_{\lambda} y] = 0$, stating that the sequence of constituent lengths does *not* start as $2q, q^{r-1}$. Expanding with (2.1b), we get, using the fact that $[v \underbrace{z \dots z}_{\lambda} yy] = [v \underbrace{z \dots z}_{\lambda+1} y] = 0$, as above,

$$\begin{aligned}
0 &= [v[y \underbrace{z \dots z}_{\lambda} y]] \\
&\approx [vy[y \underbrace{z \dots z}_{\lambda}]] \\
&\approx [v \underbrace{z \dots z}_{\lambda+2}] \cdot \left(\sum_{i=0}^{r-1} (-1)^{qi+1} \binom{qr+q-1}{qi+1} \right) \\
&= [v \underbrace{z \dots z}_{\lambda+2}].
\end{aligned}$$

Now we deal with the case when (7.3) continues as

$$\dots, q^{r-2}, 2q - p^{\beta}, q^{r-2}, 2q - p^{\gamma}, \tag{7.5}$$

with $h > \gamma > \beta$. Deflating β times, according to Proposition 3.11, we may assume β to be zero. We employ a variation of the previous argument. We use the same v, so we assume $[v \underbrace{z \dots z}_{\lambda} y] \ne 0$, where this time

$$\lambda = q(r+1) + q - p^{\gamma} - 1,$$

and prove $[v \underbrace{z \dots z}_{\lambda+2}] = 0$, a contradiction. We use the relation $[y \underbrace{z \dots z}_{\lambda} y] = 0$, stating that the sequence of constituent lengths does *not* start as $2q, q^{r-2}, 2q - p^{\gamma}$.

Expanding with (2.1b), we get

$$\begin{aligned}
0 &= [v[y\underbrace{z\dots z}_{\lambda}y]]\\
&\approx [vy[y\underbrace{z\dots z}_{\lambda}]]\\
&\approx [v\underbrace{z\dots z}_{\lambda+2}]\cdot\\
&\qquad\cdot\Bigg((-1)^1\binom{q(r+1)+q-p^\gamma-1}{1}\\
&\qquad\quad+\sum_{i=1}^{r-1}(-1)^{qi+q-p^\gamma}\binom{q(r+1)+q-p^\gamma-1}{qi+q-p^\gamma}\Bigg)\\
&= [v\underbrace{z\dots z}_{\lambda+2}],
\end{aligned}$$

as

$$\binom{q(r+1)+q-p^\gamma-1}{qi+q-p^\gamma}\equiv\binom{q(r+1)}{qi}\cdot\binom{q-p^\gamma-1}{q-p^\gamma},$$

since $0 < q-p^\gamma-1 < q-p^\gamma < q$.

Finally, we consider the case when (7.5) holds for some $\gamma < \beta$. This has no counterpart in [Car98], where one works under the assumption $\beta = 0$. This time we may deflate γ times, according to Proposition 3.11, and assume γ to be zero, so that (7.3) takes the form

$$\dots, q^{r-2}, 2q-p^\beta, q^{r-2}, 2q-1. \tag{7.6}$$

We start with the same v as before, that is, at the end of the last short constituent before the one of length $2q-p^\beta$. The pattern (7.6) amounts to say $[v\underbrace{z\dots z}_{\lambda}y]\neq 0$, where

$$\lambda = q(r+1)+q-p^\beta-1.$$

We will prove

$$0=[v\underbrace{z\dots z}_{\lambda+p^\beta+1}]=[v\underbrace{z\dots z}_{\lambda}y\underbrace{z\dots z}_{p^\beta}].$$

This would exhibit a constituent of length at most $p^\beta < q$, a contradiction.

We define $u=[v\underbrace{z\dots z}_{p^\beta-1}]$, so that in particular $[uy]=0$. We have to prove

$$[u\underbrace{z\dots z}_{\lambda+2}]=0.$$

We use the relation $[y\underbrace{z\dots z}_{\lambda}x] = 0$ which says that the second non-short constituent is of length $2q - p^\beta$. We expand from the back, according to (2.1b),

$$\begin{aligned}
0 &= [u[y\underbrace{z\dots z}_{\lambda}x]]\\
&= [u[y\underbrace{z\dots z}_{\lambda}]x] - [ux[y\underbrace{z\dots z}_{\lambda}]]\\
&\approx [u\underbrace{z\dots z}_{\lambda+1}x]\cdot\\
&\quad\cdot\left((-1)^{p^\beta-1}\binom{\lambda}{p^\beta-1} + \sum_{i=2}^{r}(-1)^{qi+p^\beta-2}\binom{\lambda}{qi+p^\beta-2}\right)\\
&\quad - [u\underbrace{z\dots z}_{\lambda+1}z]\cdot\left((-1)^{p^\beta}\binom{\lambda}{p^\beta} + \sum_{i=2}^{r}(-1)^{qi+p^\beta-1}\binom{\lambda}{qi+p^\beta-1}\right)\\
&\quad - [u\underbrace{z\dots z}_{\lambda+1}y].
\end{aligned}$$

We evaluate the binomial coefficients modulo p, according to Lucas' Theorem.

$$\begin{aligned}
(-1)^{p^\beta-1}\binom{\lambda}{p^\beta-1} &= (-1)^{p^\beta-1}\binom{q(r+1)+q-p^\beta-1}{p^\beta-1}\\
&\equiv (-1)^{p^\beta-1}\binom{q-2p^\beta+p^\beta-1}{p^\beta-1}\\
&\equiv \binom{p^\beta(p^{h-\beta}-2)}{0}(-1)^{p^\beta-1}\binom{p^\beta-1}{p^\beta-1}\\
&\equiv 1.
\end{aligned}$$

In a similar fashion

$$\begin{aligned}
(-1)^{qi+p^\beta-2}\binom{\lambda}{qi+p^\beta-2} &\equiv (-1)^i\binom{r+1}{i}\cdot(-1)^{p^\beta-2}\binom{q-2p^\beta+p^\beta-1}{p^\beta-2}\\
&\equiv (-1)^i\binom{r+1}{i},
\end{aligned}$$

and

$$(-1)^{qi+p^\beta-1}\binom{\lambda}{qi+p^\beta-1} \equiv (-1)^i\binom{r+1}{i}.$$

Also

$$\begin{aligned}
(-1)^{p^\beta}\binom{\lambda}{p^\beta} &\equiv (-1)^{p^\beta}\binom{q-2p^\beta+p^\beta}{p^\beta}\\
&\equiv (-1)^{p^\beta}\binom{p^\beta(p^{h-\beta}-2)}{p^\beta}\\
&\equiv 2.
\end{aligned}$$

Keeping in mind that

$$\sum_{i=2}^{r}(-1)^i\binom{r+1}{i} = -\left(1-\binom{r+1}{1}+\binom{r+1}{r+1}\right) = -1,$$

and that $[u\underbrace{z\dots z}_{\lambda+1}y]=0$, we obtain

$$\begin{aligned}
0 &= [u[y\underbrace{z\dots z}_{\lambda}x]]\\
&= [u\underbrace{z\dots z}_{\lambda+1}x]\cdot\left(1+\sum_{i=2}^{r}(-1)^i\binom{r+1}{i}\right)\\
&\quad - [u\underbrace{z\dots z}_{\lambda+1}z]\cdot\left(2+\sum_{i=2}^{r}(-1)^i\binom{r+1}{i}\right)\\
&\approx [u\underbrace{z\dots z}_{\lambda+1}z],
\end{aligned}$$

as required.

8. The algebras of Albert-Frank-Shalev

We now prove Step 8.

The situation is the following. We have an algebra L with initial segment of the sequence of constituent lengths of the form

$$2q, q^{r-2}, 2q-p^{\beta},$$

for some $0 \le \beta < h$, where no two-step centralizer other than the first two occurs. Here $q = p^h$ and $r = p^k$ as usual. We know from Step 6 that the sequence of constituent lengths consists of repetitions of the patterns

$$2q, q^{r-2} \qquad \text{or} \qquad 2q-p^{\beta}, q^{r-2}.$$

In view of Proposition 3.11, we may deflate β times, and assume that $\beta = 0$. We can also suppose, in view of Step 7, that another intermediate constituent appears in the sequence of constituent lengths. The sequence of constituent lengths of L thus begins like

$$2q, q^{r-2}, 2q-1, (q^{r-2}, 2q)^{\sigma-1}, q^{r-2}, 2q-1. \tag{8.1}$$

We have to show that $\sigma = p^l$ for some $l \ge 0$.

We first show that σ is odd. (8.1) implies that the commutator $[y\underbrace{z\dots z}_{\mu}y]$, of weight $\mu+2$, where $\mu = qr(\sigma+1)+2q-3$, is non-zero. We will prove that if σ is even, then $[y\underbrace{z\dots z}_{\mu}y]=0$, a contradiction. Let

$$\lambda = qr\frac{\sigma}{2} + q\frac{r+1}{2} + \frac{q-3}{2},$$

so that $2\lambda = \mu$. We expand with (2.1b)

$$\begin{aligned} 0 &= [[y \underbrace{z \dots z}_{\lambda}][y \underbrace{z \dots z}_{\lambda}]] \\ &\approx [y \underbrace{z \dots z}_{\mu} y] + \vartheta [y \underbrace{z \dots z}_{\mu+1}]. \end{aligned}$$

Now the first term of the coefficient ϑ is, up to a sign,

$$\binom{\lambda}{2q-1} = \binom{qr\frac{\sigma}{2} + q\frac{r+1}{2} + \frac{q-3}{2}}{q+q-1} \equiv 0 \pmod{p},$$

by Lucas' Theorem, as the top entry in the binomial coefficient is congruent to $\dfrac{q-3}{2} \pmod{q}$, while the bottom one is congruent to $q-1 > \dfrac{q-3}{2}$. It is not difficult to see that all terms in ϑ vanish similarly. Therefore $\vartheta = 0$, and $[y \underbrace{z \dots z}_{\mu} y] = 0$, as claimed.

We now prove that the sequence of constituent lengths of L cannot have an initial segment of the form

$$\begin{aligned} &2q, q^{r-2}, 2q-1, (q^{r-2}, 2q)^{\sigma-1}, \\ &\quad q^{r-2}, 2q-1, (q^{r-2}, 2q)^{\tau-1}, \\ &\quad q^{r-2}, 2q-1, q, \end{aligned} \tag{8.2}$$

for any $\tau < \sigma$. If there is a segment of the form (8.2), then the commutator $[y \underbrace{z \dots z}_{\mu} y]$, of weight $\mu + 2$, where

$$\mu = qr(\sigma + \tau + 1) + 3q - 4,$$

is non-zero. We will show that this commutator vanishes, thereby obtaining a contradiction. We use the relation $[y \underbrace{z \dots z}_{\lambda} y] = 0$, where

$$\lambda = qr(\tau + 1) + 2q - 3.$$

This relation states that the initial segment of the sequence of constituent lengths is *not* of the form

$$2q, q^{r-2}, 2q-1, (q^{r-2}, 2q)^{\tau-1}, q^{r-2}, 2q-1.$$

(Remember we have taken $\tau < \sigma$.) Take an appropriate non-zero homogeneous element v so that $[v[y \underbrace{z \dots z}_{\lambda} y]]$ has the appropriate weight $\mu + 2$. This means v has weight $\mu - \lambda = qr\sigma + q - 1$. Inspection shows that v lies in the middle of the last long constituent in the first line of (8.2). Therefore $[v \underbrace{z \dots z}_{q}]$ lies at the end of this long constituent, and of course $[vy] = 0$. As we start expanding with (2.1a)

$$0 = [v[y \underbrace{z \dots z}_{\lambda} y]] = [v[y \underbrace{z \dots z}_{\lambda}] y],$$

we first encounter a coefficient

$$\sum_{i=1}^{r-1} (-1)^{qi} \binom{\lambda}{qi},$$

as we go through the $r-2$ short constituents following the long one. This evaluates modulo p, according to Lucas' Theorem, to

$$\begin{aligned}\sum_{i=1}^{r-1} (-1)^{qi} \binom{qr(\tau+1)+q+q-3}{qi} &\equiv \sum_{i=1}^{r-1} (-1)^{i} \binom{r(\tau+1)+1}{i} \\ &\equiv (-1)^1 \binom{r(\tau+1)+1}{1} \\ &\equiv -1.\end{aligned}$$

We claim that all the remaining binomial coefficients in (2.1a) vanish. In fact the next one, as we go through the first intermediate constituent in the second line of (8.2), is

$$\binom{qr(\tau+1)+q+q-3}{qr+q-1}.$$

This vanishes, because the top entry is congruent to $q-3 \pmod{p}$, while the bottom one is congruent to $q-1 > q-3$. The same applies to all the remaining coefficients; as we pass the last intermediate constituent, the bottom entry in the binomial coefficient becomes congruent to $q-2 > q-3$.

Let now p^l be the highest power of p (possibly 1) dividing σ. Write $\sigma = p^l \sigma'$, so that $\sigma' \not\equiv 0 \pmod{p}$. We suppose $p^l \neq \sigma$, so that $p^l < \sigma$, and obtain a contradiction.

In fact we have shown that (8.2) does not hold for $\tau \leq p^l$. By Step 6, the sequence of constituent lengths of L must have an initial segment of the form

$$2q, q^{r-2}, 2q-1, (q^{r-2}, 2q)^{\sigma-1}, q^{r-2}, 2q-1, (q^{r-2}, 2q)^{p^l}. \tag{8.3}$$

In particular, the commutator $[y \underbrace{z \dots z}_{\mu} y]$, of weight $\mu + 2$, where

$$\mu = qr(\sigma + p^l + 1) + 2q - 3,$$

is non-zero. We will show that this commutator vanishes, thereby obtaining a contradiction. Take

$$\lambda = qr\frac{\sigma + p^l}{2} + q\frac{r+1}{2} + \frac{q-3}{2},$$

so that $2\lambda = \mu$, and expand with (2.1b) the balanced relation

$$
\begin{aligned}
0 &= [[y \underbrace{z \dots z}_{\lambda}][y \underbrace{z \dots z}_{\lambda}]] \\
&\approx [y \underbrace{z \dots z}_{\mu} y] \\
&\quad + [y \underbrace{z \dots z}_{\mu+1}] \cdot \left(\sum_{i=0}^{p^l-1} \sum_{j=2}^{r} (-1)^{qri+qj} \binom{\lambda}{qri+qj} \right).
\end{aligned}
\tag{8.4}
$$

A word of comment is in order. The summation in (8.4) comprises the terms of (2.1b) as we go through the final segment $(q^{r-2}, 2q)^{p^l}$ of (8.3). As we pass the second intermediate constituent in (8.3), the bottom entry of the relevant binomial coefficient becomes congruent to $q-1 \pmod q$, while the top entry λ is congruent to $\dfrac{q-3}{2} < q-1$. Therefore the terms we have not displayed vanish as above.

We compute the summation in (8.4). Note first

$$
\sum_{i=0}^{p^l-1} \sum_{j=2}^{r} (-1)^{qri+qj} \binom{\lambda}{qri+qj} = \sum_{i=0}^{p^l-1} (-1)^i \sum_{j=2}^{r} (-1)^j \binom{\lambda}{qri+qj}.
$$

For a given i we now have, using Lucas' Theorem,

$$
\begin{aligned}
&(-1)^i \sum_{j=2}^{r} (-1)^j \binom{\lambda}{qri+qj} = \\
&\qquad = (-1)^i \sum_{j=2}^{r} (-1)^j \binom{qr\frac{\sigma+p^l}{2} \;+\; q\frac{r+1}{2} \;+\; \frac{q-3}{2}}{qri \;+\; qj} \\
&\qquad = (-1)^i \binom{\frac{\sigma+p^l}{2}}{i} \sum_{j=2}^{r-1} (-1)^j \binom{\frac{r+1}{2}}{j} + (-1)^{i+1} \binom{\frac{\sigma+p^l}{2}}{i+1}.
\end{aligned}
\tag{8.5}
$$

Now $(r+1)/2 \le r-1$, as $r \ge 3$. Therefore

$$
\begin{aligned}
\sum_{j=2}^{r-1} (-1)^j \binom{\frac{r+1}{2}}{j} &= \sum_{j=2}^{(r+1)/2} (-1)^j \binom{\frac{r+1}{2}}{j} \\
&= -\left(1 - \frac{r+1}{2}\right) \\
&\equiv -\frac{1}{2} \pmod p.
\end{aligned}
$$

Suppose first $p^l \ne 1$. As p^l divides σ, and $i < p^l$, Lucas' Theorem yields that $\binom{\frac{\sigma+p^l}{2}}{i} \not\equiv 0 \pmod p$ only for $i = 0$, and $\binom{\frac{\sigma+p^l}{2}}{i+1} \not\equiv 0 \pmod p$ only for $i = p^l - 1$.

Therefore for $0 < i < p^l - 1$ the summation (8.5) vanishes.

For $i = 0$ the summation (8.5) yields $-\dfrac{1}{2}$, while for $i = p^l - 1$ it yields

$$(-1)^{i+1}\binom{\frac{\sigma+p^l}{2}}{i+1} = -\binom{p^l\frac{\sigma'+1}{2}}{p^l} \equiv -\frac{\sigma'+1}{2} \pmod{p}.$$

In conclusion, the summation in (8.4) evaluates to

$$-\frac{1}{2} - \frac{\sigma'+1}{2} = -\frac{\sigma'+2}{2},$$

so that (8.4) yields

$$0 = [y\underbrace{z\dots z}_{\mu}y] - \frac{\sigma'+2}{2}[y\underbrace{z\dots z}_{\mu+1}] = -\frac{\sigma'}{2}[y\underbrace{z\dots z}_{\mu}y],$$

as $[y\underbrace{z\dots z}_{\mu}x] = 0$. Since $\sigma' \not\equiv 0 \pmod{p}$, we obtain a contradiction.

It is easy to see that one obtains exactly the same result when $p^l = 1$.

9. Specialization of two-step centralizers

In this section we will prove Proposition 4.3 about specialization of centralizers.

So let x_1 and x_2 be as in the statement of the Proposition. As in Section 3, we can clearly write all two-step centralizers other than $\mathbf{F}x_1$ in the form

$$\mathbf{F}\cdot(x_2 - \alpha x_1), \tag{9.1}$$

for some $\alpha \in \mathbf{F}$. Now let λ be an indeterminate, and consider the algebra

$$L(\lambda) = L \otimes_{\mathbf{F}} \mathbf{F}(\lambda)$$

over the field $\mathbf{F}(\lambda)$. We can redefine x_1 as $\lambda^{-1}x_1$ in $L(\lambda)$. Thus $\mathbf{F}(\lambda)\cdot x_1$ and $\mathbf{F}(\lambda)\cdot x_2$ are unchanged, whereas (9.1) becomes

$$\mathbf{F}(\lambda)\cdot(x_2 - \alpha\lambda x_1), \tag{9.2}$$

with $\alpha \in \mathbf{F}$. As in Section 3 of I, we can describe $\mathrm{ad}(x_1)$ and $\mathrm{ad}(x_2)$ by introducing the following basis of $L(\lambda)$:

$$v_0 = x_1, \qquad v_1 = x_2 \qquad v_2 = [x_2x_1],$$

and for $i \geq 3$

$$v_i = \begin{cases} [v_{i-1}x_1] & \text{if } [v_{i-1}x_1] \neq 0 \\ [v_{i-1}x_2] & \text{if } [v_{i-1}x_1] = 0. \end{cases} \tag{9.3}$$

When $[v_{i-1}x_1]$ is different from zero, then $C_{L(\lambda)_1}(v_{i-1}) = \mathbf{F}(\lambda)\cdot(x_2 - \alpha\lambda x_1)$ for some $\alpha \in \mathbf{F}$, so that

$$[v_{i-1}x_2] = [v_{i-1}, x_2 - \alpha\lambda x_1 + \alpha\lambda x_1] = \alpha\lambda[v_{i-1}x_1]. \tag{9.4}$$

It follows from (9.3) and (9.4) that the coefficients of the adjoint representation of x_1 and x_2 with respect to the basis of the v_i are in $\mathbf{F}[\lambda]$. Consider the Lie subring

S of $L(\lambda)$ generated by x_1 and x_2. Then the structure constants of S are in the polynomial ring $\mathbf{F}[\lambda]$. We can thus consider the algebra over $\mathbf{F}$

$$T = S \otimes_{\mathbf{F}[\lambda]} \mathbf{F}[\lambda]/(\lambda).$$

The relations (9.3) show that T is still an algebra of maximal class over $\mathbf{F}$, which is effectively obtained by letting $\lambda = 0$ in (9.2). The process of going from L to T can thus be described in terms of the respective sequences of two-step centralizers by saying that we are leaving the two-step centralizer x_1 unchanged, and we are changing all other two-step centralizers to x_2, as claimed.

10. More than two distinct two-step centralizers

Using the result about specialization of two-step centralizers, we are now able to complete the classification, by proving Step 11, that is, if the algebra of maximal class L has three or more distinct two-step centralizers, then its constituents are long or short, so that by Step 2 L is inflated.

We first prove Step 10. Let z be any centralizer higher than the first two in order of occurrence. We can leave z fixed, and specialize all other centralizers to y. In the resulting algebra, z will thus play the role of the second two-step centralizer in order of occurrence. By Proposition 3.2, it will first occur as

$$C_{L_1}(L_{2p^n}),$$

as stated.

Now we prove Lemma 3.13. Let q be the parameter of L. Again, let z be any centralizer higher than the first two in order of occurrence. Suppose that in the sequence of two-step centralizers we have the segment

$$w_1 \underbrace{y \dots y}_{m-1} z \underbrace{y \dots y}_{n-1} w_2,$$

where $w_1, w_2 \neq y$. Therefore $\underbrace{y \dots y}_{m-1} z$ and $\underbrace{y \dots y}_{n-1} w_2$ are two constituents, so that $m, n \geq q$. Keep the second two-step centralizer x fixed, and specialize all others, including z, to y. This does not alter the parameter q. In the resulting algebra, we have now a constituent that comprises at least

$$\underbrace{y \dots y}_{m-1} y \underbrace{y \dots y}_{n-1} w_2.$$

This has thus length at least $m + n \geq 2q$. Since the length of a constituent is at most $2q$, it follows that $m = n = q$, and $w_1 = w_2 = x$. We have thus proved Lemma 3.13, and also the other related results of [CJ99].

We are now able to prove Step 11. Keep x fixed in L, and specialize all centralizers other than x to y. As remarked, we are not altering the parameter q. We obtain an algebra of maximal class L' with two distinct two-step centralizers. Because of Lemma 3.13, the first occurrence of a centralizer higher than the second one in L will give rise to a long constituent in L'. Because of Step 6, all the constituents leading to this occurrence in L are either short or long. Hence the first non-short constituent of L' is long. It follows from Step 4 that *all* constituents in

L' are either short or long. If we translate this from L' back to L, all that happens is that some of the long constituents of L' are split into two short ones in L by a two-step centralizer higher than the second one. Thus in L too all constituents are short or long. Step 11 is proved.

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