PUB. IRMA, Lille - 1993
Vol. 33, N$^{\circ}$ X
hep-th/9312112

SYMPLECTIC AND POISSON STRUCTURES OF CERTAIN MODULI SPACES

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November 30, 1993

Abstract. Let π be the fundamental group of a closed surface and G a Lie group with a biinvariant metric, not necessarily positive definite. It is shown that a certain construction due to A. Weinstein relying on techniques from equivariant cohomology may be refined so as to yield (i) a symplectic structure on a certain smooth manifold $\mathcal M(\mathcal P,G)$ containing the space $\mathrm{Hom}(\pi,G)$ of homomorphisms and, furthermore, (ii) a hamiltonian G-action on $\mathcal M(\mathcal P,G)$ preserving the symplectic structure, with momentum mapping $\mu\colon \mathcal M(\mathcal P,G) \to g^*$, in such a way that the reduced space equals the space $\mathrm{Rep}(\pi,G)$ of representations. Our approach is somewhat more general in that it also applies to twisted moduli spaces; in particular, it yields the Narasimhan-Seshadri moduli spaces of semistable holomorphic vector bundles by *symplectic reduction in finite dimensions*. This implies that, when the group G is compact, such a twisted moduli space inherits a structure of *stratified symplectic space*, and that the strata of these twisted moduli spaces have finite symplectic volume.

1991 *Mathematics Subject Classification.* 32G13, 32G15, 32S60, 58C27, 58D27, 58E15, 81T13.

Key words and phrases. Geometry of principal bundles, singularities of smooth mappings, symplectic reduction with singularities, Yang-Mills connections, stratified symplectic space, geometry of moduli spaces, representation spaces.

Introduction

Let π be the fundamental group of a closed surface and G a Lie group with a biinvariant metric, not necessarily positive definite. The symplectic structure on the smooth part of (the components of) the moduli space $\mathrm{Rep}(\pi, G) = \mathrm{Hom}(\pi, G)/G$ [2], [6] has recently been obtained by A. Weinstein by entirely finite dimensional techniques [23]. Weinstein constructed a closed equivariant 2-form on (the smooth part of) $\mathrm{Hom}(\pi, G)$ and showed by techniques from equivariant cohomology [1] that this 2-form descends to (the non-singular part of) $\mathrm{Rep}(\pi, G)$. In this paper we refine Weinstein's method by showing that in fact it yields (i) a symplectic structure on a certain smooth manifold $\mathcal{M}(\mathcal{P}, G)$ containing $\mathrm{Hom}(\pi, G)$ (as a deformation retract) and, furthermore, (ii) a hamiltonian G-action on $\mathcal{M}(\mathcal{P}, G)$ preserving the symplectic structure, with momentum mapping $\mu \colon \mathcal{M}(\mathcal{P}, G) \to g^*$, in such a way that the reduced space equals $\mathrm{Rep}(\pi, G)$. Our approach is somewhat more general in that it also applies to twisted moduli spaces, see Section 6 below. In particular, it yields the NARASIMHAN-SESHADRI [21] moduli spaces of semistable holomorphic vector bundles by *symplectic reduction, applied to a smooth finite dimensional symplectic manifold with a hamiltonian action of the unitary group*, which is *finite dimensional*. Our result, apart from being interesting in its own right, reveals some interesting and attractive geometric properties of these twisted moduli spaces. Firstly, it implies that, when the group G is compact, with reference to the decomposition of orbit types, such a twisted moduli space inherits a structure of *stratified symplectic space* in the sense of [22]; in particular, this yields a description of the behaviour the symplectic or more generally Poisson structure of these moduli spaces in their singularities (in the appropriate sense); see our paper [12] where the Poisson geometry has been worked out explicitly in some special cases. Secondly our results implies that the strata of these twisted moduli spaces have finite symplectic volume. See Section 7 below for details. The finiteness of the symplectic volume of the moduli spaces of semistable holomorphic vector bundles has been obtained also by NARASIHMAN [unpublished] by an argument involving the Quillen metric. Moreover it is likely that the momentum mapping which we found can be studied further by means of the well known convexity results, cf. [17] and the literature there. We also expect that KIRWAN'S methods [18] or suitable extensions thereof become applicable so that more calculations in the cohomology of these moduli spaces can be done.

Our approach in fact applies to arbitrary groups π with a finite presentation and yields then somewhat more general results. It may be viewed as a step towards the "grand unified theory" searched for by A. Weinstein in that, cf. Lemma 1 below, it explains the occurence of the form ϑ in [23] which there was put in "by hand".

An extension of our construction yields other cohomology classes from the characteristic ring, in fact, it also applies to integral cohomology via the cohomology of the classifying space of G. We shall make this precise at another occasion.

For the flat case, the space $\mathrm{Rep}(\pi, G)$ has been obtained as the reduced space of a certain smooth finite dimensional symplectic manifold with hamiltonian G-action also by L. Jeffrey [14]. Her construction involves gauge theory techniques. After completion of the paper I learnt that some of the same results have been obtained by her independently [15].

I am much indebted to J. Stasheff, A. Weinstein, and Dong Yan for a number of

comments which helped clarify the exposition, and to R. Sjamaar for explaining me the requisite technical details for the proof in [22] of the finiteness of the symplectic volumes of the strata of a reduced space under appropriate circumstances; in fact the proof of (7.2) below crucially relies thereupon.

1. Equivariant Maurer-Cartan calculus

Let G be a Lie group, with Lie algebra g, and let $\cdot$ be an adjoint action invariant symmetric bilinear form on g, not necessarily definite nor positive. We use the notation and terminology in [23].

For ease of exposition we recall some notions from equivariant de Rham cohomology [1], [3]. Let M be a G-manifold M, and consider its bi-differential graded algebra $(\Omega_G^{*,*}(M); d, \delta_G)$ of G-equivariant forms; here the Lie algebra g is concentrated in degree two, and $\Omega_G^{2j,k}(M)$ refers to the G-equivariant homogeneous degree j polynomial maps on g with values in the usual degree k de Rham forms $\Omega^k(M)$. The usual de Rham differential on M is G-invariant and hence induces an operator
— $d\colon \Omega_G^{2j,k}(M) \to \Omega_G^{2j,k+1}(M)$, for every $k, j \geq 0$;
the additional *equivariant* operator
— $\delta_G\colon \Omega_G^{2j,k}(M) \to \Omega_G^{2j+2,k-1}(M)$, for $k \geq 1$ and $j \geq 0$,
assigns to a G-invariant polynomial map $\alpha\colon g \to \Omega^k(M)$ the G-invariant polynomial map $\delta_G(\alpha)\colon g \to \Omega^{k-1}(M)$ which is given by

$$(\delta_G(\alpha))(X) = -i_{X_M}(\alpha(X)),$$

where X_M denotes the vector field given by the Lie algebra element X acting on M. It is clear that $dd = 0$ and a straightforward contraction and Lie derivative calculation shows that

$$d\delta_G + \delta_G d = 0, \quad \delta_G\delta_G = 0. \tag{1}$$

Consequently the operator

$$d + \delta_G$$

is a differential on the total object of $\Omega_G^{*,*}(M)$. It is clear that $(\Omega_G^{*,*}(\cdot); d, \delta_G)$ is a functor in the appropriate sense. This is all we need to know about equivariant cohomology; for our purposes it is merely an excellent tool to say something concisely which might have been technically more involved in some other language. We only mention that we could as well have defined $(\delta_G(\alpha))(X)$ by $i_{X_M}(\alpha(X))$ but we have chosen the above operator to arrive at the same formulas as in [23].

For every $q \geq 1$, the direct product G^q of q factors of G will be considered as a G-manifold, with G-action by inner automorphisms on each copy of G. In the usual way these fit together to a simplicial manifold NG having G^q in degree q, and NG inherits an obvious structure of simplicial G-manifold [4], [5]. Application of the functor $(\Omega_G^{*,*}(\cdot); d, \delta_G)$ to NG yields a cosimplicial bi-differential graded algebra $(\Omega_G^{*,*}(NG); d, \delta_G)$ — we did not indicate the cosimplicial structure explicitly. Its realization yields the *equivariant bar de Rham* tricomplex

$$\left(\Omega_G^{*,*}(G^*); \delta, d, \delta_G\right).$$

Here δ refers to the (inhomogeneous) bar complex operator. Recall (p. 116 of [19]) it is given by the formula

$$\begin{aligned}\delta(f)(a_0,\dots,a_q) = (-1)^{q+1}\,&[f(a_1,\dots,a_q)\\ &+\sum_{j=1}^{q-1}(-1)^j f(a_0,\dots,a_{j-1}a_j,\dots,a_q)\\ &+(-1)^q f(a_0,\dots,a_{q-1})]\,.\end{aligned}$$

The corresponding (inhomogeneous) bar complex operator $\delta\colon \Omega_G^{*;*}(G^q)\to\Omega_G^{*;*}(G^{q+1})$ looks like

$$\delta f = (-1)^{q+1}\sum_{j=0}^{q}(-1)^j\delta_j f;$$

here f refers to a de Rham form on G^q, $\delta_0 f$ and $\delta_q f$ are the forms on G^{q+1} arising from pulling back f from G^q via the projection mappings from G^{q+1} to G^q forgetting the first and last copy of G, respectively, while for $1\le j\le q-1$, $\delta_j f$ is the form on G^{q+1} arising from pulling back f from G^q via the map which multiplies the $(j-1)$th with the jth variable and fixes the rest. By construction, the operator δ is compatible with the other structure, and hence the operator d_G, which on elements of total *form degree* p is given by

$$d_G = \delta + (-1)^p(d+\delta_G),$$

is a differential on the totalization for $\left(\Omega_G^{*;*}(G^*);\delta,d,\delta_G\right)$. The resulting complex is the *total equivariant bar de Rham complex*. The above sign $(-1)^{q+1}$ does not occur in [23]; according to the Eilenberg-Koszul convention it is the formally appropriate sign. Since the only explicit instance of the operator δ in [23] refers to a situation where $q=1$, see (2) and (4) below, our sign convention does not affect the formulas.

For intelligibility we now recall briefly the equivariant Maurer-Cartan calculus from [23]: Denote by ω the g-valued left-invariant 1-form on G which maps each tangent vector to the left invariant vector field having that value. The corresponding right invariant form will be denoted by $\overline{\omega}$. Recall that the *triple product* $(x,y,z)\mapsto[x,y]\cdot z$ yields an alternating trilinear form on g; for our purposes, the appropriate form is

$$\tau(x,y,z)=\frac{1}{2}[x,y]\cdot z,\quad x,y,z\in g;$$

it is closed for the usual Cartan-Chevalley-Eilenberg Lie algebra cohomology operator. Its left translate is a closed invariant 3-form λ on G which by the way coincides with the right translate of τ. Notice

$$\lambda=\frac{1}{12}[\omega,\omega]\cdot\omega.$$

At the risk of making a mountain of a molehill we recall that the present conventions entail that, for arbitrary left invariant vector fields X and Y on G, we have $[\omega,\omega](X,Y)=2[X,Y]$ whence the sum of the three requisite shuffles yields

$$([\omega,\omega]\cdot\omega)(X,Y,Z)=6[X,Y]\cdot Z,$$

for arbitrary left invariant vector fields X, Y, Z, and thence the left translate λ of τ looks like stated. Next, if α is any differential form on G, we denote by α_j the pullback of α to $G \times G$ by the projection p_j to the j'th component. Let

$$\Omega = \frac{1}{2}\omega_1 \cdot \overline{\omega}_2.$$

This is an alternating 2-form on $G \times G$. Here the convention is that

$$(\omega_1 \cdot \overline{\omega}_2)(U, V) = \omega_1(U) \cdot \overline{\omega}_2(V) - \overline{\omega}_2(U) \cdot \omega_1(V),$$

that is, there is *no* factor of $\frac{1}{2}$, and the pairing of arbitrary forms will likewise involve shuffles instead of all permutations. This is the convention coming into play in [23]; it is forced by the form of the Maurer-Cartan equations in [23] and, furthermore, by the identity (4) below, and we keep this convention to arrive at the same formulas as in [23]. Finally, consider the element $\vartheta \in \Omega_G^{2,1}(G)$, that is, the linear G-invariant map $\vartheta \colon g \to \Omega^1(G)$ whose adjoint $\vartheta^\flat$, viewed as a g^*-valued 1-form on G, amounts to $\frac{1}{2}(\omega + \overline{\omega})$, combined with the adjoint $g \to g^*$ of the given 2-form; thus, when we view $X \in g$ as a constant g-valued 0-form on G,

$$\vartheta(X) = \frac{1}{2} X \cdot (\omega + \overline{\omega}).$$

We note that the forms $\lambda, \Omega, \vartheta$ are exactly half the forms denoted by the same symbols in [23]. The reason why we work here with half the forms in [23] will become clear in Sections 4 and 5 below. In fact, the form Ω will eventually yield the symplectic structures on moduli spaces.

With these preparations out of the way, the equivariant Maurer-Cartan calculus amounts to the following relations:

$$d\Omega = \delta\lambda \qquad [23]\ (3.3), \tag{2}$$

$$\delta\Omega = 0 \qquad [23]\ (3.4), \tag{3}$$

$$\delta_G\Omega = -\delta\vartheta \qquad [23]\ (4.4), \tag{4}$$

$$d\lambda = 0 \qquad [23]\ (3.1), \tag{5}$$

$$\delta_G\lambda = d\vartheta \qquad [23]\ (4.1), \tag{6}$$

$$\delta_G\vartheta = 0 \qquad [23]\ (4.3). \tag{7}$$

These identities say that (i) the form $\Omega - \lambda$ is an equivariant closed form (*not* equivariantly closed) of (total) degree 4 and that (ii) the form $Q_4 = \Omega - \lambda + \vartheta$ is an equivariantly closed element of (total) degree 4 in the total complex of the equivariant bar de Rham complex $(\Omega_G^{*,*}(G^*); d, \delta, \delta_G)$, cf. [23] (4.5).

We conclude this Section with another consequence of these relations which will be crucial in the sequel. Let $h \colon \Omega^*(g) \to \Omega^{*-1}(g)$ be an adjoint action invariant homotopy operator, so that the operator $dh + hd$ equals the identity on $\Omega^*(g)$. For example we could take the usual homotopy operator given by integration of forms along linear paths, but this will not be important for us since we shall only need the mentioned formal properties of h. Write $\rho = \exp^*(\lambda) \in \Omega^3(g)$ and let $\beta = h(\rho) \in \Omega^2(g)$. Then it is obvious that

$$d\beta = \rho \tag{8}$$

in the *de Rham* complex of g. Moreover, since h is equivariant, so is β, that is, $\beta \in \Omega^{0,2}_G(g)$, with reference to the adjoint representation. As usual we write $\exp\colon g \to G$ for the exponential mapping and $\exp^*$ for the induced map from $\Omega^{*,*}_G(G)$ to $\Omega^{*,*}_G(g)$.

Lemma 1. *There is a smooth G-equivariant map $\psi\colon g \to g^*$, uniquely determined by the requirement that $\psi(0) = 0$, whose adjoint $\psi^\sharp\colon g \to \Omega^0(g) = C^\infty(g)$, that is, $\psi^\sharp \in \Omega^{2,0}_G(g)$, satisfies*

$$d\psi^\sharp = \exp^*(\vartheta) + \delta_G(\beta).$$

To prove this we observe at first that (1) and (6) imply

$$d(\exp^*(\vartheta) + \delta_G\beta) = 0,$$

that is, $\exp^*(\vartheta) + \delta_G\beta$ is a cocycle for the de Rham operator d. In fact,

$$\begin{aligned} d(\exp^*(\vartheta) + \delta_G\beta) &= \exp^*(d\vartheta) + d\delta_G\beta \\ &= \exp^*(d\vartheta) - \delta_G(d\beta) \\ &= \exp^*(d\vartheta) - \delta_G(\rho) \\ &= \exp^*(d\vartheta - \delta_G(\lambda)) = 0. \end{aligned}$$

Hence the smooth G-equivariant map ψ whose adjoint is given by

$$\psi^\sharp = h \circ (\exp^*(\vartheta) + \delta_G(\beta))\colon g \to C^\infty(g)$$

has the asserted property. It is manifestly unique up to a constant. □

Corollary. *The value of the derivative of ψ at an arbitrary point of the centre of g equals the adjoint $g \to g^*$ of the given 2-form on g. Consequently the restriction of ψ to the centre equals itself the adjoint of this 2-form.*

Proof. Let z be a point of the centre of g. By Lemma 1, the derivative amounts to the sum of $\vartheta^\flat_z\colon g \to g^*$, the adjoint $\vartheta^\flat$ of ϑ, viewed as a g^*-valued 1-form on G, at $\exp(z) \in G$, and $(\delta_G\beta)^\flat_z\colon g \to g^*$, the adjoint $(\delta_G\beta)^\flat$ of $\delta_G\beta \in \Omega^{2,1}_G(g)$, viewed as a g^*-valued 1-form on g, at $z \in g$. Since $\vartheta^\flat$, viewed as a g^*-valued 1-form on G, amounts to $\frac{1}{2}(\omega + \overline{\omega})$, combined with the adjoint $g \to g^*$ of the given 2-form on g, the derivative $\vartheta^\flat_z$ equals the adjoint $g \to g^*$ of this 2-form. To understand the other term, recall that the vector field X_G for the adjoint representation amounts to the sum $X + \overline{X}$, where $\overline{X}$ is the right invariant vector field on G having at the identity of G the same value as $X \in g$. By construction, $\delta_G\beta$ is the equivariant linear map $\delta_G\beta\colon g \to \Omega^1(g)$ given by

$$((\delta_G\beta)(X))(Y) = -\beta(X_G, Y),$$

for $X, Y \in g$, where X_G and Y refer to the vector fields on g arising from pulling back via the exponential mapping the vector fields on G denoted by the same symbols; in particular Y should *not* be viewed as the vector field on g arising from the usual identification of a vector space with its tangent space at any point. In fact, the latter amounts to the *canonical* parallelization on g, viewed as a vector space, whereas the vector field Y arises from the parallelization given by *left translation* on G. Thus all told, we obtain the formula

$$((\delta_G \beta)(X))(Y) = \beta(\overline{X} - X, Y) = \beta(\overline{X}, Y) - \beta(X, Y),$$

for $X, Y \in g$. However, at a point of the centre, the two vector fields $\overline{X}$ and X coincide whence $(\delta_G \beta)_z^\flat \colon g \to g^*$ is zero. □

Remark. Suppose h is the standard homotopy operator given by integration of forms along linear paths in g. Then it may be shown that $\delta_G h + h \delta_G$ is zero and a calculation of L. Jeffrey [15] shows that the map ψ amounts to the adjoint of the given 2-form on all of g. In fact, in view of (6),

$$\begin{aligned} d(h \circ \exp^*(\vartheta)) &= (\mathrm{Id} - hd) \exp^*(\vartheta) \\ &= \exp^*(\vartheta) - h \exp^*(\delta_G(\lambda)) \\ &= \exp^*(\vartheta) + \delta_G h \exp^*(\lambda)) \\ &= \exp^*(\vartheta) + \delta_G(\beta), \end{aligned}$$

whence we may take $\psi = h \circ \exp^*(\vartheta)$. Given x and y in g, we then have

$$\begin{aligned} \psi(x) \cdot y &= \int_0^1 (\exp^*(\vartheta))_{tx}(x) \cdot y \\ &= \int_0^1 \vartheta_{\exp(tx)}(\exp_* x) \cdot y \\ &= \frac{1}{2} \int_0^1 (\omega + \overline{\omega})_{\exp(tx)}(\exp_* x) \cdot y \\ &= x \cdot y, \end{aligned}$$

since along the 1-parameter subgroup generated by x (i) the vector field $\exp_* x$ coincides with x and (ii) $\overline{\omega}(x)$ equals $\omega(x) = x$.

2. Forms on representation spaces

Let Π be a group; we denote the chain and cochain complexes of its inhomogeneous *reduced* normalized bar resolution $B\Pi$ [19] by $(C_*(\Pi), \partial)$ and $(C^*(\Pi), \delta)$, respectively. For $p \geq 1$, consider the evaluation map

$$E \colon \Pi^p \times \mathrm{Hom}(\Pi, G) \longrightarrow G^p \tag{9}$$

and the induced maps

$$E^* \colon \Omega_G^{*;*}(G^p) \longrightarrow C^p(\Pi) \otimes \Omega_G^{*;*}(\mathrm{Hom}(\Pi, G)). \tag{10}$$

They assemble to a morphism

$$\big(\Omega_G^{*,*}(G^*); d, \delta, \delta_G\big) \to (C^*(\Pi), \delta) \otimes \big(\Omega_G^{*,*}(\mathrm{Hom}(\Pi, G)); d, \delta_G\big) \tag{11}$$

of tricomplexes; in a given tridegree (i, j, k), it looks like

$$\Omega_G^{i,j}(G^k) \to C^k(\Pi) \otimes \Omega_G^{i,j}(\mathrm{Hom}(\Pi, G)).$$

Pairing with chains in $C_*(\Pi)$, we obtain the chain map

$$\begin{aligned}\langle \cdot, \cdot \rangle \colon (C_*(\Pi), \partial) \otimes (C^*(\Pi), \delta) \otimes \big(\Omega_G^{*,*}(\mathrm{Hom}(\Pi, G)); d, \delta_G\big) \\ \longrightarrow \big(\Omega_G^{*,*}(\mathrm{Hom}(\Pi, G)); d, \delta_G\big)\end{aligned} \tag{12}$$

which produces equivariant forms on $\mathrm{Hom}(\Pi, G)$, suitably interpreted in the singularities of $\mathrm{Hom}(\Pi, G)$; actually, the problem of singularities will *not* occur below. In particular, for a 2-chain $c \in C_2(\Pi)$, let

$$\omega_c = \langle c, E^* Q_4 \rangle. \tag{13}$$

Since (11) is a morphism of tricomplexes, this amounts to

$$\omega_c = \langle c, E^* \Omega \rangle, \tag{14}$$

and, cf. (1) and (3),

$$d\omega_c = d\langle c, E^*\Omega\rangle = \langle c, E^* d\Omega\rangle = \langle c, E^*\delta\lambda\rangle = \langle \partial c, E^*\lambda\rangle, \tag{15}$$

$$\delta_G\omega_c = \delta_G\langle c, E^*\Omega\rangle = \langle c, E^*\delta_G\Omega\rangle = \langle c, -E^*\delta\vartheta\rangle = -\langle \partial c, E^*\vartheta\rangle. \tag{16}$$

In particular, when c is closed, ω_c is closed *and* equivariantly closed; moreover, when $c = \partial b$,

$$\langle \partial b, E^*\Omega\rangle = \langle b, E^*\delta\Omega\rangle = 0,$$

cf. (2), whence the assignment of ω_c to c yields a homomorphism from $\mathrm{H}_2(\Pi)$ to the space of closed and equivariantly closed equivariant 2-forms on $\mathrm{Hom}(\Pi, G)$, cf. [23] (5.1). For $\kappa \in \mathrm{H}_2(\Pi)$, we therefore write $\omega_\kappa \in \Omega^2(\mathrm{Hom}(\Pi, G))$ for its image, to indicate that this form depends merely on the cohomology class.

Let

$$\mathcal{P} = \langle x_1, \dots, x_n; r_1, \dots, r_m \rangle$$

be a presentation of a group π, write F for the free group on the generators and N for the normal closure of the relators, so that $\pi = F/N$. The choice of generators identifies $\mathrm{Hom}(F, G)$ with G^n and, furthermore, the space $\mathrm{Hom}(\pi, G)$ with the pre-image of the identity element in G^m, for the word map

$$r = (r_1, \dots, r_m) \colon G^n \longrightarrow G^m$$

induced by the relators. Let $O \subseteq g$ be the open G-invariant subset of g where the exponential mapping from g to G is regular; notice that O contains the centre of

g. Write $\mathcal{H}(F,G)$ for the space determined by the requirement that a pull back diagram

$$\begin{array}{ccc} \mathcal{H}(\mathcal{P},G) & \xrightarrow{r} & O^m \\ {\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle\exp} \\ \mathrm{Hom}(F,G) & \xrightarrow[r]{} & G^m \end{array} \tag{17}$$

results, where the induced map from $\mathcal{H}(\mathcal{P},G)$ to O^m is still denoted by r. The space $\mathcal{H}(\mathcal{P},G)$ is a smooth manifold and the induced map η from $\mathcal{H}(\mathcal{P},G)$ to $\mathrm{Hom}(F,G) = G^n$ is a smooth codimension zero immersion whence $\mathcal{H}(\mathcal{P},G)$ has the same dimension as G^n; moreover the above injection of $\mathrm{Hom}(\pi,G)$ into $\mathrm{Hom}(F,G)$ induces a canonical injection of $\mathrm{Hom}(\pi,G)$ into $\mathcal{H}(\mathcal{P},G)$, and in this way $\mathrm{Hom}(\pi,G)$ will be viewed as a subspace of $\mathcal{H}(\mathcal{P},G)$; when we restrict to a suitable G-invariant ball in O^m, we obtain even an open submanifold of G^n but this will not be important for us.

Let c be a 2-chain of F whose image c_π in $C_2(\pi)$ under the canonical map is closed, and write $\kappa = [c_\pi] \in \mathrm{H}_2(\pi)$ for its class. We can then apply the above construction, to $\Pi = F$ and $\Pi = \pi$. With $\Pi = F$ we obtain a 2-form ω_c on $\mathrm{Hom}(F,G) = G^n$ which, by naturality, on $\mathrm{Hom}(\pi,G)$ restricts to the closed 2-form ω_κ (in the appropriate sense) corresponding to $\kappa \in \mathrm{H}_2(\pi)$. In [23], A. Weinstein raised the question whether ω_κ admits a closed extension to $\mathrm{Hom}(F,G) = G^n$. Here is a possible answer to this question, with G^n replaced by the space $\mathcal{H}(\mathcal{P},G)$.

Theorem 1. *For every class $\kappa \in \mathrm{H}_2(\pi)$, there is a 2-chain $c \in C_2(F)$ and an equivariant 2-form B on g^m determined by c and $\mathcal{P}$ so that $\omega_{c,\mathcal{P}} = \eta^*(\omega_c) - r^*B$ is a closed G-equivariant 2-form on $\mathcal{H}(\mathcal{P},G)$ extending ω_κ.*

Remark 1. The theorem does not assert that the extension is equivariantly closed. In fact, in general it will *not* be equivariantly closed. An equivariantly closed extension will be constructed in the next Section.

Lemma 2. *For every $\kappa \in \mathrm{H}_2(\pi)$, there is a 2-chain $c \in C_2(F)$ whose image $c_\pi \in C_2(\pi)$ represents $\kappa \in \mathrm{H}_2(\pi)$ and whose boundary ∂c looks like*

$$\partial c = \sum \nu_j r_j, \quad \nu_j \in \mathbf{Z}.$$

Moreover, ∂c then represents κ, the second homology group $\mathrm{H}_2(\pi)$ being identified with the kernel of the induced map from $N/[F,N]$ to $F/[F,F]$ by means of the Schur-Hopf formula.

Proof. Pick a 2-chain $c \in C_2(F)$ whose image $c_\pi \in C_2(\pi)$ represents $\kappa \in \mathrm{H}_2(\pi)$. The boundary ∂c is a finite linear combination

$$\partial c = \sum \nu_j w_j, \quad \nu_j \in \mathbf{Z},\ \ w_j \in F.$$

We assert that for a suitable choice, each w_j lies in N. In fact, ∂c lies in the kernel of the induced map from $C_1(F)$ to $C_1(\pi)$. Since $C_1(F)$ and $C_1(\pi)$ are the free abelian groups generated by $F^* = F \setminus 1$ and $\pi^* = \pi \setminus 1$, the kernel of the induced map from $C_1(F)$ to $C_1(\pi)$ is generated by the elements of N^* together with elements of the kind $[wn]-[w]$, for $w\in F$ and $n\in N$. However, for $w\in F$ and $n\in N$,

$$\partial[w|n]=[n]-[wn]+[w],$$

and $[w|n]\in C_2(F)$ goes to zero in $C_2(\pi)$. Hence a suitable modification of c if necessary yields a 2-chain $c'\in C_2(F)$ so that (i) $\partial c'$ is a sum of terms $\mu_k w_k$ where each $w_k\in N$ and (ii) whose image $c'_\pi\in C_2(\pi)$ still represents $\kappa\in \mathrm{H}_2(\pi)$. Thus we may assume that the boundary of c involves only elements of N.

Next we show that we may in fact pick c in the asserted way. In fact, each $w\in N$ is a product of conjugates yr_jy^{-1} and $zr_k^{-1}z^{-1}$, $y,z\in F$. However,

$$\partial[u|v]=[v]-[uv]+[u],\quad \partial[u|u^{-1}]=[u^{-1}]+[u],$$

whence adding suitable terms to c if necessary, we may assume ∂c is of the kind

$$\partial c=\sum\nu_jy_jr_jy_j^{-1},\quad \nu_j\in\mathbf{Z};$$

notice adding these terms does not change the image $c_\pi\in C_2(\pi)$. Finally,

$$\partial[xv|x^{-1}]=[x^{-1}]-[xvx^{-1}]+[xv],\quad \partial[x^{-1}|xv]=[xv]-[v]+[x^{-1}],$$

whence adding further terms to c if necessary, we arrive at a 2-chain in $C_2(F)$ of the asserted kind. The statement involving the Schur-Hopf formula is then obvious. □

Proof of Theorem 1. Let c be a 2-chain for F whose image $c_\pi\in C_2(\pi)$ represents $\kappa\in\mathrm{H}_2(\pi)$ and whose boundary ∂c looks like

$$\partial c=\sum\nu_jr_j,\quad \nu_j\in\mathbf{Z}.$$

By Lemma 2, such a c exists. For $j=1,\ldots,m$, let $p_j\colon G^m\to G$ be the projection onto the j'th factor, let $\lambda_j=p_j^*(\lambda)\in\Omega^3(G^m)$, and let $\Lambda=\sum\nu_j\lambda_j\in\Omega^3(G^m)$. This is a closed equivariant 3-form and, since $E\colon F\times\mathrm{Hom}(F,G)\to G$ maps (w,ϕ) to the result of application of the word map for w to $(\phi(x_1),\ldots,\phi(x_n))$,

$$\langle\partial c,E^*\lambda\rangle=r^*\Lambda$$

whence

$$d\omega_c=r^*\Lambda\in\Omega^3(G^n). \tag{18}$$

For $j=1,\ldots,m$, let $\beta_j=p_j^*(\beta)\in\Omega^2(g^m)$, and let $B=\sum\nu_j\beta_j\in\Omega^2(g^m)$ so that

$$dB=\sum\nu_jd\beta_j=\sum\nu_j\exp^*(\lambda_j)=\exp^*(\Lambda)\in\Omega^3(g^m).$$

Then on $\mathcal{H}(\mathcal{P},G)$ we have the identity $d\eta^*(\omega_c)=r^*dB$, that is,

$$d(\eta^*(\omega_c)-r^*B)=0.$$

The form $\eta^*(\omega_c)-r^*B$ is the closed equivariant extension of ω_κ we are looking for. □

Remark 2. The 2-form ω_c may be closed on a space larger than $\mathrm{Hom}(\pi,G)$. In fact, the 3-form λ is identically zero on the centre Z of G. Hence, in view of the identity (18), the naturality of the construction implies that, the space G^n being identified with $\mathrm{Hom}(F,G)$ by means of the choice of generators, the 2-form ω_c is closed on the subspace of $\mathrm{Hom}(F,G)$ consisting of homomorphisms ϕ from F to G having the property that $\phi(r_j)$ lies in Z, for $1\le j\le m$.

3. The equivariantly closed extension

Let $\kappa \in \mathrm{H}_2(\pi)$, and let $c \in C_2(F)$ be a 2-chain and B the corresponding equivariant 2-form on g^m so that $\omega_{c,\mathcal{P}} = \eta^*(\omega_c) - r^*B$ is an equivariant closed 2-form $\mathcal{H}(\mathcal{P},G)$ extending ω_κ, as in Theorem 1.

Theorem 2. *There is a smooth equivariant map $\mu\colon \mathcal{H}(\mathcal{P},G) \to g^*$ whose adjoint $\mu^\sharp\colon g \to C^\infty(\mathcal{H}(\mathcal{P},G))$ satisfies the identity*

$$\delta_G(\omega_{c,\mathcal{P}}) = d\mu^\sharp$$

on $\mathcal{H}(\mathcal{P},G)$. Consequently $\omega_{c,\mathcal{P}} - \mu^\sharp$ is an equivariantly closed form in $(\Omega_G^{*,*}(\mathcal{H}(\mathcal{P},G)); d, \delta_G)$ of total degree 2 extending ω_κ.

Thus, cf. [1] and what is said in Section 5 below, μ is a momentum mapping for the 2-form $\omega_{c,\mathcal{P}}$ on $\mathcal{H}(\mathcal{P},G)$, with reference to the obvious G-action, except that $\omega_{c,\mathcal{P}}$ is not necessarily non-degenerate.

Proof. For $j = 1, \dots, m$, let $\vartheta_j = p_j^*(\vartheta) \in \Omega_G^{2,1}(G^m)$, and let $\Theta = \sum \nu_j \vartheta_j \in \Omega_G^{2,1}(G^m)$ so that

$$\langle \partial c, E^* \vartheta \rangle = r^*\Theta \in \Omega_G^{2,1}(G^n).$$

In view of (16),

$$\begin{aligned}
\delta_G(\omega_{c,\mathcal{P}}) &= \delta_G(\eta^*(\omega_c) - r^*B) \\
&= \eta^*(\delta_G \omega_c) - r^*\delta_G B \\
&= -\eta^*(\langle \partial c, E^*\vartheta \rangle) - r^*\delta_G B \\
&= -r^*(\exp^*(\Theta) + \delta_G B) \in \Omega_G^{2,1}(\mathcal{H}(\mathcal{P},G)).
\end{aligned}$$

For $j = 1, \dots, m$, write $\psi_j\colon g^m \to g^*$ for the composite of the projection onto the j'th factor with ψ, so that its adjoint $\psi_j^\sharp\colon g \to \Omega^0(g^m) = C^\infty(g^m)$, that is, $\psi_j^\sharp \in \Omega_G^{2,0}(g^m)$, satisfies

$$d\psi_j^\sharp = (\exp^*(\vartheta_j) + \delta_G \beta_j).$$

Let

$$\Psi = \sum \nu_j \psi_j \colon g^m \to g^*;$$

this is a smooth map. By Lemma 1, its adjoint $\Psi^\sharp \in \Omega_G^{2,0}(g^m)$ satisfies

$$d\Psi^\sharp = \sum \nu_j (\exp^*(\vartheta_j) + \delta_G \beta_j) = \exp^*(\Theta) + \delta_G B \in \Omega_G^{2,1}(g^m).$$

Hence on $\mathcal{H}(\mathcal{P},G)$ we have the identity

$$\delta_G(\omega_{c,\mathcal{P}}) = -r^*(d\Psi^\sharp) = -d(r^*\Psi^\sharp)$$

which, with $\mu = -\Psi \circ r\colon \mathcal{H}(\mathcal{P},G) \to g^*$, looks like

$$\delta_G(\omega_{c,\mathcal{P}}) = d\mu^\sharp$$

as asserted. □

4. Examination of the equivariantly closed 2-form

Let $\phi \in \mathrm{Hom}(F, G) = G^n$ and suppose that each $\phi(r_j)$ lies in the centre of G. Then the composite of ϕ with the adjoint representation of G induces a structure of a π-module on g. We write g_ϕ for g, with the resulting π-module structure. By means of the *left* Fox calculus, the presentation $\mathcal{P}$ determines a chain complex

$$\mathbf{C}(\mathcal{P}, g_\phi)\colon \mathrm{C}^0(\mathcal{P}, g_\phi) \xrightarrow{\delta^0_\phi} \mathrm{C}^1(\mathcal{P}, g_\phi) \xrightarrow{\delta^1_\phi} \mathrm{C}^2(\mathcal{P}, g_\phi) \tag{4.1}$$

computing the group cohomology $\mathrm{H}^*(\pi, g_\phi)$ in degrees 0 and 1; we only mention that there are canonical isomorphisms

$$\mathrm{C}^0(\mathcal{P}, g_\phi) \cong g, \quad \mathrm{C}^1(\mathcal{P}, g_\phi) \cong g^n, \quad \mathrm{C}^2(\mathcal{P}, g_\phi) \cong g^m.$$

See e. g. our paper [10] where the details are given for the *right* Fox calculus. To explain the geometric significance of this chain complex, denote by α_ϕ the smooth map from G to $\mathrm{Hom}(F, G)$ which assigns $x\phi x^{-1}$ to $x \in G$, and write $R_\phi\colon g^n \to \mathrm{T}_\phi\mathrm{Hom}(F, G)$ and $R_{r\phi}\colon g^m \to \mathrm{T}_{r\phi}G^m$ for the corresponding operations of right translation. The tangent maps $\mathrm{T}_e\alpha_\phi$ and $\mathrm{T}_\phi r$ make commutative the diagram

$$\begin{array}{ccccc}
\mathrm{T}_eG & \xrightarrow{\mathrm{T}_e\alpha_\phi} & \mathrm{T}_\phi\mathrm{Hom}(F, G) & \xrightarrow{\mathrm{T}_\phi r} & \mathrm{T}_{r(\phi)}G^m \\
\Big\uparrow{\scriptstyle \mathrm{Id}} & & \Big\uparrow{\scriptstyle \mathrm{R}_\phi} & & \Big\uparrow{\scriptstyle \mathrm{R}_{r(\phi)}} \\
g & \xrightarrow[\delta^0_\phi]{} & g^n & \xrightarrow[\delta^1_\phi]{} & g^m.
\end{array} \tag{4.2}$$

We work here with right translation since this yields formulas consistent with existing literature on representation spaces, cf. e. g. [20]. In our paper [10] we worked with left translation since there the relationship with principal bundles is made explicit, and the usual convention in the literature is to have the structure group of a principal bundle act on the right.

The 2-form $\cdot$ on g and the homology class κ determine the alternating 2-form

$$\omega_{\kappa,\cdot,\phi}\colon \mathrm{H}^1(\pi, g_\phi) \otimes \mathrm{H}^1(\pi, g_\phi) \xrightarrow{\cup} \mathrm{H}^2(\pi, \mathbf{R}) \xrightarrow{\cap\kappa} \mathbf{R} \tag{4.3}$$

on $\mathrm{H}^1(\pi, g_\phi)$. In view of the commutativity of (4.2), right translation identifies the kernel of the derivative $\mathrm{T}_\phi r$ with the kernel of the coboundary operator δ^1_ϕ from $\mathrm{C}^1(\mathcal{P}, g_\phi)$ to $\mathrm{C}^2(\mathcal{P}, g_\phi)$, that is, with the vector space $\mathrm{Z}^1(\pi, g_\phi)$ of g_ϕ-valued 1-cocycles of π; this space does *not* depend on a specific presentation $\mathcal{P}$, whence the notation. We note that $\mathrm{C}^1(\mathcal{P}, g_\phi) = Z^1(F, g_\phi)$, the space of g_ϕ-valued 1-cocycles for F. Our present goal is to prove the following.

Theorem 4.4. *Right translation identifies the restriction of the 2-form ω_c to the kernel of the derivative $\mathrm{T}_\phi r$ with the 2-form on $\mathrm{Z}^1(\pi, g_\phi)$ obtained as the composite of $\omega_{\kappa,\cdot,\phi}$ with the projection from $\mathrm{Z}^1(\pi, g_\phi)$ to $\mathrm{H}^1(\pi, g_\phi)$.*

While this is essentially contained in [23], see also [16], we give a complete proof since our argument will clarify the role of the factor $\frac{1}{2}$ in the definition of the forms $\lambda, \Omega, \vartheta$.

For a group Π, as usual we write $\beta(\Pi)$ for the *unreduced* normalized inhomogeneous bar resolution; the *reduced* normalized inhomogeneous bar resolution $B\Pi$ arises from $\beta(\Pi)$ by dividing out the Π-action.

Lemma 4.5. *The cup-pairing* $\cup\colon Z^1(F,g_\phi)\otimes Z^1(F,g_\phi)\longrightarrow C^2(F,\mathbf{R})$ *with reference to the 2-form* $\cdot$ *on* g *is given by the formula*

$$(u\cup v)[x|y]=u(x)\cdot(\mathrm{Ad}(\phi(x))v(y)),\quad u,v\in Z^1(F,g_\phi),\ \ [x|y]\in B_2(\Pi).$$

Proof. Let Π be an arbitrary discrete group. Recall from [19] (VIII.9 Ex. 1, p. 248) that the relevant term of the *Alexander-Whitney* diagonal map $\Delta\colon\beta(\Pi)\to\beta(\Pi)\otimes\beta(\Pi)$ is given by the formula

$$\Delta(w[x|y])=w\otimes w[x|y]+w[x]\otimes wx[y]+w[x|y]\otimes wxy,$$

where $w,x,y\in\Pi$. When we apply this to $\Pi=F$ we see that, for arbitrary 1-cochains $u,v\colon\beta_1(F)\to g_\phi$, their cup product $u\cup v$, evaluated via the given G-invariant 2-form on g, amounts to the 2-cochain which assigns

$$(u\cup v)[x|y]=u(x)\cdot(\mathrm{Ad}(\phi(x))v(y))$$

to $[x|y]\in B_2(F)$. □

The chosen 2-chain c in $C_2(F)$ looks like

$$c=\sum\nu_{j,k}[x_j|x_k].$$

Define the 2-form $\omega_{c,\cup,\phi}$ on $Z^1(F,g_\phi)$ by the explicit formula

$$\omega_{c,\cup,\phi}(u,v)=\langle c,u\cup v\rangle=\sum\nu_{j,k}u(x_j)\cdot(\mathrm{Ad}(\phi(x_j))v(x_k)),\quad u,v\in Z^1(F,g_\phi).$$

Notice this 2-form will *not* be antisymmetric; only its restriction to the space $Z^1(\pi,g_\phi)$ of g_ϕ-valued 1-cocycles for π is antisymmetric. By construction, the 2-form $\omega_{\kappa,\cdot,\phi}$ is induced by the restriction of $\omega_{c,\cup,\phi}$ to the space $Z^1(\pi,g_\phi)$. Hence Theorem 4.4 will be a consequence of the following.

Lemma 4.6. *At the point* ϕ *of* $\mathrm{Hom}(F,G)$*, the 2-form* ω_c *is the antisymmetrization of* $\omega_{c,\cup,\phi}$*.*

The 2-form ω_c on $\mathrm{Hom}(F,G)$ arises from the 2-form Ω on G^2, pulled back via the evaluation map E from $F^2\times\mathrm{Hom}(F,G)$ to G^2 and evaluated at the 2-chain c. More precisely, for $x,y\in F$ fixed, consider the smooth map $\phi_{[x|y]}$ from $\mathrm{Hom}(F,G)$ to G^2 which assigns $(\phi(x),\phi(y))$ to $\phi\in\mathrm{Hom}(F,G)$ and let $\omega_{[x|y]}=\phi^*_{[x|y]}(\Omega)$; then

$$\omega_c=\sum\nu_{j,k}\omega_{[x_j|x_k]}.$$

Thus it is manifest that Lemma 4.6 is a consequence of the following.

Lemma 4.7. *For every* $x,y\in F$*, the 2-form* $\omega_{[x|y]}$ *is given by the formula*

$$\omega_{[x|y]}(u,v)=\frac{1}{2}\left(u(x)\cdot(\mathrm{Ad}(\phi(x))v(y))-(\mathrm{Ad}(\phi(x))u(y))\cdot v(x)\right),\tag{4.7.1}$$

whatever $u, v \in \mathrm{T}_\phi(\mathrm{Hom}(F,G))$, the vector space $\mathrm{T}_\phi(\mathrm{Hom}(F,G))$ being identified with the space of g_ϕ-valued 1-cocycles for F via right translation. In other words, for every $x, y \in F$, the 2-form $\omega_{[x|y]}$ amounts to the antisymmetrization of the 2-form on $\mathrm{T}_\phi(\mathrm{Hom}(F,G))$ given by the assignment

$$(u,v) \longmapsto \langle u \cup v, [x|y] \rangle.$$

Proof. Writing $\mathrm{T}_\phi = \mathrm{T}_\phi \mathrm{Hom}(F,G)$, we must calculate the composite

$$\mathrm{T}_\phi \otimes \mathrm{T}_\phi \longrightarrow \left(\mathrm{T}_{\phi(x)}G \times \mathrm{T}_{\phi(y)}G\right) \otimes \left(\mathrm{T}_{\phi(x)}G \times \mathrm{T}_{\phi(y)}G\right) \xrightarrow{\omega_1 \otimes \overline{\omega}_2 - \overline{\omega}_2 \otimes \omega_1} g \otimes g$$

and combine it with the given pairing. In other words, we must compute the two composites

$$\mathrm{T}_\phi \otimes \mathrm{T}_\phi \longrightarrow \mathrm{T}_{\phi(x)}G \otimes \mathrm{T}_{\phi(y)}G \xrightarrow{\omega \otimes \overline{\omega}} g \otimes g \overset{\cdot}{\longrightarrow} \mathbf{R}$$
$$\mathrm{T}_\phi \otimes \mathrm{T}_\phi \longrightarrow \mathrm{T}_{\phi(y)}G \otimes \mathrm{T}_{\phi(x)}G \xrightarrow{\overline{\omega} \otimes \omega} g \otimes g \overset{\cdot}{\longrightarrow} \mathbf{R}.$$

However, right translation identifies the space T_ϕ with the space of g_ϕ-valued 1-cocycles for F. Moreover a composite of the kind

$$\mathrm{T}_\phi \longrightarrow \mathrm{T}_{\phi(y)}G \xrightarrow{\overline{\omega}} g$$

amounts to the assignment of the value $v(y)$ to a g_ϕ-valued 1-cocycle v for F since $\overline{\omega}$ is the canonical *right* invariant g-valued 1-form on G; likewise, the adjoint representation Ad being viewed as a 0-form on G with values in $\mathrm{Aut}(g)$, we have $\overline{\omega} = \mathrm{Ad}\,\omega$ whence a composite of the kind

$$\mathrm{T}_\phi \longrightarrow \mathrm{T}_{\phi(x)}G \xrightarrow{\omega} g$$

amounts to the assignment of the value $\mathrm{Ad}(\phi(x^{-1}))u(x)$ to a g_ϕ-valued 1-cocycle u for F. Thus all told, for every $u, v \in \mathrm{T}_\phi$,

$$\omega_{[x|y]}(u,v) = \frac{1}{2}\left((\mathrm{Ad}(\phi(x^{-1}))u(x)) \cdot v(y) - u(y) \cdot (\mathrm{Ad}(\phi(x^{-1}))v(x))\right).$$

Since the 2-form is G-invariant, this identity is equivalent to (4.7.1), and the assertion follows. □

The proof of Theorem 4.4 is now complete.

Corollary 4.8. *Right translation identifies the restriction of the 2-form $\omega_{c,\mathcal{P}}$ to the kernel of the derivative $\mathrm{T}_\phi r$ with the 2-form on $\mathrm{Z}^1(\pi, g_\phi)$ obtained as the composite of $\omega_{\kappa,\cdot,\phi}$ with the projection from $\mathrm{Z}^1(\pi, g_\phi)$ to $\mathrm{H}^1(\pi, g_\phi)$. Consequently the alternating 2-form $\omega_{c,\mathcal{P}}$ on $\mathcal{H}(\mathcal{P}, G)$ induces the 2-form $\omega_{\kappa,\cdot,\phi}$ on $\mathrm{H}^1(\pi, g_\phi)$, whatever $\mathcal{P}$ and c.*

5. The case of a surface group

Let

$$\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r\rangle, \quad r = \Pi[x_j, y_j],$$

be the standard presentation of the fundamental group π of a closed surface Σ of genus ℓ, and let $\kappa \in \mathrm{H}_2(\pi)$ be a generator. This implies that the 2-chain $c \in C_2(F)$ has $\partial c = \pm r \in C_2(F)$, and we assume the choices have been made in such a way that $\partial c = r$. Further, suppose that the given 2-form $\cdot$ on g is non-degenerate. Write Z for the centre of G and z for its Lie algebra. By Poincaré duality in the cohomology of π, for every ϕ in the pre-image $r^{-1}(z) \subseteq \mathcal{H}(\mathcal{P}, G)$ of $z \subseteq O$, in particular, for every $\phi \in \mathrm{Hom}(\pi, G)$, the 2-form $\omega_{\kappa,\cdot,\phi}$ is then symplectic. This form is just that considered by Goldman [6]. Furthermore, the identity $\delta_G(\omega_{c,\mathcal{P}}) = d\mu^\sharp$ says that, for every $X \in g$,

$$\omega_{c,\mathcal{P}}(X_{\mathcal{H}(\mathcal{P},G)}, \cdot\,) = d(X \circ \mu),$$

that is, formally the momentum mapping property is satisfied. This together with the symplecticity of the 2-form $\omega_{\kappa,\cdot,\phi}$ at every $\phi \in r^{-1}(z)$ implies that $\omega_{c,\mathcal{P}}$ has maximal rank equal to $\dim \mathcal{H}(\mathcal{P}, G)$ at every point of the pre-image $r^{-1}(z)$ of z, in particular, at every point of $\mathrm{Hom}(\pi, G)$. In fact, the symplecticity of the 2-form $\omega_{\kappa,\cdot,\phi}$ at $\phi \in r^{-1}(z)$ implies that the 2-form $\omega_{c,\mathcal{P}}$ on the tangent space $\mathrm{T}_\phi\mathcal{H}(\mathcal{P}, G) \cong \mathrm{C}^1(\mathcal{P}, g_\phi)$, restricted to the subspace $\mathrm{Z}^1(\pi, g_\phi)$ of 1-cocycles, has degeneracy space equal to the subspace $\mathrm{B}^1(\pi, g_\phi)$ of 1-coboundaries, and the momentum mapping property then implies that the 2-form $\omega_{c,\mathcal{P}}$ on the whole space $\mathrm{C}^1(\mathcal{P}, g_\phi)$ is non-degenerate. Let $\mathcal{M}(\mathcal{P}, G)$ be the subspace of $\mathcal{H}(\mathcal{P}, G)$ where the 2-form $\omega_{c,\mathcal{P}}$ is non-degenerate; this is an open G-invariant subset containing the pre-image $r^{-1}(z)$. In other words, $\mathcal{M}(\mathcal{P}, G)$ is a smooth G-manifold and the 2-form $\omega_{c,\mathcal{P}}$ is in fact a G-invariant symplectic structure on it. Moreover, the restriction

$$\mu = -\psi \circ r \colon \mathcal{M}(\mathcal{P}, G) \longrightarrow g^*$$

is a momentum mapping in the usual sense. By the Corollary to Lemma 1, the derivative of ψ at the origin equals the adjoint of the given 2-form which is assumed non-degenerate, whence ψ is regular near the origin and the zero locus of μ coincides with that of r, that is, with the space $\mathrm{Hom}(\pi, G)$. Symplectic reduction then yields the space $\mathrm{Rep}(\pi, G)$.

6. Twisted moduli spaces

The universal central extension of π arises from the presentation $\mathcal{P}$ in the following way: Let F be the free group on the generators, N the normal closure of r in F, and $\Gamma = F\big/[F, N]$; then the kernel of the canonical projection from Γ to π is a copy of the integers, generated by $[r] = r[F, N] \in \Gamma = F\big/[F, N]$, and these combine to the *universal central extension*

$$0 \longrightarrow \mathbf{Z} \longrightarrow \Gamma \longrightarrow \pi \longrightarrow 1$$

of π. Write Z for the centre of G, let z be the Lie algebra of Z, and let $X \in z$. When G is connected, z coincides with the zentre of g but in general z equals the invariants for the induced action of the group π_0 of components of

G on the centre of g. Let $\mathrm{Hom}_X(\Gamma, G)$ denote the space of homomorphisms ϕ from Γ to G having the property that $\phi[r] = \exp(X)$; we assume X chosen so that $\mathrm{Hom}_X(\Gamma, G)$ is non-empty. We comment on the significance of this assumption below. Let $\mathrm{Rep}_X(\Gamma, G) = \mathrm{Hom}_X(\Gamma, G)/G$, the resulting *twisted moduli space* or *twisted representation space*. The choice of generators identifies the space $\mathrm{Hom}_X(\Gamma, G)$ with the pre-image of $\exp(X) \in G$, for the word map r from $G^{2\ell}$ to G induced by the relator r, and we can play a similar game as before, with the same choice of $c \in C_2(F)$ so that $\partial c = r$ represents $\kappa \in \mathrm{H}_2(\pi)$. More precisely, since the centre of G is contained in O, the space $\mathrm{Hom}_X(\Gamma, G)$ arises as the pre-image of $X \in z \subseteq O$ under the map r from $\mathcal{H}(F, G)$ to O. Furthermore, in view of the Corollary to Lemma 1, the map ψ from g to g^* is regular at every point of the centre of g, in fact, the restriction of ψ to the centre equals the adjoint of the given 2-form whence the space $\mathrm{Hom}_X(\Gamma, G)$ equals the pre-image of the adjoint $X^\sharp \in g^*$ of X under the momentum mapping μ from $\mathcal{M}(\mathcal{P}, G)$ to g^*. Consequently the space $\mathrm{Rep}_X(\Gamma, G)$ is the corresponding reduced space, for the coadjoint orbit in g^* consisting of the single point $X^\sharp$.

We now explain briefly the geometric significance of the spaces $\mathrm{Rep}_X(\Gamma, G)$ for G compact: Let $\Gamma_{\mathbf{R}}$ denote the non-connected Lie group arising from Γ when its centre $\mathbf{Z}\langle [r] \rangle$ is extended to the reals $\mathbf{R}$. A homomorphism ϕ from Γ to G having the property that $\phi[r] = \exp(X)$ extends canonically to a homomorphism Φ from $\Gamma_{\mathbf{R}}$ to G by the assignment $\Phi(x) = \phi(x)$ for $x \in \Gamma$ and $\Phi(t[r]) = \exp(tX) \in Z$, for $t \in \mathbf{R}$. This homomorphism, in turn, determines a principal G-bundle $\xi \colon P \to \Sigma$ together with a central Yang-Mills connection A on ξ, and this association in fact identifies the moduli space $N(\xi)$ of central Yang-Mills connections with a connected component of the space $\mathrm{Rep}_X(\Gamma, G)$. Details may be found in [2] for the case of connected G and in our paper [9] for the general case. The element X is in fact a topological characteristic class of the corresponding bundle ξ. However, when the structure group G is not connected, it may happen that the space $\mathrm{Rep}_X(\Gamma, G)$ is not connected and topologically inequivalent bundles may still have the same characteristic class X. In particular we see that the requirement that X be chosen in such a way that the space $\mathrm{Hom}_X(\Gamma, G)$ is non-empty is topological in nature. Moreover, it is now clear that the space of representations of Γ in G with the property that the value of $[r]$ lies in the centre of G is partitioned into a disjoint union of representation spaces of the kind $\mathrm{Rep}_{X_\xi}(\Gamma, G)$, where $X_\xi \in z$ refers to the characteristic class of a corresponding principal G-bundle ξ on Σ. The space $\mathrm{Rep}(\pi, G)$ is the special case of this construction for $X_\xi = 0$ and its connected components correspond to topologically inequivalent flat G-bundles. For example, for $G = \mathrm{U}(n)$, the Lie algebra of the centre amounts to $2\pi i\mathbf{R}$, and the possible choices for X are integral multiples of $2\pi i$. These are just the Chern classes of the corresponding bundles ξ — notice a principal $\mathrm{U}(n)$-bundle on a closed surface Σ is topologically classified by its Chern class in $\mathrm{H}^2(\pi, \pi_1(\mathrm{U}(n)))$. For such a bundle ξ, the moduli space $N(\xi)$ is homeomorphic to the NARASIMHAN-SESHADRI [21] moduli space of semistable rank n holomorphic vector bundles of degree equal to the Chern class of ξ. Thus our construction in particular yields these moduli spaces by *symplectic reduction, applied to a smooth finite dimensional symplectic manifold with a hamiltonian action of the finite dimensional Lie group* $\mathrm{U}(n)$. In [2] these spaces have been obtained only by reduction applied to the *infinite* dimensional space of all connections, with hamiltonian action of the group of gauge transformations.

7. Applications

Suppose G compact. Recall that the notion of stratified symplectic space has been introduced in [22].

Theorem 7.1. *With respect to the decomposition according to G-orbit types, the space* $\mathrm{Rep}(\pi, G)$ *and, more generally, each twisted representation space* $\mathrm{Rep}_X(\Gamma, G)$ *inherits a structure of stratified symplectic space.*

In fact, the momentum mapping μ from $\mathcal{M}(\mathcal{P}, G)$ to g^* is proper; this follows from the fact that it arises from the word mapping r from $G^{2\ell}$ to G, which is proper since $G^{2\ell}$ and G are compact. The statement of the Theorem is thus an immediate consequence of the main result of [22]. The structure of stratified symplectic space has been obtained in our paper [11] by another method.

Theorem 7.2. *Each stratum of the space* $\mathrm{Rep}(\pi, G)$ *and, more generally, each stratum of a twisted representation space* $\mathrm{Rep}_X(\Gamma, G)$ *has finite symplectic volume.*

The proof follows the same pattern as that for the argument for (3.9) in [22]. There the unreduced symplectic manifold is assumed compact. However the compactness of the zero level set suffices; in our situation the zero level set *is* compact. In fact, it suffices to prove the statement for the local model in [22] which looks like the reduced space of a unitary representation of a compact Lie group, for the corresponding unique momentum mapping having the value zero at the origin. For the local model there is no difference between (3.9) in [22] and our situation. Once the statement is established for the local model, that of Theorem 7.2 follows since the reduced space may be covered by finitely many open sets having a local model of the kind described.

We mention two other consequences:

Corollary 7.3. *There is a unique open, connected, and dense stratum.*

In fact, this follows at once from [22] (5.9). A different argument has been given in our paper [8]. Likewise [22] (5.11) entails the following, also observed in [11].

Corollary 7.4. *The reduced Poisson algebra is symplectic, that is, its only Casimir elements are the constants.*

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