arXiv:math/0206102v1 [math.DG] 10 Jun 2002

Poisson manifolds with compatible pseudo-metric and pseudo-Riemannian Lie algebras

M. Boucetta

Abstract

The notion of Poisson manifold with compatible pseudo-metric was introduced by the author in [1]. In this paper, we introduce a new class of Lie algebras which we call a pseudo-Riemannian Lie algebras. The two notions are strongly related: we prove that a linear Poisson structure on the dual of a Lie algebra has a compatible pseudo-metric if and only if the Lie algebra is a pseudo-Riemannian Lie algebra, and that the Lie algebra obtained by linearizing at a point a Poisson manifold with compatible pseudo-metric is a pseudo-Riemannian Lie algebra. Furthermore, we give some properties of the symplectic leaves of such manifolds, and we prove that every Poisson manifold with compatible metric (every Riemann-Lie algebras) is unimodular. As a final, we classify all pseudo-Riemannian Lie algebras of dimension 2 and 3.

1 Introduction

Let P be a Poisson manifold with Poisson tensor π. A pseudo-metric of signature (p,q) on the contangent bundle T^*P is a smooth symmetric contravariant 2-form $<,>$ on P such that, at each point $x \in P$, $<,>_x$ is non-degenerate on T^*_xP with signature (p,q). For any pseudo-metric $<,>$ on T^*P, we define a contravariant connection in the sens of Fernandes [2] and Vaisman [4] by

$$2 < D_\alpha\beta, \gamma > \;=\; \#_\pi(\alpha).<\beta,\gamma> + \#_\pi(\beta).<\alpha,\gamma> - \#_\pi(\gamma).<\alpha,\beta>$$

$$+ < [\alpha, \beta]_\pi, \gamma > + < [\gamma, \alpha]_\pi, \beta > + < [\gamma, \beta]_\pi, \alpha > \qquad (1)$$

where $\alpha, \beta, \gamma \in \Omega^1(P)$ and the Lie bracket $[,]_\pi$ is given by

$$\begin{aligned} [\alpha, \beta]_\pi &= L_{\#_\pi(\alpha)}\beta - L_{\#_\pi(\beta)}\alpha - d(\pi(\alpha, \beta)) \\ &= i_{\#_\pi(\alpha)}d\beta - i_{\#_\pi(\beta)}d\alpha + d(\pi(\alpha, \beta)). \end{aligned}$$

D will be called the Levi-Civita contravariant connection associated with the couple $(\pi, <, >)$. D satisfies:
1. $D_\alpha\beta - D_\beta\alpha = [\alpha, \beta]_\pi$;
2. $\#_\pi(\alpha). < \beta, \gamma > = < D_\alpha\beta, \gamma > + < \beta, D_\alpha\gamma >$.

Definition 1.1 *With the notations above, the triple $(P, \pi, <, >)$ is called a pseudo-Riemannian Poisson manifold if, for any $\alpha, \beta, \gamma \in \Omega^1(P)$,*

$$D\pi(\alpha, \beta, \gamma) = \#_\pi(\alpha).\pi(\beta, \gamma) - \pi(D_\alpha\beta, \gamma) - \pi(\beta, D_\alpha\gamma) = 0. \qquad (2)$$

When $<, >$ is positive definite we call the triple a Riemann-Poisson manifold.

Let $(\mathcal{G}, [,])$ be a Lie algebra and let a be a non-degenerate bilinear and symmetric form on $\mathcal{G}$. We define a bilinear map $A : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ by

$$2a(A_u v, w) = a([u, v], w) + a([w, u], v) + a([w, v], u) \qquad (3)$$

for any $u, v, w \in \mathcal{G}$. A satisifes:
1. $A_u v - A_v u = [u, v]$;
2. $a(A_u v, w) + a(u, A_u w) = 0$.

Definition 1.2 *With the notations above, the triple $(\mathcal{G}, [,], a)$ is called a pseudo-Riemannian Lie algebra if*

$$[A_u v, w] + [u, A_w v] = 0 \qquad (4)$$

for all $u, v, w \in \mathcal{G}$. When a is positive definite we call the triple $(\mathcal{G}, [,], a)$ a Riemann-Lie algebra.

In Section 1, we give some basic properties of a pseudo-Riemannian Poisson manifold and we prove the following theorems:

Theorem 1.1 *Let $(P, \pi, <, >)$ be a pseudo-Riemannian Poisson manifold. Then, for any point $x \in P$ such that the restriction of $<, >$ to $Ker\#_\pi(x)$ is non-degenerate, the Lie algebra $\mathcal{G}_x$ obtained by linearizing the Poisson structure at x is a pseudo-Riemannian Lie algebra.*

Theorem 1.2 *Let $\mathcal{G}$ be a Lie algebra. The dual $(\mathcal{G}^*, \pi)$ of $\mathcal{G}$ endowed with its linear Poisson structure π has a pseudo-metric $<, >$ for which the triple $(\mathcal{G}^*, \pi, <, >))$ is a pseudo-Riemannian Poisson manifold if and only if $\mathcal{G}$ is a pseudo-Riemannian Lie algebra.*

Remark. The condition on a x in Theorem 1.1 is a condition which depend only on the symplectic leaf of x that means that if x satisfies this condition, since the parallel transport associated with D preserve $Ker\#_\pi$ and $<, >$, each point on the symplectic leaf of x satisfies the condition. Moreover, any point in which the Poisson tensor vanish satisfies this condition and also any point in a Riemann-Poisson manifold.
In Section 2, we prove the following theorems:

Theorem 1.3 *Let $(P, \pi, <, >)$ be a Riemann-Poisson manifold and let S be a symplectic leaf of P. Then S is a Kähler manifold.*

Theorem 1.4 *Let $(P, \pi, <, >)$ be a Riemann-Poisson manifold and let S be a symplectic regular leaf of P. Then the holonomy group of S is finite.*

In Section 3, we prove the following theorem:

Theorem 1.5 *Every Riemann-Poisson manifold is unimodular. In particular, every Riemann-Lie algebra is unimodular.*

The proof of the following theorem is a very long calculation and we don't give it here.

Theorem 1.6 *1. The 2-dimensional abelian Lie algebra is the unique 2-dimensional pseudo-Riemann Lie algebra.*
2. The 3-dimensional Lie algebras which has a pseudo-Riemannian Lie algebra structure are:
a) The Heisenberg Lie algebra given by

$$[e_1, e_2] = e_3, \qquad [e_3, e_1] = [e_3, e_2] = 0,$$

b) A family of Lie algebras given by

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 - \alpha e_3, \quad [e_2, e_3] = 0$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ *and* $\alpha^2 + \beta\gamma \neq 0$.
Furthermore, there is no Riemann-Lie algebra structure on the Heisenberg Lie algebra, and a Lie algebra among the above familly has a structure of Riemann-Lie algebra if and only if $\alpha^2 + \beta\gamma < 0$ *and* $\gamma > \beta$.

It seems, now, that the condition of compatibility given by (3) is a nice notion of compatibility beetwen Poisson structure and Riemann structure. Moreover, the Riemann-Lie algebras, more than are examples of Riemann-Poisson manifolds, give some new examples of homogeneous Kähler manifolds (the coadjoint orbits).

2 Link beetwen pseudo-Riemannian Poisson manifolds and pseudo-Riemannian Lie algebras

2.1 Basic materiel

In this subsection, we collect the basic material which will be used throughout this paper. Many fundamental definitions and results can be found in Vaisman's monograph [4].
Let P be a Poisson manifold with Poisson tensor π. The Poisson bracket on P is given by

$$\{f_1, f_2\} = \pi(df_1, df_2), \qquad f_1, f_2 \in C^\infty(P). \tag{5}$$

We also have a bundle map $\#_\pi : T^*P \longrightarrow TP$ defined by

$$\beta(\#_\pi(\alpha)) = \pi(\alpha, \beta), \qquad \alpha, \beta \in T^*P.$$

On the space of differential 1-forms $\Omega^1(P)$, the Poisson tensor induces a Lie bracket

$$\begin{aligned} [\alpha, \beta]_\pi &= L_{\#_\pi(\alpha)}\beta - L_{\#_\pi(\beta)}\alpha - d(\pi(\alpha, \beta)) \\ &= i_{\#_\pi(\alpha)}d\beta - i_{\#_\pi(\beta)}d\alpha + d(\pi(\alpha, \beta)). \end{aligned} \tag{6}$$

For this Lie bracket and the usual Lie bracket on vector fields, the bundle map $\#_\pi$ induces a Lie algebra homomorphism $\#_\pi : \Omega^1(P) \longrightarrow \mathcal{X}(P)$:

$$\#_\pi([\alpha,\beta]_\pi) = [\#_\pi(\alpha), \#_\pi(\beta)]. \tag{7}$$

We denote, as usual, by $X_f = \#_\pi(df)$ the hamiltonian associated with the function $f \in C^\infty(P)$. We have

$$[df_1, df_2]_\pi = d\{f_1, f_2\} = L_{X_{f_1}} df_2, \qquad f_1, f_2 \in C^\infty(P). \tag{8}$$

Let $<,>$ be a pseudo-metric on the contangent bundle T^*P and D the Levi-Civita contravariant connection associated with the couple $(\pi, <,>)$.
For any $f, g, h \in C^\infty(P)$, we have

$$\begin{aligned} L_{X_f} <,> (dg, dh) &= \#_\pi(df).< dg, dh > - < L_{X_f} dg, dh > - < dg, L_{X_f} dh > \\ &= < D_{df} dg, dh > + < dg, D_{df} dh > \\ &\quad - < [df, dg]_\pi, dh > - < dg, [df, dh]_\pi > \\ &= < D_{dg} df, dh > + < dg, D_{dh} df > . \end{aligned}$$

So, we obtain, for any $f \in C^\infty(P)$ and any $\alpha, \beta \in T^*P$,

$$L_{X_f} <,> (\alpha, \beta) =< D_\alpha df, \beta > + < \alpha, D_\beta df > . \tag{9}$$

Now, we give a condition of compatibility beetwen the Poisson tensor π and the pseudo-metric $<,>$ which is weaker than the condition given in the Definition 1.1. This is possible because the Poisson tensor π has vanishing Schouten bracket $[\pi,\pi]_S$. In fact, since D has vanishing torsion and since the contravariant exterior differential d_π associated with the bracket $[,]_\pi$ is given by $d_\pi Q = -[\pi, Q]_S$, we can deduce obviously that, for any $\alpha, \beta, \gamma \in \Omega^1(P)$,

$$-[\pi,\pi]_S(\alpha,\beta,\gamma) = D\pi(\alpha,\beta,\gamma) + D\pi(\beta,\gamma,\alpha) + D\pi(\gamma,\alpha,\beta). \tag{10}$$

From this formula and a straightforward calculation, we obtain, for any $f, g, h \in C^\infty(P)$,

$$-[\pi,\pi]_S(df, dg, dh) - D\pi(dh, df, dg) = \pi(D_{df} dh, dg) + \pi(df, D_{dg} dh). \tag{11}$$

Now, we can give a condition of compatibility beetwen the Poisson tensor π and the pseudo-metric $<,>$ which is weaker than the condition given by (2).

Proposition 2.1 *Let $(P, \pi, <,>)$ be a Poisson manifold with a pseudo-metric on the cotangent bundle and D the Levi-Civita contravariant connection associated with the couple $(\pi, <,>)$. The following assertions are equivalent:*
1. The triple $(P, \pi, <,>)$ is a pseudo-Riemannian Poisson manifold.
2. For any $\alpha, \beta \in \Omega^1(P)$ and any $f \in C^\infty(P)$,

$$\pi(D_\alpha df, \beta) + \pi(\alpha, D_\beta df) = 0. \tag{12}$$

2.2 Proof of Theorem 1.1

We recall the definition of the Lie algebra obtained by linearizing a Poisson structure at a point. (See [6]).
Let (P, π) be a Poisson manifold and let $x \in P$. We denote $\mathcal{G}_x = Ker\#_\pi(x)$. For any $\alpha, \beta \in \mathcal{G}_x$, set

$$[\alpha, \beta]_x = d_x(\pi(\widetilde{\alpha}, \widetilde{\beta}))$$

where $\widetilde{\alpha}, \widetilde{\beta} \in \Omega^1(P)$ such that $\widetilde{\alpha}_x = \alpha$ and $\widetilde{\beta}_x = \beta$. $(\mathcal{G}_x, [,]_x)$ is a Lie algebra.
Let $(P, \pi, <,>)$ be a pseudo-Riemannian Poisson manifold. Fix a point $x \in P$ such that the restriction of $<,>$ to $Ker\#_\pi(x)$ is non-degenerate. We denote a the restriction of $<,>$ to $\mathcal{G}_x$ and A the bilinear 2-form given by (3).
For any $\alpha, \beta \in \mathcal{G}_x$ and $\widetilde{\alpha}, \widetilde{\beta} \in \Omega^1(P)$ such that $\widetilde{\alpha}_x = \alpha$ and $\widetilde{\beta}_x = \beta$, we claim that

$$(D_{\widetilde{\alpha}}\widetilde{\beta})_x = A_\alpha \beta.$$

In fact, for each $\gamma \in \mathcal{G}_x$ and $\widetilde{\gamma} \in \Omega^1(P)$ such that $\widetilde{\gamma}_x = \gamma$, we have

$$\begin{aligned} 2 < D_{\widetilde{\alpha}}\widetilde{\beta}, \widetilde{\gamma} > &= \#_\pi(\widetilde{\alpha}).<\widetilde{\beta}, \widetilde{\gamma}> + \#_\pi(\widetilde{\beta}).<\widetilde{\alpha}, \widetilde{\gamma}> - \#_\pi(\widetilde{\gamma}).<\widetilde{\alpha}, \widetilde{\beta}> \\ &+ <[\widetilde{\alpha}, \widetilde{\beta}]_\pi, \widetilde{\gamma}> + <[\widetilde{\gamma}, \widetilde{\alpha}]_\pi, \widetilde{\beta}> + <[\widetilde{\gamma}, \widetilde{\beta}]_\pi, \widetilde{\alpha}> . \end{aligned}$$

Since $\#_\pi(\widetilde{\alpha})_x = \#_\pi(\widetilde{\beta})_x = \#_\pi(\widetilde{\gamma})_x = 0$ and since $[\widetilde{\alpha}, \widetilde{\beta}]_\pi(x) = [\alpha, \beta]_x$, we have

$$2 < (D_{\widetilde{\alpha}}\widetilde{\beta})_x, \gamma >= 2a(A_\alpha \beta, \gamma).$$

It remains to show that $(D_{\widetilde{\alpha}}\widetilde{\beta})_x \in \mathcal{G}_x$. Indeed, for any $\mu \in \Omega^1(P)$, we have

$$\begin{aligned} \mu(\#_\pi(D_{\widetilde{\alpha}}\widetilde{\beta})) &= \pi(D_{\widetilde{\alpha}}\widetilde{\beta}, \mu) \\ &= \#_\pi(\widetilde{\alpha}).\pi(\widetilde{\beta}, \mu) - \pi(\widetilde{\beta}, D_{\widetilde{\alpha}}\mu) \\ &= \#_\pi(\widetilde{\alpha}).\pi(\widetilde{\beta}, \mu) - D_{\widetilde{\alpha}}\mu(\#_\pi(\widetilde{\beta})). \end{aligned}$$

Since $\#_\pi(\widetilde{\alpha})_x = \#_\pi(\widetilde{\beta})_x = 0$, we deduce that $(D_{\widetilde{\alpha}}\widetilde{\beta})_x \in \mathcal{G}_x$.
Now, let $f \in C^\infty(P)$ such that $d_x f = \gamma$. We have, from (12),

$$\pi(D_{\widetilde{\alpha}} df, \widetilde{\beta}) + \pi(\widetilde{\alpha}, D_{\widetilde{\beta}} df) = 0.$$

Differentiating this relation at x, we obtain

$$[A_\alpha \gamma, \beta]_x + [\alpha, A_\beta \gamma]_x = 0$$

and the theorem follows.□

2.3 Proof of Theorem 1.2

Let a be a bilinear non-degenerate symmetric form on $\mathcal{G}$ such that the triple $(\mathcal{G}, [,], a)$ is a pseudo-Riemannian Lie algebra. We define on $T^*\mathcal{G}^* = \mathcal{G}^* \times \mathcal{G}$ a pseudo-metric $<,>$ by setting

$$< (\mu, u), (\mu, v) >= a(u, v), \quad \mu \in \mathcal{G}^*, (u, v) \in \mathcal{G}^2.$$

For each vector $v \in \mathcal{G}$, we can define a linear form on $\mathcal{G}^*$ denoted also v. For any $u, v \in \mathcal{G}$ and any $\mu \in \mathcal{G}^*$,we have

$$\pi(dv, du)(\mu) = \mu([u, v]); \qquad [du, dv]_\pi = d[u, v]; \qquad D_{du} dv = d(A_u v).$$

It obvious that, in this case, (4) and (12) are equivalent.
Suppose that there is a pseudo-metric on $\mathcal{G}^*$ such that the triple $(\mathcal{G}^*, \pi, <, >)$ is a pseudo-Riemannian Poisson manifold. The Lie algebra obtained by linearizing the Poisson structure at the origin of $\mathcal{G}^*$ is $\mathcal{G}$ and the theorem follows by Theorem 1.1. □

3 Sypmlectic leaves and the holonomy of Riemann-Poisson manifold

This section is devoted to the proof of Theorem 1.3 and Theorem 1.4. Before, we need some Lemmas.

Lemma 3.1 *Let $(P, \pi, <, >)$ be a pseudo-Riemannian Poisson manifold. Let $S \subset P$ be a symplectic leaf an U an open of P. For any $\alpha, \beta \in \Omega^1(P)$, we have*

$$\#_\pi(\alpha)_{|S\cap U} = 0 \qquad or \qquad \#_\pi(\beta)_{|S\cap U} = 0 \Longrightarrow \#_\pi(D_\alpha\beta)_{|S\cap U} = 0. \tag{13}$$

D is the Levi-Civita contravariant connection associated with $(\pi, <, >)$.

Proof: We have

$$\#_\pi(D_\alpha\beta) - \#_\pi(D_\beta\alpha) = [\#_\pi(\alpha), \#_\pi(\beta)].$$

Hence $\#_\pi(\alpha) = 0$ or $\#_\pi(\beta) = 0$ gives $\#_\pi(D_\alpha\beta) = \#_\pi(D_\beta\alpha)$.
Suppose that $\#_\pi(\beta) = 0$. For any $\gamma \in \Omega^1(P)$,

$$\begin{aligned} \gamma(\#_\pi(D_\alpha\beta)) &= \pi(D_\alpha\beta, \gamma) \\ &= \#_\pi(\alpha).\pi(\beta, \gamma) - \pi(\beta, D_\alpha\gamma) = 0 \end{aligned}$$

and the lemma follows. □

Lemma 3.2 *Let $(P, \pi, <, >)$ be a Riemann-Poisson manifold. Let O be the regular open on which the rank of the Poisson tensor is locally constant. Then, we have:*
1. D is a $\mathcal{F}$-connection on O in the sens of Fernandes [2], this means that, for any $x \in O$ and any $\alpha \in T^*_xP$, we have

$$\#_\pi(\alpha) = 0 \qquad \Longrightarrow \qquad D_\alpha = 0; \tag{14}$$

2. D is a basic connection on O in the sens of Fernandes [2], this means that for any symplectic leaf $S \subset O$ and for any $\alpha, \beta \in \Omega^1(O)$ such that $\#_\pi(\beta)_{|S} = 0$, we have

$$(D_\alpha\beta)_{|S} = [\alpha, \beta]_{\pi|S}. \tag{15}$$

Proof: Let U be an open in O on which the rank of the Poisson tensor is constant. On U, the cotangent bundle splits

$$T^*P = Ker\#_\pi \oplus Ker\#_\pi^\perp$$

where $Ker\#_\pi^\perp$ is the $<, >$-orthogonal of $Ker\#_\pi$.

Let $x \in U$ and $\alpha \in T^*_x P$ such that $\#_\pi(\alpha) = 0$. Choose a 1-forme $\widetilde{\alpha}$ on U such that $\#_\pi(\widetilde{\alpha}) = 0$ and $\widetilde{\alpha}_x = \alpha$. For any $\beta \in \Omega^1(U)$ we have, by Lemma 3.1, $\#_\pi(D_{\widetilde{\alpha}}\beta) = 0$. We claim that $D_{\widetilde{\alpha}}\beta \in Ker\#_\pi^\perp$. Indeed, for any $\gamma \in Ker\#_\pi$, we have

$$< D_{\widetilde{\alpha}}\beta, \gamma >= \#_\pi(\widetilde{\alpha}).< \beta, \gamma > - < \beta, D_{\widetilde{\alpha}}\gamma >= - < \beta, D_{\widetilde{\alpha}}\gamma > .$$

Now, by the spliting of T^*P, $\beta = \beta_1 + \beta_1^\perp$ and so $< \beta, D_{\widetilde{\alpha}}\gamma >=< \beta_1, D_{\widetilde{\alpha}}\gamma >$. This quantity vanish from the definition of D, so D is a $\mathcal{F}$-connexion. Since D has a vanishing torsion, is also a basic connection.□

Lemma 3.3 *Let $(P, \pi, <, >)$ be a Riemann-Poisson manifold. Let O be the regular open on which the rank of the Poisson tensor is locally constant. Then, we have:*
1. For any $x \in O$, for any $\alpha, \beta \in Ker\#_\pi(x)$ and for any $f \in C^\infty(O)$, we have

$$L_{X_f} <, > (\alpha, \beta) = 0;$$

2. For any Casimir functions f, g, $< df, dg >$ is a Casimir function.

Proof: 1. follows from Lemma 3.2 and (9).
2. Let (f, g) be a couple of Casimir functions on P and let $h \in C^\infty(P)$. We have

$$\begin{aligned} \{h, < df, dg >\} &= X_h.< df, dg > \\ &= < D_{dh}df, dg > + < df, D_{dh}dg > \\ &= < D_{df}dh + [dh, df]_\pi, dg > + < df, D_{dg}dh + [dh, dg]_\pi > . \end{aligned}$$

Now $[dh, df]_\pi = d\{h, f\} = 0$ and $D_{df}dh$ vanish on O by Lemma 3.2. So $\{h, < df, dg >\}$ is zero on P since O is dense.□

3.1 Proof of Theorem 1.3

Let $(P, \pi, <, >)$ be a Riemannian-Poisson manifold and let $S \subset P$ be a symplectic leaf. We denote the symplectic form of S by ω_S.
For any vectors fields X, Y tangent to S, we set

$$\nabla^S_X Y = \#_\pi(D_\alpha\beta)_{|S} \tag{16}$$

where $\#_\pi(\alpha)_{|S} = X$ and $\#_\pi(\beta)_{|S} = Y$. It follows from Lemma 3.1 that ∇^S define a torsionless covariant connection on S and, obviously, we have

$$\nabla^S \omega_S = 0.$$

For any $x \in S$, we have

$$T_x^* P = Ker\#_\pi(x) \oplus (Ker\#_\pi(x))^\perp$$

and so the linear map $\#_\pi(x) : (Ker\#_\pi(x))^\perp \longrightarrow T_x S$ is an isomorphism. For any $u, v \in T_x S$, we set

$$g_S(u, v) =< \#_\pi(x)^{-1}(u), \#_\pi(x)^{-1}(v) > . \tag{17}$$

(S, g_s) is a Riemann manifold and $\nabla^S g_S = 0$ and so ∇^S is the Levi-Civita connection of g_S.
Now, it is classical (see [4]) that TS has a ∇^S-parallel complex structure $J = A(-A^2)^{-\frac{1}{2}}$, where A is given by

$$\omega_S(u, v) = g_S(Au, v).$$

So S is a Kähler manifold.□

3.2 Proof of Theorem 1.4

The linear Poisson holonomy was introduced by Ginzburg and Golubev in [3]. A more deep study of this notion was done by Fenrnandes in [2] essentially by the notion of basic connection. In the same paper, Fernandes introduces the notion of Poisson holonomy. For a regular leaf, the (linear) Poisson holonomy coincides with standart (linear) holonomy.
Let $(P, \pi, <, >)$ be a Riemann-Poisson manifold and S a regular symplectic leaf. We prove that the holonomy group of S is finite.
First step: The linear holonomy group of S is finite.
The linear holonomy group of S coincides with the linear Poisson holonomy since the leaf is regular. Now, the linear Poisson holonomy can be determined by a basic connection (see [R]). The Levi-Civita contravariant connection D is, by Lemma 3.2, a basic connection on a neighbourhood of S. Moreover, the parallel transport defined by D is given by isometries on $Ker\#_\pi$. This gives the claim.

Second step: The holonomy group of S is finite.
The holonomy group coincides with the Poisson holonomy group since the leaf is regular. The Poisson holonomy is given by the hamiltonian flows. Now, let $x \in S$ and $(q_1, \ldots, q_k, p_1, \ldots, p_k, y_1, \ldots, y_l)$ a Darboux coordinates defined on a neighbourhood U of x. We considere the submanifold N of U defined by $p = q = 0$. We define a metric g_N on T^*N by

$$g_N(dy_i, dy_j) =< dy_i, dy_j > . \tag{17}$$

By considering (N, g_N), we have according to Lemma 3.3 a well-defined notion of "transverse Riemann structure" along S and the holonomy preserve this transverse structure. Now, the elements of the holonomy group are isometries and their differentials at x are elements of linear holonomy group which is finite so the holonomy group is finite.□

4 Poisson-Riemann manifolds are unimodular

This section is devoted to the proof of Theorem 1.5.
We begin with a metric version of the modular vector field. For the details on the modular vector field see [5].
Let (P, π) be a Poisson manifold and g a Riemannian metric on P. We denote $\#_g : T^*P \longrightarrow TP$ the musical isomorphism associated with g. Also g defines a density on P which we denote μ_g. The modular vector field with respect to μ_g is given by

$$\phi_{\mu_g}(f) = div_g(X_f) = -\sum_{i=1} g(\nabla_{e_i} X_f, e_i) \tag{18}$$

where ∇ is the Levi-Civita connection associated to g, $(e_1, \ldots, e_n)$ is a local orthonormal basis of vector fields and $n = dimP$.
If $h(u, v) = g(Ju, v)$ is another Riemann metric, we have $\mu_h = \sqrt{detJ}\mu_g$ and

$$\phi_{\mu_h} = \phi_{\mu_g} - \frac{1}{2} X_{Ln(detJ)}. \tag{19}$$

Now, we define a Riemann metric $<,>$ on the cotangent bundle T^*P by

$$< \alpha, \beta >= g(\#_g(\alpha), \#_g(\beta)), \qquad \alpha, \beta \in T^*P.$$

(We remark that any Riemann metric on the cotangent bundle T^*P can be obtained in this may).
Let D be the Levi-Civita contravariant connection associated with $(\pi, <,>)$. We claim that

$$\phi_{\mu_g} = \frac{1}{2}\sum_{i=1}^{n} L_{X_f} <,> (\alpha_i, \alpha_i) = \sum_{i=1}^{n} < D_{\alpha_i} df, \alpha_i > \tag{20}$$

where $(\alpha_1, \ldots, \alpha_n)$ is a local othonormal basis of 1-forms. This fact follows easily from the following formula which is also easy to etablish:

$$L_{X_f} <,> (\alpha, \alpha) = -L_{X_f} g(\#_g(\alpha), \#_g(\beta)), \qquad \alpha, \beta \in T^*P. \tag{21}$$

Now, we considere a Riemann-Poisson manifold $(P, \pi, <,>)$ and denote $\phi_{<,>}$ the modular vector field defined by (20). We will show that $\phi_{<,>}$ is zero on P by showing that it is zero on the regular open O.
Let $x \in O$ and S the symplectic leaf of x. We considere the symplectic form ω_S of S, the Riemannian metric g_S on S given by (18) and ∇^S its Levi-Civita connection.
We have, in a neighbourhood of x,

$$T^*P = Ker\#_\pi \oplus (Ker\#_\pi)^\perp.$$

We choose $(\alpha_1, \ldots, \alpha_l)$ and $(\beta_1, \ldots, \beta_{n-l})$ two local orthonormal basis respectively of $Ker\#_\pi$ and $(Ker\#_\pi)^\perp$. We have, according to Lemma 3.2,

$$\begin{aligned} \phi_{<,>}(f)(x) &= \sum_{i=1}^{n-l} < D_{\beta_i} df, \beta_i > \\ &= \sum_{i=1}^{n-l} g_S(\nabla^S_{\#_\pi(\beta_i)} X_f, \#_\pi(\beta_i)) \\ &= -div_{g_S}(X_f)(x). \end{aligned}$$

Now, according to Theorem 1.3 there is a Riemann metric h on S such that (S, h, ω_S) is a Kähler manifold and such that the isomorphism J given by $h(u, v) = g_S(Ju, v)$ is ∇^S-parallel. It follows that $detJ$ is constant and $div_{g_S}(X_f) = div_h(X_f)$. To conclude, we recall the well-known fact that in a Kähler manifold the divergence with respect to the Kähler metric of any hamiltonian vector field is zero.

Now, let $\mathcal{G}$ be a Riemann-Lie algebra. According to Theorem 1.2, $\mathcal{G}^*$ inherits a structure of Poisson-Riemann manifold which is unimodular and the theorem follows.□

References

[1] M. Boucetta, Compatibilité des structures pseudo-riemanniennes et des structures de Poisson, C. R. Acad. Sci. Paris, t. 333, Série I, p. 763-768, 2001.
[2] R. L. Fernandes, Connections in Poisson geometry 1: holonomy and invariants, J. Diff. Geom. 54 , p. 303-366, 2000.
[3] V. Ginzburg and A. Golubev, Holonomy on Poisson Manifolds and The Modular Class, Preprint math.DG/9812153.
[4] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, Progress in Mathematics, vol. 118, Birkhäuser, Berlin, 1994.
[4] A. Weinstein, The modular Automorphism Group of a Poisson Manifold, J. Geom. Phys. 23, p. 379-394, (1997).
[5] A. Weinstein, The Local Structure of Poisson Manifolds, J. Diffrential Geometry 18, p. 523-557,1983.

Mohamed BOUCETTA
Faculté des sciences et techniques, BP 549, Marrakech, Maroc
Email: boucetta@fstg-marrakech.ac.ma