

Research Article

Solution and Stability of a Mixed Type Cubic and Quartic Functional Equation in Quasi-Banach Spaces

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We obtain the general solution and the generalized Ulam-Hyers stability of the mixed type cubic and quartic functional equation $f(x+2y)+f(x-2y)=4(f(x+y)+f(x-y))-24f(y)-6f(x)+3f(2y)$ in quasi-Banach spaces.

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1. Introduction

We recall some basic facts concerning quasiBanach space. A quasinorm is a real-valued function on X satisfying the following.

- (1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
- (2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
- (3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasinormed space if $\|\cdot\|$ is a quasinorm on X . A quasiBanach space is a complete quasinormed space. A quasinorm $\|\cdot\|$ is called a p -norm ($0 < p \leq 1$) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \quad (1.1)$$

for all $x, y \in X$. In this case, a quasiBanach space is called a p -Banach space. Given a p -norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X . By the Aoki-Rolewicz theorem [1] (see also [2]), each quasinorm is equivalent to some p -norm. Since it is much easier to work with p -norms, henceforth we restrict our attention mainly to p -norms. The stability problem of functional equations originated from a question of Ulam [3] in 1940, concerning the stability of group homomorphisms. Let (G_1, \cdot) be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [4] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta \quad (1.2)$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta \quad (1.3)$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. Rassias [5] succeeded in extending the result of Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $(\|x\|^p + \|y\|^p)$, $p \in [0, 1)$ to be unbounded. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms. A number of mathematicians were attracted to the pertinent stability results of Rassias [6], and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Rassias is called Hyers-Ulam-Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 5, 7-18]).

The following cubic functional equation, which is the oldest cubic functional equation, was introduced by the third author of this paper, Rassias [6] (in 2001):

$$f(x + 2y) + 3f(x) = 3f(x + y) + f(x - y) + 6f(y). \quad (1.4)$$

Jun and Kim [19] introduced the following cubic functional equation:

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \quad (1.5)$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function $f(x) = x^3$ satisfies the functional equation (1.5), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real

vector spaces X and Y is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables (see also [20]).

The quartic functional equation (1.6) was introduced by Rassias [21] (in 2000) and then (in 2005) was employed by Park and Bae [22] and others, such that:

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 24f(y) - 6f(x). \quad (1.6)$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.6) if and only if there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = Q(x, x, x, x)$ for all x (see also [21–29]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.6), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function. In this paper we deal with the following functional equation:

$$f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y)) - 24f(y) - 6f(x) + 3f(2y) \quad (1.7)$$

in quasiBanach spaces. It is easy to see that the function $f(x) = ax^3 + bx^4$ is a solution of the functional equation (1.7). In the present paper we investigate the general solution of functional equation (1.7) when f is a mapping between vector spaces, and we establish the generalized Hyers-Ulam-Rassias stability of the functional equation (1.7) whenever f is a mapping between two quasiBanach spaces. We only mention here the papers [30, 31] concerning the stability of the mixed type functional equations.

2. General Solution

Throughout this section, X and Y will be real vector spaces. Before proceeding to the proof of Theorem 2.3 which is the main result in this section, we shall need the following two lemmas.

Lemma 2.1. *If an even function $f : X \rightarrow Y$ satisfies (1.7), then f is quartic.*

Proof. Putting $x = y = 0$ in (1.7), we get $f(0) = 0$. Setting $x = 0$ in (1.7), by evenness of f we obtain

$$f(2y) = 16f(y) \quad (2.1)$$

for all $y \in X$. Hence (1.7) can be written as

$$f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y)) + 24f(y) - 6f(x). \quad (2.2)$$

This means that f is quartic function, which completes the proof of the lemma. \square

Lemma 2.2. *If an odd function $f : X \rightarrow Y$ satisfies (1.7), then f is a cubic function.*

Proof. Setting $x = y = 0$ in (1.7) gives $f(0) = 0$. Putting $x = 0$ in (1.7), then by oddness of f , we have

$$f(2y) = 8f(y). \quad (2.3)$$

Hence (1.7) can be written as

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x). \quad (2.4)$$

Replacing x by $x + y$ in (2.4), we obtain

$$f(x + 3y) + f(x - y) = 4f(x + 2y) - 6f(x + y) + 4f(x). \quad (2.5)$$

Substituting $-y$ for y in (2.5) gives

$$f(x - 3y) + f(x + y) = 4f(x - 2y) - 6f(x - y) + 4f(x). \quad (2.6)$$

If we subtract (2.5) from (2.6), we obtain

$$f(x + 3y) - f(x - 3y) = 4f(x + 2y) - 4f(x - 2y) - 5f(x + y) + 5f(x - y). \quad (2.7)$$

Let us interchange x and y in (2.7). Then we see that

$$f(3x + y) + f(3x - y) = 4f(2x + y) + 4f(2x - y) - 5f(x + y) - 5f(x - y). \quad (2.8)$$

With the substitution $y := x + y$ in (2.4), we have

$$f(3x + 2y) - f(x + 2y) = 4f(2x + y) - 4f(y) - 6f(x). \quad (2.9)$$

From the substitution $y := -y$ in (2.9) it follows that

$$f(3x - 2y) - f(x - 2y) = 4f(2x - y) + 4f(y) - 6f(x). \quad (2.10)$$

If we add (2.9) to (2.10), we have

$$f(3x + 2y) + f(3x - 2y) = 4f(2x + y) + 4f(2x - y) + f(x + 2y) + f(x - 2y) - 12f(x). \quad (2.11)$$

Replacing x by $2x$ in (2.7) and using (2.3), we obtain

$$f(2x + 3y) - f(2x - 3y) = 32f(x + y) - 32f(x - y) - 5f(2x + y) + 5f(2x - y). \quad (2.12)$$

Interchanging x with y in (2.12) gives the equation

$$f(3x + 2y) + f(3x - 2y) = 32f(x + y) + 32f(x - y) - 5f(x + 2y) - 5f(x - 2y). \quad (2.13)$$

If we compare (2.11) and (2.13) and employ (2.4), we conclude that

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \quad (2.14)$$

This means that f is cubic function. This completes the proof of Lemma. \square

Theorem 2.3. *A function $f : X \rightarrow Y$ satisfies (1.7) for all $x, y \in X$ if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ and a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x) + Q(x, x, x, x)$ for all $x \in X$, and that C is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Let f satisfy (1.7). We decompose f into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)) \quad (2.15)$$

for all $x \in X$. By (1.7), we have

$$\begin{aligned} f_e(x + 2y) + f_e(x - 2y) &= \frac{1}{2}[f(x + 2y) + f(-x - 2y) + f(x - 2y) + f(-x + 2y)] \\ &= 4(f_e(x + y) + f_e(x - y)) - 24f_e(y) - 6f_e(x) + 3f_e(2y) \end{aligned} \quad (2.16)$$

for all $x, y \in X$. This means that f_e satisfies in (1.7). Similarly we can show that f_o satisfies (1.7). By Lemmas 2.1 and 2.2, f_e and f_o are quartic and cubic, respectively. Thus there exists a unique function $C : X \times X \times X \rightarrow Y$ and a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f_e(x) = Q(x, x, x, x)$ and that $f_o(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. Thus $f(x) = C(x, x, x) + Q(x, x, x, x)$ for all $x \in X$. The proof of the converse is trivial. \square

3. Stability

Throughout this section, X and Y will be a uniquely two-divisible abelian group and a quasiBanach spaces respectively, and p will be a fixed real number in $[0, 1]$. We need the following lemma in the main theorems. Now before taking up the main subject, given $f : X \rightarrow Y$, we define the difference operator $D_f : X \times X \rightarrow Y$ by

$$D_f(x, y) = f(x + 2y) + f(x - 2y) - 4[f(x + y) + f(x - y)] - 3f(2y) + 24f(y) + 6f(x) \quad (3.1)$$

for all $x, y \in X$. We consider the following functional inequality:

$$\|D_f(x, y)\| \leq \phi(x, y) \quad (3.2)$$

for an upper bound $\phi : X \times X \rightarrow [0, \infty)$.

Lemma 3.1. *Let x_1, x_2, \dots, x_n be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^n x_i \right)^p \leq \sum_{i=1}^n x_i^p. \quad (3.3)$$

Theorem 3.2. *Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \rightarrow \mathbb{R}^+$ be a function such that*

$$\lim_{n \rightarrow \infty} 16^{ln} \varphi\left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}}\right) = 0 \quad (3.4)$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 16^{ilp} \varphi^p\left(0, \frac{y}{2^{li}}\right) < \infty \quad (3.5)$$

for all $y \in X$. Suppose that an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \varphi(x, y), \quad (3.6)$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n \rightarrow \infty} 16^{ln} f\left(\frac{x}{2^{ln}}\right) \quad (3.7)$$

exists for all $x \in X$ and $Q : X \rightarrow Y$ is a unique quartic function satisfying

$$\|f(x) - Q(x)\|_Y \leq \frac{K}{16} [\tilde{\varphi}_e(x)]^{1/p}, \quad (3.8)$$

where

$$\tilde{\varphi}_e(x) := \sum_{i=|l+1|/2}^{\infty} 16^{ilp} \varphi^p\left(0, \frac{x}{2^i}\right) \quad (3.9)$$

for all $x \in X$.

Proof. Let $l = 1$. By putting $x = 0$ in (3.6), we get

$$\|f(2y) - 16f(y)\|_Y \leq \varphi(0, y) \quad (3.10)$$

for all $y \in X$. Replacing y by x in (3.10) yields

$$\|f(2x) - 16f(x)\|_Y \leq \varphi(0, x) \quad (3.11)$$

for all $x \in X$. Let $\varphi(x) = \varphi(0, x)$ for all $x \in X$, then by (3.11), we get

$$\|f(2x) - 16f(x)\|_Y \leq \varphi(x) \quad (3.12)$$

for all $x \in X$. Interchanging x with $x/2^{n+1}$ in (3.12), and multiplying by 16^n it follows that

$$\left\| 16^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 16^n f\left(\frac{x}{2^n}\right) \right\|_Y \leq K16^n \varphi\left(\frac{x}{2^{n+1}}\right) \quad (3.13)$$

for all $x \in X$ and all nonnegative integers n . Since Y is p -Banach space, then by (3.13) we have

$$\begin{aligned} \left\| 16^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 16^m f\left(\frac{x}{2^m}\right) \right\|_Y^p &\leq \sum_{i=m}^n \left\| 16^{i+1}f\left(\frac{x}{2^{i+1}}\right) - 16^i f\left(\frac{x}{2^i}\right) \right\|_Y^p \\ &\leq K^p \sum_{i=m}^n 16^{ip} \varphi^p\left(\frac{x}{2^{i+1}}\right) \end{aligned} \quad (3.14)$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. Since $\varphi^p(x) = \varphi^p(0, x)$ for all $x \in X$. Therefore by (3.5) we have

$$\sum_{i=1}^{\infty} 16^{ip} \varphi^p\left(\frac{x}{2^i}\right) < \infty \quad (3.15)$$

for all $x \in X$. Therefore we conclude from (3.14) and (3.15) that the sequence $\{16^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it follows that the sequence $\{16^n f(x/2^n)\}$ converges for all $x \in X$. We define the mapping $Q : X \rightarrow Y$ by (3.7) for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3.14), we get

$$\|f(x) - Q(x)\|_Y^p \leq K^p \sum_{i=0}^{\infty} 16^{ip} \varphi^p\left(\frac{x}{2^{i+1}}\right) = \frac{K^p}{16^p} \sum_{i=1}^{\infty} 16^{ip} \varphi^p\left(\frac{x}{2^i}\right) \quad (3.16)$$

for all $x \in X$. Therefore (3.8) follows from (3.9) and (3.16). Now we show that Q is quartic. It follows from (3.4), (3.6) and (3.7)

$$\|D_Q(x, y)\|_Y = \lim_{n \rightarrow \infty} 16^n \left\| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\|_Y \leq \lim_{n \rightarrow \infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad (3.17)$$

for all $x, y \in X$. Therefore the mapping $Q : X \rightarrow Y$ satisfies (1.7). Since $Q(0) = 0$, then by Lemma 2.1 we get that the mapping $Q : X \rightarrow Y$ is quartic. To prove the uniqueness of Q , let $T : X \rightarrow Y$ be another quartic mapping satisfies (3.8). Since

$$\lim_{n \rightarrow \infty} 16^{np} \sum_{i=1}^{\infty} 16^{ip} \varphi^p \left(\frac{x}{2^{i+n}}, \frac{y}{2^{i+n}} \right) = \lim_{n \rightarrow \infty} \sum_{i=n+1}^{\infty} 16^{ip} \varphi^p \left(\frac{x}{2^i}, \frac{y}{2^i} \right) = 0 \quad (3.18)$$

for all $y \in X$ and all $x \in \{0\}$, then

$$\lim_{n \rightarrow \infty} 16^{np} \tilde{\varphi}_e \left(\frac{x}{2^n} \right) = 0 \quad (3.19)$$

for all $x \in X$. It follows from (3.8), (3.19)

$$\|Q(x) - T(x)\|_Y^p = \lim_{n \rightarrow \infty} 16^{np} \left\| f \left(\frac{x}{2^n} \right) - T \left(\frac{x}{2^n} \right) \right\|_Y^p \leq \frac{K^p}{16^p} \lim_{n \rightarrow \infty} 16^{np} \tilde{\varphi}_e \left(\frac{x}{2^n} \right) = 0 \quad (3.20)$$

for all $x \in X$. Hence $Q = T$. For $l = -1$, we obtain

$$\left\| \frac{f(2^n x)}{16^n} - f(x) \right\|_Y^p \leq \frac{K^p}{16^p} \sum_{i=0}^{\infty} \frac{\varphi^p(0, 2^i x)}{16^{ip}}, \quad (3.21)$$

from which one can prove the result by a similar technique. \square

Corollary 3.3. Let θ, r, s, u, v be nonnegative real numbers such that $s \neq 4 \neq u + v$. Suppose that an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \theta (\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s) \quad (3.22)$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \rightarrow Y$ satisfying

$$\|f(x) - Q(x)\|_Y \leq K\theta \left\{ \frac{1}{|16^p - 2^{sp}|} \right\}^{1/p} \|x\|_X^s \quad (3.23)$$

for all $x \in X$.

Proof. It follows from Theorem 3.2 that $\varphi(x, y) := \theta (\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$. \square

Theorem 3.4. Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \rightarrow [0, \infty)$ be a function such that

$$\lim_{n \rightarrow \infty} 8^{ln} \varphi \left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) = 0 \quad (3.24)$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 8^{ip} \varphi^p \left(0, \frac{y}{2^i} \right) < \infty \quad (3.25)$$

for all $y \in X$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \varphi(x, y), \quad (3.26)$$

for all $x, y \in X$. Then the limit

$$C(x) := \lim_{n \rightarrow \infty} 8^{ln} f \left(\frac{x}{2^{ln}} \right) \quad (3.27)$$

exists for all $x \in X$ and $C : X \rightarrow Y$ is a unique cubic function satisfying

$$\|f(x) - C(x)\|_Y \leq \frac{K}{24} [\tilde{\phi}_o(x)]^{1/p} \quad (3.28)$$

for all $x \in X$, where

$$\tilde{\phi}_o(x) := \sum_{i=|l+1|/2}^{\infty} 8^{ip} \varphi^p \left(0, \frac{x}{2^i} \right). \quad (3.29)$$

Proof. Let $l = 1$. Setting $x = 0$ in (3.26), we get

$$\|3f(2y) - 24f(y)\|_Y \leq \varphi(0, y) \quad (3.30)$$

for all $y \in X$. If we replace y in (3.30) by x and divide both sides of (3.30) by 3, we get

$$\|f(2x) - 8f(x)\|_Y \leq \frac{1}{3} \varphi(0, x) \quad (3.31)$$

for all $x \in X$. Let $\phi(x) = (1/3)\varphi(0, x)$ for all $x \in X$, then by (3.31), we get

$$\|f(2x) - 8f(x)\|_Y \leq \phi(x) \quad (3.32)$$

for all $x \in X$. Multiply (3.32) by 8^n and replace x by $x/2^{n+1}$, we obtain that

$$\left\| 8^{n+1} f \left(\frac{x}{2^{n+1}} \right) - 8^n f \left(\frac{x}{2^n} \right) \right\|_Y \leq K 8^n \phi \left(\frac{x}{2^{n+1}} \right) \quad (3.33)$$

for all $x \in X$ and all nonnegative integers n . Since Y is a p -Banach space, (3.33) follows that

$$\begin{aligned} \left\| 8^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 8^n f\left(\frac{x}{2^n}\right) \right\|_Y^p &\leq \sum_{i=m}^n \left\| 8^{i+1} f\left(\frac{x}{2^{i+1}}\right) - 8^i f\left(\frac{x}{2^i}\right) \right\|_Y^p \\ &\leq K^p \sum_{i=m}^n 8^{ip} \phi^p\left(\frac{x}{2^{i+1}}\right) \end{aligned} \quad (3.34)$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. Since $\phi^p(x) = (1/3^p)\varphi^p(0, x)$ for all $x \in X$. Therefore it follows from (3.25) that

$$\sum_{i=1}^{\infty} 8^{ip} \phi^p\left(\frac{x}{2^i}\right) < \infty \quad (3.35)$$

for all $x \in X$, therefore we conclude from (3.34) and (3.35) that the sequence $\{8^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{8^n f(x/2^n)\}$ converges for all $x \in X$. So one can define the mapping $C : X \rightarrow Y$ by (3.27) for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3.34), we get

$$\|f(x) - C(x)\|_Y^p \leq K^p \sum_{i=0}^{\infty} 8^{ip} \phi^p\left(\frac{x}{2^{i+1}}\right) = \frac{K^p}{8^p} \sum_{i=1}^{\infty} 8^{ip} \phi^p\left(\frac{x}{2^i}\right) \quad (3.36)$$

for all $x \in X$. Therefore (3.28) follows from (3.29) and (3.36). Now we show that C is cubic. It follows from (3.24), (3.26) and (3.27)

$$\|D_C(x, y)\|_Y = \lim_{n \rightarrow \infty} 8^n \left\| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\|_Y \leq \lim_{n \rightarrow \infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad (3.37)$$

for all $x, y \in X$. Therefore the mapping $C : X \rightarrow Y$ satisfies (1.7). Since f is an odd function, then (3.27) implies that the mapping odd. Therefore by Lemma 2.2 we get that the mapping $C : X \rightarrow Y$ is cubic. The rest of proof is similar to the proof of Theorem 3.2. \square

Corollary 3.5. *Let θ be a nonnegative real number and r, s be real numbers such that $s \neq 3 \neq u + v$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality*

$$\|D_f(x, y)\|_Y \leq \theta (\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s) \quad (3.38)$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \rightarrow Y$ satisfying

$$\|f(x) - C(x)\|_Y \leq \frac{K\theta}{3} \left\{ \frac{1}{|8^p - 2^{sp}|} \right\}^{1/p} \|x\|_X^s \quad (3.39)$$

for all $x \in X$.

Proof. It follows from (3.38) and Theorem 3.4 that $\varphi(x, y) := \theta (\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$. \square

Theorem 3.6. Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \rightarrow [0, \infty)$ be a function which satisfies

$$\lim_{n \rightarrow \infty} \left\{ \left(\frac{|l|+1}{2} \right) 16^{ln} \varphi \left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) + \left(\frac{|l|-1}{2} \right) 8^{ln} \varphi \left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) \right\} = 0 \quad (3.40)$$

for all $x, y \in X$ and

$$\sum_{i=|l+1|/2}^{\infty} \left\{ \left(\frac{|l|+1}{2} \right) 16^{ip} \varphi^p \left(0, \frac{y}{2^i} \right) + \left(\frac{|l|-1}{2} \right) 8^{ip} \varphi^p \left(0, \frac{y}{2^i} \right) \right\} < \infty \quad (3.41)$$

for all $y \in X$. Suppose that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \varphi(x, y) \quad (3.42)$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying (1.7) and

$$\|f(x) - Q(x) - C(x)\|_Y \leq \frac{K^3}{32} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)]^{1/p} + \frac{K^3}{48} [\tilde{\phi}_o(x) + \tilde{\phi}_o(-x)]^{1/p} \quad (3.43)$$

for all $x \in X$, where $\tilde{\psi}_e(x)$ and $\tilde{\phi}_o(x)$ that have been defined in (3.9) and (3.29), respectively.

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0, f_e(-x) = f_e(x)$, and $\|D_{f_e}(x, y)\| \leq (K/2)[\varphi(x, y) + \varphi(-x, -y)]$ for all $x, y \in X$. Let

$$\Phi(x, y) = \frac{K}{2} [\varphi(x, y) + \varphi(-x, -y)] \quad (3.44)$$

for all $x, y \in X$. So

$$\lim_{n \rightarrow \infty} 16^{ln} \Phi \left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) = 0 \quad (3.45)$$

for all $x, y \in X$. Since

$$\Phi^p(x, y) \leq \frac{K^p}{2^p} [\varphi^p(x, y) + \varphi^p(-x, -y)] \quad (3.46)$$

for all $x, y \in X$, then

$$\sum_{i=1}^{\infty} 16^{ip} \Phi^p \left(\frac{x}{2^{i1}}, \frac{y}{2^{i1}} \right) < \infty \quad (3.47)$$

for all $y \in X$ and all $x \in \{0\}$. Hence, in view of Theorem 3.2, there exists a unique quartic function $Q : X \rightarrow Y$ satisfying

$$\|f_e(x) - Q(x)\|_Y \leq \frac{K}{16} [\tilde{\Psi}_e(x)]^{1/p} \quad (3.48)$$

for all $x \in X$, where

$$\tilde{\Psi}_e(x) := \sum_{i=1}^{\infty} 16^{ip} \Phi^p \left(0, \frac{x}{2^i} \right). \quad (3.49)$$

We have

$$\tilde{\Psi}_e(x) \leq \frac{K^p}{2^p} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)] \quad (3.50)$$

for all $x \in X$. Therefore it follows from (3.48) that,

$$\|f_e(x) - Q(x)\|_Y \leq \frac{K^2}{32} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)]^{1/p} \quad (3.51)$$

for all $x \in X$. Let $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and $\|D_{f_o}(x, y)\| \leq \Phi(x, y)$ for all $x, y \in X$. From Theorem 3.4, it follows that there exists a unique cubic function $C : X \rightarrow Y$ satisfying

$$\|f_o(x) - C(x)\|_Y \leq \frac{K}{24} [\tilde{\Phi}_o(x)]^{1/p} \quad (3.52)$$

for all $x \in X$, where

$$\tilde{\Phi}_o(x) := \sum_{i=1}^{\infty} 8^{ip} \Phi^p \left(0, \frac{x}{2^i} \right). \quad (3.53)$$

Since

$$\tilde{\Phi}_o(x) \leq \frac{K^p}{2^p} [\tilde{\phi}_o(x) + \tilde{\phi}_o(-x)] \quad (3.54)$$

for all $x \in X$, it follows from (3.52) that,

$$\|f_o(x) - C(x)\|_Y \leq \frac{K^2}{48} [\tilde{\phi}_o(x) + \tilde{\phi}_o(-x)]^{1/p} \quad (3.55)$$

for all $x \in X$. Hence (3.43) follows from (3.51) and (3.55). \square

Corollary 3.7. Let θ, r, s be nonnegative real numbers such that $u + v, s \in (4, \infty) \cup (-\infty, 3)$. Suppose that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \theta(\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s) \quad (3.56)$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying (1.7) and

$$\|f(x) - Q(x) - C(x)\|_Y \leq \frac{K^3\theta}{3} \left\{ 3 \left[\frac{1}{|16^p - 2^{sp}|} \right]^{1/p} + \left[\frac{1}{|8^p - 2^{sp}|} \right]^{1/p} \right\} \|x\|_X^s \quad (3.57)$$

for all $x \in X$.

Proof. It follows from Theorem 3.6 that

$$\varphi(x, y) := \theta(\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s) \quad (3.58)$$

for all $x, y \in X$. □

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