A VERSION OF KAC'S LEMMA ON FIRST RETURN TIMES FOR SUSPENSION FLOWS

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Abstract. In this article we study the mean return times to a given set for suspension flows. In the discrete time setting, this corresponds to the classical version of Kac's lemma [12] that the mean of the first return time to a set with respect to the normalized probability measure is one. In the case of suspension flows we provide formulas to compute the mean return time. Positive measure sets on cross sections are also considered.In particular, this varies linearly with continuous reparametrizatons of the flow and takes into account the mean escaping time from the original set. Relation with entropy and returns to positive measure sets on cross sections is also considered.

1. Introduction

The construction of invariant measures and the study of their statistical properties are the main goals of the ergodic theory in the attempt to describe the behavior of a discrete and continuous time dynamical system. In the presence of invariant measures the celebrated Poincaré' s recurrence theorem guarantees that recurrence occurs almost everywhere and, from this qualitative statement, a natural question is to ask for quantitative results on the recurrence. Given a measure preserving map $f : (X, \mu) \to (X, \mu)$ and a positive measure set $A \subset X$ consider the *first return time* to the set A given by

$$n_A(x) = \inf\{k \geq 1 : f^k(x) \in A\}.$$

In 1947, Kac [12] established an expression to the expectation of the return time proving that

$$\int_A n_A(x)\, d\mu_A = 1,$$

where $\mu_A(\cdot) = \mu(\cdot \cap A)/\mu(A)$ denotes the normalized restriction of μ to A. Such a quantitative estimate has been very useful tool in the study of other recurrence properties (see e.g. [7, 8, 9, 15, 16] and the references therein)

In the time-continuous setting the situation is substantially different. It is clear that if a flow $(X_t)_t$ preserves a probability measure μ then it is an invariant probability measure for discrete-time transformation obtained as the time-t map $f = X_t$ of the flow and Kac' s lemma holds for (f, μ). However, this does not constitute a true time-continuous quantitative recurrence estimate. In fact, given a continuous flow $(X_t)_t$, an open set A and a point $x \in A$, the first return time of x to the set A should be considered after the time $e_A(x,t) > 0$ that orbit of the point x needs to exit A (c.f. Section 2 for the precise definition).

Date: November 25, 2014.
2010 *Mathematics Subject Classification.* Primary: 37B20 Secondary: 28D10, 37C10 .
Key words and phrases. Suspension flows, Poincaré recurrence, hitting times, Kac's lemma.

Our purpose here is to prove a time-continuous quantitative Kac-like formula for first return times for suspension flows for positive measure sets both in the cross-section and ambient spaces. Our first motivation follows from Ambrose and Kakutani [3, 4] which proved that any aperiorid flow $(Y_t)_t$ on a probability space (M, ν) without singularities is metrically isomorphic to a suspension flow $(Z_t)_t$. More precisely, there exists a measurable transformation $f : \Sigma \to \Sigma$, an f-invariant probability measure μ_Σ and a roof function $\tau : \Sigma \to \mathbb{R}_0^+$ defining a suspension flow $(Z_t)_t$ on Σ_τ that preserves $\mu = \mu_\Sigma \times Leb / \int \tau d\mu_\Sigma$, and there exists a measure preserving $\psi : (M, \nu) \to (\Sigma_\tau, \mu)$ that is a bijection (modulo zero measure sets) and satisfying $Z_t \circ \psi = \psi \circ Y_t$ and $Y_t \circ \psi^{-1} = \psi^{-1} \circ Z_t$ for all $t \geq 0$. Our second motivation is that, using finitely many tubular neighborhoods or the Hartman-Grobman theorem around singularities, smooth flows on compact manifolds that exhibit some hyperbolicity often can be modelled by suspension flows. This is true for hyperbolic flows (see e.g. [6, 14]) for geometric Lorenz and three-dimensional singular-hyperbolic flows (see e.g. [5]) and for smooth three-dimensional flows with positive topological entropy (see [11]). A final motivation is that it is expected to have a large utility and large amount of applications, including the study of large deviations for return times in cylinders, fluctuations theorems and study of the recurrence properties for Axiom A flows. Our main results below assert that the expectation of the first return time to open sets takes into account the mean return time of the roof function. Moreover, the mean return time to subsets of the global cross section can be understood as a quotient of the corresponding entropies of the flow and the first return time map on the base. The detailed statements will be given in the next section.

After this work was complete we were informed by J.-R. Chazottes of the unnoticed and very interesting work by G. Helberg [10] where the author addresses the problem of return time estimates for positive measure sets for measurable and continuous flows. Since this is a rather unknown result, never quoted, we shall describe his main results in Section 2 for completeness.

2. Suspension flows

Given a topological space Σ, a measurable map $f : \Sigma \to \Sigma$ and a roof function $\tau : \Sigma \to \mathbb{R}^+$ that is bounded away from zero consider the quotient space

$$\Sigma_\tau = \{(x, s) \in \Sigma \times \mathbb{R}^+ : 0 \leq s \leq \tau(x)\} / \sim$$

obtained by the equivalence relation that $(x, \tau(x)) \sim (f(x), 0)$ for every $x \in \Sigma$. The suspension flow $(X_t)_t$ on Σ_τ associated to (f, Σ, τ) is defined by as the 'vertical displacement' $X_t(x, s) = (x, t + s)$ whenever the expression is well defined. More precisely,

$$X_s(x, t) = \Big(f^k(x), t + s - \sum_{j=0}^{k-1} \tau(f^j(x)) \Big) \tag{2.1}$$

where $k = k(x, t, s) \geq 0$ is determined by $\sum_{j=0}^{k-1} \tau(f^j(x)) \leq t + s < \sum_{j=0}^{k} \tau(f^j(x))$. We shall refer to Σ as a cross-section to the flow. Since τ is bounded away from zero there is a natural identification between the space $\mathcal{M}_X$ of $(X_t)_t$-invariant probability measures and the space $\mathcal{M}_f(\tau)$ of f-invariant probability measures μ

so that $\tau \in L^1(\mu)$. Namely, if m denotes the Lebesgue measure in $\mathbb{R}$ then the map

$$\begin{array}{rccc} L: & \mathcal{M}_f(\tau) & \to & \mathcal{M}_X \\ & \mu & \mapsto & \bar{\mu} := \frac{(\mu\times m)|_{\Sigma_\tau}}{\int_\Sigma \tau\, d\mu} \end{array} \tag{2.2}$$

is a bijection. For that reason in some situations one reduces some ergodic properties from the suspension semiflow to the first return map to the global Poincaré section $\Sigma \times \{0\}$. For instance, while the measure induced on the cross section can detect ergodicity of the flow it fails to detect the mixing properties.

From now on, and if otherwise stated, we shall fix an f-invariant ergodic probability measure μ and let $\bar{\mu}$ denote the corresponding flow-invariant ergodic probability measure. We start by recalling Kac's qualitative recurrence estimates for measurable maps.

Theorem 2.1 (Kac's lemma)**.** *Let $f : \Sigma \to \Sigma$ be a measurable map preserving a probability measure μ. For any measurable set A with $\mu(A) > 0$ the hitting time $n_A(\cdot) : \Sigma \to \mathbb{N}$ defined by $n_A(x) = \inf\{k \geq 1 : f^k(x) \in A\}$ satisfies*

$$\int_A n_A(x)\, d\mu = 1$$

Equivalently, $\int_A n_A(x)\, d\mu_A = \frac{1}{\mu(A)}$ is inversely proportional to the measure of A.

In the end of the sixties, Helmberg [10] studied recurrence for arbitrary sets satisfying a 'boundary condition' with respect to measurable and continuous flows. In general, if a set A is such that the boundary ∂A has positive $\bar{\mu}$-measure and admits a fractal structure then it was not clear how to define properly return times to A. To overcome this difficulty Helmberg paper defined return times using (measurable) families of exit and entrance regions: for any $s > 0$ define the *exit region* A_s by $A_s = \{z \in M : \exists 0 \leq \ell < r \leq s \text{ so that } X_\ell(z) \in A \text{ and } X_r(z) \notin A\}$ and analogousy the *entrance region* $\tilde{A}_s$ by $\tilde{A}_s = \{z \in M : \exists 0 \leq \ell < r \leq s \text{ so that } X_\ell(z) \notin A \text{ and } X_r(z) \in A\}$. These are cleary decreasing families as s tends to zero. In this case, the *return time* is defined by $r_A(z) = \inf\{s > 0 : z \in \tilde{A}_s\}$ and it is proved to be almost everywhere well defined. More precisely,

Theorem 2.2. [10, Theorem 5] *Let $(X_t)_{t\geq 0}$ be a semiflow of measurable transformations on a probability space $(M, \bar{\mu})$. If $A \subset M$ is a positive $\bar{\mu}$-measure set and $t > 0$ is such that $\lim_{n\to\infty} \bar{\mu}(A_{\frac{t}{n}}) = 0$ then*

$$\lim_{s\to 0} \frac{1}{s} \int_{A_s} r_A(z)\, d\bar{\mu} = \bar{\mu}\Big(\bigcup_{r\geq 0} X_{-r}(A)\Big) - \bar{\mu}(A).$$

In particular, if $\bar{\mu}$ is ergodic then the right hand side is equal to $1 - \bar{\mu}(A)$.

The limit in the previous theorem follows by the strategy used by Helmberg of approximating return times for the flow by studying return times for individual time-s maps T_s for small $s > 0$. While the result holds under big generality e.g. for all positive measure sets A so that $\bar{\mu}(\partial A) = 0$ it provides estimate on the limiting mean return time function on the exit components A_s, that is, on points in the eminence of leaving the set A.

In our time-continuous setting of suspension flows and geometrical objects like cylinders or balls whose boundary will have always always zero. We will compute the mean return time among points in the whole set A. Taking into account the normalization given by equation (2.2) it is a natural question to understand in which sense the mean return time for suspension flows reflects its dependence on the integral of the roof function. More precisely, given a set A if one makes a reparametrization of the suspension flow leaving the set A invariant it is interesting to understand if the mean return times for both flows coincide or, if not, if they can be related to each other. Moreover, we shall address some questions on positive measures sets on the cross section Σ, which have zero $\bar{\mu}$-measure on the flow while are dynamically significant and contain non-trivial recurrence.

Let us introduce some necessary notions. Given a set $A \subset \Sigma_\tau$ define the *escape time* $e_A(\cdot,\cdot) : \Sigma_\tau \to \mathbb{R}_0^+$ to be given by

$$e_A(x,t) = \inf\{s > 0 : X_s(x,t) \notin A\}$$

for any $(x,t) \in \Sigma_\tau$. The escape time $e_A(x,t)$ is clearly zero for any point (x,t) in the complement of A. We define also the *hitting time* to A as the function $n_A : \Sigma_\tau \to \mathbb{R}_0^+$ given by

$$n_A(x,t) = \inf\{s > e_A(x,t) : X_s(x,t) \in A\}. \tag{2.3}$$

The *return time function to* A consists of the restriction of $n_A(\cdot,\cdot)$ to the set A and, clearly, $n_A(x,t) \geq e_A(x,t)$ for every $(x,t) \in A$. Alternatively, we can define the return time $\tilde{n}_A(x,t) = \inf\{s > 0 : X_{s+e_A(x,t)}(x,t) \in A\}$, in which case the functions verify $n_A = e_A + \tilde{n}_A$ and consequently to $\int_A n_A \, d\bar{\mu}$ and $\int_A \tilde{n}_A \, d\bar{\mu}$ just differ by the average escaping time $\int_A e_A \, d\bar{\mu}$. For that reason we shall restrict to the study of the previously defined hitting time function. Our first result is a version of Kac's lemma for returns to cylinder sets.

Theorem A. *Let $f : \Sigma \to \Sigma$ be a measurable invertible map on a topological space Σ preserving an ergodic probability measure μ and let $\tau : \Sigma \to \mathbb{R}_0^+$ be a μ-integrable roof function. Denote by $(X_t)_t$ the suspension semiflow over (f, μ, τ) and let $\bar{\mu}$ be the corresponding $(X_t)_t$ invariant probability measure given by* (2.2)*. Given a positive $\bar{\mu}$-measure cylinder set $A = I \times [t_1, t_2] \subset \Sigma \times \mathbb{R}^+$ in Σ_τ then*

$$\begin{aligned} \int_A n_A(x,t)\, d\bar{\mu}_A &= \int_A e_A(x,t)\, d\bar{\mu}_A + \big(1 - \bar{\mu}(A)\big) \frac{1}{\mu(I)} \int_\Sigma \tau \, d\mu \\ &= \frac{m([t_1,t_2])}{2} + \frac{1}{\mu(I)} \int_\Sigma \Big(\tau - (t_2 - t_1)\chi_I \Big)\, d\mu \end{aligned}$$

In particular, if the set A is kept unchanged and we change the roof function τ the mean return time $\int_A n_A(x,t)\, d\mu_A$ varies lineary with respect to the mean of the roof function $\int_\Sigma \tau \, d\mu$.

Proof. Let $n_I(\cdot)$ be the first return time function to I by f. Given $(x,t) \in \Sigma_\tau$ and $s \geq 0$ it follows from (2.1) that $X_s(x,t) = (f^k(x), t + s - \sum_{j=0}^{k-1} \tau(f^j(x)))$ where $k = k(x,t,s) \geq 0$ is determined by $\sum_{j=0}^{k-1} \tau(f^j(x)) \leq t + s < \sum_{j=0}^{k} \tau(f^j(x))$. Observe that if $X_s(x,t) \in A$ for $s > e_A(x,t)$ and $k = k(x,t,s)$ is defined as above then necessarily $f^k(x) \in I$. Reciprocally, if $(x,t) \in \Sigma_\tau$ and $f^k(x) \in I$ then there exists $s < \sum_{j=0}^{k} \tau(f^j(x)) - t$ such that $X_s(x,t) \in A$. Now, since we are dealing with a cylinder $A = I \times [t_1, t_2]$ then $e_A(x,t) = t_2 - t$ and so

$$n_A(x,t) = \Big[\sum_{j=0}^{n_I(x)-1} \tau(f^j(x)) - t\Big] + t_1$$
$$= \sum_{j=0}^{n_I(x)-1} \tau(f^j(x)) + e_A(x,t) - m([t_1,t_2]) \qquad (2.4)$$

for any $(x,t) \in A$. Given $n \geq 1$ consider the sets $I_n = \{x \in I : n_I(x) = n\}$ and it is clear that $I = \cup I_n$ modulo a zero measure set with respect to μ. For $0 \leq k \leq n-1$ set also $I_{n,k} = f^k(I_n)$. Since all points in $I_{n,k}$ have $n-k$ as first hitting time to A and f is invertible then $f^{-k}(I_{n,k}) = I_n$ and, consequently, by f-invariance of the measure it holds $\mu(I_{n,k}) = \mu(I_n)$. Since μ-almost every x will eventually visit the set $I \subset \Sigma$, μ is ergodic and $I_{n,k}$ is family of disjoint sets then $\mu(\cup_n \cup_{0\leq k\leq n-1} I_{n,k}) = 1$. Hence,

$$\int_\Sigma \tau \, d\mu = \sum_{n\geq 1}\sum_{k=0}^{n-1} \int_{I_{n,k}} \tau \, d\mu = \sum_{n\geq 1}\sum_{k=0}^{n-1} \int_{I_n} \tau \circ f^k \, d\mu$$
$$= \sum_{n\geq 1} \int_{I_n} \sum_{k=0}^{n-1} \tau \circ f^k \, d\mu = \int_I \sum_{k=0}^{n_I(x)-1} \tau \circ f^k \, d\mu. \qquad (2.5)$$

In consequence using equations (2.4) and (2.5) together it follows that

$$\int_A n_A(x,t) \, d\bar\mu = \frac{1}{\int_\Sigma \tau \, d\mu} \int_{t_1}^{t_2} \int_I \Big[\sum_{j=0}^{n_I(x)-1} \tau \circ f^j \Big] \, d\mu \, dt$$
$$+ \int_A e_A(x,t) \, d\bar\mu - m([t_1,t_2]) \, \bar\mu(A)$$

or, in other words,

$$\int_A n_A(x,t) \, d\bar\mu = \int_A e_A(x,t) \, d\bar\mu + m([t_1,t_2]) \big(1 - \bar\mu(A)\big). \qquad (2.6)$$

If $\bar\mu_A$ denotes as before the normalized probability measure $\frac{\bar\mu|_A(\cdot)}{\bar\mu(A)}$ the later becomes

$$\int_A n_A(x,t) \, d\bar\mu_A = \int_A e_A(x,t) \, d\bar\mu_A + \big(1 - \bar\mu(A)\big) \frac{1}{\mu(I)} \int_\Sigma \tau \, d\mu \qquad (2.7)$$

which proves the first equality in the theorem. A simple integral computation shows that $\int_{t_1}^{t_2} (t_2 - t) \, dt = m([t_1,t_2])^2/2$ leading to $\int_A e_A(x,t) \, d\bar\mu = \bar\mu(A) \, \frac{m([t_1,t_2])}{2}$. So $\int_A e_A(x,t) \, d\bar\mu_A = \frac{m([t_1,t_2])}{2}$ and replacing $\bar\mu(A) = \frac{\mu(I)}{\int \tau \, d\mu} \, m([t_1,t_2])$ in equation (2.7) we conclude

$$\int_A n_A(x,t) \, d\bar\mu_A = \frac{m([t_1,t_2])}{2} + \frac{1}{\mu(I)} \int_\Sigma \Big(\tau - (t_2 - t_1)\chi_I \Big) \, d\mu$$

where χ_I denotes the indicator function of the set $I \subset \Sigma$. This proves the second equality in the theorem. In particular, it follows from the previous expression that if the set A is kept unchanged the mean return time varies linearly with $\int_\Sigma \tau \, d\mu$. This finishes the proof of the theorem. $\square$

Some comments are in order. The first one concerns the ergodicity assumption. It is a simple consequence of Birkhoff's ergodic theorem and the integrability of the roof function that $\bar{\mu}$ is ergodic for the flow if and only if μ is ergodic for f. If ergodicity fails it follows from the ergodic decomposition theorem that for μ almost every $x \in \Sigma$ there are ergodic probability measures μ_x on Σ so that $\mu = \int \mu_x \, d\mu$. In such case $\bar{\mu} = \int \bar{\mu}_x d\bar{\mu}$ is the ergodic decompositon for $\bar{\mu}$, where $\bar{\mu}_x$ are the almost everywhere defined ergodic measures $\bar{\mu}_x = (\mu_x \times m) / \int \tau \, d\mu_x$. Using this we recover analogous expression for the mean return time as in Theorem A without the ergodicity assumption.

A second remark is related with the heigh of the cylinders A. Indeed, the role of the constants t_1 and t_2 was not important to guarantee recurrence to the set A since in our setting Σ is a global cross-section and recurrence is obtained simply by assuming the projection I has positive μ-measure. Hence, as a consequence of our strategy we deduce the following recurrence estimates for subsets of the cross-section.

Corollary 1. *Let $f : \Sigma \to \Sigma$ be a measurable invertible map on a topological space Σ, μ be an f-invariant ergodic probability measure and let $\tau \in L^1(\mu)$ be a roof function, and let $(X_t)_t$ be the suspension semiflow over preserving $\bar{\mu}$. Then, for any positive μ-measure set $A \subset \Sigma$,*

$$\int_A n_A(x, 0) \, d\mu_A = \frac{1}{\mu(A)} \int_\Sigma \tau \, d\mu$$

where μ_A of the normalization of the measure $\mu \mid_A$ in the cross-section Σ. In particular, if $f_A : A \to A$ denotes the first return time map then

$$\int_A n_A(x, 0) \, d\mu_A = \frac{h_{\mu_A}(f_A)}{h_{\bar{\mu}}(X_1)}$$

is the entropy of the first return time map quotiented by the entropy of the flow.

Proof. For simplicity reasons we shall denote also by A the set $A \times \{0\} \subset \Sigma_\tau$. it is clear that $e_A(\cdot, \cdot) \equiv 0$. Hence It follows e.g. from equation (2.4) and (2.5) (taking $I = A$) that $n_A(x, 0) = \sum_{j=0}^{n_A(x)-1} \tau(f^j(x))$ and we deduce that $\int_A n_A(x, 0) \, d\mu = \int_\Sigma \tau \, d\mu$, from which expression the result immediately follows.

In addition, on the one hand Abramov [1] proved that the entropy $h_{\mu_A}(f_A)$ of the first return time map f_A with respect to μ_A satisfies $h_\mu(f) = h_{\mu_A}(f_A) \, \mu(A)$. On the other hand, the formula established for the time-t map of the flow with relation with the base map obtained by Abramov [2]

$$h_{\bar{\mu}}(X_t) = \frac{|t| \, h_\mu(f)}{\int_\Sigma \tau \, d\mu}$$

and so

$$\int_A n_A(x, 0) \, d\mu_A = \frac{1}{\mu(A)} \int_\Sigma \tau \, d\mu = \frac{h_{\mu_A}(f_A)}{h_\mu(f)} \, \frac{h_\mu(f)}{h_{\bar{\mu}}(X_1)} = \frac{h_{\mu_A}(f_A)}{h_{\bar{\mu}}(X_1)}.$$

This finishes the proof of the corollary. □

Remark 2.3. Let us mention that the assumption of Theorem 2.2 is satisfied in the case of suspension flows and cylinder sets. In fact, if $A = I \times [t_1, t_2] \subset \Sigma_\tau$ and $s > 0$ is small then $A_s = I \times [t_2 - s, t_2]$ and $\tilde{A}_s = I \times \{t_1\}$. In particular $\bar{\mu}(A_s) \to 0$ as s

tends to zero. Since $A_s \subset A$ then the return time r_A defined in [10] coincides with the definition given in equation 2.3 and it follows that

$$\lim_{s\to 0} \frac{1}{s} \int_{A_s} r_A(z)\, d\bar{\mu} = 1 - \bar{\mu}(A).$$

Now, let us observe that the case of flows cannot be obtained directly from the discrete time setting since it reflects the escaping times and the width of the cylinders as shown by the following immediate consequence of the theorem, in which we also use the mean return times to compute entropy of a measure in the cross section.

Corollary 2.4. *Let $((X_t)_t, \bar{\mu})$ be the suspension semiflow over (f, μ, τ) as in Theorem A and assume that the roof function τ is constant. If $A = I \times [0, \tau] \subset \Sigma \times \mathbb{R}$ is a full cylinder set in Σ_τ with positive $\bar{\mu}$-measure then*

$$\int_A n_A(x,t)\, d\bar{\mu}_A = \frac{1}{\mu(I)} \int_\Sigma \left(\tau - \frac{\tau}{2}\chi_I \right) d\mu = \frac{\tau}{\mu(I)} \left(1 - \frac{\mu(I)}{2} \right).$$

Despite the fact that cylinder sets arise naturally for flows (e.g. from the tubular neighborhood theorem) sometimes we are interested in the return times to geometric objects as balls which can be used to study dimension of measures among other relevant dynamical quantities. For that reason, we will now extend our result for a more general class of sets that include balls. Let $\mathcal{A}$ be the family of closed sets whose boundary are graphs of functions over Σ. More precisely, $A \in \mathcal{A}$ if and only if there are measurable functions $h_1, h_2 : \pi_1(A) \to \mathbb{R}_0^+$ such that $A = \{(x,t) \in \Sigma_\tau : h_1(x) \le t \le h_2(x)\}$, where $\pi_1 : \Sigma_\tau \to \Sigma$ denotes the natural projection on Σ. For simplicity we shall assume $h_1(x) \le h_2(x) \le \tau(x)$ for every $x \in \Sigma$ ands denote such sets by A_{h_1,h_2}. Just as a remark, for the purpose of recurrence properties to these sets the return times for the flow coincide with the ones if one had considered sets of the form $A = \{(x,t) \in \Sigma_\tau : h_1(x) < t < h_2(x)\}$. We can now state our next result.

Theorem B. *Let $(X_t)_t$ be a suspension semiflow associated to (f, Σ, τ) and let $\bar{\mu}$ be the ergodic $(X_t)_t$-invariant probability measure associated to the f-invariant ergodic probability measure μ. Given $A = A_{h_1,h_2} \in \mathcal{A}$ with $\bar{\mu}$-positive measure then*

$$\int_A n_A(x,t)\, d\bar{\mu}_A = \int_A e_A(x,t)\, d\bar{\mu}_A + \frac{\int_I h(x)\, \tau^{n_I}(x)\, d\mu}{\int_I h(x)\, d\mu} + \frac{\int_I h(x)[h_1(f^{n_I(x)}(x)) - h_2(x)]\, d\mu}{\int_I h(x)\, d\mu}$$

where $h(x) = h_2(x) - h_1(x)$, $\tau^{n_I}(x) := \sum_{k=0}^{n_I(x)-1} \tau \circ f^k(x)$ and also

$$\int_A e_A(x,t)\, d\bar{\mu}_A = \frac{\int_I h(x) \left[\frac{h_2(x)-h_1(x)}{2}\right] d\mu}{\int_I h(x)\, d\mu}.$$

Proof. Set $A = A_{h_1,h_2}$ and $I = \pi_1(A)$ for simplicity. Given $(x,t) \in A$ then clearly the escape time is given by $e_A(x,t) = h_2(x) - t$. Moreover, if there exists $s \ge 0$ so that $X_s(x,t) = \left(f^k(x), t + s - \sum_{j=0}^{k-1} \tau(f^j(x))\right) \in A$ then $f^k(x) \in I$ and $k = k(x,t,s) \ge 0$ given as after (2.1). Therefore, similarly to the proof of Theorem A, for $\bar{\mu}$-almost every $(x,t) \in A$ we get

$$n_A(x,t) = \sum_{j=0}^{n_I(x)-1} \tau(f^j(x)) + e_A(x,t) - h_2(x) + h_1(f^{n_I(x)}(x)),$$

we write $I = \cup I_n \pmod{\mu}$ and notice, by ergodicity, the family $I_{n,k} = f^k(I_n)$ pairwise of disjoint sets verifies $\mu(\cup_n \cup_{0\le k\le n-1} I_{n,k}) = 1$. Hence, a simple computation shows that

$$\bar{\mu}(A) = \frac{1}{\int \tau \, d\mu} \int_I h(x) \, d\mu$$

and also

$$\int_A n_A(x,t) \, d\bar{\mu} = \frac{1}{\int \tau \, d\mu} \int_I h(x) \, \tau^{n_I}(x) \, d\mu + \frac{1}{\int \tau \, d\mu} \int_I \int_{h_1(x)}^{h_2(x)} [h_2(x) - t] \, dt \, d\mu$$
$$+ \frac{1}{\int \tau \, d\mu} \int_I \int_{h_1(x)}^{h_2(x)} [h_1(f^{n_I(x)}(x)) - h_2(x)] \, dt \, d\mu$$

and the result follows by simple computations. □

3. Final remarks

'Almost' cylinders. The first remark is that computations are above clearly simpler in the case where A_{h_1,h_2} has 'parallel sides' meaning $h_2(x) = h_1(x) + c$ for some $c > 0$. In this case, using again the notation $I = \pi_1(A)$ and noticing $h(x) = c$ for every $x \in I$ it follows that $\bar{\mu}(A) = c\,\mu(I)/\int \tau \, d\mu$, that $\int_A e_A(x,t) \, d\bar{\mu}_A = \frac{c}{2}$ and, using that the first return time map $f^{n_I} : I \to I$ preserves the (ergodic) normalized probability measure μ_I

$$\begin{aligned}\int_A n_A(x,t) \, d\bar{\mu}_A &= \frac{c}{2} + \frac{1}{\mu(I)} \int_\Sigma \tau \, d\mu + \frac{1}{\mu(I)} \int_I [h_1(f^{n_I(x)}(x)) - h_1(x) - c] \, d\mu \\ &= \frac{c}{2} + \frac{1}{\mu(I)} \int_\Sigma (\tau - c\chi_I) \, d\mu + \int_I [h_1(f^{n_I(x)}(x)) - h_1(x)] \, d\mu_I \\ &= \frac{c}{2} + \frac{1}{\mu(I)} \int_\Sigma (\tau - c\chi_I) \, d\mu\end{aligned}$$

Suspension semiflows. We should mention that our results hold for suspension semiflows $(X_t)_{t\ge 0}$ associated to the suspension for non-invertible maps $f : \Sigma \to \Sigma$. Given an ergodic measure μ, for $I \subset \Sigma$ we can decompose

$$\Sigma = \bigcup_{n\ge 1} [I_n \cup I_n^*] \quad (\mu_\Sigma \bmod 0)$$

where $I_n = I \cap \{n_I(\cdot) = n\}$ and $I_n^* = (\Sigma \setminus I) \cap \{n_I(\cdot) = n\}$, and all elements in the previous union are pairwise disjoint. Moreover, for every $n \ge 1$ we get $\mu(I_n^*) = \mu(f^{-1}(I_n^*)) = \mu(I_{n+1}) + \mu(I_{n+1}^*)$ and consequently $\mu(I_n^*) = \sum_{k\ge n+1} \mu(I_k)$ and so $1 = \mu(\Sigma) = \sum_{k\ge 1} k\mu(I_k) = \int n_I(\cdot) \, d\mu$. Taking $I_{n,j} = f^j(I_n)$ for $0 \le j \le n-1$, since $I_{n,j} \subset I^*_{n-j}$, it follows that

$$\begin{aligned}\mu\Big(\bigcup_{n\geq 1}\bigcup_{j=0}^{n-1} I_{n,j}\Big) &= \mu\Big(\bigcup_{n\geq 1}\bigcup_{j\geq 0} I_{n+j,j}\Big)\\ &= \sum_{n\geq 1}\Big[\mu\big(I_{n,0}\big) + \mu\Big(\bigcup_{j\geq 0} I_{n+j,j}\Big)\Big]\\ &= \sum_{n\geq 1}\big[\mu(I_n) + \mu(I^*_n)\big] = 1.\end{aligned}$$

Thus, as in equation (2.5) and obtain $\int_\Sigma \tau \; d\mu = \int_I \sum_{k=0}^{n_I(x)-1} \tau \circ f^k \; d\mu$. and the same computations as in the proof of Theorems A and B follow straightforward.

Acknowledgements. This work was supported by a CNPq-Brazil postdoctoral fellowship at University of Porto. The author is grateful to M. Bessa, F. Rodrigues and J. Rousseau for some comments on the manuscript. The author is deeply grateful to J.-R. Chazottes for his comments and for providing the reference [10]. The author is also grateful to the organizers of the event "Probability in Dynamics at UFRJ" where part of this work was developed.

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